Topological Methods in Nonlinear Analysis
Volume 54, No. 2A, 2019, 701–714
DOI: 10.12775/TMNA.2019.065

SUBSPACES OF INTERVAL MAPS RELATED TO THE TOPOLOGICAL ENTROPY

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Abstract. For $a \in [0, +\infty)$, the function space $E_{\geq a}$ ($E_{>a}$; $E_{\leq a}$; $E_{<a}$) of all continuous maps from $[0, 1]$ to itself whose topological entropies are larger than or equal to a (larger than a; smaller than or equal to a; smaller than a) with the supremum metric is investigated. It is shown that the spaces $E_{\geq a}$ and $E_{>a}$ are homeomorphic to the Hilbert space l_2 and the spaces $E_{\leq a}$ and $E_{<a}$ are contractible. Moreover, the subspaces of $E_{\leq a}$ and $E_{<a}$ consisting of all piecewise monotone maps are homotopy dense in them, respectively.

1. Introduction

One of the central topics in the study of infinite-dimensional topology is the problem which function spaces are homeomorphic to the separable infinite dimensional Hilbert space l_2 or its well-behaved subspaces. The well-known Anderson–Kadec's theorem states that the countable infinite product $\mathbb{R}^{\mathbb{N}}$ of lines is homeomorphic to l_2, see [1], [10]. Using this result, it was proved that the space of real valued maps of an infinite compact metric space with the supremum metric is homeomorphic to l_2. See [4], [14], [15] for more on this topic. Moreover, in [6], the authors proved that the function space of real valued maps of an

2010 *Mathematics Subject Classification.* Primary: 37E05, 54F65; Secondary: 54H20.
Key words and phrases. Interval maps; topological entropy; the Hilbert space l_2; homotopy dense; contractible.
X. Fan and Z. Yang was supported in part by the NNSF of China (11471202, 11971287).
J. Li and Y. Yang was supported in part by the NNSF of China (1771264, 11871188) and NSF of Guangdong Province (2018B030306024).

infinite countable metric space with the topology of pointwise convergence is homeomorphic to the subspace $c_0 = \Big\{(x_n) \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} x_n = 0\Big\}$ of $\mathbb{R}^{\mathbb{N}}$. In a series of papers, the fourth named author of the present paper and his coauthors gave a condition for the space of continuous functions from a k-space to $\mathrm{I} = [0,1]$ with the Fell hypergraph topology to be homeomorphic to the space c_0, see [16]–[19].

In the study of dynamical systems, some function spaces appear naturally. The group of measure preserving transformations of the unit interval equipped with the weak topology is homeomorphic to l_2 (see [5] and [13]). Recently, in [11] Kolyada et al. proposed the study of dynamical topology: investigating the topological properties of spaces of maps that can be described in dynamical terms. They showed in [11] that the space of transitive interval maps is contractible and uniformly locally arcwise connected, see also [12] for more detailed results. In [8], Grinc et al. discussed some topological properties of subspaces of interval maps related to the periods of periodic points.

In this paper, we will follow the idea in [11] to study subspaces of interval maps related to the topological entropy. Let $\mathrm{I} = [0,1]$ and $C(\mathrm{I})$ be the collection of continuous maps on I with the supremum metric d. For each $f \in C(\mathrm{I})$, denote by $h_{\mathrm{top}}(f)$ the topological entropy of f. For any $a \in [0,+\infty]$, let

$$E_{\geq a} = \{f \in C(\mathrm{I}) : h_{\mathrm{top}}(f) \geq a\}; \qquad E_{>a} = \{f \in C(\mathrm{I}) : h_{\mathrm{top}}(f) > a\};$$
$$E_{\leq a} = \{f \in C(\mathrm{I}) : h_{\mathrm{top}}(f) \leq a\}; \qquad E_{<a} = \{f \in C(\mathrm{I}) : h_{\mathrm{top}}(f) < a\}.$$

A map $f \in C(\mathrm{I})$ is said to be *piecewise monotone* if there exist $0 = t_0 < t_1 < \ldots < t_n = 1$ such that $f|_{[t_{i-1},t_i]}$ is monotone for every $i = 1,\ldots,n$. Similarly, we can define a map to be *piecewise linear*. We use $C^{\mathrm{PM}}(\mathrm{I})$ to denote the set of all piecewise monotone continuous maps on I and

$$E^{\mathrm{PM}}_{\leq a} = E_{\leq a} \cap C^{\mathrm{PM}}(\mathrm{I}).$$

The main results of this paper are as follows:

Theorem 1.1. *For every $a \in [0,+\infty)$, both $E_{\geq a}$ and $E_{>a}$ are homeomorphic to l_2.*

Theorem 1.2. *There exists a homotopy $H : C(\mathrm{I}) \times \mathrm{I} \to C(\mathrm{I})$ such that*

(a) $H_0 = \mathrm{id}_{C(\mathrm{I})}$;
(b) $h_{\mathrm{top}}(H_t(f)) \leq h_{\mathrm{top}}(f)$ *and* $H_t(f) \in C^{\mathrm{PM}}(\mathrm{I})$ *for any* $t \in (0,1)$ *and* $f \in C(\mathrm{I})$;
(c) $H_1(f) \equiv 0$ *for any* $f \in C(\mathrm{I})$.

Restricting the homotopy in Theorem 1.2 to $E_{\leq a}$ and $E_{<a}$, respectively, we can obtain the following corollary:

Corollary 1.3. *For every* $a \in [0, +\infty]$, $E_{\leq a}$ ($E_{<a}$, *respectively*) *is contractible and* $E^{\mathrm{PM}}_{\leq a}$ ($E^{\mathrm{PM}}_{<a}$, *respectively*) *is homotopy dense in* $E_{\leq a}$ ($E_{<a}$, *respectively*).

The paper is organized as follows. In Section 2, we recall some basic notions which we will use in the paper. Theorems 1.1 and 1.2 are proved in Sections 3 and 4, respectively.

2. Preliminaries

In this section, we recall some notions and aspects of infinite-dimensional topology and topological entropy which will be used later.

2.1. Infinite-dimensional topology. In this subsection, we give some concepts and facts on general topology and infinite-dimensional topology. For more information, we refer the reader to [7], [4], [14], [15].

Let (X, d) be a metric space. We say that

- X is *nowhere locally compact* if no non-empty open set in X is locally compact;
- X is an *absolute* (*neighbourhood*) *retract* (A(N)R, briefly) if for every metric space Y which contains X as a closed subspace, there exists a continuous map $r\colon Y \to X$ ($r : U \to X$ from a neighbourhood U of X) such that $r|_X = \mathrm{id}$;
- X has the *strong discrete approximation property* (SDAP, briefly) if for every continuous map $\varepsilon\colon X \to (0, 1)$, every compact metric space K and every continuous map $f\colon K \times \mathbb{N} \to X$, there exists a continuous map $g\colon K \times \mathbb{N} \to X$ such that $\{g(K \times \{n\}) : n \in \mathbb{N}\}$ is discrete in X and $d(f(k, n), g(k, n)) < \varepsilon(f(k, n))$ for every $(k, n) \in K \times \mathbb{N}$.

A *homotopy* on X is a continuous map $H\colon X \times \mathrm{I} \to X$, $(x, t) \mapsto H_t(x)$. The space X is said to be *contractible* if there exists a homotopy $H\colon X \times \mathrm{I} \to X$ such that $H_0 = \mathrm{id}_X$ and H_1 is a constant map. A subset A of X is called **homotopy dense** if there exists a homotopy $H\colon X \times \mathrm{I} \to X$ such that $H_0 = \mathrm{id}_X$ and $H_t(x) \in A$ for every $x \in X$ and $t \in (0, 1]$.

We will need the following important results in infinite-dimensional topology.

Proposition 2.1 ([14, Theorem 5.2.15]). *A metric space is an* AR *if and only if it is a contractible* ANR.

Theorem 2.2 ([2, 1.2.1 Proposition and Exercise 1.3.4]). *Let Y be a homotopy dense subspace of X. If X is an* ANR (*with* SDAP) *then Y is also an* ANR (*with* SDAP).

Theorem 2.3 ([2, 1.1.14 (Characterization Theorem)]). *A separable topologically complete metric space is homeomorphic to l_2 if and only if it is an* AR *with* SDAP.

Theorem 2.4 ([2, 5.5.2 Corollary]). *A convex subspace X of a separable Banach space is homeomorphic to l_2 if and only if X is topologically complete and nowhere locally compact.*

The following result must be "folklore", but we can not find a proper reference and therefore we provide a proof for the completeness.

Proposition 2.5. *The function space $C(\mathrm{I})$ is homeomorphic to l_2.*

Proof. Let $C(\mathrm{I},\mathbb{R})$ be the collection of all continuous maps from I to $\mathbb{R}$ with the standard linear structure and the supremum norm. Then $C(\mathrm{I},\mathbb{R})$ is a separable Banach space. The space $C(\mathrm{I})$ is a closed and convex subspace of $C(\mathrm{I},\mathbb{R})$. It is not hard to verify that $C(\mathrm{I})$ is nowhere locally compact. It follows from Theorem 2.4 that $C(\mathrm{I})$ is homeomorphic to l_2. □

Combining the above results, we have the following useful criterion when a subspace of $C(\mathrm{I})$ is homeomorphic to l_2.

Corollary 2.6. *A homotopy dense subspace A of $C(\mathrm{I})$ is homeomorphic to l_2 if and only if it is topologically complete and contractible.*

Proof. The necessity is clear and we only need to prove the sufficiency. By Proposition 2.5, $C(\mathrm{I})$ is homeomorphic to l_2. So by Theorem 2.3, $C(\mathrm{I})$ is an ANR with SDAP. Since A is homotopy dense in $C(\mathrm{I})$, it follows from Theorem 2.2 that A is also an ANR with SDAP. By the assumption we have A is contractible, then by Proposition 2.1, A is an AR. Finally by Theorem 2.3 again, A is homeomorphic to l_2. □

2.2. Topological entropy. Let X be a compact metric space. Denote by $\mathrm{Cov}(X)$ the family of all open covers of X. For $\alpha,\beta\in\mathrm{Cov}(X)$ and $f\in C(X)$, let

$$N(\alpha)=\min\bigg\{n\in\mathbb{N}:\text{there exist } U_1,\ldots,U_n\in\alpha \text{ such that } \bigcup_{i=1}^{n}U_i=X\bigg\};$$

$$\alpha\vee\beta=\{U\cap V:U\in\alpha,V\in\beta\},\ f^{-1}(\alpha)=\{f^{-1}(U):U\in\alpha\}$$

and

$$h_{\mathrm{top}}(f,\alpha)=\lim_{n\to\infty}\frac{\log N(\alpha\vee f^{-1}(\alpha)\vee\ldots\vee f^{-n+1}(\alpha))}{n}.$$

The *topological entropy* of a continuous map $f\colon X\to X$ is defined as

$$h_{\mathrm{top}}(f)=\sup\big\{h_{\mathrm{top}}(f,\alpha):\alpha\in\mathrm{Cov}(X)\big\}.$$

Let $f\in C(\mathrm{I})$. A family $\{J_1,\ldots,J_n\}$ of non-degenerate closed intervals is called an *n-horseshoe* if

(1) $\mathrm{int}(J_i) \cap \mathrm{int}(J_j) = \emptyset$ for all $1 \le i < j \le n$, where $\mathrm{int}(J_i)$ is the interior of J_i in I;
(2) $J_i \subset f(J_j)$ for all $1 \le i, j \le n$.

The following result can be easily obtained, see e.g. [3, Proposition VIII.8].

LEMMA 2.7. *If $f \in C(\mathrm{I})$ has an n-horseshoe, then $h(f) \ge \log n$.*

The following result was first proved by Misiurewicz, see e.g. [3, Proposition VIII.30].

THEOREM 2.8. *The entropy function $h_{\mathrm{top}}\colon C(\mathrm{I}) \to [0, +\infty]$, $f \mapsto h_{\mathrm{top}}(f)$ is lower-semicontinuous.*

COROLLARY 2.9. *For every $a \in [0, +\infty)$, $E_{>a}$ is open and $E_{\ge a}$ is a G_δ-set in $C(\mathrm{I})$.*

The convexity of $C(\mathrm{I})$ in the Banach space $C(\mathrm{I}, \mathbb{R})$ plays a key role in the proof of Proposition 2.5. The following examples show that neither $E_{\le a}$ nor $E_{>a}$ is convex in $C(\mathrm{I}, \mathbb{R})$.

EXAMPLE 2.10. Note that, for every $f \in C(\mathrm{I})$, if

$$f\left(\frac{1}{2} - x\right) = f\left(\frac{1}{2} + x\right) \quad \text{for all } x \in \left[0, \frac{1}{2}\right],$$

then f and $1 - f$ are topologically conjugate and thus $h_{\mathrm{top}}(1 - f) = h_{\mathrm{top}}(f)$. But

$$h_{\mathrm{top}}\left(\frac{1}{2} f + \frac{1}{2}(1 - f)\right) = h_{\mathrm{top}}\left(\frac{1}{2}\right) = 0.$$

It follows that $E_{>a}$ is not convex for any $a \in [0, +\infty)$.

EXAMPLE 2.11. It is well-known that, for every $f \in C(\mathrm{I})$, $h_{\mathrm{top}}(f) = 0$ if and only if all periods of f are of the form 2^n (see e.g. Proposition VIII.34 and Theorem II.14 in [3]). Let f and g be the broken line maps through the points $(0, 1)$, $(1/4, 0)$, $(1, 0)$ and the points $(0, 1/2)$, $(1/4, 0)$, $(1/2, 0)$, $(3/4, 1/2)$, $(1, 1/2)$, respectively. Then it is not hard to verify that n is a period for f or g if and only if $n = 1$ or 2. It follows that $h_{\mathrm{top}}(f) = h_{\mathrm{top}}(g) = 0$. For the convex combination $\varphi = f/2 + g/$, we have $\varphi(0) = 3/4$, $\varphi(3/4) = 1/4$ and $\varphi(1/4) = 0$. It follows that 0 is a periodic point with period 3 for φ, which implies $h_{\mathrm{top}}(\varphi) > 0$. This shows that $E_{\le 0}$ is not convex.

3. Proof Theorem 1.1

In this section, we will prove Theorem 1.1. At first, we introduce the box maps defined in [11]. Define a subset Λ of $\mathbb{R}^5$ as follows

$$\Lambda = \left\{(a_l, a_r, a_b, a_t, a_s) \in \mathbb{R}^5 : a_b < a_t, a_l, a_r \in [a_b, a_t],\ a_s \ge 20\right\}.$$

For every non-degenerate closed interval $K = [a_0, a_1]$ and $\lambda = (a_l, a_r, a_b, a_t, a_s) \in \Lambda$, the authors in [11] defined a continuous surjection $\xi_\lambda\colon K \to [a_b, a_t]$, which was called a *box map*, such that ξ_λ is piecewise linear with constant slope $a_s(a_t - a_b)/(a_1 - a_0)$, $\xi_\lambda(a_0) = a_l$ and $\xi_\lambda(a_1) = a_r$. We make this construction both from left and right, ξ_λ is increasing on the leftmost lap unless $a_l = a_t$ and decreasing on the rightmost one unless $a_r = a_t$. We choose the *meeting point* m to be on the fifth decreasing lap from the left (see Figure 1 for example). If the left and right graphs coincide, then there is no well-defined meeting point, but the graph of ξ_λ is clear.

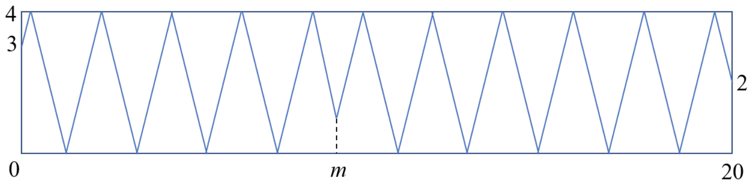

Figure 1. $K = [0, 20]$, $a_l = 3$, $a_r = 2$, $a_b = 0$, $a_t = 4$, $a_s = 20$.

Remark 3.1. Let ξ_λ be a box map on K. If $a_b = a_0$ and $a_t = a_1$, then there exist closed subintervals $J_1, \ldots, J_{[a_s-4]}$ of K with disjoint interiors such that $f(J_j) = K$ for $j = 1, \ldots, [a_s - 4]$, where $[x]$ is the greatest integer less than or equal to x. Hence, $J_1, \ldots, J_{[a_s-4]}$ form an $[a_s - 4]$-horseshoe of ξ_λ. By Lemma 2.7, $h_{\rm top}(\xi_\lambda) \geq \log([a_s - 4])$.

Following the idea in [11], for every $\alpha \geq 20$ we first construct a homotopy $\widetilde{H}^\alpha\colon C(\mathrm{I}) \times \mathrm{I} \to C(\mathrm{I})$ as follows. Fix a function $f \in C(\mathrm{I})$. First let $\widetilde{H}^\alpha_0(f) = f$. For $t \in (0, 1]$, let s be the largest non-negative integer such that $st < 1$. We obtain $s+1$ closed intervals:

$$I_i = [(i-1)t, it], \quad i = 1, \ldots, s, \ I_{s+1} = [st, 1].$$

In particular, if $t = 1$, then $s = 0$ and we have only one closed interval $I_1 = [0, 1]$. For $i = 1, \ldots, s+1$, let $\alpha_i = \max\{|I_i|, |f(I_i)|\}$, where $|J|$ is the length of a closed interval J, and

$$a^i_b = \max\{0, \min f(I_i) - 4\alpha_i\}; \qquad a^i_t = \min\{1, \max f(I_i) + 4\alpha_i\};$$
$$a^i_l = f(\min I_i); \qquad a^i_r = f(\max I_i).$$

It is not hard to verify that if $I_i \cap f(I_i) \neq \emptyset$ then

$$I_i \subset [a^i_b, a^i_t]. \tag{3.1}$$

It is clear that $\lambda^\alpha_i = (a^i_l, a^i_r, a^i_b, a^i_t, \alpha) \in \Lambda$ and then we define $\widetilde{H}^\alpha_t(f)$ on I_i as the box map $\xi_{\lambda^\alpha_i} \in C(I_i, \mathrm{I})$. So $H^\alpha_t(f)$ is well-defined for $t \in (0, 1]$. By Lemma 2.2 of [11], $\widetilde{H}^\alpha\colon C(\mathrm{I}) \times \mathrm{I} \to C(\mathrm{I})$ is a homotopy. Note that $\widetilde{H}^\alpha_0 = \mathrm{id}_{C(\mathrm{I})}$ and for every

$f \in C(\mathrm{I})$, $\widetilde{H}_1^\alpha(f)$ is the box map on I with the parameter $(f(0), f(1), 0, 1, \alpha)$. Now we construct another homotopy $\widehat{H}^\alpha\colon \widetilde{H}_1^\alpha(C(\mathrm{I})) \times \mathrm{I} \to C(\mathrm{I})$. For every $f \in C(\mathrm{I})$ and $t \in [0,1]$ we define $\widehat{H}_t^\alpha(\widetilde{H}_1^\alpha(f))$ to be the box map on I with the parameter $((1-t)f(0), (1-t)f(1), 0, 1, \alpha)$. By Lemma 2.1 of [11], $\widehat{H}^\alpha$ is continuous then it is a homotopy. It should be noticed that for every $f \in C(\mathrm{I})$, $\widehat{H}_1^\alpha(\widetilde{H}_1^\alpha(f))$ is the box map on I with the parameter $(0,0,0,1,\alpha)$. Finally, we define a homotopy $H^\alpha\colon C(\mathrm{I}) \times \mathrm{I} \to C(\mathrm{I})$ by joining $\widehat{H}^\alpha$ and $\widetilde{H}^\alpha$, that is, for every $f \in C(\mathrm{I})$, $H_t^\alpha(f) = \widetilde{H}_{2t}^\alpha(f)$ for $t \in [0, 1/2]$ and $H_t^\alpha(f) = \widehat{H}_{2(t-1/2)}^\alpha(\widetilde{H}_1^\alpha(f))$ for $t \in (1/2, 1]$.

We have the following estimate of the topological entropy of $H_t^\alpha(f)$.

Lemma 3.2. *For every $t \in (0,1]$, $\alpha \geq 20$ and $f \in C(\mathrm{I})$, we have*

$$h_{\mathrm{top}}(H_t^\alpha(f)) \geq \log([\alpha - 4]).$$

Proof. Fix $\alpha \geq 20$ and $f \in C(\mathrm{I})$. By Remark 3.1, we have $h_{\mathrm{top}}(H_t^\alpha(f)) \geq \log([\alpha-4])$ for all $t \in [1/2, 1]$. Now assume that $t \in [0, \frac{1}{2})$. By the construction of H_t^α, there exists an interval I_i and $x_0 \in I_i$ such that $f(x_0) = x_0$. By the formula (3.1), we have $I_i \subset [a_b^i, a_t^i]$. Now, by the construction of the box map on I_i, there exist closed subintervals $J_1, \dots, J_{[\alpha-4]}$ of I_i with disjoint interiors such that $H_t^\alpha(f)(J_j) = [a_b^i, a_t^i]$ for $j = 1, \dots, [\alpha-4]$. Then $J_1, \dots, J_{[\alpha-4]}$ form an $[\alpha-4]$-horseshoe of $H_t^\alpha(f)$. By Lemma 2.7, $h_{\mathrm{top}}(H_t^\alpha(f)) \geq \log([\alpha-4])$. □

We summarize the above results as follows.

Proposition 3.3. *For every $\alpha \geq 20$, there exists a homotopy $H^\alpha\colon C(\mathrm{I}) \times \mathrm{I} \to C(\mathrm{I})$ such that*:

(a) $H_0^\alpha = \mathrm{id}_{C(\mathrm{I})}$;
(b) $h_{\mathrm{top}}(H_t^\alpha(f)) \geq \log([\alpha-4])$ *for* $t \in (0,1]$ *and for every* $f \in C(I)$;
(c) $H_1^\alpha(f)$ *is the box map on* I *with the parameter* $(0,0,0,1,\alpha)$.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Fix $a \in [0, +\infty)$ and choose $\alpha \in [20, +\infty)$ such that $\log([\alpha-4]) > a$. Let H^α as in Proposition 3.3. Then both $E_{\geq a}$ and $E_{>a}$ are homotopy dense in $C(\mathrm{I})$. Using the homotopies $H^\alpha|_{E_{\geq a} \times \mathrm{I}}$ and $H^\alpha|_{E_{>a} \times \mathrm{I}}$, both $E_{\geq a}$ and $E_{>a}$ are contractible. By Corollary 2.9, both $E_{\geq a}$ and $E_{>a}$ are topologically complete. Now, using Corollary 2.6, $E_{\geq a}$ and $E_{>a}$ are homeomorphic to l_2. □

Corollary 3.4. *For every $a \in [0, +\infty)$, $E_{\geq a}$ and $E_{>a}$ are homotopy dense in $C(\mathrm{I})$. Moreover, $E_{>a} \cap E_{<+\infty}$ is homotopy dense and open in $E_{<+\infty}$.*

Proof. The former was shown in the proof of Theorem 1.1. To show the latter, we only note that the topological entropy of a piecewise monotone map is finite (see e.g. [3, Proposition VIII.18]). □

By Theorem 2.8 and Corollary 3.4, we know that the subspace $E_{+\infty} = \{f \in C(\mathrm{I})\colon h_{\mathrm{top}}(f) = +\infty\}$ is a dense G_δ-set in $C(\mathrm{I})$. But the following question remains open.

Problem 3.5. Is $E_{+\infty}$ homeomorphic to l_2?

In Proposition 3.3, for every $f \in C(\mathrm{I})$ and $t \in (0,1]$, $H_t^\alpha(f)$ is piecewise monotone and then it has finite topological entropy. So we can not use the method in the beginning of this section to construct a proper homotopy to show that $E_{+\infty}$ is contractible. Another important fact is that there is no continuous selection of fixed points.

Proposition 3.6. *There does not exist a continuous map $\phi\colon C(\mathrm{I}) \to \mathrm{I}$ such that $\phi(f)$ is a fixed point of f for every $f \in C(\mathrm{I})$.*

Proof. Suppose that $\phi\colon C(\mathrm{I}) \to \mathrm{I}$ is such a map. Choose $x_0 \in \mathrm{I}\setminus\{0, 1, \phi(\mathrm{id}_{\mathrm{I}})\}$. Let $l_n\colon \mathrm{I} \to \mathrm{I}$ be the broken line map through the points $(0, 1/n)$, (x_0, x_0) and $(1, 1-1/n)$. Then $l_n \to \mathrm{id}_{\mathrm{I}}$ in $C(\mathrm{I})$ as $n \to \infty$. Since l_n has a unique fixed point x_0, $\phi(l_n) = x_0 \not\to \phi(\mathrm{id}_{\mathrm{I}})$ as $n \to \infty$. So ϕ is not continuous, which is a contradiction. □

4. Proof of Theorem 1.2

In this section we construct the homotopy in Theorem 1.2, which is done by connecting three homotopies. Inspired by [8] and [9], we introduce the following concept. Let f and $\bar{f} \in C([a,b], \mathrm{I})$. We say that $\bar{f}$ is made from f by *procedure of making constant pieces* (PMCP, briefly) if there exists a sequence of open intervals $\{U_n\}_{n=1}^{\infty}$ of $[a,b]$ in the relative topology such that

$$\overline{f}|_{I\setminus \bigcup_{n=1}^{\infty} U_n} = f|_{I\setminus \bigcup_{n=1}^{\infty} U_n}$$

and $\overline{f}|_{U_n}$ is constant for every $n \in \mathbb{N}$. It should be noticed that our definition here is more general than the one in [8]. We will need the following result which was proved in [9, Lemma 5].

Lemma 4.1. *Let $f \in C(\mathrm{I})$. If $\overline{f}$ is made from f by* PMCP*, then*

$$h_{\mathrm{top}}(\overline{f}) \le h_{\mathrm{top}}(f).$$

For every $c \in \mathrm{I}$, the map $\max\{f(x), c\}$ can be thought to be made from f by PMCP. For every $f \in C([a,b], \mathrm{I})$, let

$$\begin{aligned} M(f) &= \max\{f(x) : x \in [a,b]\};\\ c_1(f) &= \min\{x \in [a,b] : f(x) = M(f)\};\\ c_2(f) &= \max\{x \in [a,b] : f(x) = M(f)\}.\end{aligned}$$

Now we define $\widetilde{f}\colon [a,b]\to \mathrm{I}$ as follows

$$\widetilde{f}(x)=\begin{cases}\max\{f(t): a\le t\le x\}, & x\in[a,c_1(f)],\\ M(f), & x\in[c_1(f),c_2(f)],\\ \max\{f(t): x\le t\le b\}, & x\in[c_2(f),b].\end{cases}$$

First we have the following lemma.

LEMMA 4.2. *For any* $f,g\in C([a,b],\mathrm{I})$*, we have*

(a) $\widetilde{f}$ *is made from* f *by PMCP and it is in* $C^{\mathrm{PM}}([a,b],\mathrm{I})$*;*
(b) $\widetilde{f}(a)=f(a)$*,* $\widetilde{f}(b)=f(b)$ *and* $\widetilde{f}([a,b])\subset f([a,b])$*;*
(c) $d(\widetilde{f},\widetilde{g})\le d(f,g)$*;*
(d) *if* $c\in(a,b)$ *and* $\varepsilon>0$ *satisfy either*

$$\max f|_{[a,c]}-\min f|_{[a,c]}<\varepsilon \quad or \quad \max f|_{[c,b]}-\min f|_{[c,b]}<\varepsilon,$$

that is, the amplitude of f *on* $[a,c]$ *or on* $[c,b]$ *is smaller than* ε*, then*

$$d\Big(\widetilde{f},\widetilde{f|_{[a,c]}}\cup\widetilde{f|_{[c,b]}}\Big)<\varepsilon.$$

PROOF. Parts (a) and (b) are obvious. We only need to show (c) and (d).

(c) We note that, for any maps $h,k\colon J\to\mathrm{I}$,

$$\big|\sup\{h(x):x\in J\}-\sup\{k(x):x\in J\}\big|\le\sup\{|h(x)-k(x)|:x\in J\}. \tag{4.1}$$

It follows that (c) holds in the case $[c_1(f),c_2(f)]\cap[c_1(g),c_2(g)]\ne\emptyset$. For the case $[c_1(f),c_2(f)]\cap[c_1(g),c_2(g)]=\emptyset$, without loss of generality, we assume that $c_2(f)<c_1(g)$. For $x\in[a,b]\setminus[c_2(f),c_1(g)]$ using the formula (4.1), we have that

$$|\widetilde{f}(x)-\widetilde{g}(x)|\le d(f,g).$$

If $x\in(c_2(f),c_1(g))$ and $\widetilde{f}(x)\ge\widetilde{g}(x)$, then

$$0\le\widetilde{f}(x)-\widetilde{g}(x)\le f(c_2(f))-g(c_2(f))\le d(f,g).$$

If $x\in(c_2(f),c_1(g))$ and $\widetilde{g}(x)>\widetilde{f}(x)$, then

$$0<\widetilde{g}(x)-\widetilde{f}(x)\le g(c_1(g))-f(c_1(g))\le d(f,g).$$

Hence (c) holds in the case $[c_1(f),c_2(f)]\cap[c_1(g),c_2(g)]=\emptyset$.

(d) Without loss of generality, we assume that $\max f|_{[a,c]}-\min f|_{[a,c]}<\varepsilon$. By (b), $h=\widetilde{f|_{[a,c]}}\cup\widetilde{f|_{[c,b]}}\in C([a,b],\mathrm{I})$.

Case 1. $c\in[c_1(f),c_2(f)]$. By the assumption, we have $M(f)-\varepsilon<f(c)\le M(f)$. It follows that

$$M(f)-\varepsilon<f(c)\le h(x)\le M(f)=\widetilde{f}(x),\quad x\in[c_1(f),c_2(f)].$$

Hence

$$\big|\widetilde{f}(x)-h(x)\big|<\varepsilon,\quad x\in[c_1(f),c_2(f)].$$

Moreover, it is trivial that

$$\widetilde{f}(x) = h(x) \quad \text{for } x \in [a,b] \setminus [c_1(f), c_2(f)].$$

Hence $d(\widetilde{f}, h) < \varepsilon$.

Case 2. $c \in [a, c_1(f)]$. In this case,

$$\widetilde{f}(x) = h(x), \quad x \in [c_1(f), b].$$

Moreover, by the assumption in (d),

$$|\widetilde{f}(x) - h(x)| < \varepsilon, \quad x \in [a, c].$$

Furthermore, for every $x \in [c, c_1(f)]$,

$$h(x) = \widetilde{f|_{[c,b]}}(x) \le \widetilde{f}(x) < \widetilde{f|_{[c,b]}}(x) + \varepsilon = h(x) + \varepsilon.$$

Therefore, $d(\widetilde{f}, h) < \varepsilon$.

Case 3. $c \in [c_2(f), b]$. By the assumption, $M(f) - \varepsilon < f(c) \le M(f)$. It follows that

$$M(f) - \varepsilon < f(c) \le f(c_2(f|_{[c,b]})) \le M(f). \tag{4.2}$$

Note that

$$\widetilde{f}(x) = \widetilde{f|_{[c,b]}}(x) \le f(c_2(f|_{[c,b]})), \quad x \in [c_2(f|_{[c,b]}), b].$$

Moreover, using this and the formula (4.2), we have

$$|\widetilde{f}(x) - h(x)| < \varepsilon, \quad x \in [a, c_2(f|_{[c,b]})].$$

So in this case we also have $d(\widetilde{f}, h) < \varepsilon$. □

Using the above, we can give the first homotopy.

Lemma 4.3. *There exists a homotopy $H^1 \colon C(\mathrm{I}) \times \mathrm{I} \to C(\mathrm{I})$ such that*

(a) $H^1_0 = \mathrm{id}_{C(\mathrm{I})}$;
(b) $h_{\mathrm{top}}(H^1_t(f)) \le h_{\mathrm{top}}(f)$ *and* $H^1_t(f) \in C^{\mathrm{PM}}(\mathrm{I})$ *for* $t \in (0,1]$ *and* $f \in C(\mathrm{I})$.

Proof. In the same way as in the construction of the homotopy H^α in Section 3, let $H^1_0 = \mathrm{id}_{C(\mathrm{I})}$, and for $t \in (0,1]$, let s be the largest non-negative integer such that $st < 1$. We can obtain $s+1$ closed intervals:

$$I_i = [(i-1)t, it], \quad i = 1, \cdots, s, I_{s+1} = [st, 1].$$

The integer s and the interval I_i are also denoted by $s(t)$ and I^t_i if necessary. We define H^1_t such that, for every $f \in C(\mathrm{I})$ and $i = 1, \ldots, s+1$,

$$H^1_t(f)|_{I_i} = \widetilde{f|_{I_i}}.$$

Using Lemma 4.2 (b), $H^1 \colon C(\mathrm{I}) \times \mathrm{I} \to C(\mathrm{I})$ is well-defined. Trivially, it satisfies (a). From Lemmas 4.1 and 4.2 (a) it follows that it satisfies (b).

It remains to verify that $H^1\colon C(\mathrm{I}) \times \mathrm{I} \to C(\mathrm{I})$ is continuous. At first, we show that $H^1(f, \cdot\,)$ is continuous for every fixed $f \in C(\mathrm{I})$. For every $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$|x_1 - x_2| < \delta \quad \text{implies} \quad |f(x_1) - f(x_2)| < \frac{\varepsilon}{2}. \tag{4.3}$$

Now, for every $t_0 \in \mathrm{I}$, we verify that there exists $\delta(t_0) \in (0, \delta]$ such that

$$|t - t_0| < \delta(t_0) \quad \text{implies} \quad d(H^1(f,t), H^1(f,t_0)) < \varepsilon, \tag{4.4}$$

which shows that $H^1(f, \cdot\,)$ is continuous.

If $t_0 = 0$, we let $\delta(t_0) = \delta$. For every $t \in (0, \delta)$ and i, from Lemma 4.2 (b) it follows that

$$\widetilde{f|_{I_i}}(I_i) \subset f(I_i).$$

Since $|I_i| \le t < \delta$, using (4.3), we have $|f(I_i)| < \varepsilon$. Thus (4.4) holds.

If $t_0 \in (0,1]$, choose $\delta(t_0) \in (0, \delta)$ small enough such that for every $t \in \mathrm{I} \cap (t_0 - \delta(t_0), t_0 + \delta(t_0))$, we have $|s(t_0) - s(t)| < 2$ and $(s(t_0) + 2)\delta(t_0) < \delta$. Then all points $\{it, jt_0\}$ divide I into closed intervals $\{J_j\}$. Let

$$G = \bigcup \widetilde{f|_{J_j}} \in C(\mathrm{I}).$$

Then, for every i, $I_i^{t_0}$ is either a union of the two closed intervals in $\{J_j\}$ or just a closed interval in $\{J_j\}$. If the former holds, then by the choice of $\delta(t_0)$ and the formula (4.3), the amplitude of f in one of the two closed intervals is smaller than $\varepsilon/2$. Using Lemma 4.2 (d), we have that

$$d\Big(H^1(f,t_0)|_{I_i^{t_0}}, G|_{I_i^{t_0}}\Big) < \frac{\varepsilon}{2}.$$

If the later holds, then $H^1(f,t_0)|_{I_i^t} = G|_{I_i^t}$ and hence the above formula also holds. Thus,

$$d\big(H^1(f,t_0), G\big) < \frac{\varepsilon}{2}.$$

Similarly, we have that

$$d\big(H^1(f,t), G\big) < \frac{\varepsilon}{2}.$$

Hence the formula (4.4) holds.

By Lemma 4.2 (c), we can obtain that $d(H^1(f,t), H^1(g,t)) \le d(f,g)$. In combination with the continuity of H^1 on t, we have that $H^1\colon C(\mathrm{I}) \times \mathrm{I} \to C(\mathrm{I})$ is jointly continuous. □

The second homotopy we need is the following.

Lemma 4.4. *There exists a homotopy $H^2\colon C(\mathrm{I}) \times \mathrm{I} \to C(\mathrm{I})$ satisfying*:

(a) $H_0^2 = \mathrm{id}_{C(\mathrm{I})}$;

(b) $h_{\mathrm{top}}(H_t^2(f)) \le h_{\mathrm{top}}(f)$ *and* $H_t^2(C^{\mathrm{PM}}(\mathrm{I})) \subset C^{\mathrm{PM}}(\mathrm{I})$ *for any* $t \in (0,1]$ *and* $f \in C(\mathrm{I})$;

(c) $H_1^2(f)$ *is a constant map for any* $f \in C(\mathrm{I})$.

Proof. For every $f \in C(\mathrm{I})$, let

$$M(f) = \max\{f(x) : x \in \mathrm{I}\}, \qquad m(f) = \min\{f(x) : x \in \mathrm{I}\}.$$

Then $M, m\colon C(\mathrm{I}) \to \mathrm{I}$ are continuous. Using them, we can define our homotopy as follows

$$H^2(f,t)(x) = \max\{f(x), (M(f) - m(f))t + m(f))\}.$$

Then it is not hard to verify that $H^2\colon C(\mathrm{I}) \times \mathrm{I} \to C(\mathrm{I})$ is continuous and it satisfies (a) and (c). Moreover, $H^2(f,t)$ is made from f by PMCP. It follows from Lemma 4.1 that H^2 also satisfies (b). □

The third homotopy $H^3\colon \mathrm{I}^2 \to \mathrm{I}$ is defined as

$$H^3(s,t) = (1-t)s,$$

which is a homotopy between the identical map and the constant map 0 in I. Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Define $H\colon C(\mathrm{I}) \times \mathrm{I} \to C(\mathrm{I})$ as

$$H(f,t) = \begin{cases} H^1(f,3t), & t \in [0,1/3), \\ H^2(H^1(f,1), 3t-1), & t \in [1/3,2/3), \\ H^3(H^2(H^1(f,1),1), 3t-2), & t \in [2/3,1]. \end{cases}$$

Since $H^2(H^1(f,1),1)$ is a constant map, the homotopy H is well-defined. Note that $h_{\mathrm{top}}(c) = 0$ for every constant map c. By Lemmas 4.3 and 4.4, it is easy to see that $H\colon C(\mathrm{I}) \times \mathrm{I} \to C(\mathrm{I})$ is the homotopy as required. □

It follows from Corollary 3.4 and Theorem 2.8 that $E_{\le a}$ is nowhere dense and closed in the space $E_{<+\infty}$. Hence

$$E_{<+\infty} = \bigcup_{n\in\mathbb{N}} E_{\le n}$$

is not topologically complete. Therefore, $E_{<+\infty}$ is not homeomorphic to l_2. It is natural to put the following problem:

Problem 4.5. Does there exist $a \in (0,+\infty)$ such that $E_{<a}$ is homeomorphic to l_2?

For every $a \in [0,+\infty)$, by Corollary 1.3, we know that $E_{\le a}$ is contractible. By Theorem 2.8, $E_{\le a}$ is a closed subset of $C(\mathrm{I})$ and hence it is topologically complete. But the following problem is still open.

Problem 4.6. Is $E_{\le a}$ homeomorphic to l_2 for every $a \in [0,+\infty)$? In particular, is $E_{\le 0}$ homeomorphic to l_2?

By Anderson–Kadec's theorem, l_2 is homeomorphic to $\mathbb{R}^{\mathbb{N}}$, then it is also homeomorphic to $s = (-1,1)^{\mathbb{N}}$. Let

$$\begin{aligned} Q &= [-1,1]^{\mathbb{N}}, \\ \Sigma &= \{(x_n) \in Q : \sup |x_n| < 1\}, \\ P_{\prec 2^\infty} &= \{f \in C(\mathrm{I}) : \text{there exists } n \in \mathbb{N} \text{ such that} \\ &\qquad\qquad \text{the set of periods of } f \text{ is } \{2^i : 0 \le i \le n\}\}. \end{aligned}$$

Using these symbols, we have the following problem:

PROBLEM 4.7. For every $a \in (0,+\infty]$, does there exist a homeomorphism $h\colon E_{\le a} \to s \times Q$ such that $h(E_{<a}) = s \times \Sigma$? Does there exist a homeomorphism $h\colon E_{\le 0} \to s \times Q$ such that $h(P_{\prec 2^\infty}) = s \times \Sigma$?

Acknowledgments. The authors would like to thank the anonymous referees for the careful reading and helpful suggestions.

REFERENCES

[1] R.D. ANDERSON, *Hilbert space is homeomorphic to the countable infinite product of lines*, Bull. Amer. Math. Soc. **72** (1966), 515–519.

[2] T. BANAKH, T. RADUL AND M. ZARICHNYI, *Absorbing Sets in Infinite-Dimensional Manifolds*, Mathematical Studies Monograph Series, vol. 1, VNTL Publishers, L'viv, 1996.

[3] L.S. BLOCK AND W.A. COPPEL, *Dynamics in One Dimension*, Lecture Notes in Mathematics, vol. 1513, Springer–Verlag, Berlin, 1992.

[4] A. CHIGOGIDZE, *Infinite dimensional topology and shape theory*, Handbook of Geometric Topology, North-Holland, Amsterdam, 2002, pp. 307–371.

[5] T. DOBROWOLSKI, *Examples of topological groups homeomorphic to* l_2^f, Proc. Amer. Math. Soc. **98** (1986), no. 2, 303–311.

[6] T. DOBROWOLSKI, W. MARCISZEWSKI AND J. MOGILSKI, *On topological classification of function spaces* $C_p(X)$ *of low Borel complexity*, Trans. Amer. Math. Soc. **328** (1991), no. 1, 307–324.

[7] R. ENGELKING, *General Topology*, second ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989, Translated from the Polish by the author.

[8] M. GRINČ, R. HRIC AND L. SNOHA, *The structure of the space* $C(I,I)$ *from the point of view of Sharkovsky stratification*, Topology **39** (2000), no. 5, 937–946.

[9] V. JIMÉNEZ LÓPEZ AND L. SNOHA, *All maps of type* 2^∞ *are boundary maps*, Proc. Amer. Math. Soc. **125** (1997), no. 6, 1667–1673.

[10] M.Ĭ. KADEC', *Topological equivalence of all separable Banach spaces*, Dokl. Akad. Nauk SSSR **167** (1966), 23–25.

[11] S. KOLYADA, M. MISIUREWICZ AND L. SNOHA, *Spaces of transitive interval maps*, Ergodic Theory Dynam. Systems **35** (2015), no. 7, 2151–2170.

[12] S. KOLYADA, M. MISIUREWICZ AND L. SNOHA, *Loops of transitive interval maps*, Dynamics and Numbers, Contemp. Math., vol. 669, Amer. Math. Soc., Providence, RI, 2016, pp. 137–154.

[13] N.T. NHU, *The group of measure preserving transformations of the unit interval is an absolute retract*, Proc. Amer. Math. Soc. **110** (1990), no. 2, 515–522.

[14] J. van Mill, *Infinite-Dimensional Topology*, North-Holland Mathematical Library, vol. 43, North-Holland Publishing Co., Amsterdam, 1989, Prerequisites and introduction.

[15] J. van Mill, *The Infinite-Dimensional Topology of Function Spaces*, North-Holland Mathematical Library, vol. 64, North-Holland Publishing Co., Amsterdam, 2001.

[16] H. Yang, Z. Yang and Y. Zheng, *Topological classification of function spaces with the Fell topology* IV, Topology Appl. **228** (2017), 222–235.

[17] Z. Yang, L. Chen and Y. Zheng, *Topological classification of function spaces with the Fell topology* III, Topology Appl. **197** (2016), 112–132.

[18] Z. Yang and P. Yan, *Topological classification of function spaces with the Fell topology* I, Topology Appl. **178** (2014), 146–159.

[19] Z. Yang, Y. Zheng and J. Chen, *Topological classification of function spaces with the Fell topology* II, Topology Appl. **187** (2015), 82–96.

Manuscript received September 18, 2018

accepted February 3, 2019

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