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INTEGRABILITY OF THE DERIVATIVE OF SOLUTIONS TO A SINGULAR ONE-DIMENSIONAL PARABOLIC PROBLEM

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To the memory of Professor Marek Burnat

Abstract. We study integrability of the derivative of a solution to a singular one-dimensional parabolic equation with initial data in $W^{1,1}$. In order to avoid additional difficulties we consider only the periodic boundary conditions. The problem we study is a gradient flow of a convex, linear growth variational functional. We also prove a similar result for the elliptic companion problem, i.e. the time semidiscretization.

1. Introduction

We study a one-dimensional parabolic equation

$$\begin{aligned} &u_t = (W_p(u_x))_x, \quad (x,t) \in Q_T := \mathbb{T} \times (0,T), \\ &u(x,0) = u_0(x), \quad x \in \mathbb{T}, \end{aligned} \tag{1.1}$$

where $W\colon \mathbb{R} \to \mathbb{R}$, $W_p = dW(p)/dp$ and $\mathbb{T}$ is a flat one-dimensional torus, which we identify with $[0,1)$. In other words, for the sake of simplicity we consider the periodic boundary conditions, but the same argument with little change applies to the zero Neumann data.

Equation (1.1) is formally a gradient flow of the following functional,

$$\mathcal{E}(u) = \begin{cases} \displaystyle\int_{\mathbb{T}} W(u_x)\,dx & \text{for } u \in W^{1,1}(\mathbb{T}), \\ +\infty & \text{for } u \in L^2(\mathbb{T}) \setminus W^{1,1}(\mathbb{T}). \end{cases}$$

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Our main assumption on W, apart from convexity, is the linear growth of W.

We also consider a companion of this equation, namely, the time semidiscretization of (1.1),

$$\frac{1}{h}(u - f) = (W_p(u_x))_x \quad \text{in } \mathbb{T}. \tag{1.2}$$

Even though it makes sense to consider $u_0 \in \mathrm{BV}(\mathbb{T})$ for equation (1.1) we study here the propagation of regularity, i.e. we show that the integrability of du_0/dx (denoted by $u_{0,x}$) implies that the derivative of the weak solution is also integrable, $u_x(\,\cdot\,,t) \in L^1(\mathbb{T})$, see Theorem 4.2 in Section 4. Apparently, such results are not known in the general context. We are only aware of the paper by Bellettini *et al.* [4], on the parabolic minimal surface equation, for which the authors show that the solutions are eventually regularized, i.e. there is a positive waiting time. We stress that our assumptions on W are more general, since we need only convexity and the linear growth. The precise formulation of these conditions is in the statement of Theorem 4.2.

What we prove in Theorem 4.2 shows that equation (1.1) does not create singularities like jumps. Such a result is known in a multidimensional setting for $W(p) = |p|$. In particular, the jumps present in the data persist, see [6], and Hölder continuity of the data propagates, [7]. We also note that our method is essentially restricted to one dimension. We are not able to address the same question in higher dimensions.

Our Theorem 3.1 is a companion result on a closely related elliptic problem (1.2). But we prove it first, because it is slightly simpler than Theorem 4.2. Here, in equation (1.2) f plays the role of initial conditions, hence $f \in L^p(\mathbb{T})$, $p \geq 1$ implies only that $u \in \mathrm{BV}(\mathbb{T})$. Since (1.2) is the time semidiscretization of (1.1), then integrability of the derivative of solutions following from integrability of the derivative of f is not surprising. A similar statement for a domain in $\mathbb{R}^N$ is proved by Beck *et al.* in [2], but for smooth nonlinearities corresponding to functionals with linear growth. In the setting of [2] the smooth dependence of W on p is important for the argument. In [3], in a similar setting the Lipschitz continuity of minimizers is shown.

If $W(p) = |p|$, the we can offer an additional comment about solutions to (1.2), which is the Euler–Lagrange equation for the Rudin–Osher–Fatemi functional, see [14]. We can say that if data are regular, in this case $f \in W^{1,1}(\mathbb{T})$, then we cannot detect edges, understood as jumps of u solutions to (1.2), because jumps may not be created.

Both of our results can be expressed as no singularity formation. They are both obtained with the same technique depending on the insight into the structure of $L^1(\mathbb{T})$. The necessary preliminary results are presented in Section 2. Namely, if function g belongs to $L^1(\mathbb{T})$, then it automatically enjoys a better integrability, see Lemma 2.1 and [13, §2.1]. In our setting g is the derivative of data, i.e. $g = f_x$ in the case of equation (1.2) or $g = (u_0)_x$ in the case of parabolic equation (1.1). In fact, we show that this better integrability of derivatives of data is passed to the derivatives of solutions, see Theorems 4.2 and 3.1.

We show first the desired estimates for solutions to the regularized problems either elliptic or parabolic. The passage to the limit requires weak compactness in L^1 and the Pettis theorem. In order to show that the limit of solutions to the regularized problems are actually solutions to the original equation we depend on the theory of monotone operators, i.e. Minty's trick.

In Section 3 we first prove our result for the elliptic problem. For this purpose we study solutions to a regularized problem. The parabolic problem, treated in Theorem 4.2, requires an additional step, as compared with the elliptic equation, and this is why we deal with this in the last section. Section 4 is closed with a remark on finite extinction (or rather stopping) time, which is common to the problems we consider, if W has a singularity at $p = 0$.

2. Preliminaries

We gather here our assumptions on W and we present necessary information about the structure of the space $L^1(\Omega)$ for any $\Omega \subset \mathbb{R}^N$.

2.1. Conditions on W and functional $\mathcal{E}$. Throughout the paper, we assume that W is an even, convex function with linear growth at infinity, i.e.

$$\lim_{t\to\infty} \frac{W(t)}{t} = W^+, \qquad \lim_{t\to\infty} \frac{W(-t)}{t} = W^-. \tag{2.1}$$

In the above formula, $W^\pm$ are positive numbers. Without loss of generality we could assume that

$$W^+ = W^- = W^\infty > 0. \tag{2.2}$$

Indeed, one could consider $\widetilde{W}(p) = W(p) + (W^- - W^+)p/2$ in place of W. This modified $\widetilde{W}$ does not change neither (1.1) nor (1.2).

We will not impose any further restrictions W. Here are some examples,

$$|p|, \qquad |p+1| + |p-1|, \qquad \sqrt{1+p^2}, \qquad |p| + \sqrt{1+p^2}.$$

We note that functional $\mathcal{E}$ is defined naturally on the space $W^{1,1}(\mathbb{T})$. However, in general $\mathcal{E}$ is not lower semicontinuous on $W^{1,1}(\mathbb{T})$ with respect to the L^2 topology, unless W is piecewise linear, see [9], [12]. The lower semicontinuous envelope or the relaxation of $\mathcal{E}$, denoted by $\overline{\mathcal{E}}$, is naturally well-defined on $\mathrm{BV}(\mathbb{T})$. For $u \in \mathrm{BV}(\mathbb{T})$ we write,

$$\overline{\mathcal{E}}(u) = \inf\Big\{ \varliminf_{n\to\infty} \mathcal{E}(u_n) : u_n \to u \text{ in } L^2\Big\}. \tag{2.3}$$

We know that (see [1, Theorem 5.47])

$$\overline{\mathcal{E}}(u) = \int_{\mathbb{T}} W(u_x)\,dx + W^\infty \int_{\mathbb{T}} |D^s u|. \tag{2.4}$$

Here, $Du = u_x \llcorner \mathcal{L}^1 + D^s u$ is a decomposition of measure Du into the absolutely continuous part with respect to the Lebesgue measure and the part singular to it.

2.2. The useful structure of $L^1(\Omega)$. Here, we recall the information on $L^1(\Omega)$ needed to derive estimates on solutions to (1.1) and (1.2).

Lemma 2.1. *Let us suppose that $f \in L^1(\Omega)$, then there exists a smooth, convex function $\Phi\colon \mathbb{R} \to \mathbb{R}$ such that, $\lim_{|x|\to\infty} \Phi(x)/|x| = \infty$ and*

$$\int_\Omega \Phi(f)\,dx < \infty. \tag{2.5}$$

Proof. By [13, §1.2, Corollary 3], there exists a convex function $\widetilde{\Phi}$ such that

$$\lim_{|x|\to\infty} \widetilde{\Phi}(x)/|x| = \infty \quad \text{and} \quad \int_\Omega \widetilde{\Phi}(f)\,dx < \infty.$$

From now on, we will use that

$$\widetilde{\Phi} \text{ is decreasing on } (-\infty, 0] \text{ and } \widetilde{\Phi} \text{ is increasing on } [0, +\infty).$$

In other words, 0 is minimum point of $\widetilde{\Phi}$. We can achieve that by adding to $\widetilde{\Phi}$ a linear function ap for a properly chosen real a.

Now, for all $\delta > 0$, we define

$$\widehat{\Phi}_\delta(p) = \begin{cases} \widetilde{\Phi}(p-\delta) & \text{for } p > \delta, \\ \widetilde{\Phi}(p+\delta) & \text{for } p < -\delta, \\ \widetilde{\Phi}(0) & \text{for } |p| \le \delta. \end{cases}$$

Once we have it, we take $\Phi = \widehat{\Phi}_\delta * \phi_\delta$, for any $\delta < 1$, where ϕ_δ is the standard, positive mollifier kernel with $\operatorname{supp}\phi_1 \subset B(0,1)$ and $\max \phi = \phi(0)$. It is easy to see that $\Phi(p)/|p| \to +\infty$ as $|p| \to +\infty$.

Now, we check that

$$\Phi(p) \le C_0 \widetilde{\Phi}(p) + C_1, \tag{2.6}$$

where $C_0 = \phi(0)/\delta$. For $p > 1$ we see that

$$\Phi(p) \le \frac{1}{\delta} \int_{\mathbb{R}} \widehat{\Phi}_\delta(q) \phi\bigg(\frac{p-q}{\delta}\bigg)\,dq \le \frac{\phi(0)}{\delta} \int_{p-\delta}^{p+\delta} \widehat{\Phi}_\delta(p+\delta)\,dq \le C_0 \widetilde{\Phi}(p).$$

Similar inequality holds for $p < -1$.

If $|p| \le \delta$, then

$$\Phi(p) \le C_0 \widetilde{\Phi}(0) \le \widetilde{\Phi}(p) + C_1,$$

where $C_1 = C_0 \max\{1, \widetilde{\Phi}(0)\}$. Thus, (2.6) holds. Since we established (2.6), we conclude that (2.5) holds, too.

We recall that a family $\mathcal{F}$ of integrable functions is *uniformly integrable* if and only if

(i) $\sup_{f\in\mathcal{F}} \int_\Omega |f|\,d\mu = c < \infty$, and

(ii) $\lim_{\mu(A)\to 0} \int_A |f|\,d\mu = 0$ uniformly with respect to $f \in \mathcal{F}$.

Let us introduce the notation

$$\mathcal{G}(v) := \int_D \Phi(v(x))\,dx,$$

where $D = \mathbb{T}$ or $D = Q_T$. The Pettis Theorem immediately implies the following fact.

LEMMA 2.2. *If a sequence $\mathcal{F} = \{f_k\}_{k=0}^\infty \subset L^1(D)$ satisfies*

$$\mathcal{G}(f_k) \le M < \infty, \quad k \in \mathbb{N},$$

then we can select a subsequence f_{k_m} converging weakly in $L^1(D)$ to $f \in L^1(D)$.

We address now the question of the limit passage in $\mathcal{G}$ or $\mathcal{E}$.

LEMMA 2.3. *Let us suppose that $f_n \in L^1(D)$, where $D \subset \mathbb{R}^d$, $d \ge 1$, satisfy the following bound*

$$\int_D \Phi(f_n)\,dx \le M,$$

where Φ is as in Lemma 2.1, and $f_n \rightharpoonup f$ in $L^1(D)$. Then,

$$\varliminf_{n\to\infty} \int_D \Phi(f_n(x))\,dx \ge \int_D \Phi(f(x))\,dx.$$

PROOF. Due to the convexity of Φ, this function is the envelope of a family of straight lines,

$$\Phi(p) = \sup_{\alpha\in I} \ell_\alpha(p).$$

Thus, for any index α we have $\Phi(p) \ge \ell_\alpha(p) = a_\alpha p + b_\alpha$ and

$$\varliminf_{n\to\infty} \int_D \mathcal{G}(f_n(x))\,dx \ge \varliminf_{n\to\infty} \int_D \ell_\alpha(f_n(x))\,dx = \int_D a_\alpha f(x)\,dx + b_\alpha|D|,$$

because any constant a_α may be identified with a continuous functional over $L^1(D)$. Thus,

$$\varliminf_{n\to\infty} \int_D \mathcal{G}(f_n(x))\,dx \ge \int_D \ell_\alpha(f(x))\,dx.$$

After having taken the supremum over $\alpha \in I$ we reach the claim. □

REMARK 2.4. We need this lemma only when $d = 1$ or $d = 2$.

3. The elliptic problem of time semidiscretization

We first deal with integrability of solutions to the following elliptic problem,

$$\frac{1}{h}(u - f) = (W_p(u_x))_x \quad \text{in } \Omega, \tag{3.1}$$

augmented with either the periodic for $\Omega = \mathbb{T}$ or the Neumann boundary conditions, when $\Omega = (0,1)$. However, the argument will be provided for $\Omega = \mathbb{T}$.

First of all, we have to settle the meaning of a solution to (3.1). If we assume that f is in $L^2(\mathbb{T})$, then (3.1) is formally the Euler-Lagrange equation of the following functional,

$$\mathcal{E}(u) + \frac{1}{2h}\int_{\mathbb{T}} (u - f)^2\,dx.$$

However, due to the lack of lower semicontinuity of $\mathcal{E}$ in general, we could understand solutions to (3.1) as minimizers, which are the only critical points here, to

$$\mathcal{F}_f(u) = \overline{\mathcal{E}}(u) + \frac{1}{2h}\int_{\mathbb{T}} (u - f)^2\,dx,$$

where $\overline{\mathcal{E}}$ is the lower semicontinuous envelope of $\mathcal{E}$ defined in (2.3), cf. (2.4). If u is a minimizer of $\mathcal{F}_f$, then this fact just implies that

$$|Du|(\mathbb{T}) \le \frac{1}{\alpha}\|f\|_{L^2}^2, \qquad \|u\|_{L^2} \le 4\|f\|_{L^2}^2.$$

But this is not sufficient to deduce that $u_x \in L^1(\mathbb{T})$.

If we wish to establish integrability of the derivative of the solution to (3.1), we have to proceed differently. Since we expect that $u \in W^{1,1}(\mathbb{T})$, we can define the appropriate notion of a solution. We say that a function $u \in W^{1,1}(\mathbb{T})$ is a *weak solution* to (3.1) if there exists $\xi \in L^\infty(\mathbb{T})$, $\xi_x \in L^2(\mathbb{T})$ such that $\xi(x) \in \partial W(u_x(x))$ for almost every $x \in \mathbb{T}$ and the following identity

$$\int_{\mathbb{T}} \left(\frac{1}{h}(u - f)\varphi + \xi\varphi_x\right) dx = 0$$

holds for all $\varphi \in C^\infty(\mathbb{T})$. We notice that since $C^\infty(\mathbb{T})$ is dense in $W^{1,1}(\mathbb{T})$ and $W^{1,1}(\mathbb{T}) \subset L^2(\mathbb{T})$, we can take test functions from $W^{1,1}(\mathbb{T})$.

Theorem 3.1. *Let us assume that W is convex and the assumption* (2.1)–(2.2) *holds. If $f \in W^{1,1}$ and $h > 0$, then there exists a unique weak solution to* (3.1)*, u. The distributional derivative of u is an element of $L^1(\mathbb{T})$. Moreover,*

$$\mathcal{G}(u_x) \le \mathcal{G}(f_x), \tag{3.2}$$

where Φ is given by Lemma 2.1 *for f_x.*

Proof. In order to obtain existence of solutions, we regularize the equation by adding the εu_{xx} term and smoothing out the nonlinearity, $W^\varepsilon(p) = (W * \rho_\varepsilon)(p)$, where ρ_ε is the standard symmetric mollifying kernel. Thus, we consider,

$$\frac{1}{h}(u^\varepsilon - f) = W_p^\varepsilon(u_x^\varepsilon)_x + \varepsilon u_{xx}^\varepsilon, \quad x \in \mathbb{T}. \tag{3.3}$$

We shall say that a function $u^\varepsilon \in W^{1,2}(\mathbb{T})$ is a *weak solution* to (3.3) if the following identity holds,

$$\int_{\mathbb{T}} \left(\frac{1}{h}(u^\varepsilon - f)\varphi + (W_p^\varepsilon(u_x^\varepsilon) + \varepsilon u_x^\varepsilon)\varphi_x \right) dx = 0 \tag{3.4}$$

for all $\varphi \in C^\infty(\mathbb{T})$. In formula (3.4) we require that φ is smooth, but since $C^\infty(\mathbb{T})$ is dense in $W^{1,2}(\mathbb{T})$ we may use u^ε as a test function.

We notice that equation (3.4) is the Euler-Lagrange equation for the functional

$$\mathcal{F}_f^\varepsilon(u) = \int_{\mathbb{T}} \left(\frac{1}{2h}(u-f)^2 + W^\varepsilon(u_x) + \frac{\varepsilon}{2}u_x^2 \right) dx.$$

Since $\mathcal{F}_f^\varepsilon$ is strictly convex and lower semicontinuous on $W^{1,2}(\mathbb{T})$, we immediately conclude the existence and the uniqueness of minimizers, $u^\varepsilon \in W^{1,2}(\mathbb{T})$. Since W^ε is smooth, we immediately conclude that u^ε satisfies (3.4).

Due to the linear growth of W the derivative W_{pp}^ε is bounded and $W_{pp}^\varepsilon + \varepsilon \geq \varepsilon$. Hence it is easy to deduce higher regularity of u^ε, i.e. $u^\varepsilon \in W^{2,2}(\mathbb{T})$, because

$$\frac{1}{h}(u^\varepsilon - f) = (W_{pp}^\varepsilon(u_x^\varepsilon) + \varepsilon)u_{xx}^\varepsilon.$$

We set

$$\xi^\varepsilon = W_p^\varepsilon(u_x^\varepsilon), \tag{3.5}$$

we notice that $\xi^\varepsilon \in W^{1,2}(\mathbb{T})$. Since W^ε is convex, then its derivative is a monotone function. If we combine it with the linear growth of W, then we notice,

$$\xi^\varepsilon(x) \in [-W^\infty, W^\infty]. \tag{3.6}$$

We have to deduce that the family $\{u^\varepsilon\}$ is relatively weakly compact in $W^{1,1}(\mathbb{T})$. The main point is establishing the existence of a subsequence $\{u_x^\varepsilon\}$ converging weakly in $L^1(\mathbb{T})$. For this purpose, we use Lemma 2.1 guaranteeing that (2.5) holds, i.e. $\mathcal{G}(f_x) < \infty$. Once we have Φ, we multiply both sides of (3.3) by $\Phi''(u_x^\varepsilon)u_{xx}^\varepsilon \in L^2(\mathbb{T})$. After integration over $\mathbb{T}$ and integration by parts, we come to

$$\int_{\mathbb{T}} (f_x \Phi'(u_x^\varepsilon) - u_x^\varepsilon \Phi'(u_x^\varepsilon)) \geq 0.$$

Now, the convexity of Φ gives us,

$$\int_{\mathbb{T}} \Phi(f_x)\, dx - \int_{\mathbb{T}} \Phi(u_x^\varepsilon)\, dx \geq \int_{\mathbb{T}} \Phi'(u_x^\varepsilon)(f_x - u_x^\varepsilon).$$

Combining these two inequalities yields,

$$\mathcal{G}(u_x^\varepsilon) \equiv \int_{\mathbb{T}} \Phi(u_x^\varepsilon) \leq \int_{\mathbb{T}} \Phi(f_x)\, dx \equiv \mathcal{G}(f_x). \tag{3.7}$$

Now, we can use Lemma 2.2 to deduce the weak convergence in $L^1(\mathbb{T})$ of u_x^ε to $u_x \in L^1(\mathbb{T})$ as $\varepsilon \to 0$. In the next step, Lemma 2.3 guarantees the lower semicontinuity of $\mathcal{G}$ and $\mathcal{E}$ with respect to weak convergence in $L^1(\mathbb{T})$. Thus, we reach the bound (3.2).

Now, we want to show that u is indeed a weak solution to (3.1), i.e. we have to find ξ stipulated by the definition of a weak solution and to show that it has the desired properties. For each $\varepsilon > 0$ we have at our disposal, solutions u^ε to (3.4) and ξ^ε defined by (3.5). We notice that due to (3.6) ξ^ε converges (possibly after extracting a subsequence) weakly* in $L^\infty(\mathbb{T})$ to ξ and $\xi(x) \in [-W^\infty, W^\infty]$ almost everywhere.

We know that u^ε_x converges weakly in $L^1(\mathbb{T})$ and we assumed that the test function φ in (3.4) is in $C^\infty(\mathbb{T})$. Thus, in order to be able to pass to the limit in (3.4) we need to know that $\varepsilon \int_\mathbb{T} u^\varepsilon_x \varphi_x \, dx$ goes to zero as $\varepsilon \to 0$. Indeed, since u^ε is a minimizer of $\mathcal{F}^\varepsilon_f$, then we notice

$$\varepsilon \|u^\varepsilon_x\|^2_{L^2} \le \mathcal{F}^\varepsilon_f(u^\varepsilon) \le \mathcal{F}^\varepsilon_f(0) = \int_\mathbb{T} \left[\frac{1}{2h} f^2 + W^\varepsilon(0)\right] dx \le \frac{\|f\|^2_{L^2}}{2h} + W(1) =: C_\mathcal{F}.$$

Thus,

$$\varepsilon \left| \int_\mathbb{T} u^\varepsilon_x \varphi_x \, dx \right| \le \varepsilon \|u^\varepsilon_x\|_{L^2} \|\varphi_x\|_{L^2} \le \varepsilon^{1/2} \sqrt{C_\mathcal{F}} \to 0.$$

Finally, after passing to the limit in (3.4), we obtain the following identity,

$$\int_\mathbb{T} \left(\frac{1}{h}(u - f)\varphi + \xi \varphi_x\right) dx = 0 \tag{3.8}$$

for all $\varphi \in C^\infty(\mathbb{T})$. The density of $C^\infty(\mathbb{T})$ in $W^{1,1}(\mathbb{T})$ and the embedding $W^{1,1}(\mathbb{T}) \subset L^2(\mathbb{T})$ imply that we may take test functions from $W^{1,1}(\mathbb{T})$ in (3.8).

It is important to notice that (3.8) implies that $\xi \in W^{1,2}$. Indeed, due to (3.8) the weak derivative of ξ is $(u - f)/h$, hence our claim follows.

Now, it remains to show that $\xi(x) \in \partial W(u_x(x))$ for almost every $x \in \mathbb{T}$. Indeed, from the construction of u^ε we know that for any $w \in W^{1,1}(\mathbb{T})$ we have

$$\int_\mathbb{T} W^\varepsilon(w_x) \, dx \ge \int_\mathbb{T} \xi^\varepsilon (w_x - u^\varepsilon_x) \, dx + \int_\mathbb{T} W^\varepsilon(u^\varepsilon_x) \, dx. \tag{3.9}$$

We want to calculate the limit of both sides taking into account that

$$\xi^\varepsilon \overset{*}{\rightharpoonup} \xi \quad \text{in } L^\infty(\mathbb{T}) \quad \text{and} \quad u^\varepsilon_x \rightharpoonup u_x \quad \text{in } L^1(\mathbb{T}). \tag{3.10}$$

In order to proceed we have to take a close look at each term in (3.9).

Due to the locally uniform convergence of W^ε to W and the Lebesgue dominated convergence theorem we deduce that

$$\lim_{\varepsilon \to 0} \int_\mathbb{T} W^\varepsilon(w_x) \, dx = \int_\mathbb{T} W(w_x) \, dx. \tag{3.11}$$

Next, we note that the Jensen inequality gives us $W^\varepsilon(p) \ge W(p)$. Hence, Lemma 2.3 yields,

$$\varliminf_{\varepsilon \to 0} \int_\mathbb{T} W^\varepsilon(u^\varepsilon_x) \, dx \ge \varliminf_{\varepsilon \to 0} \int_\mathbb{T} W(u^\varepsilon_x) \, dx \ge \int_\mathbb{T} W(u_x) \, dx. \tag{3.12}$$

Finally, we look at $\int_{\mathbb{T}} \xi^\varepsilon u_x^\varepsilon$ in (3.9). We use (3.4), where we take u^ε for a test function. Thus, we obtain

$$-\int_{\mathbb{T}} \xi^\varepsilon u_x^\varepsilon = \int_{\mathbb{T}} \varepsilon |u_x^\varepsilon|^2 + \frac{1}{h}\int_{\mathbb{T}} (u^\varepsilon - f)u^\varepsilon\, dx.$$

If we use this information, then (3.9) takes the following form,

$$\int_{\mathbb{T}} W^\varepsilon(w_x)\, dx \geq \int_{\mathbb{T}} \xi^\varepsilon w_x\, dx + \int_{\mathbb{T}} \varepsilon |u_x^\varepsilon|^2\, dx + \frac{1}{h}\int_{\mathbb{T}} (u^\varepsilon - f)u^\varepsilon\, dx + \int_{\mathbb{T}} W^\varepsilon(u_x^\varepsilon)\, dx.$$

After dropping the positive term $\int_{\mathbb{T}} \varepsilon |u_x^\varepsilon|^2\, dx$ on the RHS and taking the liminf, using (3.10), (3.11) and (3.12), we arrive at

$$\int_{\mathbb{T}} W(w_x)\, dx \geq \int_{\mathbb{T}} \xi w_x\, dx + \frac{1}{h}\int_{\mathbb{T}} (u - f)u\, dx + \int_{\mathbb{T}} W(u_x)\, dx.$$

We use (3.8) again, we reach

$$\int_{\mathbb{T}} W(w_x)\, dx \geq \int_{\mathbb{T}} \xi(w_x - u_x)\, dx + \int_{\mathbb{T}} W(u_x)\, dx. \tag{3.13}$$

Relaying on (3.13), $u_x \in L^1(\mathbb{T})$, due to Lemma 3.3 below, we deduce that $\xi(x) \in \partial W(u_x)$ almost everywhere. Thus, indeed $u \in W^{1,1}(\mathbb{T})$ is a weak solution to (3.1). Moreover, (3.7) and Lemma 2.3 imply that

$$\int_{\mathbb{T}} \Phi(u_x)\, dx \leq \int_{\mathbb{T}} \Phi(f_x)\, dx. \qquad \square$$

Before we state Lemma 3.3 we observe that our argument shows that

COROLLARY 3.2. *If u is a solution constructed in the previous theorem, then $-\xi_x \in \partial\overline{\mathcal{E}}(u)$.*

PROOF. We will see that $-\xi_x$ is an element of the subdifferential $\partial\overline{\mathcal{E}}(u)$. We know that for $u \in W^{1,1}$, it is true that $\mathcal{E}(u) = \overline{\mathcal{E}}(u)$. If $w \in \mathrm{BV}(\mathbb{T})$, then $w = v + \psi$, where $w_x = v_x$, $w_x \in L^1(\mathbb{T})$ and $\psi_x = 0$ $\mathcal{L}^1$-almost everywhere in $\mathbb{T}$. Then,

$$\overline{\mathcal{E}}(w) = \overline{\mathcal{E}}(v + \psi) = \mathcal{E}(v) + \int_{\mathbb{T}} W^\infty |D^s\psi|.$$

Moreover, ξ the weak* limit of ξ^ε with values in $[-W^\infty, W^\infty]$ satisfies the same constraint. Since $D^s\psi = \sigma|D^s\psi|$, where $|\sigma| = 1$ $|D^s\psi|$-almost everywhere, then

$$\int_{\mathbb{T}} W^\infty |D^s\psi| - \xi D^s\psi = \int_{\mathbb{T}} (W^\infty - \xi\sigma)|D^s\psi| \geq 0,$$

because $(W^\infty - \xi\sigma)(x) \geq 0$ for $|D^s\psi|$-almost every $x \in \mathbb{T}$.

Combining the available information, we obtain,

$$\begin{aligned}\overline{\mathcal{E}}(w) - \overline{\mathcal{E}}(u) &= \mathcal{E}(v) - \mathcal{E}(u) + \int_{\mathbb{T}} W^\infty |D^s\psi| \\ &\geq \int_{\mathbb{T}} \xi(v_x - u_x)\,dx + \int_{\mathbb{T}} \xi D^s\psi \\ &= -\int_{\mathbb{T}} \xi_x(v-u)\,dx - \int_{\mathbb{T}} \xi_x\psi\,dx = -\int_{\mathbb{T}} \xi_x(w-u)\,dx.\end{aligned}$$

In other words, $-\xi_x \in \partial\overline{\mathcal{E}}(u)$. □

Lemma 3.3. *Let us assume that* $\xi \in W^{1,2}(\mathbb{T})$ *is such that* $\xi(x) \in [-W^\infty, W^\infty]$ *and* (3.13) *holds for all* $w \in W^{1,1}(\mathbb{T})$. *Then,* $\xi(x) \in \partial W(u_x(x))$ *for almost all* $x \in \mathbb{T}$.

Proof. We will construct special test functions $h \in W^{1,1}(\mathbb{T})$. For any $x_1, x_2 \in \mathbb{T}$ and $\alpha, \varepsilon > 0$ we set,

$$h(x) = \begin{cases} \alpha(x - x_1) & \text{for } x \in (x_1 - \varepsilon, x_1 + \varepsilon), \\ \alpha\varepsilon & \text{for } x \in (x_1 + \varepsilon, x_2 - \varepsilon), \\ -\alpha(x - x_2) & \text{for } x \in (x_2 - \varepsilon, x_2 + \varepsilon), \\ -\alpha\varepsilon & \text{for } x \in \mathbb{T} \setminus ((x_1 - \varepsilon, x_2 + \varepsilon). \end{cases}$$

Of course, we assume that $2\varepsilon < |x_1 - x_2|$. By definition, $h \in W^{1,1}(\mathbb{T})$. In our notation we suppress the dependence of h on x_1, x_2 α, ε.

We stick $w = u + h$ into formula (3.13). The result is

$$\begin{aligned}(3.14)\qquad &\int_{x_1-\varepsilon}^{x_1+\varepsilon} W(u_x(s) + \alpha) - W(u_x(s))\,ds \\ &\qquad + \int_{x_2-\varepsilon}^{x_2+\varepsilon} W(u_x(s) - \alpha) - W(u_x(s))\,ds \\ &\qquad\qquad \geq \alpha \int_{x_1-\varepsilon}^{x_1+\varepsilon} \xi(s)\,ds - \alpha \int_{x_2-\varepsilon}^{x_2+\varepsilon} \xi(s)\,ds.\end{aligned}$$

For each $\alpha > 0$ there is a full measure set $A_\alpha \subset \mathbb{T}$ such that for all $y \in A_\alpha$ we have

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{y-\varepsilon}^{y+\varepsilon} W(u_x(s) + \alpha) - W(u_x(s))\,ds = W(u_x(y) + \alpha) - W(u_x(y)).$$

We take any sequence $0 < \alpha_k$ converging to zero and the corresponding set A_{α_k}. Subsequently, we take any x_1, $x_2 \in A_0 = \bigcap_{k=1}^{\infty} A_{\alpha_k}$. Then, we divide both sides of (3.14) by 2ε and pass to the limit. In this way we obtain,

$$W(u_x(x_1)+\alpha_k) - W(u_x(x_1)) + W(u_x(x_2)-\alpha_k) - W(u_x(x_2)) \geq \alpha_k(\xi(x_1) - \xi(x_2)),$$

for $x_1, x_2 \in A_0$. Now, we divide both sides of this inequality by α_k and pass to the limit. Since W is a Lipschitz continuous function having one sided derivatives, then we obtain,

$$W_p^+(u_x(x_1)) - W_p^-(u_x(x_2)) \geq \xi(x_1) - \xi(x_2). \tag{3.15}$$

Here $W_p^+(y)$ (resp. $W_p^-(y)$) denotes the right (resp. left) derivative of W at y.

Let us us suppose that there exists $x_1 \in \mathbb{T}$ such that

$$\xi(x_1) > \max\{\omega : \omega \in \partial W(u_x(x_1)\} \equiv W_p^+(u_x(x_1)). \tag{3.16}$$

Since ξ is continuous and set A_0 has the full measure so it is dense, we may assume that $x_1 \in A_0$.

We notice that (3.15) and (3.16) combined imply

$$W_p^+(u_x(x_1)) - W_p^-(u_x(x_2)) > W_p^+(u_x(x_1)) - \xi(x_2).$$

Hence for all x_2 in A_0 we have

$$\xi(x_2) > W_p^-(u_x(x_2)). \tag{3.17}$$

A similar reasoning may be performed, when

$$\xi(x_2) < \min\{\omega : \omega\in\partial W(u_x(x_2)\} \equiv W_p^-(u_x(x_2)).$$

Let us notice that if ξ satisfies (3.13) and b is a real constant, then $\xi - b$ satisfies (3.13) too. Indeed, if ψ is an element of $W^{1,1}(\mathbb{T})$, then

$$\int_{\mathbb{T}} (\xi - b)\psi_x \, dx = \int_{\mathbb{T}} \xi \psi_x \, dx.$$

Let us define $b_0 = \sup\{\xi(x) - W_p^+(u_x(x)) : x \in A_0\}$. Due to the continuity of ξ and the linear growth of W the number b_0 is finite. Since we assumed (3.16), then b_0 is positive.

Let us consider shifts $\xi - b$, where $b \in (0, b_0)$. If for all such shifts we have that

$$\xi(x_1) - b > W_p^-(u_x(x_1)), \quad \text{for all } x_1 \in A_0,$$

then due to continuity of ξ we will have

$$\xi(x_1) - b_0 \in \partial W(u_x(x_1)), \quad \text{for all } x_1 \in A_0$$

hence our claim follows after redefining ξ.

If on the other hand there is $b \in (0, b_0)$ such that there is $x_1 \in A_0$ such that $\xi(x_2) - b < W_p^-(u_x(x_1))$, then due to the definition of b_0 we have $\xi(x_2) - b > W_p^+(u_x(x_2))$. Thus, we reached a contradiction with (3.17). Our claim follows.

4. Integrability of the derivative of solutions to the evolution problem

In this section we study the integrability of the space derivative of solutions to the following evolution problem,

$$(4.1)\qquad \begin{aligned} &u_t = (W_p(u_x))_x, \quad (x,t) \in Q_T := \mathbb{T} \times (0,T), \\ &u(x,0) = u_0(x), \quad x \in \mathbb{T}. \end{aligned}$$

We assume here the periodic boundary conditions, but the same argument applies to the homogeneous Neumann data. The initial value, u_0, is in $W^{1,1}(\mathbb{T})$.

The question we address here is as follows: let us suppose that $u_0 \in W^{1,1}(\mathbb{T})$, is it true that $u(t) \in W^{1,1}(\mathbb{T})$ for almost every $t > 0$? We give an affirmative answer below. This means that, in general, equation (4.1) does not create singularities like jumps.

A relatively simple way to address the question of existence of solutions is by using the nonlinear semigroup theory by Kōmura. It is based on the observation that (4.1) is formally a gradient flow of $\mathcal{E}$. For this purpose we have to consider $\overline{\mathcal{E}}$, the lower semicontinuous envelope of $\mathcal{E}$ defined by formula (2.3), see also (2.4), in place of $\mathcal{E}$.

Proposition 4.1. *Let us suppose that W is convex and even, with the linear growth, i.e.* (2.1) *holds. If $u_0 \in \mathrm{BV}(\mathbb{T})$, then there is a unique function $u\colon [0,\infty) \to L^2(\mathbb{T})$, such that*

(a) *for all $t > 0$ we have $u(t) \in D(\partial\overline{\mathcal{E}}(u(t)))$;*
(b) $u \in L^\infty(0,\infty;\mathrm{BV}(\mathbb{T}))$;
(c) $-du/dt \in \partial\overline{\mathcal{E}}(u(t))$ *almost everywhere on* $(0,\infty)$;
(d) $u(0) = u_0$.

In addition, u has the right derivative for all $t \in (0,\infty)$ and

$$\frac{d^+u}{dt} + (\partial\overline{\mathcal{E}}(u(t)))^o = 0, \quad \text{for a.e. } t \in (0,\infty),$$

where $(\partial\overline{\mathcal{E}}(u(t)))^o$ is the minimal section of $\partial\overline{\mathcal{E}}(u(t)))$, i.e. the element of $\partial\overline{\mathcal{E}}(u(t)))$ with the smallest norm.

Proof. Due to the convexity and lower semicontinuity of $\overline{\mathcal{E}}$ with respect to the L^2 convergence, this fact follows immediately from [5, Theorem 3.2]. □

This theorem has a drawback. Namely, in order to make this result meaningful, we have to identify the subdifferential of $\overline{\mathcal{E}}$. We would like to contrast it with our main result, stated below.

Theorem 4.2. *Let us suppose that $W\colon \mathbb{R} \to \mathbb{R}$ is convex with the linear growth,* (2.1) *holds and $u_0 \in W^{1,1}$. Then, there is a unique weak solution to* (4.1)*, i.e. there are $u \in L^\infty(0,\infty;W^{1,1}(\mathbb{T}))$, $u_t \in L^2(0,\infty;L^2(\mathbb{T}))$,*

$\xi \in L^\infty(0,\infty;L^\infty(\mathbb{T}))$ *such that*

$$\int_{\mathbb{T}} (u_t(x,t)\varphi(x) + \xi(x,t)\varphi_x(x))\,dx = 0 \tag{4.2}$$

for almost every $t > 0$*, for all* $\phi \in C^\infty(\mathbb{T})$ *and* $\xi(x,t) \in \partial W(u_x(x,t))$ *for almost every* $(x,t) \in Q_T$*. In particular,* $\mathcal{E}(u(t)) = \overline{\mathcal{E}}(u(t))$*. Moreover,* $\mathcal{E}(u(t)) \le \mathcal{E}(u_0)$*.*

The proof of this result will be performed in several steps. Before we engage into it, we will make few comments. In order to construct by approximation solutions to the elliptic problem (1.2), we had to resolve the following issues:

(1) Making sure that the limiting function u has the desired integrability properties, see (3.2).

(2) Making sure that the limiting function u is indeed a weak solution, i.e. the limit ξ of $\xi^\varepsilon = W_p^\varepsilon(u_x^\varepsilon)$ is indeed an element of $\partial W(u_x)$. For this we used Minty's trick.

There is another reason for regularization of solutions. This is necessary to give meaning to the following simple informal argument. If we multiply (4.1) by $\Phi''(u_x)u_{xx}$, where Φ is convex, then the right-hand-side will take the form

$$\Phi''(u_x)W''(u_x)u_{xx}^2$$

and it will have a sign. A rigorous approach requires regularization.

In order to resolve these issues for the parabolic problem (4.1), we will proceed in a similar way, i.e. we will consider an auxiliary problem, whose initial conditions are regular,

$$\begin{aligned} u_t^\varepsilon &= (W_p(u_x^\varepsilon))_x, && (x,t) \in Q_T, \\ u^\varepsilon(x,0) &= (u_0 * \rho_\varepsilon)(x), && x \in \mathbb{T}, \end{aligned} \tag{4.3}$$

where $u_0 * \rho_\varepsilon$ is the convolution with the standard mollifying kernel ρ_ε.

We recall the basic existence result for (4.3).

PROPOSITION 4.3 ([10, Theorem 1]). *Let us assume that W satisfies hypotheses of Theorem 4.2. If $u_0 \in \mathrm{BV}(\mathbb{T})$ and $(u_0)_x \in \mathrm{BV}(\mathbb{T})$, then there exists a unique weak solution u to (4.3). More precisely, $u_x \in L^\infty(0,T;\mathrm{BV}(\mathbb{T}))$, $u_t \in L^2(Q_T)$ and there is $\xi \in L^2(0,T;W^{1,2}(\mathbb{T}))$ satisfying the (4.2). Moreover, $\xi(x,t) \in \partial W(u_x)$ for almost every $(x,t) \in Q_T$.*

In order to underline the dependence of solutions, obtained in this way, on the mollifying parameter ε, we will denote them by u^ε and ξ^ε. However, the result above is not sufficient for establishing estimates on solutions, which require prior regularization of W. For this purpose, we have to recall the problem, which led to Proposition 4.3, see [10],

$$\begin{aligned} u_t^{\varepsilon,\gamma} &= (W_p^\gamma(u_x^{\varepsilon,\gamma}))_x + \gamma u_{xx}^{\varepsilon,\gamma}, && (x,t) \in Q_T, \\ u^{\varepsilon,\gamma}(x,0) &= (u_0 * \rho_\varepsilon)(x), && x \in \mathbb{T}, \end{aligned} \tag{4.4}$$

where $W^\gamma = W * \rho_\gamma$ and ρ_γ is the standard mollifier kernel. By the classical theory, see [8], solutions $u^{\varepsilon,\gamma}$ to (4.4) are smooth.

We wish to proceed as in the proof of Theorem 3.1. For this purpose, we fix Φ corresponding to $u_{0,x}$, see Lemma 2.1. With its help we will establish additional estimates of solutions to (4.3).

Lemma 4.4. *Let us suppose that u^ε is a unique weak solution to* (4.3) *and Φ corresponding to $u_{0,x}$ is given by Lemma* 2.1. *Then,*

$$\mathcal{G}(u^\varepsilon_x(\cdot,t)) \le \mathcal{G}\bigg(\frac{d}{dx}(u^\varepsilon_0)\bigg) \le \mathcal{G}\bigg(\frac{d}{dx}u_0\bigg).$$

Proof. We multiply both sides of (4.4) by $\Phi''(u^{\varepsilon,\gamma}_x)u^{\varepsilon,\gamma}_{xx}$ and integrate over $\mathbb{T}$ to obtain,

$$\int_{\mathbb{T}} u^{\varepsilon,\gamma}_t (\Phi'(u^{\varepsilon,\gamma}_x))_x\,dx = \int_{\mathbb{T}} (W^\gamma_{pp}(u^{\varepsilon,\gamma}_x) + \gamma)\Phi''(u^{\varepsilon,\gamma}_x)|u^{\varepsilon,\gamma}_{xx}|^2\,dx \ge 0.$$

Positivity of the right-hand-side (RHS) is guaranteed by the convexity of W^γ and Φ. Integration by parts of the left-hand-side (LHS) above yields,

$$\frac{d}{dt}\int_{\mathbb{T}} \Phi(u^{\varepsilon,\gamma}_x)\,dx \le 0,$$

where the boundary terms dropped out due to the periodic boundary conditions.

After integrating in time over $(0,T)$ and recalling the definition of $\mathcal{G}$ we obtain,

$$\mathcal{G}(u^{\varepsilon,\gamma}_x(\,\cdot\,,t)) \le \mathcal{G}(u^\varepsilon_{0,x}).$$

From [10] we know that

$$u^{\varepsilon,\gamma}_x \text{ converges to } u^\varepsilon_x \text{ strongly in } L^p(0,T;L^q(\mathbb{T})),\ p\ge 1 \text{ and a.e. in } Q_T, \tag{4.5}$$

thus $\mathcal{G}(u^\varepsilon_x(\,\cdot\,,t)) \le \mathcal{G}(u^\varepsilon_{0,x})$. Since Φ is convex, then the Jensen inequality gives us

$$\mathcal{G}(u^\varepsilon_{0,x}) \le \mathcal{G}(u_{0,x}). \tag{4.6}$$

Indeed, if we recall that

$$v^\varepsilon(x) = \frac{1}{\varepsilon}\int_{\mathbb{T}} v(y)\varphi\bigg(\frac{x-y}{\varepsilon}\bigg)\,dy,$$

where $\varphi \in C^\infty(\mathbb{T})$ is the standard mollifier kernel, then we have

$$v^\varepsilon(x) = \int_{\mathbb{T}} v(y)\,d\mu^\varepsilon_x(y),$$

where $\mu^\varepsilon_x = (1/\varepsilon)\varphi((x-y)/\varepsilon)\,dy$ is a probability measure. If we combine the above observation with the Jensen inequality, then we obtain,

$$\Phi(u^\varepsilon_{0,x}) = \Phi\bigg(\int_{\mathbb{T}} u_{0,x}(y)\,d\mu^\varepsilon_x(y)\bigg) \le \int_{\mathbb{T}} \Phi(u_{0,x}(y))\,d\mu^\varepsilon_x(y).$$

Integrating over $\mathbb{T}$ yields (4.6). □

Now, in order to pass to the limit with ε, we need further estimates for this purpose.

Lemma 4.5. *Suppose that u^ε is a unique weak solution to* (4.3)*, then*

$$\int_{Q_T} (u_t^\varepsilon(x,t))^2\,dxdt + \int_{\mathbb{T}} W(u_x^\varepsilon(x,t))\,dx \le \int_{\mathbb{T}} W(u_{0,x}^\varepsilon(x))\,dx. \tag{4.7}$$

Proof. We multiply equation (4.4) by $u_t^{\varepsilon,\gamma}$ and integrate over Q_T. Integrating the RHS by parts yields,

$$\int_{Q_T} |u_t^{\varepsilon,\gamma}|^2\,dxdt + \int_{Q_T} \frac{\partial}{\partial t}\left(\frac{\gamma}{2}\,|u_x^{\varepsilon,\gamma}|^2 + W^\gamma(u_x^{\varepsilon,\gamma})\right)dxdt = 0.$$

Performing the integration over $(0,T)$ leads us to,

$$\begin{aligned}\int_{Q_T} |u_t^{\varepsilon,\gamma}|^2\,dxdt + \int_{\mathbb{T}} \Big(\frac{\gamma}{2}\,|u_x^{\varepsilon,\gamma}(x,t)|^2 + W^\gamma(u_x^{\varepsilon,\gamma}(x,t))\Big)\,dx \\ = \int_{\mathbb{T}} \Big(\frac{\gamma}{2}\,u_{0,x}^\varepsilon(x)|^2 + W^\gamma(u_{0,x}^\varepsilon(x))\Big)\,dx.\end{aligned} \tag{4.8}$$

The RHS converges to

$$\int_{\mathbb{T}} W^\gamma(u_{0,x}^\varepsilon(x))\,dx \quad \text{as } \gamma\to 0.$$

We may drop

$$\frac{1}{2}\int_{\mathbb{T}} \gamma|u_x^{\varepsilon,\gamma}(x,t)|^2\,dx$$

on the LHS. The lower semicontinuity of the L^2 norm yields

$$\varliminf_{\gamma\to 0^+} \int_{Q_T} |u_t^{\varepsilon,\gamma}|^2\,dx\,dt \ge \int_{Q_T} |u_t^\varepsilon|^2\,dx\,dt.$$

Now, when regularizing W, we notice that the averaging of a convex function, performed in the convolution gives us $W(p) \le W^\varepsilon(p)$ for all $p \in \mathbb{R}$. As a result we arrive at

$$\int_{\mathbb{T}} W(u_x^{\varepsilon,\gamma}(x,t)) \le \int_{\mathbb{T}} W^\gamma(u_x^{\varepsilon,\gamma}(x,t)) \le M.$$

We again use (4.5) to conclude that

$$\lim_{\gamma\to 0^+} \int_{\mathbb{T}} W(u_x^{\varepsilon,\gamma})(x,t)\,dx = \int_{\mathbb{T}} W(u_x^\varepsilon)(x,t)\,dx \quad \text{a.e. } t>0.$$

Combining these gives the desired result. □

We notice that Lemma 4.5 immediately implies that

$$u_t^\varepsilon \rightharpoonup u_t \quad \text{in } L^2(Q_T) \text{ as } \varepsilon\to 0. \tag{4.9}$$

We know that ξ^ε postulated by Proposition 4.3 satisfies

$$\xi^\varepsilon(x,t) \in \partial W(u_x^\varepsilon(\,\cdot\,,t)) \subset [-W^\infty, W^\infty].$$

Here, the last inclusion is obtained by the argument, which gave us (3.6). Hence, we deduce that there is a subsequence (not relabeled) such that

$$\xi^\varepsilon \rightharpoonup \xi \quad \text{in } L^2(Q_T) \quad \text{and} \quad \xi^\varepsilon \overset{*}{\rightharpoonup} \xi \quad \text{in } L^\infty(Q_T). \tag{4.10}$$

Using the argument from [11, Theorem 2.1, p. 2292] we can show that

$$\xi^\varepsilon(\,\cdot\,,t) \overset{*}{\rightharpoonup} \xi(\,\cdot\,,t) \quad \text{in } L^\infty(\mathbb{T}) \text{ for a.e. } t>0.$$

We may repeat the argument of [10], [12] to claim that

$$u^\varepsilon \text{ converges to } u \text{ in } L^p(0,T;L^q(\Omega)), \quad p,q\in(1,\infty), \tag{4.11}$$

hence $\|u^\varepsilon(\,\cdot\,,t)-u(\,\cdot\,,t)\|_{L^q}\to 0$ for a lmost every $t>0$. However, the key issue is convergence of u^ε_x.

We notice that due to Lemmas 4.4 and 2.2, we can select a subsequence $\{u^{\varepsilon_k}_x\}_{k=1}^\infty$ such that $u^{\varepsilon_k}_x$ converges weakly in $L^1(Q_T)$ to u_x and, if we fix $t>0$, there is a subsequence (not relabeled) such that $u^{\varepsilon_k}_x(\,\cdot\,,t)$ converges weakly in $L^1(\Omega)$ to $u_x(\,\cdot\,,t)$. However, copying the argument from [11, Theorem 2.1, p. 2292] leads us to the following statement:

Lemma 4.6. *There is a sequence u^k, $k\in\mathbb{N}$ such that $u^k_x \rightharpoonup u_x$ in $L^1(Q_T)$ and, for almost all $t>0$, $u^k_x(\,\cdot\,,t)\rightharpoonup u_x(\,\cdot\,,t)$ in $L^1(\Omega)$.*

Here is an immediate conclusion from Lemmas 4.6 and 2.3:

Corollary 4.7. *If u_x is a weak limit in $L^1(Q_T)$ of the sequence u^n_x, then*

$$\mathcal{G}(u(\,\cdot\,,t))\le M<\infty \quad \textit{and} \quad \mathcal{E}(u(\,\cdot\,,t))\le\mathcal{E}(u_0) \quad \textit{for a.e. } t>0.$$

Now, we claim that u with ξ is a weak solution to (4.1). If we inspect (4.2), the weak form of (4.1), and integrate it over $(0,T)$, assuming that $\phi\in C^\infty_0(Q_T)$, then we will see

$$\int_{Q_T} u^\varepsilon_t(x,t)\phi(x,t)\,dx\,dt + \int_{Q_T}\xi^\varepsilon(x,t)\phi_x(x,t)\,dx\,dt = 0. \tag{4.12}$$

The stated above weak convergence of u^ε_t and ξ^ε gives us,

$$\int_{Q_T} u_t(x,t)\phi(x,t)\,dx\,dt + \int_{Q_T}\xi(x,t)\phi_x(x,t)\,dx\,dt = 0.$$

We can localize it by arguing like in [11, Theorem 2.1, p. 2292],

$$\int_{\mathbb{T}} u_t(x,t)\psi(x)\,dx + \int_{\mathbb{T}}\xi(x,t)\psi_x(x)\,dx = 0 \quad \text{for a.e. } t>0 \text{ and all } \psi\in C^\infty(\mathbb{T}).$$

Now, it remains to show that $\xi(x,t)\in\partial W(u_x(x,t))$ for almost every (x,t) in Q_T. Indeed, from the construction of u^ε we know that, for any $w\in W^{1,1}$ and for almost every $t>0$, we have

$$\int_{\mathbb{T}} W(w_x(x))\,dx \ge \int_{\mathbb{T}}\xi^\varepsilon(x,t)(w_x(x)-u^\varepsilon_x(x,t))\,dx + \int_{\mathbb{T}} W(u^\varepsilon_x(x,t))\,dx. \tag{4.13}$$

In order to use (4.10) and Lemma 4.6 we multiply (4.13) by $\psi \geq 0$ and $\psi \in C_0^\infty(0,T)$ and integrate over $(0,T)$. We get,

$$\int_{Q_T} \psi W(w_x)\,dx\,dt \geq \int_{Q_T} \psi \xi^\varepsilon (w_x - u_x^\varepsilon)\,dx\,dt + \int_{Q_T} \psi W(u_x^\varepsilon)\,dx\,dt.$$

Due to Lemma 2.3

$$\varliminf_{n\to\infty} \int_{Q_T} \psi W(u_x^\varepsilon)\,dx\,dt \geq \int_{Q_T} \psi W(u_x)\,dx\,dt.$$

If we use u^ε as a test function in (4.12), then we reach,

$$\int_{Q_T} \xi^\varepsilon u_x^\varepsilon\,dx\,dt = -\int_{Q_T} u_t^\varepsilon u^\varepsilon\,dx\,dt.$$

Since sequence u^ε is bounded in $W^{1,1}(Q_T)$, then u^ε converges strongly to u in $L^2(Q_T)$, (possibly after extracting a subsequence). Combining this with (4.9) yields,

$$\lim_{n\to\infty} \int_{Q_T} u_t^\varepsilon u^\varepsilon\,dx\,dt = \int_{Q_T} u_t u\,dx\,dt.$$

Thus, we have reached

$$\begin{aligned}\int_{Q_T} \psi W(w_x)\,dx\,dt &- \int_{Q_T} \psi W(u_x)\,dx\,dt \\ &\geq \int_{Q_T} \psi(\xi w_x + u u_t)\,dx\,dt = \int_{Q_T} \psi \xi (w_x - u_x)\,dx\,dt,\end{aligned}$$

where we use (4.12) again in the last equality. Since $\psi \geq 0$ was arbitrary, we deduce that

$$\int_{\mathbb{T}} W(w_x)\,dx - \int_{\mathbb{T}} W(u_x)\,dx \geq \int_{\mathbb{T}} \xi(w_x - u_x)\,dx. \tag{4.14}$$

Here, we notice that since $C^\infty(\mathbb{T})$ is dense in $W^{1,1}(\mathbb{T})$ we can take u as (4.12).

Now, by applying Lemma 3.3 we deduce that $\xi(x,t) \in \partial W(u_x(x,t))$ almost everywhere in Q_T. Thus, we finalized the construction of a weak solution to (4.1) satisfying the desired bound. Now, we notice that the solution we have constructed satisfies the properties stipulated by Proposition 4.1; hence we deduce uniqueness of solutions. This concludes the proof of Theorem 4.2. □

We also notice that, in fact, in Theorem 4.2 we have constructed solutions in the sense of Proposition 4.1.

4.1. Common properties of solutions. Since we made rather weak assumptions on the nonlinearity W, we should not expect too many common features of solutions. The property, which draws attention, when dealing with the total variation flow is the finite stopping time of solutions, i.e. at some time instance the solution stops moving having reached a terminal state. In this section we will relate the finite stopping time to the lack of differentiability of W at $p = 0$. The behavior of W for large arguments does not seem to matter.

Theorem 4.8. *Let us suppose that $u_0 \in W^{1,1}(\Omega)$ and W is such that at all points p, the one-sided derivatives of W, at p are greater or equal to $\alpha > 0$. Then, for all $t \geq T_{\rm ext}$, we have*

$$u(t) \equiv \overline{u}_0, \quad \textit{where} \quad \overline{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0\,dx \quad \textit{and} \quad T_{\rm ext} \leq C_p\|u_0\|_{L^2},$$

and C_p is the constant in the Poincaré inequality.

Proof. Observe that the average of solutions is preserved due to the boundary conditions. We denote this average by $\overline{u}$. We compute $d\|u-\overline{u}\|^2_{L^2}/dt$, while integrating by parts

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u(x,t)-\overline{u}|^2\,dx &= \int_\Omega (u-\overline{u})u_t\,dx = \int_\Omega (u-\overline{u})(W_p(u_x))_x \\ &= -\int_\Omega W_p(u_x)u_x = -\int_\Omega |W_p(u_x)|\operatorname{sgn} u_x \cdot u_x\,dx.\end{aligned}$$

We used here the monotonicity of W_p, which implies that

$$W_p(u_x)u_x = |W_p(u_x)||u_x|.$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u(x,t)-\overline{u}|^2\,dx \leq -\int_\Omega \alpha|u_x|\,dx \leq -C_p^{-1}\|u-\overline{u}\|_{L^2}.$$

Here, we used the Poincaré's inequality, $\|u-\overline{u}\|_{L^2} \leq C_p\|u_x\|_{L^1}$. We conclude that

$$\frac{d}{dt}\|u-\overline{u}\|_{L^2} \leq -C_p,$$

what implies that $T_{\rm ext} \leq C_p\|u_0\|_{L^2}$. $\square$

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