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ON THE CATEGORICITY THEOREM IN $L_{\omega_1\omega}$

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Let T be a countable theory in $L_{\omega_1\omega}$. For each infinite cardinal κ we denote by $I(\kappa, T)$ the number of pairwise non-isomorphic models of T in κ. In this paper we shall prove the following theorem:

THEOREM 1. If $I(\omega_1, T)=1$ and the models of T in ω_1 are $L_{\omega_1\omega}$-homogeneous (for the definition see [1]) then $I(\kappa, T)=1$ for all $\kappa>\omega$.

At first sight it may seem that the theorem is just a special case of Corollary 1 to Theorem 32 in [1]. However the κ-categoricity of T is defined there not to be $I(\kappa, T)=1$ but $I(\kappa, T)\leqq 1$. So the conclusion of our theorem is stronger than that of the corollary for elementary classes of $L_{\omega_1\omega}$. Unlike in $L_{\omega\omega}$ theories, the $L_{\omega_1\omega}$-homogeneity of the models of T in ω_1 does not simply follow from the ω_1-categoricity: as proved in [5], there is a countable theory in $L_{\omega_1\omega}$ which is ω_1-categorical but whose models in ω_1 are not $L_{\omega_1\omega}$-homogeneous. Nevertheless, as far as I know, it seems to be still an open question, whether Theorem 1 holds without the assumption of homogeneity of the models. With a similar proof to that of Theorem 1 we can also get the following stronger version:

THEOREM 2. Let $(K, <)$ be an $(\omega, L_{\omega_1\omega})$-good class of structures (for the definition see [2] or [3]). If $I(\omega_1, K)=1$ and the models of T in ω_1 are $L_{\omega_1\omega}$-homogeneous, then $I(\kappa, K)=1$ for all $\kappa>\omega$.

Without homogeneity of the models in ω_1 Theorem 2 does not hold: as S. Shelah showed, under MA+$\neg$CH there is an $(\omega, L_{\omega_1\omega})-$good class of structures, which is κ categorical for all $\kappa<2^\omega$ but contains no structure with cardinality $>2^\omega$ (see [4]). Clearly "$(\omega, L_{\omega_1\omega})$-good class" in Theorem 2 can not be replaced by "PC class in $L_{\omega_1\omega}$": simply consider and $L_{\omega_1\omega}$-theory T' with $I(\omega_1, T')\neq 0$ and $I(\omega_2, T')=0$ and let $K=\{M\upharpoonright L_0 | M\models T'\}$ for the empty language L_0. The notations we use here is standard and/or to be found e.g. in [1], [2] or [3].

Let T be as in Theorem 1. As in [6] we may assume that $I(\omega, T)=1$ and

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there is a countable fragment L* of $L_{\omega_1\omega}$ containing T such that for every $M, N \models T$ and for every $\bar{a}\in|M|$, $\bar{b}\in|N|$, $(M, \bar{a})\equiv_{L_{\omega_1\omega}}(N, \bar{b})$ if and only if $(M, \bar{a})\equiv_{L_{\omega_1\omega}}(N, \bar{b})$. In particular the models of T in ω_1 are L*-homogeneous. By Theorem 32 in [1] and its corollary it follows that $I(\kappa, T)\leqq 1$ for all $\kappa>\omega$ and that all uncountable models of T are L*-homogeneous. A model M is said to be L*-*locally-universal*, if for every N, N' with $N\prec_{L^*}M$, $N\prec_{L^*}N'$, $\|N\|<\|M\|$ and $\|N'\|\leq\|M\|$ there is an L*-elementary embedding of N' in M over N.

LEMMA 3. Every uncountable model of T is L*-locally-universal.

PROOF. Let M an uncountable model of T. Let N and N' be as in the definition of L*-locally-universality. Since $I(\kappa, T)\leq 1$ for all $\kappa\geq\omega$, we may assume $\|N'\|=\|M\|$ and $N'\cong M$. So by L*-homogeneity of M we can extend the L*-elementary mapping id_N to an isomorphism from N' to M by a back-and-forth construction. □

Let (*) be the following property on M:

(*) For every $N\prec_{L^*}M$ such that $\|N\|<\|M\|$ and for every $a\in|M|\setminus|N|$, there are $\|M\|$-many elements of $|M|\setminus|N|$, which satisfy the L*-type of a over $|N|$.

LEMMA 4. Every model of T in ω_1 satisfies (*).

PROOF. Let M be a model of T in ω_1 and let N be a countable L*-elementary submodel of M with $a\in|M|\setminus|N|$. Since the models of T in ω_1 are L*-homogeneous, the theory

$$T'=\{\varphi(a_1, \cdots, a_n)|\varphi\in L^*, n\in\omega, a_1, \cdots, a_n\in|N|, M\models\varphi[a_1, \cdots, a_n]\}$$

is ω_1-categorical. So by Theorem 45 in [1] there are only countably many L*-types over $|N|$ realized in M. Let N' be a countable model such that $N\prec_{L^*}N'\prec_{L^*}M$ and for every $c\in|M|\setminus|N'|$, there are ω_1-many elements of $|M|\setminus|N|$, which satisfy the L*-type of c over $|N|$. By Satz 9. 8 in 2 (which is a slight modification of Lemma 3. 2. 8 in [4]) there is a countable $N_1\models T$ such that $N\prec_{L^*}N_1$ and for every $c\in|N_1|\setminus|N|$ there are $N_2, N_3\models T$ with $N\prec_{L^*}N_2\prec_{L^*}N_3\prec_{L^*}N_1$, $c\notin|N_3|$ and $(N_3, |N_2|)\cong(N', |N|)$. Since M is L^*-locally-universal, we can assume $N_1<_{L^*}M$. It follows that there are N'', $N'''\models T$ such that $N\prec_{L^*}N''\prec_{L^*}N'''\prec_{L^*}M$, $a\notin|N'''|$ and $(N''', |N''|)\cong(N', |N|)$. Again by the L*-locally-universality of M, $(M, |N'''|, |N''|)\cong(M, |N'|, |N|)$. So by the definition of N', there are ω_1-many elements of $|M|\setminus|N''|$, which satisfy the L*-type of a over $|N''|\supseteq|N|$. □

LEMMA 5. If an uncountable model M of T is L*-homogeneous and satisfies (*), then it satisfies:

(**) For every $N, N' \models T$ with $N \prec_{L^*} N' \prec_{L^*} M$ and $\|N'\| < \|M\|$ and for every $a \in |M| \setminus |N|$, there is $b \in |M| \setminus |N'|$ and an automorphism $f: M \to M$ over N such that $f(b)=a$.

PROOF. By (*) and $\|N'\| < \|M\|$, there exists $b \in |M| \setminus |N'|$ which satisfies the L^*-type of a over N. By L*-homogeneity of M, the L*-elementary mapping $\{(c, c) | c \in |N|\} \cup \{(b, a)\}$ can be extended to an automorphism of M by a back-and-forth construction. □

LEMMA 6. If an nucountable model M of T satisfies (**), then there is $N \models T$ such that $M \prec_{L^*} N$, $M \neq N$ and $M \cong N$.

PROOF. Let $\kappa = |M|$. Since $I(\kappa, T)=1$ by Corollary 1 to Theorem 32 in [1], we only need to show that there is $M' \prec_{L^*} M$ such that $M' \neq M$ and $\|M'\| = \kappa$. Let $M'_0 \prec_{L^*} M$ be such that $\|M'_0\| < \kappa$ and $a \in |M| \setminus |M'|$. By (**) we can construct an L^*-elementary chain of L*-elementary submodels M'_α, $\alpha < \kappa$ of M such that $M'_\alpha \subsetneqq M'_\beta$ for $\alpha < \beta < \kappa$ with $|M'_\alpha| < \kappa$ and $a \notin |M'_\alpha|$ for $\alpha < \kappa$. Let $M' = \bigcup_{\alpha<\kappa} M'_\alpha$. □

LEMMA 7. Every uncountable model of T satisfies (*).

PROOF. Assume, by way of contradiction, that there are uncountable models of T, which don't satisfy (*). Let M be such a model with the smallest $\kappa = \|M\|$. Clearly κ is then a successor cardinal. Let $N \prec_{L^*} M$ and $a \in |M| \setminus |N|$ such that $\|N\| < \|M\|$ and there are only less than κ-many elements of $|M| \setminus |N|$, which satisfy the L*-type of a over $|N|$. Let $<^{M^*}$ be a well-ordering on $|M|$ of type κ such that $|N|$ is an initial segment with respect to it. Let $M^* = (M, I^{M^*}, a^{M^*}, <^{M^*})$, where $I^{M^*} = |N|$ and $a^{M^*} = a$. Let T^* be the countable $L_{\omega_1\omega}$-theory in $S^* = L(M^*)$ consisting of:

(a) φ, for $\varphi \in T$
(b) "$<$ is a linear ordering"
(c) "I is an initial segment with respect to $<$"
(d) $\forall x_i \cdots \forall x_n (I(x_i) \wedge \cdots \wedge I(x_n) \to (\varphi(x_i, \cdots, x_n) \leftrightarrow \varphi^I(x_i, \cdots, x_n)))$, for $\varphi \in L^*$
(e) $\neg I(a)$
(f) $\exists x \forall y ([\bigwedge_{n\in\omega} \bigwedge_{\varphi(y,x_0,\cdots,x_n)\in L^*} \forall x_0 \cdots \forall x_n I(x_0) \wedge \cdots \wedge I(x_n)$
$\to (\varphi(y, x_0, \cdots, x_n) \leftrightarrow \varphi(a, x_0, \cdots,)))] \to y < x)$

Clearly $M^* \models T.^*$ Since κ is a regular cardinal, by Theorem 28 in [1] there is a model N^* of T^*, which is ω_1-like ordered with respect to $<^{N^*}$. Let $N=N^*\restriction L(T)$ and $N'=N^*\restriction I^{N^*}\restriction L(T)$. Then $\|N\|=\omega_1$, $N\models T$, $\|N'\|=\omega$, $N'\prec_{L^*}N$, $a^{N^*}\in|N|\setminus|N'$ and there are only countaly many elements of N', which satisfy the L*-type of a^{N^*} over N'. This contradicts Lemma 4. □

Now let us prove Theorem 1. By Corollary 1 to Theorem 32 in [1] we only need to check the existence of models of T in each uncountable κ. Assuming that there are models of T in μ for all $\omega_1\leq\mu<\kappa$, we will show that there are also models of T in κ. If κ is a limit cardinal, we can easily construct a strictly increasing L*-elementary chain $(M_\alpha)_{\alpha<\kappa}$ of models of T with $\sup\{\|M\|\,|\alpha<\kappa\}=\kappa$. The union of these models is then a model of T in κ. Suppose κ is a successor cardinal, say $\kappa=\lambda^+$. Let M be a model of T in λ. By Lemmas 3, 5 and 7 there is a proper L*-elementary extension of M. So we can again construct a strictly increasing L*-elementary chain of models of T, whose union is model of T in κ.

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