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ON q-PSEUDOCONVEX OPEN SETS IN A COMPLEX SPACE

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In a series of (perhaps not widely known) papers T. Kiyosawa ([1], [2], [3], [4], [5]) introduced and developed the notion of Levi q-convexity. Here we show how to use this notion to improve one of his results ([2] Th. 2) (for a different extension, see [7]). To state and prove our results, we recall few definitions.

Let M be a complex manifold of dimension n; a real C^2 function u on M is said to be q-convex at a point P of M if the hermitian from $L(u)(P)=\sum_{i,j}\left(\frac{\partial^2 u}{\partial z_i\partial\bar{z}_j}\right)\times(P)a_i\bar{a}_j$, $z_1, \cdots, z_n$ local coordinates around P, has at least $n-q+1$ strictly positive eigenvalues; we say that u is Levi q-convex at P if either $(du)_P=0$ and u is q-convex at P or $(du)_P\neq 0$ and the restriction of $L(u)(P)$ to the hyperplane $\left\{\sum_i\left(\frac{\partial u}{\partial z_i}\right)(P)a_i=0\right\}$ has at least $n-q$ strictly positive eigenvalues. Let X be a complex space, $A\in X$, and $f: X\rightarrow \boldsymbol{R}$ a C^2 function; we say that f is q-convex (or Levi q-convex) at A if there is a neighborhood V of A in X, a closed embbedding $p: V\rightarrow U$ with U open subset of an euclidean space, a C^2 function u on U such that $f|V=u\circ p$ and u is q-convex (or respectively Levi q-convex) at $P=p(A)$. It is well-known that a q convex function is Levi q convex and that both notions do not depend upon the choice of charts and local coordinates; for any fixed choice of charts and local coordinates we will call $L(u)(P)$ the Levi form of u at P and of f at A.

An open subset D of a complex space X is said to have regular Levi q-convex boundary if we can take a covering $\{V_i\}$ of a neighbourhood of the boundary bD of D with closed embeddings $p_i: V_i\rightarrow U_i$, U_i open in an euclidean space and C^2 functions f_i on U_i with $V_i\cap D=\{x\in V: f_i\circ p_i(x)<0\}$ and such that if $x\in V_i\cap V_j$, there is a neighborhood A of x in $V_i\cap V_j$ such that on $A(f_i\circ p_i)|A=f_{ij}(f_j\circ p_j)|A$ with $f_{ij}>0$, $f_{ij}\in C^2$ on A. The last condition is always satisfied for a domain D defined locally by Levi q-convex functions s_i if the set of points of bD at which either ds_i vanishes or X is singular is discrete.

A complex space X is called q-complete if it has a C^2 q-convex exhausting function f; if f is both q-convex and weakly plurisubharmonic, X is called very

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strongly q-convex (in the sense of T. Ohsawa [6]).

Now we can state our results.

THEOREM. *Let D be a regular Levi q-convex open subset of a complex space X. Then there exist a neighbourhood V of the boundary bD and a q-convex real function t such that $D\cap V=\{x\in V: t(x)<0\}$.*

COROLLARY. *Let X be a very strongly q-convex space and D an open subset of X with regular Levi q-convex boundary. Then D is q-complete.*

Compare the corollary with the main result in [7].

PROOF of the theorem. Note that the proof of [2] Theorem 2 goes on verbatim even if D is not relatively compact in X. The quoted result gives a neighbourhood W of bD and a Levi q-convex function g in W such that $D\cap W=\{x\in W: g(x)<0\}$. Consider a strictly positive real function v on W. Set $t=ge^{vg}$. Since g vanishes on bD, the Levi form of t at a point y in bD is propertional to the Levi form at y of e^{cg}, with $c=g(y)$. Hence if $g(y)$ is sufficiently high, t is q-convex at $y\in bD$ ([3] Prop. 2 or [5] Lemma 2); how big must be $g(y)$ depend only from the eigenvalues of the Levi form of g at y; hence the same costant works also in a neighbourhood of y. Let $\{V_n\}$, $\{U_n\}$ be locally finite coverings of W with V_n relatively compact in U_n, $\{U_n\}$ fine enough (in particular with local charts on which g may be find constants $c_n>0$ such that if $u<c_n$ on V_n, $t=ge^{ug}$ is q-convex at every point of bD, hence in a neighbourhood V of bD. Q. E. D.

PROOF of the corollary. By the theorem we may find an open neighbourhood V of bD and a real C^2 q-convex function f on V such that $V\cap D=\{x\in V: f(x)<0\}$. Let W be an open neighbourhood of bD with closure contained in V. Note that the function $s:=-f^{-1}$ is q-convex on $V\cap D$ and goes to infinity near bD. Let u be a real non-negative C^2 function on U with support contaiand in $V\cap D$, $u=1$ in $W\cap D$. We may consider us as a function on D setting $(us)(x)=0$ if $x\notin V$. Take an exhaustive, positive, q-convex function h on X. Take an increasing sequence $\{K_n\}$ or compact subset of X, with union X and a sequence $\{c_n\}$ of strictly positive real numbers. Take a C^2 function b: $\boldsymbol{R}\rightarrow\boldsymbol{R}$ with $b(t)=0$ for $t\leqslant -1$, $b(t)\geqslant c_j$ for $j\leqslant t\leqslant j+1$ and $b'(t)>0$ for $t>-1$. Set

$$g(t)=\int_{-\infty}^{t} b(x)dx$$

and set $F=g\circ h$. For every $P\in X$ and any choice of local coordinates, we have $L(F)(P)\geqslant b(h(P))L(g)(P)$. Hence we may choose the constants c_j with $c\geqslant j$ and such that $F+s$ is q-convex on $(D\backslash W)\cap K_j$ for every j. Since F is plurisubharmonic, $F+s$ is q-convex on D. If $\{x_n\}$ is a sequence in D without accumulation points in X, then $\{F(x_n)\}$ and $\{F(x_n)+s(x_n)\}$ are unbounded on $\{x_n\}$. The function s is unbounded on every sequence of points in D converging to a point in bD, hence $F+s$ is an exhaustion function on D. Q. E. D.

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