Topological **M**ethods in **N**onlinear **A**nalysis
Volume 52, No. 2, 2018, 631–664
DOI: 10.12775/TMNA.2018.025

ON SPECTRAL CONVERGENCE FOR SOME PARABOLIC PROBLEMS WITH LOCALLY LARGE DIFFUSION

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Abstract. In this paper, which is a sequel to [1], we extend the spectral convergence result from [5] to a larger class of singularly perturbed families of scalar linear differential operators. This also extends the Conley index continuation principles from [1].

1. Introduction

In the important paper [5], Carvalho and Pereira approached a problem previously considered by Fusco [6] from the point of view of spectral convergence. Specifically, they considered a family of linear differential operators $u \mapsto -(a_\varepsilon u_x)_x$ on the interval $]0,1[$ with boundary conditions

$$(\mathrm{S}_\varepsilon) \qquad \begin{cases} \rho u - (1-\rho) a_\varepsilon u_x = 0, & x = 0, \\ \sigma u + (1-\sigma) a_\varepsilon u_x = 0, & x = 1, \end{cases}$$

and made the following

Assumption 1.1. $n \in \mathbb{N}$, $\varepsilon_0 \in \,]0,\infty]$, $(e_j)_{j\in[1..n]}$, $(l_j)_{j\in[0..n]}$, $(b_j)_{j\in[0..n]}$ are sequences of positive constants and $(l'_j)_{j\in[0..n]}$, $(b'_j)_{j\in[0..n]}$ are sequences of positive functions defined on $]0,\varepsilon_0[$ such that $l'_j(\varepsilon) > l_j$ and $b'_j(\varepsilon) > b_j$ for $j \in$

2010 *Mathematics Subject Classification.* Primary: 35P99, 35K57; Secondary: 37B30, 35B25.

Key words and phrases. Spectral convergence; localized large diffusion; singular perturbations; Conley index.

$[0\,.\,.\,n]$ and $\varepsilon \in \,]0, \varepsilon_0[$ and for some $q \in \,]0,1[$ and all $j \in [0\,.\,.\,n]$, $l'_j(\varepsilon) - l_j = O(\varepsilon^q)$ and $b'_j(\varepsilon) - b_j = O(\varepsilon^q)$ as $\varepsilon \to 0$.

$(a_\varepsilon)_{\varepsilon \in]0,\varepsilon_0[}$ is a family of positive C^2 functions defined on $[0,1]$ and $(x_j)_{j\in[0..n]}$ is a strictly increasing sequence in $[0,1]$ with $x_0 = 0$ with $x_n = 1$ and such that for each $j \in [1\,.\,.\,n]$

$$\begin{aligned} a_\varepsilon(x) &\geq \frac{e_j}{\varepsilon}, && \text{for } x_{j-1} + \varepsilon l'_{j-1} \leq x \leq x_j - \varepsilon l'_j, \\ a_\varepsilon(x) &\geq \varepsilon b_j, && \text{for } x_j - \varepsilon l'_j \leq x \leq x_j + \varepsilon l'_j, \\ a_\varepsilon(x) &\leq \varepsilon b'_j, && \text{for } x_j - \varepsilon l_j \leq x \leq x_j + \varepsilon l_j. \end{aligned}$$

Here $x_0 - \varepsilon l'_0 = x_0 - \varepsilon l_0 = 0$ and $x_n + \varepsilon l'_n = x_n + \varepsilon l_n = 1$.

(Note that Carvalho and Pereira write ν instead of ε and $(a_j)_{j\in[0..n]}$ and $(a'_j)_{j\in[0..n]}$ instead of $(b_j)_{j\in[0..n]}$ and $(b'_j)_{j\in[0..n]}$, respectively.)

The above differential operator generates, for $\varepsilon \in \,]0, \varepsilon_0[$, a linear operator A_ε in $L^2(0,1)$ which has a simple spectrum $(\lambda_{l,\varepsilon})_l$, $\lambda_{l,\varepsilon} < \lambda_{l+1,\varepsilon}$, and corresponding appropriately normalized eigenfunctions $(\varphi_{l,\varepsilon})_l$. The authors proved the existence of a linear operator A_0 on $\mathbb{R}^n$ which has simple spectrum $(\lambda_{l,0})_{l\in[1..n]}$, $\lambda_{l,0} < \lambda_{l+1,0}$, and corresponding appropriately normalized eigenvectors $(\varphi_{l,0})_{l\in[1..n]}$ such that, for $\varepsilon \to 0$, $\lambda_{l,\varepsilon} \to \lambda_{l,0}$ for $l \in [1\,.\,.\,n]$ and $\lambda_{l,\varepsilon} \to \infty$ for $l > n$. Moreover, for $l \in [1\,.\,.\,n]$, in some sense, $\varphi_{l,\varepsilon} \to \varphi_{l,0}$ as $\varepsilon \to 0$. Cf [5, Lemma 3.1 and Theorem 3.5] for the precise statement of these results.

As remarked in [5], the above results continue to hold, with some modifications and essentially the same proofs if, in the above hypothesis, we remove the 'wells' close to the boundary points.

Some impressive spectral convergence results were also obtained in the papers [2] and [4]. In [2] Carvalho extends some of the spectral convergence results from [5] to rectangular domains in $\mathbb{R}^2$ and to domains whose intersection with a vertical strip in $\mathbb{R}^2$ is a rectangle. Moreover, he also discusses, for the Neumann case without wells and $n = 2$, an extension of Assumption 1.1 under which the spectral convergence result from [5] continues to hold, cf. Section 4.

In the paper [4], Carvalho and Cuminato prove a spectral convergence result for genuinely higher dimensional domains which are composed of a finite number of subdomains (cells) and their boundaries (membranes). Here, diffusion is large in the cells and small on the membranes.

In the above papers the spectral convergence results are used to obtain information on asymptotic dynamics for a corresponding family of semilinear parabolic equations with dissipative nonlinearities. In particular, existence and upper semicontinuity of attractors as well as structural stability results, as $\varepsilon \to 0$, are established.

In the interesting recent paper [3], a general functional analytic framework is developed for the study of singularly perturbed semilinear parabolic equations with finite dimensional limit problem using invariant manifold techniques. That paper also contains various practical examples.

In the recent paper [1] we considered the abstract parabolic problem

$$\dot{u} = -A_\varepsilon u + f_\varepsilon(u) \tag{1.1}$$

on $H^1(0,1)$ and a corresponding limit problem

$$\dot{z} = -A_0 z + f_0(z) \tag{1.2}$$

on $\mathbb{R}^n$. Here, A_ε, $\varepsilon \in \,]0, \varepsilon_0[$, and A_0 are as above while f_ε, $\varepsilon \in \,]0, \varepsilon_0[$, and f_0 are Nemitski operators generated by nonlinearities satisfying an appropriate condition [1, Assumption 4.2]. Let us note that no dissipativeness conditions on the nonlinearities are imposed.

Using the spectral convergence theorem from [5] together with some nonlinear convergence and compactness results we proved singular Conley index and homology index braid continuation principles for the families of local semiflows generated by (1.1) and (1.2), cf. [1, Theorem 4.5].

In this paper we introduce a linear hypothesis, Assumption 2.1 (respectively, Assumption 4.1), which is more general than the hypothesis from [5] (respectively, the condition discussed in [2]), cf. Sections 2 and 4.

It turns out that a spectral convergence result can be proved under this more general assumption, cf. Theorem 2.6 (resp. Theorem 4.2), extending the corresponding results from [5] (resp. [2]).

Now, using Theorem 2.6 (resp. Theorem 4.2) instead of results from [5], we extend, in Section 5, the Conley index continuation principles from [1] to the present more general situation, cf Theorem 5.3 (resp. Remark 5.4).

In this paper, $\mathbf{m}$ is the one-dimensional Lebesgue measure.

2. The spectral convergence result

In this section we state and discuss our main hypothesis, Assumption 2.1. Under this assumption, we define a family $(A_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$ of linear operators and the limit operator A_0. Then we state Theorem 2.6, which is the main result of this paper.

We begin by stating our linear assumption:

Assumption 2.1.

(1a) $n \in \mathbb{N}$, $\varepsilon_0 \in \,]0, \infty]$;

(1b) $(a_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$ is a family of continuous positive functions defined on $[0,1]$;

(1c) $(x_j)_{j\in[0..n]}$ is a strictly increasing sequence in $[0,1]$, with $x_0 = 0$ and $x_n = 1$, $(\tau_j)_{j\in[0..n]}$ is a sequence in $]0,\infty[$ and $\xi'_{j,\varepsilon}$, $\xi_{j,\varepsilon}$, $j \in [1..n]$ and $\zeta_{j,\varepsilon}$, $\zeta'_{j,\varepsilon}$, $j \in [0..n-1]$ are families in $]0,1[$ with $x_{j-1} < \zeta_{j-1,\varepsilon} < \zeta'_{j-1,\varepsilon} < \xi'_{j,\varepsilon} < \xi_{j,\varepsilon} < x_j$, $j \in [1..n]$, $\varepsilon \in \,]0, \varepsilon_0[$.

(1d) If $(\Gamma_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$ is any of the families:

$$([\xi'_{j,\varepsilon},\zeta'_{j,\varepsilon}])_{\varepsilon\in]0,\varepsilon_0[},\quad ([x_0,\zeta'_{0,\varepsilon}])_{\varepsilon\in]0,\varepsilon_0[},\quad ([\xi'_{n,\varepsilon},x_n])_{\varepsilon\in]0,\varepsilon_0[},$$

$j\in[1\,..\,n-1]$, then $\mathbf{m}(\Gamma_\varepsilon)\to 0$ as $\varepsilon\to 0$.

(2a) If $(\Gamma_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$ is any of the families:

$$([\zeta'_{j,\varepsilon},\xi'_{j+1,\varepsilon}])_{\varepsilon\in]0,\varepsilon_0[},\quad ([\zeta_{j,\varepsilon},\zeta'_{j,\varepsilon}])_{\varepsilon\in]0,\varepsilon_0[},\quad j\in[0\,..\,n-1]\,,$$

or else the family $([\xi'_{j,\varepsilon},\xi_{j,\varepsilon}])_{\varepsilon\in]0,\varepsilon_0[}$, $j\in[1\,..\,n]$, then

$$\frac{\inf_{\Gamma_\varepsilon} a_\varepsilon}{\mathbf{m}(\Gamma_\varepsilon)}\to\infty\quad\text{as }\varepsilon\to 0.$$

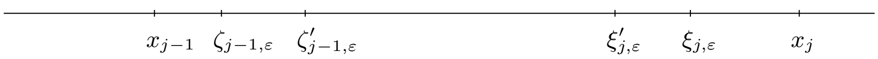
(2b) For each $j\in[0\,..\,n]$ and $\varepsilon\in\,]0,\varepsilon_0[$, set $\Gamma_{j,\varepsilon}=[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]$, if $j\in[1\,..\,n-1]$, $\Gamma_{0,\varepsilon}=[x_0,\zeta_{0,\varepsilon}]$ and $\Gamma_{n,\varepsilon}=[\xi_{n,\varepsilon},x_n]$. Then

$$\frac{\inf_{\Gamma_{j,\varepsilon}} a_\varepsilon}{\mathbf{m}(\Gamma_{j,\varepsilon})}\to\tau_j\quad\text{and}\quad\frac{\sup_{\Gamma_{j,\varepsilon}} a_\varepsilon}{\mathbf{m}(\Gamma_{j,\varepsilon})}\to\tau_j\quad\text{as }\varepsilon\to 0.$$

Notation. In this and the next section, we write, for $j\in[1\,..\,n]$,

$$K_{j,\varepsilon}=[\zeta_{j-1,\varepsilon},\xi_{j,\varepsilon}]\,,\quad K'_{j,\varepsilon}=[\zeta'_{j-1,\varepsilon},\xi'_{j,\varepsilon}]\,,\quad K_j=[x_{j-1},x_j]\,,\quad L_j=\mathbf{m}(K_j).$$

Remark 2.2. For $j\in[1\,..\,n]$ we have the following picture:



Let $j\in[1\,..\,n]$ be arbitrary. Since $\mathbf{m}(K'_{j,\varepsilon})\to L_j>0$ as $\varepsilon\to 0$, part (2a) of Assumption 2.1 implies that $a_\varepsilon\to\infty$ for $\varepsilon\to 0$, uniformly in $K'_{j,\varepsilon}$. Moreover, by part (2b), on the small intervals $[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]$ around x_j, a_ε is of the same order as the measure of these intervals so $a_\varepsilon\to 0$ for $\varepsilon\to 0$, uniformly in $[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]$. Finally, there is some transitional behavior on the remaining small intervals $[\xi'_{j,\varepsilon},\xi_{j,\varepsilon}]$ and $[\zeta_{j,\varepsilon},\zeta'_{j,\varepsilon}]$ around x_j, as a_ε is of lower order than the measure of these intervals.

The following result further clarifies the above assumption. It provides a necessary and sufficient condition, in terms of the relative positions of the partition points x_j, $\xi_{j,\varepsilon}$, $\xi'_{j,\varepsilon}$, $\zeta_{j,\varepsilon}$ and $\zeta'_{j,\varepsilon}$, for the existence of diffusion coefficient functions a_ε such that Assumption 2.1 holds.

Proposition 2.3. *If Assumption* 2.1 *holds, then*

$$\frac{\mathbf{m}([\xi'_{j,\varepsilon},\xi_{j,\varepsilon}])+\mathbf{m}([\zeta_{j,\varepsilon},\zeta'_{j,\varepsilon}])}{\mathbf{m}([\xi_{j,\varepsilon},\zeta_{j,\varepsilon}])}\to 0,\quad\textit{as }\varepsilon\to 0,\ j\in[1\,..\,n-1]\tag{2.1}$$

and

$$\frac{\mathbf{m}([\zeta_{0,\varepsilon},\zeta'_{0,\varepsilon}])}{\mathbf{m}([x_0,\zeta_{0,\varepsilon}])}\to 0\quad\textit{and}\quad\frac{\mathbf{m}([\xi'_{n,\varepsilon},\xi_{n,\varepsilon}])}{\mathbf{m}([\xi_{n,\varepsilon},x_n])}\to 0\quad\textit{as }\varepsilon\to 0.\tag{2.2}$$

Conversely, if parts (1a), (1c) *and* (1d) *of Assumption* 2.1 *together with estimates* (2.1) *and* (2.2) *hold, then there is a family* $(a_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$*, such that parts* (1b), (2a) *and* (2b) *of that assumption are also satisfied. In addition, we may assume that each function* a_ε *can be extended to a* C^∞*-function defined on all of* $\mathbb{R}$.

Proof. If Assumption 2.1 holds, then, by (2a),

$$\frac{a_\varepsilon(\zeta_{j,\varepsilon})}{\mathbf{m}([\zeta_{j,\varepsilon},\zeta'_{j,\varepsilon}])}\to\infty \quad\text{and}\quad \frac{a_\varepsilon(\xi_{j,\varepsilon})}{\mathbf{m}([\xi'_{j,\varepsilon},\xi_{j,\varepsilon}])}\to\infty \quad \text{as } \varepsilon\to 0,\ j\in[1\,.\,.\,n-1]$$

and

$$\frac{a_\varepsilon(\zeta_{0,\varepsilon})}{\mathbf{m}([\zeta_{0,\varepsilon},\zeta'_{0,\varepsilon}])}\to\infty \quad\text{and}\quad \frac{a_\varepsilon(\xi_{n,\varepsilon})}{\mathbf{m}([\xi'_{n,\varepsilon},\xi_{n,\varepsilon}])}\to\infty \quad \text{as } \varepsilon\to 0,$$

while, by (2b),

$$\frac{a_\varepsilon(\zeta_{j,\varepsilon})}{\mathbf{m}([\xi_{j,\varepsilon},\zeta_{j,\varepsilon}])}\to\tau_j \quad\text{and}\quad \frac{a_\varepsilon(\xi_{j,\varepsilon})}{\mathbf{m}([\xi_{j,\varepsilon},\zeta_{j,\varepsilon}])}\to\tau_j \quad \text{as } \varepsilon\to 0,\ j\in[1\,.\,.\,n-1]$$

and

$$\frac{a_\varepsilon(\zeta_{0,\varepsilon})}{\mathbf{m}([0,\zeta_{0,\varepsilon}])}\to\tau_0 \quad\text{and}\quad \frac{a_\varepsilon(\xi_{n,\varepsilon})}{\mathbf{m}([\xi_{n,\varepsilon},1])}\to\tau_n \quad \text{as } \varepsilon\to 0.$$

These estimates imply estimates (2.1) and (2.2). Conversely, suppose that parts (1a), (1c) and (1d) of Assumption 2.1 together with estimates (2.1) and (2.2) hold. For $j\in[1\,.\,.\,n]$ let $h_j\colon\ ]0,\varepsilon_0[\ \to\mathbb{R}$ be an arbitrary positive function with $h_j(\varepsilon)\to\infty$ as $\varepsilon\to 0$. Then, for each $\varepsilon\in\ ]0,\varepsilon_0[$ there is a uniquely determined continuous function $\widetilde{a}_\varepsilon\colon [0,1]\to\mathbb{R}$ such that: for each $j\in[1\,.\,.\,n]$, $\widetilde{a}_\varepsilon(x)=h_j(\varepsilon)$ on $K'_{j,\varepsilon}$; for each $j\in[1\,.\,.\,n-1]$, $\widetilde{a}_\varepsilon(x)=\tau_j\cdot\mathbf{m}([\xi_{j,\varepsilon},\zeta_{j,\varepsilon}])$ on $[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]$; $\widetilde{a}_\varepsilon(x)=\tau_0\cdot\mathbf{m}([0,\zeta_{0,\varepsilon}])$ on $[0,\zeta_{0,\varepsilon}]$, $\widetilde{a}_\varepsilon(x)=\tau_n\cdot\mathbf{m}([\xi_{n,\varepsilon},x_n])$ on $[\xi_{n,\varepsilon},x_n]$, and $\widetilde{a}_\varepsilon$ is affine on $[\xi'_{j,\varepsilon},\xi_{j,\varepsilon}])$ for $j\in[0\,.\,.\,n-1]$ and on $[\zeta_{j,\varepsilon},\zeta'_{j,\varepsilon}])$ for $j\in[1\,.\,.\,n]$. With this choice of $(\widetilde{a}_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$ and estimates (2.1) and (2.2) it is easily proved that parts (1b), (2a) and (2b) of Assumption 2.1 also hold. Each function $\widetilde{a}_\varepsilon$ is constant on $[0,\zeta_{0,\varepsilon}]$ and $[\xi_{n,\varepsilon},x_n]$ so it can be extended to a continuous function (again denoted by $\widetilde{a}_\varepsilon$) defined on all of $\mathbb{R}$. Choose a family $(b_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$ in $]0,\infty[$ with $b_\varepsilon<\inf\limits_{[0,1]}\widetilde{a}_\varepsilon$ for $\varepsilon\in\ ]0,\varepsilon_0[$ and such that $b_\varepsilon/\mathbf{m}(\Gamma_\varepsilon)\to 0$ and $b_\varepsilon/\mathbf{m}(\Gamma_{j,\varepsilon})\to 0$ as $\varepsilon\to 0^+$, where $(\Gamma_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$ is any family occurring in Assumption 2.1 (1d) and (2a), and $(\Gamma_{j,\varepsilon})_{\varepsilon\in]0,\varepsilon_0[}$, $j\in[0\,.\,.\,n]$, is any family occurring in Assumption 2.1 (2b). Applying to the function $\widetilde{a}_\varepsilon$ the usual smoothing procedure via mollifiers, we obtain, for every $\varepsilon\in\ ]0,\varepsilon_0[$ a smooth function a_ε on $\mathbb{R}$, which differs from $\widetilde{a}_\varepsilon$ by at most b_ε on $[0,1]$. We see that with this choice of the family $(a_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$, parts (1b), (2a) and (2b) of Assumption 2.1 also hold. □

Remark 2.4. If Assumption 1.1 is satisfied, then define $\zeta_{j,\varepsilon}=x_j+\varepsilon l_j$, $\zeta'_{j,\varepsilon}=x_j+\varepsilon l'_j(\varepsilon)$ for $j\in[0\,.\,.\,n-1]$ and $\xi_{j,\varepsilon}=x_j-\varepsilon l_j$, $\xi'_{j,\varepsilon}=x_j-\varepsilon l'_j(\varepsilon)$ for $j\in[1\,.\,.\,n]$. Moreover, set $\tau_j=(b_j/2l_j)$ for $j\in[1\,.\,.\,n-1]$ and $\tau_j=(b_j/l_j)$ for

$j \in \{0,n\}$. With these definitions, taking $\varepsilon_0 \in]0,\infty[$ smaller, if necessary, it is easy to prove that Assumption 2.1 holds.

Remark 2.5. On the other hand, Assumption 2.1 may hold without Assumption 1.1 being satisfied. In fact, let $n \in \mathbb{N}$, $(x_j)_{j\in[0..n]}$ be a strictly increasing sequence in $[0,1]$, with $x_0 = 0$ and $x_n = 1$ and $(\tau_j)_{j\in[0..n]}$ be a sequence in $]0,\infty[$. For each $\varepsilon \in]0,\infty[$ define $\zeta_{j,\varepsilon} = x_j + \varepsilon^2$, $\zeta'_{j,\varepsilon} = x_j + \varepsilon^2 + \varepsilon^3$, for $j \in [0\,.\,.\,n-1]$, and $\xi_{j,\varepsilon} = x_j - \varepsilon^2$ and $\xi'_{j,\varepsilon} = x_j - \varepsilon^2 - \varepsilon^3$, for $j \in [1\,.\,.\,n]$. Choose $\varepsilon_0 \in]0,\infty[$ such that $x_{j-1} < \zeta_{j-1,\varepsilon} < \zeta'_{j-1,\varepsilon} < \xi'_{j,\varepsilon} < \xi_{j,\varepsilon} < x_j$, $j \in [1\,.\,.\,n]$, $\varepsilon \in]0,\varepsilon_0[$. If $(\Gamma_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$ is any of the following families:

$$([\xi'_{j,\varepsilon},\zeta'_{j,\varepsilon}])_{\varepsilon\in]0,\varepsilon_0[},\quad ([x_0,\zeta'_{0,\varepsilon}])_{\varepsilon\in]0,\varepsilon_0[},\quad ([\xi'_{n,\varepsilon},x_n])_{\varepsilon\in]0,\varepsilon_0[},\quad j\in[1\,.\,.\,n-1]\,,$$

it is easy to show that $\mathbf{m}(\Gamma_\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence (1a), (1c) and (1d) of Assumption 2.1 are satisfied. Moreover,

$$\frac{\mathbf{m}([\xi'_{j,\varepsilon},\xi_{j,\varepsilon}]) + \mathbf{m}([\zeta_{j,\varepsilon},\zeta'_{j,\varepsilon}])}{\mathbf{m}([\xi_{j,\varepsilon},\zeta_{j,\varepsilon}])} = \frac{2\varepsilon^3}{2\varepsilon^2} = \varepsilon \to 0, \quad \text{as } \varepsilon \to 0,\ j \in [1\,.\,.\,n-1],$$

and

$$\frac{\mathbf{m}([\zeta_{0,\varepsilon},\zeta'_{0,\varepsilon}])}{\mathbf{m}([x_0,\zeta_{0,\varepsilon}])} = \frac{\mathbf{m}([\xi'_{n,\varepsilon},\xi_{n,\varepsilon}])}{\mathbf{m}([\xi_{n,\varepsilon},x_n])} = \frac{\varepsilon^3}{\varepsilon^2} = \varepsilon \to 0 \quad \text{as } \varepsilon \to 0.$$

It follows from Proposition 2.3 that there is a family $(a_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$, such that parts (1b), (2a) and (2b) of Assumption 2.1 are also satisfied.

Suppose that, with this choice of $(x_j)_{j\in[0..n]}$ and the family $(a_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$, Assumption 1.1 from [5] is satisfied and fix a $j \in [1\,.\,.\,n-1]$. Then, in particular, there is a constant $l = l_j$ and a positive function a'_j of $\varepsilon \in]0,\varepsilon_0[$ such that $a'_j(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $a_\varepsilon(x) \le a'_j(\varepsilon)$ for $x \in [x_j - \varepsilon l, x_j + \varepsilon l]$ (simply take $a'_j(\varepsilon) := \varepsilon b'_j(\varepsilon)$, where b'_j is as in Assumption 1.1). Now there is an $\varepsilon_1 \in]0,\varepsilon_0[$ such that $\zeta'_{j-1,\varepsilon} = x_{j-1} + \varepsilon^2 + \varepsilon^3 < x_j - \varepsilon l < x_j - \varepsilon^2 - \varepsilon^3 = \xi'_{j,\varepsilon} < x_j$ for $\varepsilon \in]0,\varepsilon_1[$. For each such ε choose a point y_ε with $x_j - \varepsilon l < y_\varepsilon < \xi'_{j,\varepsilon}$. Then, on the one hand, $a_\varepsilon(y_\varepsilon) \ge \inf_{[\zeta'_{j-1,\varepsilon},\xi'_{j,\varepsilon}]} a_\varepsilon \to \infty$ as $\varepsilon \to 0$ and, on the other hand, $a_\varepsilon(y_\varepsilon) \le a'_j(\varepsilon) \to 0$ as $\varepsilon \to 0$. This is a contradiction.

For the rest of this section we suppose that Assumption 2.1 holds.

In the sequel we write $H^1 = H^1(0,1)$ and $L^2 = L^2(0,1)$. Let $\widetilde{b}_\varepsilon \colon H^1 \times H^1 \to \mathbb{R}$ be the bilinear form defined by

$$\widetilde{b}_\varepsilon(u,v) = \int_0^1 a_\varepsilon \cdot u' \cdot v'\,dx, \quad u,v \in H^1.$$

We fix numbers ρ, $\sigma \in [0,1]$.

For $\varepsilon \in]0,\varepsilon_0[$ let A_ε be the set of all pairs (u,w) with $u \in H^1$ and $w \in L^2$ such that $a_\varepsilon u \in H^1$, $\rho u(0) - (1-\rho)a_\varepsilon(0)u'(0) = \sigma u(1) + (1-\sigma)a_\varepsilon(1)u'(1) = 0$ and $w = -(a_\varepsilon u)'$.

It is known that A_ε is (the graph of) a densely defined nonnegative self-adjoint linear operator in L^2.

Let $\langle\,\cdot\,,\,\cdot\,\rangle_{L^2}$ be the standard scalar product on L^2 and define the bilinear form $b_\varepsilon\colon Z\times Z\to\mathbb{R}$ by

$$b_\varepsilon(u,v)=\widetilde{b}_\varepsilon(u,v)+\delta_0u(0)v(0)+\delta_1u(1)v(1),\quad u,v\in Z.$$

Here we have the following cases:

(a) $\rho<1$ and $\sigma<1$. Then $Z:=H^1$, $\delta_0:=\rho/(1-\rho)$ and $\delta_1:=\sigma/(1-\sigma)$;

(b) $\rho=1$ and $\sigma<1$. Then $Z:=\{\,u\in H^1\mid u(0)=0\,\}$, $\delta_0:=0$ and $\delta_1:=\sigma/(1-\sigma)$;

(c) $\rho<1$ and $\sigma=1$. Then $Z:=\{\,u\in H^1\mid u(1)=0\,\}$, $\delta_0:=\rho/(1-\rho)$ and $\delta_1:=0$;

(d) $\rho=1$ and $\sigma=1$. Then $Z:=\{\,u\in H^1\mid u(0)=u(1)=0\,\}$, $\delta_0:=0$ and $\delta_1:=0$.

It follows that the operator A_ε is generated by the pair $(b_\varepsilon,\langle\,\cdot\,,\cdot\,\rangle_{L^2})$. This means that $(u,w)\in A_\varepsilon$ if and only if $u\in Z$, $w\in L^2$ and $b_\varepsilon(u,v)=\langle w,v\rangle_{L^2}$ for all $v\in Z$. Let $(\lambda_{l,\varepsilon})_l$ denote the increasing sequence of eigenvalues of A_ε (which are all simple).

Now define the 'limit' bilinear form $b_0\colon\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ by

$$b_0(y,z)=\widetilde{\tau}_0y_1z_1+\sum_{j=1}^{n-1}\tau_j(y_{j+1}-y_j)(z_{j+1}-z_j)+\widetilde{\tau}_ny_nz_n,$$

where

$$\widetilde{\tau}_0=\frac{\tau_0\rho^2+\delta_0\tau_0^2(1-\rho)^2}{(\rho+\tau_0(1-\rho))^2}\quad\text{and}\quad\widetilde{\tau}_n=\frac{\tau_n\sigma^2+\delta_1\tau_n^2(1-\sigma)^2}{(\sigma+\tau_n(1-\sigma))^2},$$

and the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}}$ on $\mathbb{R}^n$ by

$$\langle y,z\rangle_{\mathbb{L}}=\sum_{j=1}^{n}L_jy_jz_j,\quad y=(y_1,\dots,y_n),\,z=(z_1,\dots,z_n)\in\mathbb{R}^n.$$

(2.3) Let $A_0\colon\mathbb{R}^n\to\mathbb{R}^n$ be the linear map defined by the pair $(b_0,\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}})$.

The matrix representation of A_0 in terms of the standard basis on $\mathbb{R}^n$ is given as $\widetilde{A}=\mathbb{L}^{-1}B$, where $\mathbb{L}=\operatorname{diag}(L_1,\dots,L_n)$ and B is the matrix

$$\begin{pmatrix}
m_1 & r_1 & 0 & 0 & 0 & \cdots & 0 & 0\\
r_1 & m_2 & r_2 & 0 & 0 & \cdots & 0 & 0\\
0 & r_2 & m_3 & r_3 & 0 & \cdots & 0 & 0\\
0 & 0 & r_3 & m_4 & r_4 & \cdots & 0 & 0\\
\vdots & \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & r_{n-3} & m_{n-2} & r_{n-2} & 0\\
0 & 0 & \cdots & 0 & 0 & r_{n-2} & m_{n-1} & r_{n-1}\\
0 & 0 & \cdots & 0 & 0 & 0 & r_{n-1} & m_n
\end{pmatrix},$$

with

$$m_1 = \widetilde{\tau}_0 + \tau_1, \qquad m_n = \widetilde{\tau}_n + \tau_{n-1},$$
$$m_k = \tau_{k-1} + \tau_k, \qquad k \in [2\,.\,.\,n-1]$$

and $r_k = -\tau_k$, $k \in [1\,.\,.\,n-1]$. Since the matrix $\widetilde{A}$ is tridiagonal with non-zero product of the off-diagonal entries, it has n distinct eigenvalues.

It follows that the map A_0 is $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}}$-symmetric and all of its eigenvalues are simple. Denote by $(\lambda_{l,0})_{l\in[1..n]}$ the increasing sequence of eigenvalues of A_0.

We can now state the main result of this paper.

Theorem 2.6. *With the above notation and definitions the following assertions hold:*

(a) $\lambda_{n+1,\varepsilon} \to \infty$ *as* $\varepsilon \to 0$.
(b) *For each* $l \in [1\,.\,.\,n]$, $\lambda_{l,\varepsilon} \to \lambda_{l,0}$ *as* $\varepsilon \to 0$.
(c) *There is a family* $(\widehat{\varphi}_{l,\varepsilon})_{(l,\varepsilon)\in[1..n]\times[0,\varepsilon_0[}$ *such that if* $(l,\varepsilon) \in [1\,.\,.\,n] \times]0,\varepsilon_0[$ *then* $\widehat{\varphi}_{l,\varepsilon}$ *is an eigenfunction of* A_ε *corresponding to* $\lambda_{l,\varepsilon}$ *with* $\|\widehat{\varphi}_{l,\varepsilon}\|_{L^2} = 1$, *if* $l \in [1\,.\,.\,n]$ *then* $\widehat{\varphi}_{l,0}$ *is an eigenvector of* A_0 *corresponding to* $\lambda_{l,0}$ *with* $\|\widehat{\varphi}_{l,0}\|_{\mathbb{L}} = 1$ *and such that for* $j \in [1\,.\,.\,n]$

$$\sup_{x\in K_{j,\varepsilon}} |\widehat{\varphi}_{l,\varepsilon}(x) - \widehat{\varphi}_{l,0,j}| \to 0, \qquad \text{as } \varepsilon \to 0,$$

where $\widehat{\varphi}_{l,0,j}$ *is the* j*-th component of the vector* $\widehat{\varphi}_{l,0} \in \mathbb{R}^n$. *Moreover, there is an* $\varepsilon' \in]0,\varepsilon_0[$ *such that*

$$\sup_{\varepsilon\in[0,\varepsilon']}\ \sup_{l\in[1..n]}\ \sup_{x\in[0,1]} |\widehat{\varphi}_{l,\varepsilon}(x)| < \infty.$$

3. Proof of the main result

This section is devoted to the proof of Theorem 2.6, which is accomplished through a series of technical lemmas. Although we follow, in spirit, the proof of the spectral convergence result from [5], the various technical differences require a detailed treatment.

For the rest of this section we suppose that Assumption 2.1 holds.

Lemma 3.1. *If* $M \in]0,\infty[$, $I \subset [0,1]$ *is a compact interval,* $a\colon I \to \mathbb{R}$ *is a continuous positive function and* $\varphi \in H^1$ *is such that* $\int_I a\cdot(\varphi')^2\,dx \le M$, *then*

$$|\varphi(x) - \varphi(y)|^2 \le M\,\frac{\mathbf{m}(I)}{\inf_I a}, \quad x,y \in I.$$

Proof. For $x, y \in I$,

$$|\varphi(x)-\varphi(y)|^2 = \bigg|\int_y^x \varphi'\,dx\bigg|^2 \le \bigg(\int_I |\varphi'|\,dx\bigg)^2 \le \int_I (\varphi')^2\,dx \int_I 1^2\,dx$$
$$\le \frac{1}{\inf\limits_I a}\int_I a\cdot(\varphi')^2\,dx\cdot\mathbf{m}(I) \le M\,\frac{\mathbf{m}(I)}{\inf\limits_I a}.$$

This proves the lemma. $\square$

For each $\varepsilon \in]0, \varepsilon_0[$ and $j \in [1..n]$ define $\psi_{j,\varepsilon}\colon [0,1] \to \mathbb{R}$ as the uniquely determined continuous function such that

(a) if $j \in [2..n-1]$, then $\psi_{j,\varepsilon}(x) = 1$ for $x \in [\zeta_{j-1,\varepsilon}, \xi_{j,\varepsilon}]$, $\psi_{j,\varepsilon}(x) = 0$ for $x \notin [\xi_{j-1,\varepsilon}, \zeta_{j,\varepsilon}]$ and $\psi_{j,\varepsilon}$ is affine on each of the intervals $[\xi_{j-1,\varepsilon}, \zeta_{j-1,\varepsilon}]$ and $[\xi_{j,\varepsilon}, \zeta_{j,\varepsilon}]$.

(b) $\psi_{1,\varepsilon}(x) = 1$ for $x \in [\zeta_{0,\varepsilon}, \xi_{1,\varepsilon}]$, $\psi_{1,\varepsilon}(x) = 0$ for $x \notin [0, \zeta_{1,\varepsilon}]$, $\psi_{1,\varepsilon}(x) = 1 + (\rho/(\rho\zeta_{0,\varepsilon} + (1-\rho)a_\varepsilon(0)))(x - \zeta_{0,\varepsilon})$ for $x \in [0, \zeta_{0,\varepsilon}]$ and affine on the interval $[\xi_{1,\varepsilon}, \zeta_{1,\varepsilon}]$.

(c) $\psi_{n,\varepsilon}(x) = 1$ for $x \in [\zeta_{n-1,\varepsilon}, \xi_{n,\varepsilon}]$, $\psi_{n,\varepsilon}(x) = 0$ for $x \notin [\xi_{n-1,\varepsilon}, x_n]$, $\psi_{n,\varepsilon}(x) = 1 - (\sigma/(\sigma(1-\xi_{n,\varepsilon}) + (1-\sigma)a_\varepsilon(1)))(x - \xi_{n,\varepsilon})$ for $x \in [\xi_{n,\varepsilon}, 1]$ and affine on the interval $[\xi_{n-1,\varepsilon}, \zeta_{n-1,\varepsilon}]$.

Notice that $\psi_{j,\varepsilon} \in Z$ for all $j \in [1..n]$ and $\varepsilon \in]0, \varepsilon_0[$.

For each $\varepsilon \in]0, \varepsilon_0[$, let W_ε be the span of the functions $\psi_{j,\varepsilon}$, $j \in [1..n]$, i.e. the n-dimensional subspace of Z given by

$$W_\varepsilon = \left\{ \sum_{j=1}^n u_j \psi_{j,\varepsilon} \;\middle|\; u_j \in \mathbb{R},\ \text{for } j \in [1..n] \right\}.$$

LEMMA 3.2. *There exists a $C_1' \in]0, \infty[$ and an $\varepsilon_1' \in]0, \varepsilon_0[$ such that*

$$\frac{\widetilde{b}_\varepsilon(u,u)}{\|u\|_{L^2}^2} \le C_1' \quad \textit{for all } \varepsilon \in]0, \varepsilon_1'] \textit{ and all } u \in W_\varepsilon \textit{ with } u \neq 0.$$

PROOF. There is a $c \in]0, \infty[$ and an $\varepsilon_1 \in]0, \varepsilon_0[$ such that

$$c \le \min_{j \in [1..n]} (\xi_{j,\varepsilon} - \zeta_{j-1,\varepsilon}), \quad \text{for all } \varepsilon \in]0, \varepsilon_1].$$

Let $\varepsilon \in]0, \varepsilon_1]$ and let $u \in W_\varepsilon$ be arbitrary with $\|u\|_{L^2}^2 = 1$. Hence $u = \sum_{j=1}^n u_j \psi_{j,\varepsilon}$ with $u_j \in \mathbb{R}$, for $j \in [1..n]$. Thus

$$1 = \|u\|_{L^2}^2 \ge \sum_{j=1}^n \int_{\zeta_{j-1,\varepsilon}}^{\xi_{j,\varepsilon}} u^2\,dx = \sum_{j=1}^n u_j^2 (\xi_{j,\varepsilon} - \zeta_{j-1,\varepsilon}) \ge c \sum_{j=1}^n u_j^2,$$

so $|u_j| \le c^{-1/2}$ for all $j \in [1..n]$. Notice that $u'(x) = 0$ for $x \in \bigcup_{j=1}^n [\zeta_{j-1,\varepsilon}, \xi_{j,\varepsilon}]$.

Moreover, for $j \in [1..n-1]$ and $x \in [\xi_{j,\varepsilon}, \zeta_{j,\varepsilon}]$,

$$u(x) = u_j \psi_{j,\varepsilon}(x) + u_{j+1} \psi_{j+1,\varepsilon}(x)$$

with $|\psi_{j,\varepsilon}'(x)| = |\psi_{j+1,\varepsilon}'(x)| = (1/(\zeta_{j,\varepsilon} - \xi_{j,\varepsilon}))$.

For $x \in [0, \zeta_{0,\varepsilon}]$, $u(x) = u_1\psi_{1,\varepsilon}(x)$ and for $x \in [\xi_{n,\varepsilon}, 1]$, $u(x) = u_n\psi_{n,\varepsilon}(x)$. It follows that

$$\begin{aligned}
\widetilde{b}_\varepsilon(u,u) &= \int_0^1 a_\varepsilon \cdot (u')^2\,dx = \sum_{j=1}^n \int_{x_{j-1}}^{x_j} a_\varepsilon \cdot (u')^2\,dx \\
&= \sum_{j=1}^n \bigg(\int_{x_{j-1}}^{\zeta_{j-1,\varepsilon}} a_\varepsilon \cdot (u')^2\,dx + \int_{\xi_{j,\varepsilon}}^{x_j} a_\varepsilon \cdot (u')^2\,dx \bigg) \\
&= \sum_{j=0}^{n-1} \int_{x_j}^{\zeta_{j,\varepsilon}} a_\varepsilon \cdot (u')^2\,dx + \sum_{j=1}^n \int_{\xi_{j,\varepsilon}}^{x_j} a_\varepsilon \cdot (u')^2\,dx \\
&= \int_0^{\zeta_{0,\varepsilon}} a_\varepsilon \cdot (u')^2\,dx + \sum_{j=1}^{n-1} \int_{\xi_{j,\varepsilon}}^{\zeta_{j,\varepsilon}} a_\varepsilon \cdot (u')^2\,dx + \int_{\xi_{n,\varepsilon}}^1 a_\varepsilon \cdot (u')^2\,dx.
\end{aligned}$$

Now

$$\begin{aligned}
\int_0^{\zeta_{0,\varepsilon}} a_\varepsilon \cdot (u')^2\,dx &= \int_0^{\zeta_{0,\varepsilon}} a_\varepsilon \cdot (u_1\psi'_{1,\varepsilon})^2\,dx \\
&\le \int_0^{\zeta_{0,\varepsilon}} \Big(\sup_{[0,\zeta_{0,\varepsilon}]} a_\varepsilon \Big) \frac{\rho^2}{c\,(\rho\zeta_{0,\varepsilon} + (1-\rho)a_\varepsilon(0))^2}\,dx \\
&= \bigg(\frac{1}{\zeta_{0,\varepsilon}} \sup_{[0,\zeta_{0,\varepsilon}]} a_\varepsilon \bigg) \frac{\rho^2}{c(\rho + (1-\rho)(a_\varepsilon(0)/\zeta_{0,\varepsilon}))^2} \to \frac{\tau_0\rho^2}{c(\rho + (1-\rho)\tau_0)^2},
\end{aligned}$$

as $\varepsilon \to 0$.

$$\begin{aligned}
\sum_{j=1}^{n-1} \int_{\xi_{j,\varepsilon}}^{\zeta_{j,\varepsilon}} a_\varepsilon \cdot (u')^2\,dx &= \sum_{j=1}^{n-1} \int_{\xi_{j,\varepsilon}}^{\zeta_{j,\varepsilon}} a_\varepsilon \cdot \big(u_j\psi'_{j,\varepsilon} + u_{j+1}\psi'_{j+1,\varepsilon}\big)^2\,dx \\
&\le \sum_{j=1}^{n-1} \int_{\xi_{j,\varepsilon}}^{\zeta_{j,\varepsilon}} \Big(\sup_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} a_\varepsilon \Big) \frac{4}{c(\zeta_{j,\varepsilon} - \xi_{j,\varepsilon})^2}\,dx \\
&= \sum_{j=1}^{n-1} \Big(\sup_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} a_\varepsilon \Big) \frac{4}{c(\zeta_{j,\varepsilon} - \xi_{j,\varepsilon})} \to \frac{4}{c} \sum_{j=1}^{n-1} \tau_j,
\end{aligned}$$

as $\varepsilon \to 0$, and

$$\begin{aligned}
\int_{\xi_{n,\varepsilon}}^1 a_\varepsilon \cdot (u')^2\,dx &= \int_{\xi_{n,\varepsilon}}^1 a_\varepsilon \cdot \big(u_n\psi'_{n,\varepsilon}\big)^2\,dx \\
&\le \int_{\xi_{n,\varepsilon}}^1 \Big(\sup_{[\xi_{n,\varepsilon},1]} a_\varepsilon \Big) \frac{\sigma^2}{c(\sigma(1-\xi_{n,\varepsilon}) + (1-\sigma)a_\varepsilon(1))^2}\,dx \\
&= \bigg(\frac{1}{(1-\xi_{n,\varepsilon})} \sup_{[\xi_{n,\varepsilon},1]} a_\varepsilon \bigg) \frac{\sigma^2}{c(\sigma + (1-\sigma)(a_\varepsilon(1)/(1-\xi_{n,\varepsilon})))^2} \\
&\to \frac{\tau_n\sigma^2}{c(\sigma + (1-\sigma)\tau_n)^2},
\end{aligned}$$

as $\varepsilon \to 0$. These estimates prove the assertion. $\square$

Notation. For $C \in \,]0,\infty[$ and $\varepsilon \in \,]0,\varepsilon_0[$ let $\widetilde{B}_{\varepsilon,C}$ be the set of all $u \in H^1$ such that $\widetilde{b}_\varepsilon(u,u) + \|u\|^2_{L^2} \le C^2$.

LEMMA 3.3. *The following two assertions hold*:

(a) *There exist an* $\varepsilon_2' \in \,]0,\varepsilon_0[$ *and a* $C_2' \in \,]0,\infty[$ *such that for every* $v \in H^1$ *and every* $\varepsilon \in \,]0,\varepsilon_2']$

$$\sup_{x,y\in[0,1]} |v(x)-v(y)| \le C_2' \widetilde{b}_\varepsilon(v,v)^{1/2} \tag{3.1}$$

and

$$\sup_{x\in[0,1]} |v(x)| \le C_2'(\widetilde{b}_\varepsilon(v,v) + \|v\|^2_{L^2})^{1/2}. \tag{3.2}$$

(b) *Let* $M \in \,]0,\infty[$ *be arbitrary. For each* $j \in [1\,.\,.\,n]$ *we have*

$$\sup_{v\in\widetilde{B}_{\varepsilon,M}} \sup_{x,y\in K_{j,\varepsilon}} |v(x)-v(y)| \to 0, \quad \text{as } \varepsilon \to 0. \tag{3.3}$$

PROOF. By our assumptions there are an $\varepsilon_1 \in \,]0,\varepsilon_0[$ and a $C_1 \in \,]0,\infty[$ such that, for $\varepsilon \in \,]0,\varepsilon_1]$,

$$\frac{\mathbf{m}(\Gamma_\varepsilon)}{\inf\limits_{\Gamma_\varepsilon} a_\varepsilon} \le C_1,$$

where Γ_ε is any of the $\ell = 5n$ intervals $[x_{j-1},\zeta_{j-1,\varepsilon}]$, $[\zeta_{j-1,\varepsilon},\zeta'_{j-1,\varepsilon}]$, $[\zeta'_{j-1,\varepsilon},\xi'_{j,\varepsilon}]$, $[\xi'_{j,\varepsilon},\xi_{j,\varepsilon}]$, $[\xi_{j,\varepsilon},x_j]$, $j \in [1\,.\,.\,n]$. Thus, whenever $\varepsilon \in \,]0,\varepsilon_1]$ and $v \in H^1$, it follows from Lemma 3.1 that

$$\operatorname{diam} v(\Gamma_\varepsilon) \le (C_1 \widetilde{b}_\varepsilon(v,v))^{1/2}.$$

The above ℓ intervals can be ordered to form a sequence $(I_j)_{j\in[1..\ell]}$ such that for $j \in [1\,.\,.\,\ell-1]$ the endpoint of I_j is the initial point of I_{j+1}. Consequently,

$$\operatorname{diam} v([0,1]) \le 5n(C_1 \widetilde{b}_\varepsilon(v,v))^{1/2},$$

so

$$|v(x)| \le |v(y)| + 5n(C_1\widetilde{b}_\varepsilon(v,v))^{1/2}, \quad x,\, y \in [0,1]\,,$$

which implies

$$|v(x)| \le C_2(\widetilde{b}_\varepsilon(v,v) + \|v\|^2_{L^2})^{1/2}, \quad x \in [0,1]\,,$$

where $C_2 = 1 + 5nC_1^{1/2}$. These estimates prove part (a) of the lemma. Now, let $M \in \,]0,\infty[$ be arbitrary and for each $\varepsilon \in \,]0,\varepsilon_0[$ let β_ε be the maximum of all the values

$$M\left(\frac{\mathbf{m}(\Gamma_\varepsilon)}{\inf\limits_{\Gamma_\varepsilon} a_\varepsilon}\right)^{1/2},$$

where Γ_ε is any of the intervals $[\zeta_{j-1,\varepsilon},\zeta'_{j-1,\varepsilon}]$, $[\zeta'_{j-1,\varepsilon},\xi'_{j,\varepsilon}]$, $[\xi'_{j,\varepsilon},\xi_{j,\varepsilon}]$, $j \in [1\,.\,.\,n]$. For $j \in [1\,.\,.\,n]$ Lemma 3.1 implies that

$$\sup_{v\in\widetilde{B}_{\varepsilon,M}} \sup_{x,y\in K_{j,\varepsilon}} |v(x)-v(y)| \le 3\beta_\varepsilon.$$

Now Assumption 2.1 implies that $\beta_\varepsilon \to 0$ as $\varepsilon \to 0$. This proves the second part of the lemma. □

Lemma 3.4. *Let $(\varepsilon_m)_m$ be a null sequence in $]0,\varepsilon_0[$. Let $(u_m)_m$, $(v_m)_m$ be sequences in H^1 such that $u_m \in \widetilde{B}_{\varepsilon_m,M}$ and $v_m \in \widetilde{B}_{\varepsilon_m,M'}$ for some M, $M' \in]0,\infty[$ and all $m \in \mathbb{N}$. Let $(\gamma_{j,m})_{j\in[1..n],m\in\mathbb{N}}$ be such that $\gamma_{j,m} \in K_{j,\varepsilon_m}$ for $m \in \mathbb{N}$ and $j \in [1\,..\,n]$. Then*

$$\langle u_m, v_m\rangle_{L^2} - \sum_{j=1}^n L_j u_m(\gamma_{j,m}) v_m(\gamma_{j,m}) \to 0, \quad \text{as } m \to \infty.$$

Proof. For each $m \in \mathbb{N}$ we have

$$\begin{aligned}\int_0^1 u_m v_m\,dx &= \sum_{j=1}^n \int_{x_{j-1}}^{x_j} u_m v_m\,dx \\ &= \sum_{j=1}^n \int_{x_{j-1}}^{\zeta_{j-1,\varepsilon_m}} u_m v_m\,dx + \sum_{j=1}^n \int_{K_{j,\varepsilon_m}} u_m v_m\,dx + \sum_{j=1}^n \int_{\xi_{j,\varepsilon_m}}^{x_j} u_m v_m\,dx.\end{aligned}$$

It follows from Lemma 3.3 that all functions u_m and v_m are uniformly bounded by a common constant C. Thus Assumption 2.1 implies that

$$\sum_{j=1}^n \int_{x_{j-1}}^{\zeta_{j-1,\varepsilon_m}} u_m v_m\,dx + \sum_{j=1}^n \int_{\xi_{j,\varepsilon_m}}^{x_j} u_m v_m\,dx \to 0, \quad \text{as } m \to \infty. \tag{3.4}$$

For each $j \in [1\,..\,n]$,

$$\begin{aligned}\int_{K_{j,\varepsilon_m}} u_m v_m\,dx = {}& \mathbf{m}(K_{j,\varepsilon_m}) u_m(\gamma_{j,m}) v_m(\gamma_{j,m}) \\ &+ \int_{K_{j,\varepsilon_m}} \big(u_m v_m - u_m(\gamma_{j,m}) v_m(\gamma_{j,m})\big)\,dx.\end{aligned}$$

For $x \in K_{j,\varepsilon_m}$ we have

$$\begin{aligned}&|u_m(x)v_m(x) - u_m(\gamma_{j,m})v_m(\gamma_{j,m})| \\ &\qquad \le |u_m(x) - u_m(\gamma_{j,m})| \cdot |v_m(x)| + |v_m(x) - v_m(\gamma_{j,m})| \cdot |u_m(\gamma_{j,m})| \\ &\qquad \le C(|u_m(x) - u_m(\gamma_{j,m})| + |v_m(x) - v_m(\gamma_{j,m})|).\end{aligned}$$

Again, Lemma 3.3 implies that

$$\sup_{x\in K_{j,\varepsilon_m}} (|u_m(x) - u_m(\gamma_{j,m})| + |v_m(x) - v_m(\gamma_{j,m})|) \to 0, \quad \text{as } m \to \infty.$$

Therefore

$$\int_{K_{j,\varepsilon_m}} \big(u_m v_m - u_m(\gamma_{j,m}) v_m(\gamma_{j,m})\big)\,dx \to 0, \quad \text{as } m \to \infty. \tag{3.5}$$

Moreover, it follows from Assumption 2.1 that $\mathbf{m}(K_{j,\varepsilon_m}) - L_j \to 0$, as $m \to \infty$, for each $j \in [1\,..\,n]$. This together with (3.4) and (3.5) implies the assertion of the lemma. □

Corollary 3.5. *Let $M' \in \,]0,\infty[$ and $(\varepsilon_m)_m$ be a null sequence in $]0,\varepsilon_0[$. Let $(v_m)_m$ and $(\gamma_{j,m})_m$, $j \in [1\,..\,n]$, be sequences such that $v_m \in \widetilde{B}_{\varepsilon_m,M'}$ and $\gamma_{j,m} \in K_{j,\varepsilon_m}$ for $m \in \mathbb{N}$ and $j \in [1\,..\,n]$. Then, for each $j \in [1\,..\,n]$,*

$$\langle \psi_{j,\varepsilon_m}, v_m \rangle_{L^2} - L_j v_m(\gamma_{j,m}) \to 0, \quad \text{as } m \to \infty.$$

Proof. Lemma 3.2 and the fact that the functions $u_m = \psi_{j,\varepsilon_m}$, $j \in [1\,..\,n]$, $m \in \mathbb{N}$, are nonnegative and bounded by 1 imply that $u_m \in \widetilde{B}_{\varepsilon_m,M}$ for some constant $M \in \,]0,\infty[$ and for all $m \in \mathbb{N}$. Hence the assumptions of Lemma 3.4 are satisfied. Now that lemma implies that, for each $j \in [1\,..\,n]$,

$$\int_0^1 \psi_{j,\varepsilon_m} v_m \, dx - \sum_{l=1}^n L_l \psi_{j,\varepsilon_m}(\gamma_{l,m}) v_m(\gamma_{l,m}) \to 0, \quad \text{as } m \to \infty.$$

The definition of the map ψ_{j,ε_m}, $m \in \mathbb{N}$, implies that $\psi_{j,\varepsilon_m}(\gamma_{l,m}) = 1$ if $j = l$ and $\psi_{j,\varepsilon_m}(\gamma_{l,m}) = 0$ otherwise and so

$$\sum_{l=1}^n L_l \psi_{j,\varepsilon_m}(\gamma_{l,m}) v_m(\gamma_{l,m}) = L_j v_m(\gamma_{j,m})$$

and this completes the proof. □

Lemma 3.6. *Let $\varepsilon_2' \in \,]0,\varepsilon_0[$ be as in Lemma 3.3. Then for every $M \in \,]0,\infty[$ there is an $\varepsilon_3' = \varepsilon_3'(M) \in \,]0,\varepsilon_2']$ such that $v \notin W_\varepsilon^\perp$ for all $v \in \widetilde{B}_{\varepsilon,M}$ with $\|v\|_{L^2} = 1$ and $\varepsilon \in \,]0,\varepsilon_3']$. (Here, the orthogonal complement is taken with respect to the L^2-scalar product.)*

Proof. Suppose the conclusion of the lemma does not hold. Then, for some $M \in \,]0,\infty[$, there exists a null sequence $(\varepsilon_m)_m$ in $]0,\varepsilon_2']$ such that for each $m \in \mathbb{N}$ there exists a $v_m \in \widetilde{B}_{\varepsilon_m,M} \cap W_{\varepsilon_m}^\perp$ with $\|v_m\|_{L^2} = 1$. Thus $\langle v_m, \psi_{j,\varepsilon_m} \rangle_{L^2} = 0$ for all $m \in \mathbb{N}$ and $j \in [1\,..\,n]$.

For each $j \in [1\,..\,n]$ choose $\gamma_j \in \,]x_{j-1}, x_j[$ independently of $m \in \mathbb{N}$. Then there exists an $m_0 \in \mathbb{N}$ such that $\gamma_j \in K_{j,\varepsilon_m}$ for all $j \in [1\,..\,n]$ and $m \geq m_0$. Now Corollary 3.5 implies that, for each $j \in [1\,..\,n]$,

$$v_m(\gamma_j) \to 0, \quad \text{as } m \to \infty$$

and so Lemma 3.3 implies that

$$v_m(x) \to 0 \quad \text{as } m \to \infty \text{ for each } x \in \,]0,1[\, \setminus \bigcup_{j=1}^{n-1} \{x_j\}.$$

Moreover, it follows from Lemma 3.3 that there exists an $m_1 \in \mathbb{N}$ such that the functions v_m, for all $m \geq m_1$, are pointwise bounded by the same constant. This implies that

$$\int_0^1 v_m^2 \, dx \to 0 \quad \text{as } m \to \infty.$$

However, this is a contradiction as $\int_0^1 v_m^2 \, dx = 1$ for all $m \in \mathbb{N}$. □

Lemma 3.7. *The following statements hold*:

(a) $\lambda_{n+1,\varepsilon} \to \infty$ *as* $\varepsilon \to 0$.

(b) *There exists an* $\varepsilon_4' \in \,]0, \varepsilon_0[$ *and a* $C_3' \in \,]0, \infty[$ *such that*

$$\lambda_{n,\varepsilon} \le C_3' \quad \textit{for all } \varepsilon \in \,]0, \varepsilon_4'].$$

Proof. For each positive integer p and $\varepsilon \in \,]0, \varepsilon_0[$ let $U_{p,\varepsilon}$ be the span of the eigenfunctions $\varphi_{l,\varepsilon}$, for $l \in [1\,.\,.\,p]$. Moreover, let $U_{0,\varepsilon} = \{0\} \subset L^2$. If the first assertion is not true, then there is a null sequence $(\varepsilon_m)_m$ in $]0, \varepsilon_0[$ such that $(\lambda_{n+1,\varepsilon_m})_m$ is bounded by some $C \in \,]0, \infty[$.

We claim that $U_{n+1,\varepsilon_m} \cap W_{\varepsilon_m}^{\perp} = \{0\}$ for all $m \in \mathbb{N}$ large enough. If this is not true, then there is a subsequence $(\varepsilon_m^1)_m$ of $(\varepsilon_m)_m$ such that for each $m \in \mathbb{N}$ there is a v_m in $U_{n+1,\varepsilon_m^1} \cap W_{\varepsilon_m^1}^{\perp}$ with $\|v_m\|_{L^2} = 1$. It easily follows that $b_{\varepsilon_m^1}(v_m, v_m) \le C$ so, (as $\widetilde{b}_{\varepsilon_m^1}(v_m, v_m) \le b_{\varepsilon_m^1}(v_m, v_m)$), $v_m \in \widetilde{B}_{\varepsilon_m^1,K}$ for all $m \in \mathbb{N}$, where $K^2 = C + 1$. However, this contradicts Lemma 3.6 and the claim is proved.

The claim implies that $n + 1 \le n$, a contradiction which implies the first assertion. Let D be the set of all nonnegative integers ℓ_1 such that, for some $\widehat{\varepsilon} \in \,]0, \varepsilon_0[$ the eigenvalue family $(\lambda_{\ell_1,\varepsilon})_{\varepsilon \in]0,\widehat{\varepsilon}]}$ is bounded by some $C_1 \in \,]0, \infty[$. Let ℓ be the supremum of D if D is nonempty and $\ell = 0$ otherwise. From what we have proved so far, we have $\ell \le n$. If $\ell < n$, then $U_{\ell,\varepsilon}^{\perp} \cap W_\varepsilon \ne \{0\}$ for each $\varepsilon \in \,]0, \varepsilon_0[$ and so there is a $w_\varepsilon \ne 0$ lying in $U_{\ell,\varepsilon}^{\perp} \cap W_\varepsilon$. It follows from Lemmas 3.2 and 3.3 that there exist a constant $C' \in \,]0, \infty[$ and an $\varepsilon' \in \,]0, \varepsilon_0[$ such that

$$\begin{aligned}\lambda_{\ell+1,\varepsilon} &= \inf_{w \in Z \setminus \{0\},\, w \in U_{\ell,\varepsilon}^{\perp}} \frac{b_\varepsilon(w, w)}{\|w\|_{L^2}^2} \le \frac{b_\varepsilon(w_\varepsilon, w_\varepsilon)}{\|w_\varepsilon\|_{L^2}^2} \\ &\le \frac{\widetilde{b}_\varepsilon(w_\varepsilon, w_\varepsilon) + \delta_0 w_\varepsilon(0)^2 + \delta_1 w_\varepsilon(1)^2}{\|w_\varepsilon\|_{L^2}^2} \le C'\end{aligned}$$

for all $\varepsilon \in \,]0, \varepsilon']$. This shows in particular, that D is nonempty. Moreover, this also shows that $\ell + 1 \in D$, a contradiction proving that $\ell = n$. Since D is nonempty and finite, we have $\ell \in D$. This proves the second assertion. $\square$

In the sequel

$$\text{(3.6)} \quad \begin{array}{l}\text{for each } \varepsilon \in \,]0, \varepsilon_0[\text{ fix an arbitrary } L^2\text{-orthonormal sequence } (\varphi_{l,\varepsilon})_l \\ \text{such that } \varphi_{l,\varepsilon} \text{ is an eigenfunction of } A_\varepsilon \text{ corresponding to } \lambda_{l,\varepsilon},\ l \in \mathbb{N}.\end{array}$$

Lemma 3.8. *Let* $(\varepsilon_m)_m$ *be a null sequence in* $]0, \varepsilon_0[$ *and* $(\gamma_{j,m})_m$ *be a (double) sequence with* $\gamma_{j,m} \in K_{j,\varepsilon_m}$, *for* $m \in \mathbb{N}$ *and* $j \in [1\,.\,.\,n]$. *For each* $i, j \in [1\,.\,.\,n]$, *we then have*

(a) $\langle \psi_{j,\varepsilon_m}, \varphi_{i,\varepsilon_m} \rangle_{L^2} - L_j \varphi_{i,\varepsilon_m}(\gamma_{j,m}) \to 0$ *as* $m \to \infty$.

(b) $\sum_{j=1}^{n} L_j \varphi_{i,\varepsilon_m}(\gamma_{j,m}) \varphi_{k,\varepsilon_m}(\gamma_{j,m}) \to \delta_{i,k}$ *as* $m \to \infty$.

Proof. This follows from Lemma 3.7, Corollary 3.5 and Lemma 3.4. $\square$

Notation. For each $\varepsilon \in]0, \varepsilon_0[$, define $\Psi_\varepsilon \colon W_\varepsilon \to \mathbb{R}^n$ by

$$\Psi_\varepsilon(u) := \widehat{u} := (u_j)_{j\in[1..n]}, \quad \text{for } u = \sum_{j=1}^n u_j \psi_{j,\varepsilon} \in W_\varepsilon.$$

Consider the $n \times n$ matrix $B_\varepsilon = (b_{i,j,\varepsilon})_{i,j=1}^n$ given by

$$b_{i,j,\varepsilon} = \langle \psi_{i,\varepsilon}, \psi_{j,\varepsilon} \rangle_{L^2}, \quad \text{for } i, j \in [1..n].$$

Assume that

(3.7) $(\alpha_{j,\varepsilon})_{(j,\varepsilon)\in[1..n]\times]0,\varepsilon_0[}$ is an arbitrary family such that $\alpha_{j,\varepsilon} \in K_{j,\varepsilon}$, for $(j,\varepsilon) \in [1..n] \times]0,\varepsilon_0[$.

Let $\|\cdot\|_{\mathbb{L}}$ be the norm on $\mathbb{R}^n$ induced by the scalar product $\langle\cdot,\cdot\rangle_{\mathbb{L}}$. In what follows $\langle\cdot,\cdot\rangle$ (respectively, $\|\cdot\|$) denotes the canonical inner product (respectively, the induced norm) on $\mathbb{R}^n$. Let a, $b \in]0,\infty[$ be such that

$$a\|z\|_{\mathbb{L}} \le \|z\| \le b\|z\|_{\mathbb{L}}, \quad \text{for all } z \in \mathbb{R}^n.$$

Lemma 3.9. *Let $\varepsilon_4' \in]0,\varepsilon_0[$ be as in Lemma 3.7. There is an $\varepsilon_5' \in]0,\varepsilon_4']$ such that for each $\varepsilon \in]0,\varepsilon_5']$, there are constants c_ε, $C_\varepsilon \in]0,\infty[$ such that*

$$c_\varepsilon\|\Psi_\varepsilon(u)\|_{\mathbb{L}} \le \|u\|_{L^2} \le C_\varepsilon\|\Psi_\varepsilon(u)\|_{\mathbb{L}}, \quad u \in W_\varepsilon.$$

Moreover, $c_\varepsilon \to 1$, $C_\varepsilon \to 1$ as $\varepsilon \to 0$.

Proof. Let $\varepsilon \in]0,\varepsilon_4']$ and $u = \sum_{j=1}^n u_{j,\varepsilon}\psi_{j,\varepsilon} \in W_\varepsilon$. Hence

$$\int_0^1 u^2\,dx = \sum_{l,p=1}^n u_{l,\varepsilon}u_{p,\varepsilon}\int_0^1 \psi_{l,\varepsilon}\psi_{p,\varepsilon}\,dx = \sum_{l,p=1}^n u_{l,\varepsilon}u_{p,\varepsilon}\langle\psi_{l,\varepsilon},\psi_{p,\varepsilon}\rangle_{L^2} = \langle B_\varepsilon\widehat{u},\widehat{u}\rangle.$$

Let $(\varepsilon_m)_m$ be an arbitrary null sequence in $]0,\varepsilon_0[$. For each $m \in \mathbb{N}$ and $j \in [1..n]$ define $\gamma_{j,m} = \alpha_{j,\varepsilon_m}$ and $v_m = \psi_{j,\varepsilon_m}$. It follows as in the proof of Corollary 3.5 that there is a constant $M \in]0,\infty[$ such that $v_m \in \widetilde{B}_{\varepsilon_m,M}$ for all $m \in \mathbb{N}$. Now Corollary 3.5 implies that for each i, $j \in [1..n]$,

$$\langle\psi_{i,\varepsilon_m},\psi_{j,\varepsilon_m}\rangle_{L^2} - L_i\psi_{j,\varepsilon_m}(\alpha_{i,\varepsilon_m}) \to 0, \quad \text{as } m \to \infty$$

and so $b_{i,j,\varepsilon_m} = \langle\psi_{i,\varepsilon_m},\psi_{j,\varepsilon_m}\rangle_{L^2} \to L_i\delta_{i,j}$ as $m \to \infty$. Therefore, for each $i, j \in [1..n]$,

$$b_{i,j,\varepsilon} = \langle\psi_{i,\varepsilon},\psi_{j,\varepsilon}\rangle_{L^2} \to L_i\delta_{i,j}, \quad \text{as } \varepsilon \to 0. \tag{3.8}$$

Recall that $\mathbb{L} = \operatorname{diag}(L_1,\dots,L_n)$. It follows from formula (3.8) that $\|B_\varepsilon - \mathbb{L}\|_{L(\mathbb{R}^n,\mathbb{R}^n)} \to 0$ as $\varepsilon \to 0$. Therefore,

$$|\langle B_\varepsilon\widehat{u},\widehat{u}\rangle - \langle\mathbb{L}\widehat{u},\widehat{u}\rangle| \le \|B_\varepsilon - \mathbb{L}\|_{L(\mathbb{R}^n,\mathbb{R}^n)}\|\widehat{u}\|^2 \le b^2\|B_\varepsilon - \mathbb{L}\|_{L(\mathbb{R}^n,\mathbb{R}^n)}\|\widehat{u}\|_{\mathbb{L}}^2. \tag{3.9}$$

Define $\beta_\varepsilon := b^2\|B_\varepsilon - \mathbb{L}\|_{L(\mathbb{R}^n,\mathbb{R}^n)}$. Therefore $\beta_\varepsilon \to 0$ as $\varepsilon \to 0$. Therefore, $\beta_\varepsilon < 1$ for some $\varepsilon_5' \in\,]0,\varepsilon_4']$ and all $\varepsilon \in\,]0,\varepsilon_5']$. Moreover, it follows from (3.9) that

$$(1-\beta_\varepsilon)\|\widehat{u}\|_{\mathbb{L}}^2 = \langle \mathbb{L}\widehat{u},\widehat{u}\rangle - \beta_\varepsilon\|\widehat{u}\|_{\mathbb{L}}^2 \le \langle B_\varepsilon\widehat{u},\widehat{u}\rangle \le \langle \mathbb{L}\widehat{u},\widehat{u}\rangle + \beta_\varepsilon\|\widehat{u}\|_{\mathbb{L}}^2 = (1+\beta_\varepsilon)\|\widehat{u}\|_{\mathbb{L}}^2.$$

Define $c_\varepsilon^2 = 1-\beta_\varepsilon$ and $C_\varepsilon^2 = 1+\beta_\varepsilon$ for $\varepsilon \in\,]0,\varepsilon_5']$. Thus

$$c_\varepsilon\|\Psi_\varepsilon(u)\|_{\mathbb{L}} = c_\varepsilon\|\widehat{u}\|_{\mathbb{L}} \le \|u\|_{L^2} \le C_\varepsilon\|\widehat{u}\|_{\mathbb{L}} = C_\varepsilon\|\Psi_\varepsilon(u)\|_{\mathbb{L}},$$

with $c_\varepsilon \to 1$, $C_\varepsilon \to 1$ as $\varepsilon \to 0$. □

Notation. Define the $n\times n$ matrix $G_\varepsilon = (g_{i,j,\varepsilon})_{i,j=1}^n$ by $g_{i,j,\varepsilon} = \langle \varphi_{i,\varepsilon},\psi_{j,\varepsilon}\rangle_{L^2}$ for i, $j \in [1..n]$ and $\varepsilon \in\,]0,\varepsilon_0[$. Clearly,

$$G_\varepsilon\Psi_\varepsilon(u) = (\langle u,\varphi_{i,\varepsilon}\rangle_{L^2})_{i\in[1..n]}, \quad \varepsilon \in\,]0,\varepsilon_0[\,,\ u \in W_\varepsilon. \tag{3.10}$$

Lemma 3.10. *There exists an $\varepsilon_6' \in\,]0,\varepsilon_5']$ and for each $k \in [1..n]$ there exists a family $(v_{k,\varepsilon})_{\varepsilon\in]0,\varepsilon_6']}$ such that $v_{k,\varepsilon} \in W_\varepsilon$, $\|v_{k,\varepsilon}\|_{L^2} = 1$ for $\varepsilon \in\,]0,\varepsilon_6']$ and*

$$\langle v_{k,\varepsilon},\varphi_{i,\varepsilon}\rangle = 0 \quad \textit{for } i \ne k.$$

Moreover, if (3.7) holds, then $v_{k,\varepsilon}(\alpha_{j,\varepsilon}) - \varphi_{k,\varepsilon}(\alpha_{j,\varepsilon}) \to 0$ as $\varepsilon \to 0$.

Proof. Let i, $j \in [1..n]$. It follows from Corollary 3.5 that

$$\langle \psi_{j,\varepsilon},\varphi_{i,\varepsilon}\rangle_{L^2} - L_j\varphi_{i,\varepsilon}(\alpha_{j,\varepsilon}) \to 0, \quad \text{as } \varepsilon \to 0.$$

Lemmas 3.7 and 3.3 imply that $\sup\limits_{\varepsilon\in]0,\varepsilon_4']}\sup\limits_{x\in[0,1]}|\varphi_{k,\varepsilon}(x)| < \infty$. Therefore for each $k \in [1..n]$ we have

$$\langle \psi_{j,\varepsilon},\varphi_{i,\varepsilon}\rangle_{L^2}\varphi_{k,\varepsilon}(\alpha_{j,\varepsilon}) - L_j\varphi_{i,\varepsilon}(\alpha_{j,\varepsilon})\varphi_{k,\varepsilon}(\alpha_{j,\varepsilon}) \to 0, \quad \text{as } \varepsilon \to 0$$

and so

$$\sum_{j=1}^n\langle \psi_{j,\varepsilon},\varphi_{i,\varepsilon}\rangle_{L^2}\varphi_{k,\varepsilon}(\alpha_{j,\varepsilon}) - \sum_{j=1}^n L_j\varphi_{i,\varepsilon}(\alpha_{j,\varepsilon})\varphi_{k,\varepsilon}(\alpha_{j,\varepsilon}) \to 0, \quad \text{as } \varepsilon \to 0.$$

Thus Lemma 3.8 implies that

$$\sum_{j=1}^n\langle \psi_{j,\varepsilon},\varphi_{i,\varepsilon}\rangle_{L^2}\varphi_{k,\varepsilon}(\alpha_{j,\varepsilon}) \to \delta_{i,k}, \quad \text{as } \varepsilon \to 0. \tag{3.11}$$

Define the $n\times n$ matrix $\widetilde{G}_\varepsilon = (\widetilde{g}_{i,j,\varepsilon})_{i,j=1}^n$ by $\widetilde{g}_{i,j,\varepsilon} = \varphi_{j,\varepsilon}(\alpha_{i,\varepsilon})$ for i, $j \in [1..n]$ and $\varepsilon \in\,]0,\varepsilon_0[$. Since

$$\sum_{j=1}^n\langle \psi_{j,\varepsilon},\varphi_{i,\varepsilon}\rangle_{L^2}\varphi_{k,\varepsilon}(\alpha_{j,\varepsilon}) = \sum_{j=1}^n g_{i,j,\varepsilon}\widetilde{g}_{j,k,\varepsilon}$$

for all i, $k \in [1..n]$, it follows that $\|G_\varepsilon\widetilde{G}_\varepsilon - \mathrm{Id}\,\|_{L(\mathbb{R}^n,\mathbb{R}^n)} \to 0$ as $\varepsilon \to 0$, where Id is the identity matrix. Hence

$$\det G_\varepsilon\cdot\det\widetilde{G}_\varepsilon \to 1, \quad \text{as } \varepsilon \to 0.$$

Since the entries of the matrices G_ε and $\widetilde{G}_\varepsilon$ are uniformly bounded in $\varepsilon \in \,]0, \varepsilon_4']$, it follows that there exist a constant $C \in \,]0, \infty[$ and an $\varepsilon_6' \in \,]0, \varepsilon_5']$ such that

$$\det G_\varepsilon \geq C, \quad \text{for all } \varepsilon \in \,]0, \varepsilon_6'].$$

This implies by the formula for inverse matrices that

(3.12) $G_\varepsilon \colon \mathbb{R}^n \to \mathbb{R}^n$ is bijective and there is a constant $C_4' \in \,]0, \infty[$ such that $\|G_\varepsilon^{-1}\|_{\mathrm{op}} \leq C_4'$, for all $\varepsilon \in \,]0, \varepsilon_6']$,

where $\|\cdot\|_{\mathrm{op}}$ is operator norm with respect to the norm $\|\cdot\|_{\mathbb{L}}$ in $\mathbb{R}^n$. In particular, given $k \in [1\,..\,n]$ and $\varepsilon \in \,]0, \varepsilon_6']$ there exists a unique $\widehat{u}_\varepsilon = \widehat{u}_{k,\varepsilon} \in \mathbb{R}^n$ with $G_\varepsilon \widehat{u}_\varepsilon = e_k$, where e_k is the k-th vector of the canonical basis of $\mathbb{R}^n$. Set

$$\widetilde{u}_\varepsilon = \widetilde{u}_{k,\varepsilon} = (\varphi_{k,\varepsilon}(\alpha_{j,\varepsilon}))_{j \in [1..n]}.$$

It follows from (3.11) that $G_\varepsilon \widetilde{u}_\varepsilon \to e_k$ as $\varepsilon \to 0$ and so $G_\varepsilon(\widehat{u}_\varepsilon - \widetilde{u}_\varepsilon) \to 0$ as $\varepsilon \to 0$. Notice that

$$\|\widehat{u}_\varepsilon - \widetilde{u}_\varepsilon\|_{\mathbb{L}} \leq \|G_\varepsilon^{-1}\|_{\mathrm{op}} \|G_\varepsilon(\widehat{u}_\varepsilon - \widetilde{u}_\varepsilon)\|_{\mathbb{L}} \leq C_4' \|G_\varepsilon(\widehat{u}_\varepsilon - \widetilde{u}_\varepsilon)\|_{\mathbb{L}}.$$

Therefore $\|\widehat{u}_\varepsilon - \widetilde{u}_\varepsilon\|_{\mathbb{L}} \to 0$ as $\varepsilon \to 0$. Notice that Lemma 3.8 implies that $\|\widetilde{u}_\varepsilon\|_{\mathbb{L}} \to 1$ as $\varepsilon \to 0$. Hence $\|\widehat{u}_\varepsilon\|_{\mathbb{L}} \to 1$ as $\varepsilon \to 0$. This and Lemma 3.9 imply that $\|u_\varepsilon\|_{L^2} \to 1$ as $\varepsilon \to 0$, where $u_\varepsilon = \Psi_\varepsilon^{-1}(\widehat{u}_\varepsilon)$. Since

$$\frac{\widehat{u}_\varepsilon}{\|u_\varepsilon\|_{L^2}} - \widehat{u}_\varepsilon = \frac{1 - \|u_\varepsilon\|_{L^2}}{\|u_\varepsilon\|_{L^2}} \widehat{u}_\varepsilon,$$

it follows that

$$\frac{\widehat{u}_\varepsilon}{\|u_\varepsilon\|_{L^2}} - \widehat{u}_\varepsilon \to 0, \quad \text{as } \varepsilon \to 0$$

and so

$$\frac{\widehat{u}_\varepsilon}{\|u_\varepsilon\|_{L^2}} - \widetilde{u}_\varepsilon \to 0, \quad \text{as } \varepsilon \to 0.$$

Set $v_\varepsilon = v_{k,\varepsilon} = u_\varepsilon / \|u_\varepsilon\|_{L^2}$. It follows that

$$\langle v_\varepsilon, \varphi_{i,\varepsilon} \rangle = \left\langle \sum_{j=1}^n \frac{\widehat{u}_{\varepsilon,j}}{\|u_\varepsilon\|_{L^2}} \psi_{j,\varepsilon}, \varphi_{i,\varepsilon} \right\rangle = \frac{1}{\|u_\varepsilon\|_{L^2}} \sum_{j=1}^n \widehat{u}_{\varepsilon,j} \langle \psi_{j,\varepsilon}, \varphi_{i,\varepsilon} \rangle = \frac{1}{\|u_\varepsilon\|_{L^2}} \delta_{i,k}.$$

In particular, $\langle v_\varepsilon, \varphi_{i,\varepsilon} \rangle = 0$ for $i \neq k$. Moreover, for each $j \in [1\,..\,n]$

$$v_\varepsilon(\alpha_{j,\varepsilon}) = \frac{1}{\|u_\varepsilon\|_{L^2}} \, \widehat{u}_{\varepsilon,j}.$$

Hence $v_\varepsilon(\alpha_{j,\varepsilon}) - \widetilde{u}_{\varepsilon,j} \to 0$ as $\varepsilon \to 0$ and so $v_\varepsilon(\alpha_{j,\varepsilon}) - \varphi_{k,\varepsilon}(\alpha_{j,\varepsilon}) \to 0$ as $\varepsilon \to 0$. $\square$

Lemma 3.11. *Let $\varepsilon_4' \in \,]0, \varepsilon_0[$ be as in Lemma 3.7 and let $(u_\varepsilon)_{\varepsilon \in]0, \varepsilon_4']}$ be such that $u_\varepsilon \in W_\varepsilon$ and $\|u_\varepsilon\|_{L^2} = 1$ for each $\varepsilon \in \,]0, \varepsilon_4']$. Then*

$$b_\varepsilon(u_\varepsilon, u_\varepsilon) - b_0(\Psi_\varepsilon(u_\varepsilon), \Psi_\varepsilon(u_\varepsilon)) \to 0, \quad \text{as } \varepsilon \to 0.$$

Proof. Set $\widehat{u}_\varepsilon = \Psi_\varepsilon(u_\varepsilon)$, where $u_\varepsilon = \sum_{j=1}^n \widehat{u}_{\varepsilon,j}\psi_{j,\varepsilon} \in W_\varepsilon$. Thus, $\widehat{u}_\varepsilon = (\widehat{u}_{\varepsilon,j})_{j\in[1..n]}$. We have

$$\begin{aligned}
b_\varepsilon(u_\varepsilon,u_\varepsilon) &= \int_0^{\zeta_{0,\varepsilon}} a_\varepsilon\cdot(u_\varepsilon')^2\,dx + \sum_{j=1}^{n-1}\int_{\xi_{j,\varepsilon}}^{\zeta_{j,\varepsilon}} a_\varepsilon\cdot(u_\varepsilon')^2\,dx \\
&\quad + \int_{\xi_{n,\varepsilon}}^1 a_\varepsilon\cdot(u_\varepsilon')^2\,dx + \delta_0 u_\varepsilon(0)^2 + \delta_1 u_\varepsilon(1)^2 \\
&\le \sup_{[0,\zeta_{0,\varepsilon}]} a_\varepsilon \int_0^{\zeta_{0,\varepsilon}} (u_\varepsilon')^2\,dx + \sum_{j=1}^{n-1} \sup_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} a_\varepsilon \int_{\xi_{j,\varepsilon}}^{\zeta_{j,\varepsilon}} (u_\varepsilon')^2\,dx \\
&\quad + \sup_{[\xi_{n,\varepsilon},1]} a_\varepsilon \int_{\xi_{n,\varepsilon}}^1 (u_\varepsilon')^2\,dx + \delta_0 u_\varepsilon(0)^2 + \delta_1 u_\varepsilon(1)^2 \\
&= \sup_{[0,\zeta_{0,\varepsilon}]} a_\varepsilon \int_0^{\zeta_{0,\varepsilon}} (\widehat{u}_{\varepsilon,1}\psi_{1,\varepsilon}')^2\,dx \\
&\quad + \sum_{j=1}^{n-1} \sup_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} a_\varepsilon \int_{\xi_{j,\varepsilon}}^{\zeta_{j,\varepsilon}} (\widehat{u}_{\varepsilon,j}\psi_{j,\varepsilon}' + \widehat{u}_{\varepsilon,j+1}\psi_{j+1,\varepsilon}')^2\,dx \\
&\quad + \sup_{[\xi_{n,\varepsilon},1]} a_\varepsilon \int_{\xi_{n,\varepsilon}}^1 (\widehat{u}_{\varepsilon,n}\psi_{n,\varepsilon}')^2\,dx + \delta_0 u_\varepsilon(0)^2 + \delta_1 u_\varepsilon(1)^2.
\end{aligned}$$

Notice that

$$\begin{aligned}
\psi_{1,\varepsilon}'(x) &= \frac{\rho}{\rho\zeta_{0,\varepsilon} + (1-\rho)a_\varepsilon(0)} && \text{for } x\in[0,\zeta_{0,\varepsilon}], \\
\psi_{j,\varepsilon}'(x) &= -\frac{1}{\zeta_{j,\varepsilon}-\xi_{j,\varepsilon}} && \text{for } x\in[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}] \text{ and } j\in[1\,..\,n-1], \\
\psi_{j+1,\varepsilon}'(x) &= \frac{1}{\zeta_{j,\varepsilon}-\xi_{j,\varepsilon}} && \text{for } x\in[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}] \text{ and } j\in[1\,..\,n-1]
\end{aligned}$$

and

$$\psi_{n,\varepsilon}'(x) = -\frac{\sigma}{\sigma(1-\xi_{n,\varepsilon}) + (1-\sigma)a_\varepsilon(1)} \quad \text{for } x\in[\xi_{n,\varepsilon},1].$$

Moreover,

$$u_\varepsilon(0) = \widehat{u}_{\varepsilon,1}\psi_{1,\varepsilon}(0) = \widehat{u}_{\varepsilon,1}\bigg(1 - \frac{\rho\zeta_{0,\varepsilon}}{\rho\zeta_{0,\varepsilon} + (1-\rho)a_\varepsilon(0)}\bigg) = \widehat{u}_{\varepsilon,1}\frac{(1-\rho)a_\varepsilon(0)}{\rho\zeta_{0,\varepsilon} + (1-\rho)a_\varepsilon(0)}$$

and

$$u_\varepsilon(1) = \widehat{u}_{\varepsilon,n}\frac{(1-\sigma)a_\varepsilon(1)}{\sigma(1-\xi_{n,\varepsilon}) + (1-\sigma)a_\varepsilon(1)}.$$

Therefore $b_\varepsilon(u_\varepsilon,u_\varepsilon) \le c_\varepsilon$, where

$$c_\varepsilon = \frac{\sup_{[0,\zeta_{0,\varepsilon}]} a_\varepsilon}{\zeta_{0,\varepsilon}} \frac{\rho^2}{(\rho + (1-\rho)a_\varepsilon(0)/\zeta_{0,\varepsilon})^2}\,\widehat{u}_{\varepsilon,1}^2 + \sum_{j=1}^{n-1} \frac{\sup_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} a_\varepsilon}{\zeta_{j,\varepsilon}-\xi_{j,\varepsilon}}\,(\widehat{u}_{\varepsilon,j+1} - \widehat{u}_{\varepsilon,j})^2$$

$$
\begin{aligned}
&+ \frac{\sup\limits_{[\xi_{n,\varepsilon},1]} a_\varepsilon}{1-\xi_{n,\varepsilon}} \frac{\sigma^2}{(\sigma+(1-\sigma)a_\varepsilon(1)/(1-\xi_{n,\varepsilon}))^2}\,\widehat{u}_{\varepsilon,n}^2 \\
&+ \delta_0\left(\frac{(1-\rho)a_\varepsilon(0)/\zeta_{0,\varepsilon}}{\rho+(1-\rho)a_\varepsilon(0)/\zeta_{0,\varepsilon}}\right)^2 \widehat{u}_{\varepsilon,1}^2 + \delta_1\left(\frac{(1-\sigma)a_\varepsilon(1)/(1-\xi_{n,\varepsilon})}{\sigma+(1-\sigma)a_\varepsilon(1)/(1-\xi_{n,\varepsilon})}\right)^2 \widehat{u}_{\varepsilon,n}^2 \\
={}& (\tau_0+h_{2,\varepsilon})\left(\frac{\rho^2}{(\rho+(1-\rho)\tau_0)^2}+h_{3,\varepsilon}\right)\widehat{u}_{\varepsilon,1}^2 + \sum_{j=1}^{n-1}(\tau_j+h_{1,j,\varepsilon})(\widehat{u}_{\varepsilon,j+1}-\widehat{u}_{\varepsilon,j})^2 \\
&+ (\tau_n+h_{4,\varepsilon})\left(\frac{\sigma^2}{(\sigma+(1-\sigma)\tau_n)^2}+h_{5,\varepsilon}\right)\widehat{u}_{\varepsilon,n}^2 \\
&+ \delta_0\left(\frac{(1-\rho)^2\tau_0^2}{(\rho+(1-\rho)\tau_0)^2}+h_{6,\varepsilon}\right)\widehat{u}_{\varepsilon,1}^2 + \delta_1\left(\frac{(1-\sigma)^2\tau_n^2}{(\sigma+(1-\sigma)\tau_n)^2}+h_{7,\varepsilon}\right)\widehat{u}_{\varepsilon,n}^2
\end{aligned}
$$

so

$$
\begin{aligned}
c_\varepsilon &= \frac{\tau_0\rho^2}{(\rho+(1-\rho)\tau_0)^2}\,\widehat{u}_{\varepsilon,1}^2 + \sum_{j=1}^{n-1}\tau_j(\widehat{u}_{\varepsilon,j+1}-\widehat{u}_{\varepsilon,j})^2 + \frac{\tau_n\sigma^2}{(\sigma+(1-\sigma)\tau_n)^2}\,\widehat{u}_{\varepsilon,n}^2 \\
&\quad + \frac{\delta_0(1-\rho)^2\tau_0^2}{(\rho+(1-\rho)\tau_0)^2}\,\widehat{u}_{\varepsilon,1}^2 + \frac{\delta_1(1-\sigma)^2\tau_n^2}{(\sigma+(1-\sigma)\tau_n)^2}\,\widehat{u}_{\varepsilon,n}^2 + h_{8,\varepsilon} \\
&= \widetilde{\tau}_0\widehat{u}_{\varepsilon,1}^2 + \sum_{j=1}^{n-1}\tau_j(\widehat{u}_{\varepsilon,j+1}-\widehat{u}_{\varepsilon,j})^2 + \widetilde{\tau}_n\widehat{u}_{\varepsilon,n}^2 + h_{8,\varepsilon},
\end{aligned}
$$

with $h_{1,j,\varepsilon} \to 0$, $j \in [1..n-1]$, and $h_{i,\varepsilon} \to 0$, $i \in [2..8]$, as $\varepsilon \to 0$. This follows from Assumption 2.1, the hypothesis that $\|u_\varepsilon\|_{L^2} = 1$, for $\varepsilon \in]0, \varepsilon_4']$, and Lemma 3.9. Similarly, working with 'inf' instead of 'sup', we show that

$$
b_\varepsilon(u_\varepsilon,u_\varepsilon) \geq \widetilde{\tau}_0\widehat{u}_{\varepsilon,1}^2 + \sum_{j=1}^{n-1}\tau_j(\widehat{u}_{\varepsilon,j+1}-\widehat{u}_{\varepsilon,j})^2 + \widetilde{\tau}_n\widehat{u}_{\varepsilon,n}^2 + h_{9,\varepsilon},
$$

with $h_{9,\varepsilon} \to 0$ as $\varepsilon \to 0$. Therefore

$$
b_\varepsilon(u_\varepsilon,u_\varepsilon) - \left(\widetilde{\tau}_0\widehat{u}_{\varepsilon,1}^2 + \sum_{j=1}^{n-1}\tau_j(\widehat{u}_{\varepsilon,j+1}-\widehat{u}_{\varepsilon,j})^2 + \widetilde{\tau}_n\widehat{u}_{\varepsilon,n}^2\right) \to 0, \quad \text{as } \varepsilon \to 0. \tag{3.13}
$$

Now estimate (3.13) and the definition of b_0 and $\widehat{u}_\varepsilon$ imply the assertion. □

COROLLARY 3.12. *Let $\varepsilon_6' \in]0, \varepsilon_0[$ be as in Lemma 3.10 and $k \in [1..n]$ be arbitrary. Then*

$$
\begin{gathered}
\{\, b_\varepsilon(u,u) \mid u \in W_\varepsilon,\ \|u\|_{L^2} = 1,\ u \in U_{k-1,\varepsilon}^\perp \,\} \neq \emptyset, \\
\{\, b_0(\Psi_\varepsilon(u),\Psi_\varepsilon(u)) \mid u \in W_\varepsilon,\ \|u\|_{L^2} = 1,\ u \in U_{k-1,\varepsilon}^\perp \,\} \neq \emptyset,
\end{gathered}
$$

for all $\varepsilon \in]0, \varepsilon_6']$. Moreover, the following holds:

$$
\begin{aligned}
&\inf\{b_\varepsilon(u,u) \mid u \in W_\varepsilon,\ \|u\|_{L^2} = 1,\ u \in U_{k-1,\varepsilon}^\perp\} \\
&\quad - \inf\{b_0(\Psi_\varepsilon(u),\Psi_\varepsilon(u)) \mid u \in W_\varepsilon,\ \|u\|_{L^2} = 1,\ u \in U_{k-1,\varepsilon}^\perp\} \to 0, \quad \textit{as } \varepsilon \to 0. \ \square
\end{aligned}
$$

Lemma 3.13. *Let $\varepsilon_6' \in \,]0,\varepsilon_0[$ and for $k \in [1\,.\,.\,n]$ let the family $(v_{k,\varepsilon})_{\varepsilon\in]0,\varepsilon_6']}$ be as in Lemma* 3.10. *Then*

$$\lambda_{k,\varepsilon} - \inf\{\, b_0(\Psi_\varepsilon(u),\Psi_\varepsilon(u)) \mid u \in W_\varepsilon,\ \|u\|_{L^2} = 1,\ u \in U_{k-1,\varepsilon}^{\perp}\,\} \to 0,$$

as $\varepsilon \to 0$, and $\lambda_{k,\varepsilon} - b_\varepsilon(v_{k,\varepsilon},v_{k,\varepsilon}) \to 0$, as $\varepsilon \to 0$.

Proof. Lemma 3.10 implies that $\{\, b_0(\Psi_\varepsilon(u),\Psi_\varepsilon(u)) \mid u \in W_\varepsilon,\ \|u\|_{L^2} = 1,$ $u \in U_{k-1,\varepsilon}^{\perp}\,\} \neq \emptyset$ for all $\varepsilon \in \,]0,\varepsilon_6']$. It follows from Lemma 3.10, choosing first $\alpha_{j,\varepsilon} = \xi_{j,\varepsilon}$ for $(j,\varepsilon) \in [1\,.\,.\,n]\times\,]0,\varepsilon_0[$ and then $\alpha_{j,\varepsilon} = \zeta_{j-1,\varepsilon}$ for $(j,\varepsilon) \in [1\,.\,.\,n]\times\,]0,\varepsilon_0[$, that

$$\begin{aligned} v_{k,\varepsilon}(\xi_{j,\varepsilon}) - \varphi_{k,\varepsilon}(\xi_{j,\varepsilon}) &\to 0 \quad \text{as } \varepsilon \to 0,\ j \in [1\,.\,.\,n], \\ v_{k,\varepsilon}(\zeta_{j,\varepsilon}) - \varphi_{k,\varepsilon}(\zeta_{j,\varepsilon}) &\to 0 \quad \text{as } \varepsilon \to 0,\ j \in [0\,.\,.\,n-1]. \end{aligned} \tag{3.14}$$

Thus

$$\begin{aligned} b_\varepsilon(\varphi_{k,\varepsilon},\varphi_{k,\varepsilon}) &\geq \int_0^{\zeta_{0,\varepsilon}} a_\varepsilon\cdot(\varphi_{k,\varepsilon}')^2\,dx + \sum_{j=1}^{n-1}\int_{\xi_{j,\varepsilon}}^{\zeta_{j,\varepsilon}} a_\varepsilon\cdot(\varphi_{k,\varepsilon}')^2\,dx \\ &\quad + \int_{\xi_{n,\varepsilon}}^1 a_\varepsilon\cdot(\varphi_{k,\varepsilon}')^2\,dx + \delta_0\varphi_{k,\varepsilon}(0)^2 + \delta_1\varphi_{k,\varepsilon}(1)^2 \\ &\geq \inf_{[0,\zeta_{0,\varepsilon}]} a_\varepsilon \int_0^{\zeta_{0,\varepsilon}} (\varphi_{k,\varepsilon}')^2\,dx + \sum_{j=1}^{n-1} \inf_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} a_\varepsilon \int_{\xi_{j,\varepsilon}}^{\zeta_{j,\varepsilon}} (\varphi_{k,\varepsilon}')^2\,dx \\ &\quad + \inf_{[\xi_{n,\varepsilon},1]} a_\varepsilon \int_{\xi_{n,\varepsilon}}^1 (\varphi_{k,\varepsilon}')^2\,dx + \delta_0\varphi_{k,\varepsilon}(0)^2 + \delta_1\varphi_{k,\varepsilon}(1)^2. \end{aligned}$$

Thus

$$\begin{aligned} b_\varepsilon(\varphi_{k,\varepsilon},\varphi_{k,\varepsilon}) &\geq \frac{\inf_{[0,\zeta_{0,\varepsilon}]} a_\varepsilon}{\zeta_{0,\varepsilon}} \left(\int_0^{\zeta_{0,\varepsilon}} \varphi_{k,\varepsilon}'\,dx\right)^2 \\ &\quad + \sum_{j=1}^{n-1} \frac{\inf_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} a_\varepsilon}{\zeta_{j,\varepsilon}-\xi_{j,\varepsilon}} \left(\int_{\xi_{j,\varepsilon}}^{\zeta_{j,\varepsilon}} \varphi_{k,\varepsilon}'\,dx\right)^2 \\ &\quad + \frac{\inf_{[\xi_{n,\varepsilon},1]} a_\varepsilon}{1-\xi_{n,\varepsilon}} \left(\int_{\xi_{n,\varepsilon}}^1 \varphi_{k,\varepsilon}'\,dx\right)^2 + \delta_0\varphi_{k,\varepsilon}(0)^2 + \delta_1\varphi_{k,\varepsilon}(1)^2 \\ &= \frac{\inf_{[0,\zeta_{0,\varepsilon}]} a_\varepsilon}{\zeta_{0,\varepsilon}} \big(\varphi_{k,\varepsilon}(\zeta_{0,\varepsilon}) - \varphi_{k,\varepsilon}(0)\big)^2 \\ &\quad + \sum_{j=1}^{n-1} \frac{\inf_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} a_\varepsilon}{\zeta_{j,\varepsilon}-\xi_{j,\varepsilon}} \big(\varphi_{k,\varepsilon}(\zeta_{j,\varepsilon}) - \varphi_{k,\varepsilon}(\xi_{j,\varepsilon})\big)^2 \\ &\quad + \frac{\inf_{[\xi_{n,\varepsilon},1]} a_\varepsilon}{1-\xi_{n,\varepsilon}} \big(\varphi_{k,\varepsilon}(1) - \varphi_{k,\varepsilon}(\xi_{n,\varepsilon})\big)^2 + \delta_0\varphi_{k,\varepsilon}(0)^2 + \delta_1\varphi_{k,\varepsilon}(1)^2. \end{aligned}$$

Define $h_{1,j,\varepsilon}$, $j \in [1\,..\,n]$, $h_{2,j,\varepsilon}$, $j \in [0\,..\,n-1]$ $h_{3,j,\varepsilon}$, $j \in [1\,..\,n-1]$, $h_{4,\varepsilon}$ and $h_{5,\varepsilon}$ such that $\varphi_{k,\varepsilon}(\xi_{j,\varepsilon}) = v_{k,\varepsilon}(\xi_{j,\varepsilon}) + h_{1,j,\varepsilon}$, $\varphi_{k,\varepsilon}(\zeta_{j,\varepsilon}) = v_{k,\varepsilon}(\zeta_{j,\varepsilon}) + h_{2,j,\varepsilon}$,

$$\frac{\inf\limits_{[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]} a_\varepsilon}{\zeta_{j,\varepsilon} - \xi_{j,\varepsilon}} = \tau_j + h_{3,j,\varepsilon}, \quad \frac{\inf\limits_{[0,\zeta_{0,\varepsilon}]} a_\varepsilon}{\zeta_{0,\varepsilon}} = \tau_0 + h_{4,\varepsilon}, \quad \frac{\inf\limits_{[\xi_{n,\varepsilon},1]} a_\varepsilon}{1 - \xi_{n,\varepsilon}} = \tau_n + h_{5,\varepsilon}.$$

Assumption 2.1 and (3.14) imply that $h_{1,j,\varepsilon} \to 0$, $h_{2,j,\varepsilon} \to 0$, $h_{3,j,\varepsilon} \to 0$, $h_{4,\varepsilon} \to 0$ and $h_{5,\varepsilon} \to 0$ as $\varepsilon \to 0$. Therefore

$$\begin{aligned} b_\varepsilon(\varphi_{k,\varepsilon}, \varphi_{k,\varepsilon}) &\geq (\tau_0 + h_{4,\varepsilon})(v_{k,\varepsilon}(\zeta_{0,\varepsilon}) + h_{2,0,\varepsilon} - \varphi_{k,\varepsilon}(0))^2 \\ &\quad + \sum_{j=1}^{n-1} (\tau_j + h_{3,j,\varepsilon})(v_{k,\varepsilon}(\zeta_{j,\varepsilon}) + h_{2,j,\varepsilon} - v_{k,\varepsilon}(\xi_{j,\varepsilon}) - h_{1,j,\varepsilon})^2 \\ &\quad + (\tau_n + h_{5,\varepsilon})(\varphi_{k,\varepsilon}(1) - v_{k,\varepsilon}(\xi_{n,\varepsilon}) - h_{1,n,\varepsilon})^2 + \delta_0 \varphi_{k,\varepsilon}(0)^2 + \delta_1 \varphi_{k,\varepsilon}(1)^2 \\ &= \tau_0(v_{k,\varepsilon}(\zeta_{0,\varepsilon}) - \varphi_{k,\varepsilon}(0))^2 + \sum_{j=1}^{n-1} \tau_j (v_{k,\varepsilon}(\zeta_{j,\varepsilon}) - v_{k,\varepsilon}(\xi_{j,\varepsilon}))^2 \\ &\quad + \tau_n(\varphi_{k,\varepsilon}(1) - v_{k,\varepsilon}(\xi_{n,\varepsilon}))^2 + \delta_0 \varphi_{k,\varepsilon}(0)^2 + \delta_1 \varphi_{k,\varepsilon}(1)^2 + h_{6,\varepsilon}, \end{aligned}$$

where $h_{6,\varepsilon} \to 0$ as $\varepsilon \to 0$. Notice that if $\rho = 1$, then $\delta_0 = 0$ and $v_{k,\varepsilon}(0) = \varphi_{k,\varepsilon}(0) = 0$ and if $\sigma = 1$, then $\delta_1 = 0$ and $v_{k,\varepsilon}(1) = \varphi_{k,\varepsilon}(1) = 0$.

Suppose $\rho < 1$, then $\delta_0 = \rho/(1-\rho)$. Notice that the function

$$\alpha \mapsto \tau_0(v_{k,\varepsilon}(\zeta_{0,\varepsilon}) - \alpha)^2 + \frac{\rho}{1-\rho}\,\alpha^2$$

assumes its minimum at the point $\alpha_\varepsilon = \tau_0(1-\rho)v_{k,\varepsilon}(\zeta_{0,\varepsilon})/(\rho + \tau_0(1-\rho))$. Since

$$v_{k,\varepsilon}(0) = v_{k,\varepsilon}(\zeta_{0,\varepsilon})\left(1 - \frac{\rho\zeta_{0,\varepsilon}}{\rho\zeta_{0,\varepsilon} + (1-\rho)a_\varepsilon(0)}\right) = v_{k,\varepsilon}(\zeta_{0,\varepsilon})\frac{(1-\rho)a_\varepsilon(0)/\zeta_{0,\varepsilon}}{\rho + (1-\rho)a_\varepsilon(0)/\zeta_{0,\varepsilon}},$$

it follows from Assumption 2.1 and Lemma 3.10 that $\alpha_\varepsilon = v_{k,\varepsilon}(0) + h_{7,\varepsilon}$, with $h_{7,\varepsilon} \to 0$ as $\varepsilon \to 0$. Therefore,

$$\begin{aligned} \tau_0(v_{k,\varepsilon}(\zeta_{0,\varepsilon}) - \varphi_{k,\varepsilon}(0))^2 + \delta_0\varphi_{k,\varepsilon}(0)^2 &\geq \tau_0(v_{k,\varepsilon}(\zeta_{0,\varepsilon}) - \alpha_\varepsilon)^2 + \delta_0\alpha_\varepsilon^2 \\ &= \tau_0(v_{k,\varepsilon}(\zeta_{0,\varepsilon}) - v_{k,\varepsilon}(0))^2 + \delta_0 v_{k,\varepsilon}(0)^2 + h_{8,\varepsilon}, \end{aligned}$$

with $h_{8,\varepsilon} \to 0$ as $\varepsilon \to 0$. Similarly if $\sigma < 1$ then

$$\tau_n(v_{k,\varepsilon}(\xi_{n,\varepsilon}) - \varphi_{k,\varepsilon}(1))^2 + \delta_1\varphi_{k,\varepsilon}(1)^2 \geq \tau_n(v_{k,\varepsilon}(\xi_{n,\varepsilon}) - v_{k,\varepsilon}(1))^2 + \delta_1 v_{k,\varepsilon}(1)^2 + h_{9,\varepsilon},$$

with $h_{9,\varepsilon} \to 0$ as $\varepsilon \to 0$. Thus in all cases we have,

$$\begin{aligned} b_\varepsilon(\varphi_{k,\varepsilon}, \varphi_{k,\varepsilon}) &\geq \tau_0(v_{k,\varepsilon}(\zeta_{0,\varepsilon}) - \varphi_{k,\varepsilon}(0))^2 + \sum_{j=1}^{n-1} \tau_j(v_{k,\varepsilon}(\zeta_{j,\varepsilon}) - v_{k,\varepsilon}(\xi_{j,\varepsilon}))^2 \\ &\quad + \tau_n(\varphi_{k,\varepsilon}(1) - v_{k,\varepsilon}(\xi_{n,\varepsilon}))^2 + \delta_0\varphi_{k,\varepsilon}(0)^2 + \delta_1\varphi_{k,\varepsilon}(1)^2 + h_{6,\varepsilon} \\ &\geq \tau_0(v_{k,\varepsilon}(\zeta_{0,\varepsilon}) - v_{k,\varepsilon}(0))^2 + \sum_{j=1}^{n-1} \tau_j(v_{k,\varepsilon}(\zeta_{j,\varepsilon}) - v_{k,\varepsilon}(\xi_{j,\varepsilon}))^2 \end{aligned}$$

$$
\begin{aligned}
&+ \tau_n(v_{k,\varepsilon}(\xi_{n,\varepsilon}) - v_{k,\varepsilon}(1))^2 + \delta_0 v_{k,\varepsilon}(0)^2 + \delta_1 v_{k,\varepsilon}(1)^2 + h_{10,\varepsilon} \\
&= \widetilde{\tau}_0 v_{k,\varepsilon}(\zeta_{0,\varepsilon})^2 + \sum_{j=1}^{n-1} \tau_j (v_{k,\varepsilon}(\zeta_{j,\varepsilon}) - v_{k,\varepsilon}(\xi_{j,\varepsilon}))^2 + \widetilde{\tau}_n v_{k,\varepsilon}(\xi_{n,\varepsilon})^2 + h_{11,\varepsilon} \\
&= \widetilde{\tau}_0 \widehat{v}_{k,\varepsilon,1}^2 + \sum_{j=1}^{n-1} \tau_j (\widehat{v}_{k,\varepsilon,j+1} - \widehat{v}_{k,\varepsilon,j})^2 + \widetilde{\tau}_n \widehat{v}_{k,\varepsilon,n}^2 + h_{11,\varepsilon},
\end{aligned}
$$

with $h_{10,\varepsilon} \to 0$ and $h_{11,\varepsilon} \to 0$ as $\varepsilon \to 0$. Here, we write $\widehat{v}_{k,\varepsilon} = \Psi_\varepsilon(v_{k,\varepsilon})$ and $\widehat{v}_{k,\varepsilon,l}$ is the l-th component of $\widehat{v}_{k,\varepsilon} \in \mathbb{R}^n$. By Lemma 3.11,

$$
\begin{aligned}
\widetilde{\tau}_0 \widehat{v}_{k,\varepsilon,1}^2 + \sum_{j=1}^{n-1} \tau_j (\widehat{v}_{k,\varepsilon,j+1} - \widehat{v}_{k,\varepsilon,j})^2 + \widetilde{\tau}_n \widehat{v}_{k,\varepsilon,n}^2 &= b_0(\Psi_\varepsilon(v_{k,\varepsilon}), \Psi_\varepsilon(v_{k,\varepsilon})) \\
&= b_\varepsilon(v_{k,\varepsilon}, v_{k,\varepsilon}) + h_{12,\varepsilon}
\end{aligned}
$$

with $h_{12,\varepsilon} \to 0$ as $\varepsilon \to 0$. Thus,

$$
b_\varepsilon(\varphi_{k,\varepsilon}, \varphi_{k,\varepsilon}) - h_{13,\varepsilon} \geq b_\varepsilon(v_{k,\varepsilon}, v_{k,\varepsilon}) \tag{3.15}
$$

with $h_{13,\varepsilon} \to 0$ as $\varepsilon \to 0$. For $\varepsilon \in]0, \varepsilon_0[$ small enough and for all $k \in [1..n]$ we have

$$
\begin{aligned}
b_\varepsilon(\varphi_{k,\varepsilon}, \varphi_{k,\varepsilon}) &= \inf\{\, b_\varepsilon(\varphi, \varphi) \mid \varphi \in Z, \|\varphi\|_{L^2} = 1,\ \varphi \in U_{k-1,\varepsilon}^\perp \,\} \\
&\leq \inf\{\, b_\varepsilon(u, u) \mid u \in W_\varepsilon, \|u\|_{L^2} = 1,\ u \in U_{k-1,\varepsilon}^\perp \,\}.
\end{aligned}
$$

It follows from (3.15) that

$$
\begin{aligned}
b_\varepsilon(\varphi_{k,\varepsilon}, \varphi_{k,\varepsilon}) - h_{13,\varepsilon} &\geq b_\varepsilon(v_{k,\varepsilon}, v_{k,\varepsilon}) \\
&\geq \inf\{\, b_\varepsilon(u, u) \mid u \in W_\varepsilon, \|u\|_{L^2} = 1,\ u \in U_{k-1,\varepsilon}^\perp \,\}.
\end{aligned}
$$

Since $b_\varepsilon(\varphi_{k,\varepsilon}, \varphi_{k,\varepsilon}) = \lambda_{k,\varepsilon}$, we conclude that

$$
\lambda_{k,\varepsilon} - \inf\{\, b_\varepsilon(u, u) \mid u \in W_\varepsilon, \|u\|_{L^2} = 1,\ u \in U_{k-1,\varepsilon}^\perp \,\} \to 0, \quad \text{as } \varepsilon \to 0,
$$

and $\lambda_{k,\varepsilon} - b_\varepsilon(v_{k,\varepsilon}, v_{k,\varepsilon}) \to 0$, as $\varepsilon \to 0$. Now Corollary 3.12 completes the proof. □

Lemma 3.14. *Let $\varepsilon_6' \in]0, \varepsilon_0[$ be as in Lemma 3.10. Let $(\varepsilon_m)_m$ be a null sequence in $]0, \varepsilon_6']$ and suppose that there exists a sequence $(z_l)_{l \in [1..n]}$ in $\mathbb{R}^n$ such that for each $l \in [1..n]$ and $j \in [1..n]$,*

$$
\sup_{x \in K_{j,\varepsilon_m}} |\varphi_{l,\varepsilon_m}(x) - z_{l,j}| \to 0, \quad \text{as } m \to \infty.
$$

Here $z_l = (z_{l,j})_{j \in [1..n]} \in \mathbb{R}^n$. Then $(z_l)_{l \in [1..n]}$ is an $\langle \cdot, \cdot \rangle_{\mathbb{L}}$-orthonormal sequence. Define $Y_0 = \{0\} \subset \mathbb{R}^n$ and for each $p \in [1..n]$, let Y_p be the span of the vectors z_l, for $l \in [1..p]$. Moreover, let $Y_p^\perp$, $p \in [0..n]$, be the $\langle \cdot, \cdot \rangle_{\mathbb{L}}$-orthogonal *complement of* Y_p. *Then, for each* $k \in [1\,.\,.\,n]$,

$$\begin{aligned}(3.16)\quad &\inf\{\, b_0(\Psi_{\varepsilon_m}(u), \Psi_{\varepsilon_m}(u)) \mid u \in W_{\varepsilon_m},\ \|u\|_{L^2} = 1,\ u \in U_{k-1,\varepsilon_m}^{\perp}\,\}\\ &- \inf\{\, b_0(y,y) \mid y \in \mathbb{R}^n,\ \|y\|_{\mathbb{L}} = 1 \text{ and } y \in Y_{k-1}^{\perp}\,\} \to 0, \quad \text{as } m \to \infty.\end{aligned}$$

Moreover, $\lambda_{k,\varepsilon_m} \to b_0(z_k, z_k)$, *as* $m \to \infty$.

PROOF. Let k, $j \in [1\,.\,.\,n]$. It follows that

$$(3.17)\qquad \varphi_{k,\varepsilon_m}(\alpha_{j,\varepsilon_m}) \to z_{k,j}, \quad \text{as } m \to \infty.$$

Lemma 3.8 part (b) with $\gamma_{j,m} = \alpha_{j,\varepsilon_m}$, for $j \in [1\,.\,.\,n]$ and $m \in \mathbb{N}$, implies that $\|z_k\|_{\mathbb{L}} = 1$ and $\langle z_k, z_l\rangle_{\mathbb{L}} = 0$ for all k, $l \in [1\,.\,.\,n]$ with $k \neq l$. Therefore $(z_l)_{l\in[1..n]}$ is an $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}}$-orthonormal sequence.

Now let $k \in [1\,.\,.\,n]$ be arbitrary. For each $m \in \mathbb{N}$ define

$$\begin{aligned}E_m &= \{\, y \in \mathbb{R}^n \mid \|y\|_{\mathbb{L}} = 1 \text{ and } y \in Y_{k-1}^{\perp}\,\},\\ \widetilde{E}_m &= \{\, \widetilde{y} \in \mathbb{R}^n \mid \exists\ u \in W_{\varepsilon_m},\ \|u\|_{L^2} = 1,\ u \in U_{k-1,\varepsilon_m}^{\perp} \text{ and } \widetilde{y} = \Psi_{\varepsilon_m}(u)\,\}.\end{aligned}$$

Note that all these sets are nonempty. It is clear that the (constant) family $(E_m)_{m\in\mathbb{N}}$ is bounded. Lemma 3.9 implies that the family $(\widetilde{E}_m)_{m\in\mathbb{N}}$ is also bounded. Notice that the map $\mathcal{F}\colon \mathbb{R}^n \to \mathbb{R}$, $y \mapsto b_0(y,y)$ is bounded below and Lipschitzian on bounded subset of $\mathbb{R}^n$.

Thus, in order to establish (3.16), we only have to show that

(3.18) for each $(\widetilde{y}_m)_m \in \prod_{m\in\mathbb{N}} \widetilde{E}_m$ there is a $(y_m)_m \in \prod_{m\in\mathbb{N}} E_m$ such that $\widetilde{y}_m - y_m \to 0$ as $m \to \infty$ and, conversely, that for each $(y_m)_m \in \prod_{m\in\mathbb{N}} E_m$ there is a $(\widetilde{y}_m)_m \in \prod_{m\in\mathbb{N}} \widetilde{E}_m$ such that $\widetilde{y}_m - y_m \to 0$ as $m \to \infty$.

Consider first the case $k = 1$. In this case for each $m \in \mathbb{N}$

$$\begin{aligned}E_m &= \{\, y \in \mathbb{R}^n \mid \|y\|_{\mathbb{L}} = 1\,\},\\ \widetilde{E}_m &= \{\, \widetilde{y} \in \mathbb{R}^n \mid \text{there exists a } u \in W_{\varepsilon_m},\ \|u\|_{L^2} = 1 \text{ and } \widetilde{y} = \Psi_{\varepsilon_m}(u)\,\}.\end{aligned}$$

Let $(\widetilde{y}_m)_m \in \prod_{m\in\mathbb{N}} \widetilde{E}_m$. Then, for each $m \in \mathbb{N}$, there is a $u_m \in W_{\varepsilon_m}$ with $\|u_m\|_{L^2} = 1$ and $\widetilde{y}_m = \Psi_{\varepsilon_m}(u_m)$. It follows that $\widetilde{y}_m \neq 0$.

Since $\|u_m\|_{L^2} = 1$, it follows from Lemma 3.9 that $\|\Psi_{\varepsilon_m}(u_m)\|_{\mathbb{L}} \to 1$, as $m \to \infty$ and so $\|\widetilde{y}_m\|_{\mathbb{L}} \to 1$, as $m \to \infty$. Set $y_m = \widetilde{y}_m / \|\widetilde{y}_m\|_{\mathbb{L}}$, for $m \in \mathbb{N}$. Hence, $(y_m)_m \in \prod_{m\in\mathbb{N}} E_m$ and

$$(3.19)\qquad y_m - \widetilde{y}_m \to 0, \quad \text{as } m \to \infty,$$

proving the first half of (3.18). Conversely, let $(y_m)_m \in \prod_{m\in\mathbb{N}} E_m$. Then, for each $m \in \mathbb{N}$, $y_m \in \mathbb{R}^n$ with $\|y_m\|_{\mathbb{L}} = 1$. Let $w_m = \Psi_{\varepsilon_m}^{-1}(y_m)$, $m \in \mathbb{N}$. It follows that $w_m \neq 0$.

Since $\|y_m\|_{\mathbb{L}} = 1$, it follows from Lemma 3.9 that $\|w_m\|_{L^2} \to 1$ as $m \to \infty$. For each $m \in \mathbb{N}$ define $u_m = w_m/\|w_m\|_{L^2}$. It follows that $u_m \in W_{\varepsilon_m}$, $\|u_m\|_{L^2} = 1$. Set $\widetilde{y}_m = \Psi_{\varepsilon_m}(u_m)$. Then

$$\widetilde{y}_m - y_m = \frac{1}{\|w_m\|_{L^2}} y_m - y_m, \quad m \in \mathbb{N},$$

and this implies that

$$\widetilde{y}_m - y_m \to 0, \quad \text{as } m \to \infty, \tag{3.20}$$

proving the second half of (3.18). This proves (3.18) and thus (3.16) for $k = 1$.

Now let $k \in [2\,..\,n]$. Let $(\widetilde{y}_m)_m \in \prod_{m\in\mathbb{N}} \widetilde{E}_m$. Then, for each $m \in \mathbb{N}$ there is a $u_m \in W_{\varepsilon_m}$ with $\|u_m\|_{L^2} = 1$, $u_m \in U^{\perp}_{k-1,\varepsilon_m}$ and $\widetilde{y}_m = \Psi_{\varepsilon_m}(u_m)$. Define

$$\widehat{y}_m = \widetilde{y}_m - \sum_{l=1}^{k-1} \langle \Psi_{\varepsilon_m}(u_m), z_l\rangle_{\mathbb{L}} z_l.$$

Since $(z_l)_{l\in[1..k-1]}$ is an orthonormal basis of Y_{k-1} with respect to the inner product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}}$, it follows that $\widehat{y}_m \in Y^{\perp}_{k-1}$. Moreover,

$$\|\widehat{y}_m - \widetilde{y}_m\|^2_{\mathbb{L}} = \sum_{l=1}^{k-1} |\langle \Psi_{\varepsilon_m}(u_m), z_l\rangle_{\mathbb{L}}|^2.$$

Now, for $l \in [1\,..\,k-1]$ Lemma 3.4 with $\gamma_{j,m} = \alpha_{j,\varepsilon_m}$, for $j \in [1\,..\,n]$ and $m \in \mathbb{N}$ implies that

$$\langle u_m, \varphi_{l,\varepsilon_m}\rangle_{L^2} - \sum_{j=1}^{n} L_j(\Psi_{\varepsilon_m}(u_m))_j \varphi_{l,\varepsilon_m}(\alpha_{j,\varepsilon_m}) \to 0, \quad \text{as } m \to \infty.$$

Since $\langle u_m, \varphi_{l,\varepsilon_m}\rangle_{L^2} = 0$ and for $j \in [1\,..\,n]$, $\varphi_{l,\varepsilon_m}(\alpha_{j,\varepsilon_m}) - z_{l,j} \to 0$, as $m \to \infty$, it follows that

$$\sum_{j=1}^{n} L_j(\Psi_{\varepsilon_m}(u_m))_j z_{l,j} \to 0, \text{ as } m \to \infty$$

so $\langle \Psi_{\varepsilon_m}(u_m), z_l\rangle_{\mathbb{L}} \to 0$, as $m \to \infty$. Thus $\|\widehat{y}_m - \widetilde{y}_m\|^2_{\mathbb{L}} \to 0$, as $m \to \infty$. Since $\|u_m\|_{L^2} = 1$, it follows from Lemma 3.9 that $\|\Psi_{\varepsilon_m}(u_m)\|_{\mathbb{L}} \to 1$, as $m \to \infty$ and so $\|\widetilde{y}_m\|_{\mathbb{L}} \to 1$, as $m \to \infty$. Therefore, $\|\widehat{y}_m\|_{\mathbb{L}} \to 1$, as $m \to \infty$. In particular, $\widehat{y}_m \neq 0$ for some $m_0 \in \mathbb{N}$ and all $m \geq m_0$. Set $y_m = \widehat{y}_m/\|\widehat{y}_m\|_{\mathbb{L}}$, if $m \geq m_0$ and $y_m = y_{m_0}$ otherwise. Hence, $(y_m)_m \in \prod\limits_{m\in\mathbb{N}} E_m$ and

$$y_m - \widetilde{y}_m \to 0, \quad \text{as } m \to \infty, \tag{3.21}$$

proving the first half of (3.18).

Now let $(y_m)_m \in \prod\limits_{m\in\mathbb{N}} E_m$ be arbitrary. Then for each $m \in \mathbb{N}$, $y_m \in \mathbb{R}^n$ with $\|y_m\|_{\mathbb{L}} = 1$ and $y_m \in Y^{\perp}_{k-1}$.

For each $m \in \mathbb{N}$, let ϱ_m be the vector in $\mathbb{R}^n$ such that $\rho_{m,l} = 0$ for $l \in [1\,..\,k-1]$ and $\rho_{m,l} = \langle \Psi^{-1}_{\varepsilon_m}(y_m), \varphi_{l,\varepsilon_m}\rangle_{L^2}$, for $l \in [k\,..\,n]$.

Set $w_m = \Psi_{\varepsilon_m}^{-1}(G_{\varepsilon_m}^{-1}(\varrho_m))$, $m \in \mathbb{N}$. Thus $G_{\varepsilon_m}(\Psi_{\varepsilon_m}(w_m)) = \varrho_m$, $m \in \mathbb{N}$. On the other hand, in view of (3.10), for each $l \in [1..n]$ we have $(G_{\varepsilon_m}(\Psi_{\varepsilon_m}(w_m)))_l = \langle w_m, \varphi_{l,\varepsilon_m}\rangle_{L^2}$. Therefore $\langle w_m, \varphi_{l,\varepsilon_m}\rangle_{L^2} = 0$ for $l \in [1..k-1]$ and

$$\langle w_m, \varphi_{l,\varepsilon_m}\rangle_{L^2} = \langle \Psi_{\varepsilon_m}^{-1}(y_m,), \varphi_{l,\varepsilon_m}\rangle_{L^2}, \quad \text{for } l \in [k..n].$$

This implies that $w_m \in U_{k-1,\varepsilon_m}^{\perp}$.

In order to show that $\Psi_{\varepsilon_m}(w_m) - y_m \to 0$ as $m \to \infty$ it is enough, by (3.12), to prove that $G_{\varepsilon_m}(\Psi_{\varepsilon_m}(w_m) - y_m) \to 0$ as $m \to \infty$. Notice that $G_{\varepsilon_m}(y_m) = G_{\varepsilon_m}(\Psi_{\varepsilon_m}(\Psi_{\varepsilon_m}^{-1}(y_m)))$ so

$$(G_{\varepsilon_m}(y_m))_l = \langle \Psi_{\varepsilon_m}^{-1}(y_m), \varphi_{l,\varepsilon_m}\rangle_{L^2}, \quad \text{for } l \in [1..n].$$

Thus $(G_{\varepsilon_m}(\Psi_{\varepsilon_m}(w_m) - y_m))_l = 0$ for $l \in [k..n]$. For $l \in [1..k-1]$ we have

$$(G_{\varepsilon_m}(\Psi_{\varepsilon_m}(w_m) - y_m))_l = 0 - \langle \Psi_{\varepsilon_m}^{-1}(y_m), \varphi_{l,\varepsilon_m}\rangle_{L^2}$$

and

$$\langle \Psi_{\varepsilon_m}^{-1}(y_m), \varphi_{l,\varepsilon_m}\rangle_{L^2} - \sum_{j=1}^{n} L_j(\Psi_{\varepsilon_m}(\Psi_{\varepsilon_m}^{-1}(y_m)))_j \varphi_{l,\varepsilon_m}(\alpha_{j,\varepsilon_m}) \to 0, \quad \text{as } m \to \infty.$$

Since $\varphi_{l,\varepsilon_m}(\alpha_{j,\varepsilon_m}) \to z_{l,j}$ as $m \to \infty$ and $\|y_m\|_{\mathbb{L}} = 1$, it follows that

$$\sum_{j=1}^{n} L_j(\Psi_{\varepsilon_m}(\Psi_{\varepsilon_m}^{-1}(y_m)))_j \varphi_{l,\varepsilon_m}(\alpha_{j,\varepsilon_m}) - \sum_{j=1}^{n} L_j y_{m,j} z_{l,j} \to 0, \text{ as } m \to \infty.$$

Now $\sum_{j=1}^{n} L_j y_{m,j} z_{l,j} = \langle y_m, z_l\rangle_{\mathbb{L}} = 0$. Therefore, $\langle \Psi_{\varepsilon_m}^{-1}(y_m), \varphi_{l,\varepsilon_m}\rangle_{L^2} \to 0$ as $m \to \infty$. We conclude that $\Psi_{\varepsilon_m}(w_m) - y_m \to 0$ as $m \to \infty$.

Since $\|y_m\|_{\mathbb{L}} = 1$, it follows that $\|\Psi_{\varepsilon_m}(w_m)\|_{\mathbb{L}} \to 1$ as $m \to \infty$. And this implies that $\|\Psi_{\varepsilon_m}^{-1}(\Psi_{\varepsilon_m}(w_m))\|_{L^2} \to 1$ as $m \to \infty$. Thus $\|w_m\|_{L^2} \to 1$ as $m \to \infty$. In particular, $w_m \neq 0$ for some $m_0 \in \mathbb{N}$ and all $m \geq m_0$.

For $m \geq m_0$ define $u_m = w_m/\|w_m\|_{L^2}$ and for $m < m_0$ let u_m be an arbitrary element of $W_{\varepsilon_m} \cap U_{k-1,\varepsilon_m}^{\perp}$ with $\|u_m\|_{L^2} = 1$ (e.g. $u_m = v_{k,\varepsilon_m}$). It follows that $u_m \in W_{\varepsilon_m}$, $\|u_m\|_{L^2} = 1$, $u_m \in U_{k-1,\varepsilon_m}^{\perp}$ for $m \in \mathbb{N}$. Set $\widetilde{y}_m = \Psi_{\varepsilon_m}(u_m)$. Then, for $m \geq m_0$,

$$\widetilde{y}_m - y_m = \Psi_{\varepsilon_m}\left(\frac{w_m}{\|w_m\|_{L^2}}\right) - y_m = \left(\frac{\Psi_{\varepsilon_m}(w_m)}{\|w_m\|_{L^2}} - \Psi_{\varepsilon_m}(w_m\right) + (\Psi_{\varepsilon_m}(w_m) - y_m)$$

and this implies that

$$\widetilde{y}_m - y_m \to 0, \quad \text{as } m \to \infty, \tag{3.22}$$

proving the second half of (3.18). This proves (3.18) and thus (3.16).

By estimate (3.17) and Lemma 3.10 together with the definition of Ψ_ε we have, for $j \in [1..n]$

$$(\Psi_{\varepsilon_m}(v_{k,\varepsilon_m}))_j \to z_{k,j}, \quad \text{as } m \to \infty.$$

Hence, $b_0(\Psi_{\varepsilon_m}(v_{k,\varepsilon_m}),\Psi_{\varepsilon_m}(v_{k,\varepsilon_m})) - b_0(z_k,z_k) \to 0$, as $m\to\infty$. This together with Lemma 3.11 implies that

$$b_{\varepsilon_m}(v_{k,\varepsilon_m},v_{k,\varepsilon_m}) - b_0(z_k,z_k) \to 0, \quad \text{as } m\to\infty.$$

Now, Lemma 3.13 implies that

$$\lambda_{k,\varepsilon_m} - b_0(z_k,z_k) \to 0, \qquad \text{as } m\to\infty.$$

The proof is complete. □

Lemma 3.15. *Let $(\varepsilon_m)_m$ be a null sequence in $]0,\varepsilon_0[$ and suppose that there exists a sequence $(z_l)_{l\in[1..n]}$ in $\mathbb{R}^n$ such that, for all $l\in[1..n]$ and $j\in[1..n]$,*

$$\sup_{x\in K_{j,\varepsilon_m}} |\varphi_{l,\varepsilon_m}(x) - z_{l,j}| \to 0, \quad \text{as } m\to\infty.$$

For each $k\in[1..n]$ consider the following statement (P_k)*:*

(P_k) *For each $l\in[1..k]$, z_l is an eigenvector corresponding to $\lambda_{l,0}$.*

Then (P_k) holds for each $k\in[1..n]$. Moreover, for each $k\in[1..n]$,

$$\lambda_{k,\varepsilon_m} \to \lambda_{k,0}, \quad \text{as } m\to\infty.$$

Proof. Let $k\in[1..n]$. Lemma 3.13 implies that

$$\lim_{m\to\infty}\big(\lambda_{k,\varepsilon_m} - \inf\{\, b_0(\Psi_{\varepsilon_m}u,\Psi_{\varepsilon_m}u) \mid u\in W_{\varepsilon_m},\ \|u\|_{L^2}=1,\ u\in U^{\perp}_{k-1,\varepsilon_m}\,\}\big)=0. \tag{3.23}$$

Now the estimates (3.16) and (3.23) imply that

$$\lambda_{k,\varepsilon_m} - \inf\{\, b_0(y,y) \mid y\in\mathbb{R}^n,\ \|y\|_{\mathbb{L}}=1,\ y\in Y^{\perp}_{k-1}\,\} \to 0, \quad \text{as } m\to\infty.$$

Therefore, Lemma 3.14 implies

$$b_0(z_k,z_k) = \inf\{\, b_0(y,y) \mid y\in\mathbb{R}^n,\ \|y\|_{\mathbb{L}}=1,\ y\in Y^{\perp}_{k-1}\,\}. \tag{3.24}$$

Now let $k=1$. Then $\inf\{\, b_0(y,y) \mid y\in\mathbb{R}^n,\ \|y\|_{\mathbb{L}}=1\,\} = \lambda_{1,0}$. Estimate (3.24) implies that $b_0(z_1,z_1)=\lambda_{1,0}$. It follows that z_1 is an eigenvector corresponding to $\lambda_{1,0}$. Thus (P_1) holds.

Let $k\in[2..n]$ be such that (P_{k-1}) holds. This implies that

$$\lambda_{k,0} = \inf\{\, b_0(y,y) \mid y\in\mathbb{R}^n,\ \|y\|_{\mathbb{L}}=1,\ y\in Y^{\perp}_{k-1}\,\}.$$

Moreover, estimate (3.24) implies that $b_0(z_k,z_k)=\lambda_{k,0}$. Therefore z_k is an eigenvector corresponding to $\lambda_{k,0}$. Thus (P_k) holds for all $k\in[1..n]$ and the lemma is proved. □

Lemma 3.16. *For every null sequence $(\varepsilon_m)_m$ in $]0,\varepsilon_0[$ there are a subsequence $(\varepsilon^1_m)_m$ of $(\varepsilon_m)_m$ and a sequence $(z_l)_{l\in[1..n]}$ in $\mathbb{R}^n$ such that, for each $l\in[1..n]$ and $j\in[1..n]$,*

$$\sup_{x\in K_{j,\varepsilon^1_m}} |\varphi_{l,\varepsilon^1_m}(x) - z_{l,j}| \to 0, \quad \text{as } m\to\infty.$$

PROOF. Since $\sup_{m\in\mathbb{N}}\sup_{x\in[0,1]}|\varphi_{l,\varepsilon_m}(x)| < \infty$ for each $l \in [1\,..\,n]$, there exists a subsequence $(\varepsilon_m^1)_m$ of $(\varepsilon_m)_m$ and $z_l = (z_{l,j})_{j\in[1..n]} \in \mathbb{R}^n$ such that for each $l, j \in [1\,..\,n]$

$$\varphi_{l,\varepsilon_m^1}(\alpha_{j,\varepsilon_m^1}) \to z_{l,j}, \quad \text{as } m \to \infty. \tag{3.25}$$

Now estimate (3.25) and Lemma 3.3 imply that, for each l, $j \in [1\,..\,n]$,

$$\sup_{x\in K_{j,\varepsilon_m^1}} |\varphi_{l,\varepsilon_m^1}(x) - z_{l,j}| \to 0, \quad \text{as } m \to \infty. \qquad \square$$

COROLLARY 3.17. *For each null sequence $(\varepsilon_m)_m$ in $]0,\varepsilon_0[$ there exist a subsequence $(\varepsilon_m^1)_m$ of $(\varepsilon_m)_m$ and an $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}}$-orthonormal sequence $(z_l)_{l\in[1..n]}$ such that, for each $l \in [1\,..\,n]$, z_l is an eigenvector of A_0 corresponding to $\lambda_{l,0}$ and such that for each $j \in [1\,..\,n]$*

$$\sup_{x\in K_{j,\varepsilon_m^1}} |\varphi_{l,\varepsilon_m^1}(x) - z_{l,j}| \to 0, \ \text{as } m \to \infty,$$

where $z_{l,j}$ is the j-th component of the vector z_l.

PROOF. This follows from Lemma 3.16, Lemma 3.14 and statement (P_n) from Lemma 3.15. $\square$

PROOF OF THEOREM 2.6. Notice that part (a) of the theorem was established in Lemma 3.7.

To prove part (c) fix an arbitrary $\mathbb{L}$-orthonormal sequence $(\varphi_{l,0})_{l\in[1..n]}$ in $\mathbb{R}^n$ such that $\varphi_{l,0}$ is an eigenvector of A_0 corresponding to $\lambda_{l,0}$. Fix $l \in [1\,..\,n]$. There exists a $j_l \in [1\,..\,n]$ such that $\varphi_{l,0,j_l} \neq 0$. Define $\widehat{\varphi}_{l,0} = \nu_l\varphi_{l,0}$, where $\nu_l = 1$ if $\varphi_{l,0,j_l} > 0$ and $\nu_l = -1$ if $\varphi_{l,0,j_l} < 0$.

We claim that there exists an $\widehat{\varepsilon}_l \in\,]0,\varepsilon_0[$ such that $\varphi_{l,\varepsilon}(\zeta_{j_l,\varepsilon}) \neq 0$ for all $\varepsilon \in\,]0,\widehat{\varepsilon}_l[$. Indeed, suppose the claim does not hold. Then there exists a null sequence $(\varepsilon_m)_m$ in $]0,\varepsilon_0[$ such $\varphi_{l,\varepsilon_m}(\zeta_{j_l,\varepsilon_m}) = 0$ for each $m \in \mathbb{N}$. It follows from Corollary 3.17 that there exist a subsequence $(\varepsilon_m^1)_m$ of $(\varepsilon_m)_m$ and an $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}}$-orthonormal sequence $(z_k)_{k\in[1..n]}$ such that, for each $k \in [1\,..\,n]$, z_k is an eigenvector of A_0 corresponding to $\lambda_{k,0}$ and such that for each $j \in [1\,..\,n]$

$$\sup_{x\in K_{j,\varepsilon_m^1}} |\varphi_{k,\varepsilon_m^1}(x) - z_{k,j}| \to 0, \quad \text{as } m \to \infty,$$

so, in particular, $0 \equiv \varphi_{l,\varepsilon_m}(\zeta_{j_l,\varepsilon_m}) \to z_{l,j_l}$ as $m \to \infty$. Thus $z_{l,j_l} = 0$ a contradiction, as $z_l = \pm\varphi_{l,0}$. This proves the claim. Set $\widehat{\varepsilon} = \min_{l\in[1..n]} \widehat{\varepsilon}_l$.

Let $\varepsilon \in\,]0,\varepsilon_0[$. Define $\widehat{\varphi}_{l,\varepsilon} = (\operatorname{sign}\varphi_{l,\varepsilon}(\zeta_{j_l,\varepsilon}))\varphi_{l,\varepsilon}$ if $l \in [1\,..\,n]$ and $\varepsilon \in\,]0,\widehat{\varepsilon}[$ and $\widehat{\varphi}_{l,\varepsilon} = \varphi_{l,\varepsilon}$ otherwise.

It follows that for each $\varepsilon \in\,]0,\varepsilon_0[$, $(\widehat{\varphi}_{l,\varepsilon})_l$ is a L^2-orthonormal sequence such that $\widehat{\varphi}_{l,\varepsilon}$ is an eigenfunction of A_ε corresponding to $\lambda_{l,\varepsilon}$, $l \in \mathbb{N}$. We claim that for all l, $j \in [1\,..\,n]$

$$\sup_{x \in K_{j,\varepsilon}} |\widehat{\varphi}_{l,\varepsilon}(x) - \widehat{\varphi}_{l,0,j}| \to 0, \qquad \text{as } \varepsilon \to 0.$$

Suppose the claim does not hold. Then there exist $\widehat{l}, \widehat{j} \in [1\,..\,n]$, a null sequence $(\varepsilon_m)_m$ in $]0, \widehat{\varepsilon}[$ and $\beta > 0$ such that

$$\sup_{x \in K_{\widehat{j},\varepsilon_m}} |\widehat{\varphi}_{\widehat{l},\varepsilon_m}(x) - \widehat{\varphi}_{\widehat{l},0,\widehat{j}}| > \beta \quad \text{for all } m \in \mathbb{N}. \tag{3.26}$$

An application of Corollary 3.17 to the family $(\widehat{\varphi}_{l,\varepsilon})_{(l,\varepsilon)\in[1..n]\times]0,\varepsilon_0[}$ shows that there exist a subsequence $(\varepsilon_m^1)_m$ of $(\varepsilon_m)_m$ and an $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}}$-orthonormal sequence $(z_l)_{l\in[1..n]}$ such that, for each $l \in [1\,..\,n]$, z_l is an eigenvector of A_0 corresponding to $\lambda_{l,0}$ and for each $j \in [1\,..\,n]$

$$\sup_{x \in K_{j,\varepsilon_m^1}} |\widehat{\varphi}_{l,\varepsilon_m^1}(x) - z_{l,j}| \to 0, \quad \text{as } m \to \infty.$$

In particular, $\widehat{\varphi}_{\widehat{l},\varepsilon_m^1}(\zeta_{j_{\widehat{l}},\varepsilon_m^1}) \to z_{\widehat{l},j_{\widehat{l}}}$ as $m \to \infty$. The definition of $\widehat{\varphi}_{\widehat{l},\varepsilon_m^1}$ implies that $\widehat{\varphi}_{\widehat{l},\varepsilon_m^1}(\zeta_{j_{\widehat{l}},\varepsilon_m^l}) > 0$ for all $m \in \mathbb{N}$. Since $z_{\widehat{l},j_{\widehat{l}}} \neq 0$, it follows that $z_{\widehat{l},j_{\widehat{l}}} > 0$ and so $z_{\widehat{l}} = \widehat{\varphi}_{\widehat{l},0}$. Hence, in particular,

$$\sup_{x \in K_{\widehat{j},\varepsilon_m^1}} |\widehat{\varphi}_{\widehat{l},\varepsilon_m^1}(x) - \widehat{\varphi}_{\widehat{l},0,\widehat{j}}| \to 0, \quad \text{as } m \to \infty,$$

but this contradicts (3.26).

The last statement of part (c) follows from Lemma 3.3. This completes the proof of part (c) of the theorem.

The arbitrariness of the sequence $(\varepsilon_m)_m$ in part (c) and Lemma 3.14 imply part (b) of the theorem. The proof is complete. □

4. A case without wells at the boundary

We will now formulate Assumption 2.1 for the case in which there are no wells close to the boundary points. This simply means that we remove in that hypothesis the points $\zeta_{0,\varepsilon}$, $\zeta'_{0,\varepsilon}$, $\xi_{n,\varepsilon}$, $\xi'_{n,\varepsilon}$ and the numbers τ_0 and τ_n:

Assumption 4.1.

(1a) $n \in \mathbb{N}$, $\varepsilon_0 \in\,]0, \infty]$;

(1b) $(a_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$ is a family of continuous positive functions defined on $[0, 1]$;

(1c) $(x_j)_{j\in[0..n]}$ is a strictly increasing sequence in $[0, 1]$, with $x_0 = 0$ and $x_n = 1$, $(\tau_j)_{j\in[1..n-1]}$ is a sequence in $]0, \infty[$ and $\xi'_{j,\varepsilon}$, $\xi_{j,\varepsilon}$ and $\zeta_{j,\varepsilon}$, $\zeta'_{j,\varepsilon}$, $j \in [1\,..\,n-1]$ are families in $]0, 1[$ with $x_0 < \xi'_{1,\varepsilon} < \xi_{1,\varepsilon} < x_1$, $x_{j-1} < \zeta_{j-1,\varepsilon} < \zeta'_{j-1,\varepsilon} < \xi'_{j,\varepsilon} < \xi_{j,\varepsilon} < x_j$, $j \in [2\,..\,n-1]$ and $x_{n-1} < \zeta_{n-1,\varepsilon} < \zeta'_{n-1,\varepsilon} < x_n$, $\varepsilon \in\,]0, \varepsilon_0[$.

(1d) If $(\Gamma_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$ is the family $([\xi'_{j,\varepsilon}, \zeta'_{j,\varepsilon}])_{\varepsilon\in]0,\varepsilon_0[}$, $j \in [1\,..\,n-1]$, then $\mathbf{m}(\Gamma_\varepsilon) \to 0$ as $\varepsilon \to 0$.

(2a) If $(\Gamma_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$ is any of the families:

$$([\zeta'_{j,\varepsilon},\xi'_{j+1,\varepsilon}])_{\varepsilon\in]0,\varepsilon_0[},\quad j\in[1\,.\,.\,n-2]\ (\text{if } n\geq 3),$$
$$([\zeta_{j,\varepsilon},\zeta'_{j,\varepsilon}])_{\varepsilon\in]0,\varepsilon_0[},\quad ([\xi'_{j,\varepsilon},\xi_{j,\varepsilon}])_{\varepsilon\in]0,\varepsilon_0[},\quad j\in[1\,.\,.\,n-1]\,,$$

$([x_0,\xi'_{1,\varepsilon}])_{\varepsilon\in]0,\varepsilon_0[}$ or else the family $([\zeta'_{n-1,\varepsilon},x_n])_{\varepsilon\in]0,\varepsilon_0[}$, then

$$\frac{\inf\limits_{\Gamma_\varepsilon} a_\varepsilon}{\mathbf{m}(\Gamma_\varepsilon)}\to\infty\quad\text{as } \varepsilon\to 0.$$

(2b) For each $j\in[1\,.\,.\,n-1]$ and $\varepsilon\in\,]0,\varepsilon_0[$, set $\Gamma_{j,\varepsilon}=[\xi_{j,\varepsilon},\zeta_{j,\varepsilon}]$. Then

$$\frac{\inf\limits_{\Gamma_{j,\varepsilon}} a_\varepsilon}{\mathbf{m}(\Gamma_{j,\varepsilon})}\to\tau_j\quad\text{and}\quad\frac{\sup\limits_{\Gamma_{j,\varepsilon}} a_\varepsilon}{\mathbf{m}(\Gamma_{j,\varepsilon})}\to\tau_j\qquad\text{s } \varepsilon\to 0.$$

Notation. In this section, for $n=2$ we write $K_{1,\varepsilon}=[x_0,\xi_{1,\varepsilon}]$ and $K_{2,\varepsilon}=[\zeta_{1,\varepsilon},x_2]$. For $n\geq 3$ we write

$$K_{1,\varepsilon}=[x_0,\xi_{1,\varepsilon}]\,,\quad K_{j,\varepsilon}=[\zeta_{j-1,\varepsilon},\xi_{j,\varepsilon}]\,,\quad j\in[2\,.\,.\,n-1]\,,\quad K_{n,\varepsilon}=[\zeta_{n-1,\varepsilon},x_n]\,.$$

Moreover,

$$K_j=[x_{j-1},x_j]\,,\quad L_j=\mathbf{m}(K_j),\ \text{for } j\in[1\,.\,.\,n].$$

Again we write $H^1=H^1(0,1)$ and $L^2=L^2(0,1)$. Let $b_\varepsilon=\widetilde{b}_\varepsilon\colon H^1\times H^1\to\mathbb{R}$ be the bilinear form defined by

$$b_\varepsilon(u,v)=\widetilde{b}_\varepsilon(u,v)=\int_0^1 a_\varepsilon\cdot u'\cdot v'\,dx,\quad u,v\in H^1.$$

We also assume the Neumann case: for $\varepsilon\in\,]0,\varepsilon_0[$ let A_ε be the set of all pairs (u,w) with $u\in H^1$ and $w\in L^2$ such that $a_\varepsilon u\in H^1$, $u'(0)=u'(1)=0$ and $w=-(a_\varepsilon u)'$.

As before, A_ε is (the graph of) a densely defined nonnegative self-adjoint linear operator in L^2. Again let $\langle\,\cdot\,,\,\cdot\,\rangle_{L^2}$ be the standard scalar product on L^2.

Define the 'limit' bilinear form $b_0\colon\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ by

$$b_0(y,z)=\sum_{j=1}^{n-1}\tau_j(y_{j+1}-y_j)(z_{j+1}-z_j)$$

and the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}}$ on $\mathbb{R}^n$ by

$$\langle y,z\rangle_{\mathbb{L}}=\sum_{j=1}^{n}L_jy_jz_j,\quad y=(y_1,\dots,y_n),\ z=(z_1,\dots,z_n)\in\mathbb{R}^n.$$

Let $A_0\colon\mathbb{R}^n\to\mathbb{R}^n$ be the linear map defined by the pair $(b_0,\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}})$.

It follows that the map A_0 is $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}}$-symmetric and all of its eigenvalues are simple. Denote by $(\lambda_{l,0})_{l\in[1..n]}$ the increasing sequence of eigenvalues of A_0.

We can now state the spectral convergence theorem for the case in question.

Theorem 4.2. *Let Assumption* 4.1 *be satisfied. With the above notation and definitions the following assertions hold*:

(a) $\lambda_{n+1,\varepsilon} \to \infty$ *as* $\varepsilon \to 0$.
(b) *For each* $l \in [1\,.\,.\,n]$, $\lambda_{l,\varepsilon} \to \lambda_{l,0}$ *as* $\varepsilon \to 0$.
(c) *There is a family* $(\widehat{\varphi}_{l,\varepsilon})_{(l,\varepsilon)\in[1..n]\times[0,\varepsilon_0[}$ *such that if* $(l,\varepsilon) \in [1\,.\,.\,n] \times\,]0,\varepsilon_0[$ *then* $\widehat{\varphi}_{l,\varepsilon}$ *is an eigenfunction of* A_ε *corresponding to* $\lambda_{l,\varepsilon}$ *with* $\|\widehat{\varphi}_{l,\varepsilon}\|_{L^2} = 1$, *if* $l \in [1\,.\,.\,n]$ *then* $\widehat{\varphi}_{l,0}$ *is an eigenvector of* A_0 *corresponding to* $\lambda_{l,0}$ *with* $\|\widehat{\varphi}_{l,0}\|_{\mathbb{L}} = 1$ *and such that, for* $j \in [1\,.\,.\,n]$,

$$\sup_{x\in K_{j,\varepsilon}} |\widehat{\varphi}_{l,\varepsilon}(x) - \widehat{\varphi}_{l,0,j}| \to 0, \ \text{as } \varepsilon \to 0,$$

where $\widehat{\varphi}_{l,0,j}$ *is the* j*-th component of the vector* $\widehat{\varphi}_{l,0} \in \mathbb{R}^n$. *Moreover, there is an* $\varepsilon' \in\,]0,\varepsilon_0[$ *such that*

$$\sup_{\varepsilon\in[0,\varepsilon']}\ \sup_{l\in[1..n]}\ \sup_{x\in[0,1]} |\widehat{\varphi}_{l,\varepsilon}(x)| < \infty.$$

Proof. The proof is very similar to (and simpler than) the proof of Theorem 2.6. We omit the details. □

Remark 4.3. With appropriate modifications, other boundary conditions may also be treated, cf also the corresponding discussion in [5].

Carvalho in [2] considers the following

Assumption 4.4. Let $\varepsilon_0 \in\,]0,\infty]$. Let l_1 be a positive constant and $l_1' > l_1$ be a positive function of $\varepsilon \in\,]0,\varepsilon_0[$ such that

$$\lim_{\varepsilon\to 0} l_1'(\varepsilon) = l_1.$$

Let a_1, a_1' and e_1, e_2 be positive functions of $\varepsilon \in\,]0,\varepsilon_0[$ such that

$$a_1(\varepsilon) \le a_1'(\varepsilon), \quad \text{for } \varepsilon \in\,]0,\varepsilon_0[,$$
$$\lim_{\varepsilon\to 0} a_1'(\varepsilon) = \lim_{\varepsilon\to 0} a_1(\varepsilon) = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} e_i(\varepsilon) = \infty, \quad \text{for } i = 1,\, 2.$$

Let $(a_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$ be a family of positive C^2 functions defined on $[0,1]$ and $x_1 \in\,]0,1[$ be such that

$$\begin{aligned} a_\varepsilon(x) &\ge e_1(\varepsilon), && \text{for } x_0 \le x \le x_1 - \varepsilon l_1', \\ a_\varepsilon(x) &\ge e_2(\varepsilon), && \text{for } x_1 + \varepsilon l_1' \le x \le x_2, \\ a_\varepsilon(x) &\ge a_1(\varepsilon), && \text{for } x_1 - \varepsilon l_1' \le x \le x_1 + \varepsilon l_1', \\ a_\varepsilon(x) &\le a_1'(\varepsilon), && \text{for } x_1 - \varepsilon l_1 \le x \le x_1 + \varepsilon l_1. \end{aligned}$$

Here $x_0 = 0$ and $x_2 = 1$. Assume that $l_1' - l_1 = o(1)$ and and $a_1' - a_1 = o(1)$. Assume also that

(a) $a_1'(\varepsilon)/(\varepsilon \min\{e_1(\varepsilon), e_2(\varepsilon)\}) = o(1)$,
(b) $\varepsilon^2 \min\{e_1(\varepsilon), e_2(\varepsilon)\}/a_1(\varepsilon) = O(1)$, and

(c) there exists an $a_1 \in \mathbb{R}$ such that

$$\liminf_{\varepsilon\to 0} \frac{a_1(\varepsilon)}{2\varepsilon l_1} = \limsup_{\varepsilon\to 0} \frac{a_1'(\varepsilon)}{2\varepsilon l_1} = a_1.$$

Under this hypothesis, Carvalho states without proof an analogue of Theorem 4.2 for $n = 2$ (with $\tau_1 = a_1$), see [2, Lemma 2.1 and the assumptions preceding equation (2.5)].

We will now show that Carvalho's Assumption 4.4 implies our Assumption 4.1 (with $n = 2$) but not necessarily vice versa.

Suppose Assumption 4.4 holds. For each $\varepsilon \in]0, \varepsilon_0[$, define

$$\xi'_{1,\varepsilon} = x_1 - \varepsilon l_1'(\varepsilon), \qquad \xi_{1,\varepsilon} = x_1 - \varepsilon l_1,$$
$$\zeta'_{1,\varepsilon} = x_1 + \varepsilon l_1'(\varepsilon), \qquad \zeta_{1,\varepsilon} = x_1 + \varepsilon l_1.$$

Choosing ε_0 smaller, if necessary, we see that parts (1a), (1b) and (1c) of Assumption 4.1 hold.

Notice that $\mathbf{m}([\xi'_{1,\varepsilon}, \zeta'_{1,\varepsilon}]) = 2\varepsilon l_1'(\varepsilon) \to 0$ as $\varepsilon \to 0$, so (1d) in Assumption 4.1 holds. Moreover, $\mathbf{m}([\zeta_{1,\varepsilon}, \zeta'_{1,\varepsilon}]) = \mathbf{m}([\xi'_{1,\varepsilon}, \xi_{1,\varepsilon}]) = \varepsilon(l_1'(\varepsilon) - l_1)$.

For each $\varepsilon \in]0, \varepsilon_0[$, let $\Gamma_\varepsilon = [\zeta_{1,\varepsilon}, \zeta'_{1,\varepsilon}]$ or $\Gamma_\varepsilon = [\xi'_{1,\varepsilon}, \xi_{1,\varepsilon}]$. Then $a_\varepsilon(x) \geq a_1(\varepsilon)$ for all $x \in \Gamma_\varepsilon$, so $\inf\limits_{x\in\Gamma_\varepsilon} a_\varepsilon(x) \geq a_1(\varepsilon)$ and so

$$\frac{\inf\limits_{\Gamma_\varepsilon} a_\varepsilon}{\mathbf{m}(\Gamma_\varepsilon)} \geq \frac{a_1(\varepsilon)}{\mathbf{m}(\Gamma_\varepsilon)} = \frac{a_1(\varepsilon)}{\varepsilon(l_1'(\varepsilon) - l_1)}.$$

Therefore,

$$\liminf_{\varepsilon\to 0} \frac{\inf\limits_{\Gamma_\varepsilon} a_\varepsilon}{\mathbf{m}(\Gamma_\varepsilon)} \geq \liminf_{\varepsilon\to 0} \frac{a_1(\varepsilon)}{\varepsilon(l_1'(\varepsilon) - l_1)} = \liminf_{\varepsilon\to 0} \left(\frac{a_1(\varepsilon)}{2\varepsilon l_1} \frac{2l_1}{(l_1'(\varepsilon) - l_1)} \right).$$

Since

$$\liminf_{\varepsilon\to 0} \frac{a_1(\varepsilon)}{2\varepsilon l_1} = a_1 \quad \text{and} \quad \lim_{\varepsilon\to 0} \frac{2l_1}{(l_1'(\varepsilon) - l_1)} = \infty,$$

it follows that $\liminf\limits_{\varepsilon\to 0} a_1(\varepsilon)/(2\varepsilon l_1) = \infty$ and this implies that

$$\liminf_{\varepsilon\to 0} \frac{\inf\limits_{\Gamma_\varepsilon} a_\varepsilon}{\mathbf{m}(\Gamma_\varepsilon)} = \infty, \quad \text{so} \quad \lim_{\varepsilon\to 0} \frac{\inf\limits_{\Gamma_\varepsilon} a_\varepsilon}{\mathbf{m}(\Gamma_\varepsilon)} = \infty.$$

Now for $\Gamma_\varepsilon = [x_0, \xi'_\varepsilon]$ we have $\inf\limits_{\Gamma_\varepsilon} a_\varepsilon \geq e_1(\varepsilon) \to \infty$ and $\mathbf{m}(\Gamma_\varepsilon) \to x_1 - x_0 = x_1 > 0$ as $\varepsilon \to 0$ so

$$\frac{\inf\limits_{\Gamma_\varepsilon} a_\varepsilon}{\mathbf{m}(\Gamma_\varepsilon)} \to \infty \quad \text{as } \varepsilon \to 0.$$

Analogously, the same statement holds for $\Gamma_\varepsilon = [\zeta'_\varepsilon, x_2]$. This proves property (2a) of Assumption 4.1.

Now, for each $\varepsilon \in]0, \varepsilon_0[$, let $\Gamma_\varepsilon = [\xi_{1,\varepsilon}, \zeta_{1,\varepsilon}]$. Then $\mathbf{m}(\Gamma_\varepsilon) = 2\varepsilon l_1$. Moreover,

$$a_\varepsilon(x) \geq a_1(\varepsilon), \quad \text{for all } x \in \Gamma_\varepsilon \tag{4.1}$$

and so,

$$\liminf_{\varepsilon\to0}\frac{\inf\limits_{\Gamma_\varepsilon} a_\varepsilon}{\mathbf{m}(\Gamma_\varepsilon)}\geq \liminf_{\varepsilon\to0}\frac{a_1(\varepsilon)}{2\varepsilon l_1}=a_1.$$

Furthermore

$$a_\varepsilon(x)\leq a_1'(\varepsilon),\quad \text{for all } x\in\Gamma_\varepsilon \tag{4.2}$$

and so,

$$\limsup_{\varepsilon\to0}\frac{\inf\limits_{\Gamma_\varepsilon} a_\varepsilon}{\mathbf{m}(\Gamma_\varepsilon)}\leq \limsup_{\varepsilon\to0}\frac{a_1'(\varepsilon)}{2\varepsilon l_1}=a_1.$$

This implies that

$$\lim_{\varepsilon\to0}\frac{\inf\limits_{\Gamma_\varepsilon} a_\varepsilon}{\mathbf{m}(\Gamma_\varepsilon)}=a_1.$$

It follows from (4.1) and (4.2) that

$$a_1(\varepsilon)\leq\sup_{x\in\Gamma_\varepsilon}a_\varepsilon(x)\leq a_1'(\varepsilon).$$

Hence

$$a_1=\liminf_{\varepsilon\to0}\frac{a_1(\varepsilon)}{2\varepsilon l_1}\leq\liminf_{\varepsilon\to0}\frac{\sup\limits_{\Gamma_\varepsilon} a_\varepsilon}{\mathbf{m}(\Gamma_\varepsilon)}\leq\limsup_{\varepsilon\to0}\frac{\sup\limits_{\Gamma_\varepsilon} a_\varepsilon}{\mathbf{m}(\Gamma_\varepsilon)}\leq\limsup_{\varepsilon\to0}\frac{a_1'(\varepsilon)}{2\varepsilon l_1}=a_1.$$

Thus,

$$\lim_{\varepsilon\to0}\frac{\sup\limits_{\Gamma_\varepsilon} a_\varepsilon}{\mathbf{m}(\Gamma_\varepsilon)}=a_1$$

so (2b) of Assumption 4.1 holds with $\tau_1=a_1$. Thus Assumption 4.1 holds.

Now an easy adaptation of the example given in Remark 2.5 shows that, for $n=2$ and an arbitrary choice of $x_1\in\,]0,1[$, $\varepsilon_0\in\,]0,\infty]$ and $\tau_1\in\,]0,\infty[$ there is a family $(a_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$ of C^∞-functions such that, for $\xi_{1,\varepsilon}'=x_1-\varepsilon^2-\varepsilon^3$, $\xi_{1,\varepsilon}=x_1-\varepsilon^2$, $\zeta_{1,\varepsilon}=x_1+\varepsilon^2$ and $\zeta_{1,\varepsilon}'=x_1+\varepsilon^2+\varepsilon^3$, Assumption 4.1 is satisfied, but Assumption 4.4 cannot be satisfied with this choice of x_1 and the family $(a_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$.

5. Conley index continuation

In this section we will briefly extend the Conley index continuation results from [1] to the present more general case. We assume the reader's familiarity with the paper [1]. Moreover, recall that we suppose that Assumption 2.1 is satisfied with the ensuing definitions and notation of Sections 2 and 3.

For each $\varepsilon\in\,]0,\varepsilon_0[$ define $H^\varepsilon=L^2$, $\langle\,\cdot\,,\cdot\,\rangle_{H^\varepsilon}=\langle\,\cdot\,,\cdot\,\rangle_{L^2}$ and A_ε as in Section 2. Define also $H^0=\mathbb{R}^n$, $\langle\,\cdot\,,\cdot\,\rangle_{H^0}=\langle\,\cdot\,,\cdot\,\rangle_{\mathbb{L}}$ and A_0 as in (2.3).

Consider the following norms:

$$\begin{aligned}&\|u\|_\varepsilon^2:=b_\varepsilon(u,u)+\|u\|_{L^2}^2,\quad \varepsilon\in\,]0,\varepsilon_0[,\ u\in H^1(0,1),\\ &\|u\|_0^2:=b_0(u,u)+\|u\|_{\mathbb{L}}^2,\quad u\in\mathbb{R}^n.\end{aligned}\tag{5.1}$$

Notice that for each $\varepsilon \in \,]0,\varepsilon_0[$, $H_1^\varepsilon = H^1(0,1)$ and $|\cdot|_{H_1^\varepsilon} = \|\cdot\|_\varepsilon$. Moreover, $H_1^0 = \mathbb{R}^n$ and $|\cdot|_{H_1^0} = \|\cdot\|_0$.

Theorem 5.1. *There exists an $\varepsilon_7' \in \,]0,\varepsilon_0[$ and a family $(J_\varepsilon)_{\varepsilon\in]0,\varepsilon_7']}$ such that the family*

$$(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon, J_\varepsilon)_{\varepsilon\in[0,\varepsilon_7']}$$

satisfies condition (FSpec).

Proof. This is proved exactly like [1, Theorem 4.1]. □

For the rest of this section assume the nonlinear convergence hypothesis [1, Assumption 4.2].

Let $\varepsilon \in \,]0,\varepsilon_0[$. Note that each $u \in H^1(0,1)$ is (uniquely represented by) a continuous function. Hence the map $\widehat{g}_\varepsilon(u)\colon [0,1] \to \mathbb{R}$ defined by

$$\widehat{g}_\varepsilon(u)(x) = g_\varepsilon(x, u(x)), \quad x \in [0,1]\,,$$

is continuous and bounded. Moreover, $\widehat{g}_\varepsilon(u)$ is Lebesgue measurable and so it lies in $L^2(0,1)$. Therefore, for each $\varepsilon \in \,]0,\varepsilon_0[$, we obtain a well defined map $f_\varepsilon\colon H^1(0,1) \to L^2$ given by $f_\varepsilon(u) = \widehat{g}_\varepsilon(u)$, $u \in H^1(0,1)$. Moreover, define $f_0\colon \mathbb{R}^n \to \mathbb{R}^n$ by $f_0(u) = (f_0(u)_1, \ldots, f_0(u)_n)$, where

$$f_0(u)_j = \frac{1}{L_j}\int_{K_j} g_0(x, u_j)\,dx,$$

$u = (u_1, \ldots, u_n)$, for $j \in [1\,.\,.\,n]$.

Theorem 5.2. *Let $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon, J_\varepsilon)_{\varepsilon\in[0,\varepsilon_7']}$ be as Theorem* 5.1. *There exists an $\varepsilon_8' \in \,]0,\varepsilon_7']$ such that the family $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_8']}$ satisfies condition* (Conv).

Proof. This is proved exactly as [1, Theorem 4.4] except that we use Lemma 3.3 in place of [1, Lemma 4.3]. □

Now consider, for each $\varepsilon \in \,]0,\varepsilon_8']$, the abstract parabolic equation

$$\dot{u} = -A_\varepsilon u + f_\varepsilon(u) \tag{5.2}$$

on $H^1(0,1)$. This equation generates a local semiflow π_ε on $H^1(0,1)$. Equation (5.2) is an abstract formulation of the boundary value problem

$$\begin{cases} u_t = (a_\varepsilon u_x)_x + g_\varepsilon(x,u), & 0 < x < 1,\ t > 0, \\ \rho u - (1-\rho)a_\varepsilon u_x = 0, & x = 0,\ t > 0, \\ \sigma u + (1-\sigma)a_\varepsilon u_x = 0, & x = 1,\ t > 0. \end{cases} \tag{$\mathrm{E}_\varepsilon, \mathrm{S}_\varepsilon$}$$

Moreover, we may also consider the system of ordinary diferential equations

$$\dot{z} = -A_0 z + f_0(z) \tag{5.3}$$

on $\mathbb{R}^n$. This system generates a local (semi)flow π_0 on $\mathbb{R}^n$.

Now, exactly as in [1], we obtain the following

Theorem 5.3. *The Conley index and homology index braid continuation results,* [1, Theorems 2.4 and 2.5], *hold for the family* $(\pi_\varepsilon)_{\varepsilon\in[0,\varepsilon'_8]}$.

Remark 5.4. Theorem 5.3 remains valid (with the same proof) if we replace Assumption 2.1 by Assumption 4.1 and Theorem 2.6 by Theorem 4.2.

Acknowledgements. We thank the referees for their suggestions which have led to a significant improvement of the paper.

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Manuscript received July 4, 2017

accepted December 10, 2017

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