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ON THE WELL-POSEDNESS OF DIFFERENTIAL MIXED QUASI-VARIATIONAL-INEQUALITIES

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Abstract. We discuss the well-posedness and the well-posedness in the generalized sense of differential mixed quasi-variational inequalities ((DMQVIs), for short) in Hilbert spaces. This gives us an outlook to the convergence analysis of approximating sequences of solutions for (DMQVIs). Using these concepts we point out the relation between metric characterizations and well-posedness of (DMQVIs). We also prove that the solution set of (DMQVIs) is compact, if problem (DMQVIs) is well-posed in the generalized sense.

1. Introduction

Let X be a Hilbert space whose norm and scalar product are $\|\cdot\|_X$ and $\langle\cdot\,,\cdot\rangle_X$, respectively. The norm convergence is denoted by $\to$ and the weak convergence by $\rightharpoonup$. Let $0 < T < +\infty$ and $I := [0,T]$. Recall that the Hilbert space $L^2(I;X)$ is endowed with the scalar product defined by

$$\langle u_1,u_2\rangle_{L^2(I;X)} := \int_0^T \langle u_1(t),u_2(t)\rangle_X\,dt, \quad \text{for all } u_1,u_2\in L^2(I;X).$$

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Set $W(X) := \{x \in L^2(I;X) : \dot{x} \in L^2(I;X)\}$. Here $\dot{x}$ stands for the generalized derivative of x, i.e.

$$\int_0^T \dot{x}(t)\phi(t)\,dt = -\int_0^T x(t)\dot{\phi}(t)\,dt, \quad \text{for all } \phi \in C_0^\infty(I).$$

It is well known that $W(X)$ is a Hilbert space with the scalar product

$$\langle x_1, x_2\rangle_{W(X)} := \langle x_1, x_2\rangle_{L^2(I;X)} + \langle \dot{x}_1, \dot{x}_2\rangle_{L^2(I;X)}, \quad \text{for all } x_1, x_2 \in W(X),$$

and it is densely and continuously embedded in $C(I;X)$.

Given the Hilbert spaces X and V, we state the following differential mixed quasi-variational inequality ((DMQVI), for short):

$$\begin{cases} \dot{x} \in \Phi(x,u), \\ u \in \mathrm{SOL}(S(\,\cdot\,), G(x,\,\cdot\,), \phi), \\ \Gamma(x(0), x(T)) = 0, \end{cases} \tag{1.1}$$

where the set-valued Nemytskiĭ operator $\Phi\colon L^2(I;X) \times L^2(I;V) \rightrightarrows L^2(I;X)$ is defined by

$$\Phi(x,u)(t) := \psi(t, x(t), u(t)), \quad t \in I,$$

with a set-valued mapping $\psi\colon I \times X \times V \rightrightarrows X$. The notation $\mathrm{SOL}(S(\,\cdot\,), G(x,\,\cdot\,), \phi)$ stands for the solution set of the following mixed quasi-variational inequality: Find $u \in S(u)$ such that

$$\langle G(x,u), v-u\rangle_{L^2(I,V)} + \phi(v) - \phi(u) \ge 0, \quad \text{for all } v \in S(u).$$

In the statement above, $S\colon \overline{K} \rightrightarrows \overline{K}$ is a set-valued mapping with

$$\overline{K} := \{u \in L^2(I;V) : u(t) \in K \text{ for a.e. } t \in I\}$$

for a fixed nonempty closed convex set $K \subseteq V$, whereas $G\colon L^2(I,X) \times L^2(I,V) \to L^2(I,V)$ is defined by

$$G(x,u)(t) := g(t, x(t), u(t)), \quad t \in I,$$

with $g\colon I \times X \times V \to V$, $\Gamma\colon X \times X \to X$, and $\phi\colon L^2(V) \to \mathbb{R}$ is a convex functional $\not\equiv +\infty$.

The differential variational inequalities ((DVIs), for short) are useful for the study of models involving both dynamics and constraints in the form of inequalities. They arise in many applications: electrical circuits with ideal diodes, Coulomb friction problems for contacting bodies, economical dynamics, dynamic traffic networks. Pang and Stewart [26], [27] established the existence, uniqueness, and Lipschitz dependence of solutions subject to boundary conditions for (DVIs) in finite dimensional spaces. Han and Pang investigated a class of differential quasi-variational inequalities in [11], and Li, Huang and O'Regan [18] studied a class of differential mixed variational inequalities in finite dimensional spaces. Gwinner [8] obtained an equivalence result between (DVIs) and projected dynamical systems. In [9] he also proved a stability property for (DVIs) by using the monotonicity method of Browder and Minty, and Mosco set convergence. Chen and Wang [4] studied dynamic Nash equilibrium problems which have the formulation of differential mixed quasi-variational inequalities. Elastoplastic contact problems can also be incorporated into (DMQVIs) formulation because general dynamic processes in the nonsmooth unilateral contact problems are governed by quasi-variational inequalities. A numerical study for nonsmooth contact problems with Tresca friction can be found in [10], Liu, Loi and Obukhovskii [19] studied the existence and global bifurcation for periodic solutions of a class of (DVIs) by using the topological degree theory for multivalued maps and the method of guiding functions. For more details about (DVIs) we refer to [3], [30], [12], [22]–[21].

Recently, for such types of problems, Liu, Zeng and Motreanu [22]–[21] proved the existence of solutions for differential variational inequalities in Banach spaces. However, the approximation of solutions has not yet been studied in infinite dimensional spaces. Based on this, we focus on the concept of well-posedness and establish the existence and approximation of solutions for (DVIs) in the infinite dimensional setting. The classical concept of well-posedness for a minimization problem, as known from Tykhonov [29], requires the existence and uniqueness of the solution and the convergence of every minimizing sequence toward the unique solution. However, in many specific situations, the solution may not be unique. Thus, the concept of well-posedness in the generalized sense was introduced meaning the existence of solutions and the convergence of some subsequence of every minimizing sequence toward a solution. Many works are devoted to extending the concept of well-posedness in optimization problems [1], [5], variational inequalities [7], [6], [32], fixed point problems [17], equilibrium problems [13], [15], [25], inclusion problems [2], mixed quasi-variational-hemivariational inequalities [24].

In the present paper we introduce the notions of well-posedness and well-posedness in the generalized sense for (DMQVIs). Relying on them, we derive metric characterizations of well-posedness of (DMQVI). Here we provide criteria of well-posedness and well-posedness in the generalized sense for problem (DMQVI), in particular for (DMVI), in terms of approximate sequences, which can be regarded as solutions of corresponding regularized problems. In the case of highly nonsmooth problems in infinite dimensional spaces, for instance quasi-variational inequalities in various function spaces, it is definitely easier to check the relevant conditions for approximate problems, sometimes in a finite dimensional framework, exhibiting more regularity. For a systematic presentation of the use of regularized method, approximation and convergence in studying a wide range of nonlinear evolution and elliptic problems we refer to the monograph [31]. We also prove that the solution set of (DMQVI) is compact if the problem is well-posed in the generalized sense.

2. Preliminaries and well-posedness of (DMQVIs)

We state the relevant definitions.

Definition 2.1. The measure of noncompactness for a nonempty subset A of a Banach space E is defined by

$$\mu(A) := \inf\bigg\{\varepsilon > 0 : A = \bigcup_{i=1}^{n} A_i,\ \mathrm{diam}(A_i) < \varepsilon,\ i = 1, \ldots, n\bigg\},$$

where $\mathrm{diam}(A_i)$ denotes the diameter of the set A_i.

Definition 2.2. Let X be a Banach space and $A\colon X \to X^*$. We say that A is upper-hemicontinuous, if for all $u, v, w \in X$ the functional

$$t \mapsto \langle A(u + tv), w\rangle_{X^* \times X}$$

is upper semicontinuous on $[0, 1]$.

Definition 2.3. The Hausdorff distance between nonempty subsets A, B of the Banach space E is defined by

$$\mathcal{H}(A, B) := \max\{e(A, B), e(B, A)\},$$

where $e(A, B) := \sup\limits_{a\in A} d(a, B)$ with $d(a, B) := \inf\limits_{b\in B} \|a - b\|_E$.

If $\{A_n\}$ is a sequence of nonempty subsets of E, we say that A_n converges to A in the sense of Hausdorff metric if $\mathcal{H}(A_n, A) \to 0$. In order to investigate the well-posedness for (DMQVIs), we recall the definitions of Painlevé–Kuratowski limits.

Definition 2.4. The Painlevé–Kuratowski strong limit inferior and (sequential) weak limit superior of a sequence $\{A_n\} \subset E$ are defined by

$$\begin{aligned} \text{s-}\liminf A_n &:= \{x \in E : \exists\, x_n \in A_n,\ n \in \mathbb{N},\ \text{with } x_n \to x \text{ in } E\},\\ \text{w-}\limsup A_n &:= \{x \in E : \exists\, n_k \uparrow +\infty, \exists\, x_{n_k} \in A_{n_k},\ k \in \mathbb{N},\\ &\qquad \text{with } x_{n_k} \rightharpoonup x \text{ in } E\}. \end{aligned}$$

Definition 2.5 [14]. A multi-valued mapping $A\colon E \rightrightarrows Y$ between Banach spaces E, Y is called

(a) (s, w)-closed if $\text{w-}\limsup A(x_n) \subseteq A(x)$ as $x_n \to x$ in E;
(s, s)-closed if in the preceding inclusion the strong convergence replaces the weak convergence;

(b) (s, s)-lower semicontinuous if $A(x) \subseteq \text{s-}\liminf A(x_n)$ as $x_n \to x$ in E;

(c) (s, w)-subcontinuous if for every sequence $\{x_n\}$ strongly converging in E, every sequence $\{y_n\} \subset Y$ with $y_n \in A(x_n)$ has a weakly convergent subsequence in Y.

Remark 2.6. It follows from [14, Theorem 1.1.4] that a multi-valued mapping being (s, w)-upper semicontinuous and with closed values is sequentially (s, w)-closed.

Next we turn to well-posedness for (DMQVIs) (see (1.1)).

Definition 2.7. A sequence $\{(x_n, u_n)\}$ in $W(X) \times L^2(I, V)$ is called an approximating sequence for (DMQVIs) if there exists a sequence $\varepsilon_n \to 0^+$ as $n \to \infty$ such that

$$(2.1)\qquad \begin{cases} d_{L^2(I,X)}(\dot{x}_n, \Phi(x_n, u_n)) \le \varepsilon_n, \\ d_{L^2(I,V)}(u_n, S(u_n)) \le \varepsilon_n, \\ \langle G(x_n, u_n), v - u_n\rangle_{L^2(I,V)} + \phi(v) - \phi(u_n) \\ \qquad\qquad \ge -\varepsilon_n \|v - u_n\|_{L^2(I,V)}, \quad \text{for all } v \in S(u_n), \\ \|\Gamma(x_n(0), x_n(T))\|_X \le \varepsilon_n. \end{cases}$$

Definition 2.8. Problem (DMQVIs) is said to be strongly well-posed if it has a unique solution (x_0, u_0) and every approximating sequence $\{(x_n, u_n)\}$ strongly converges to (x_0, u_0).

Definition 2.9. Problem (DMQVIs) is said to be strongly well-posed in the generalized sense if the solution set Σ of (DMQVIs) is nonempty and every approximating sequence $\{(x_n, u_n)\}$ has a subsequence which strongly converges to some point of Σ.

For every $\varepsilon > 0$, we set

$$(2.2)\qquad \begin{aligned} \Omega(\varepsilon) = \big\{&(x, u) \in W(X) \times L^2(I, V) : \\ & d_{L^2(I,X)}(\dot{x}, \Phi(x, u)) \le \varepsilon,\ d_{L^2(V)}(u, S(u)) \le \varepsilon, \\ & \langle G(x, u), v - u\rangle_{L^2(V)} + \phi(v) - \phi(u) \ge -\varepsilon\|v - u\|_{L^2(V)}, \\ & \text{for all } v \in S(u) \text{ and } \|\Gamma(x(0), x(T))\|_X \le \varepsilon\big\}. \end{aligned}$$

3. Basic assumptions

This short section sets forth the basic assumptions needed in the sequel. Remarks below subsequent results point out the possible separate use of these assumptions.

First, we list the following group of assumptions:

(A_1) $\Phi \colon W(X) \times L^2(I, V) \rightrightarrows L^2(I, X)$ is sequentially (s,w)-closed and sequentially (s,w)-subcontinuous;

(A_2) $S\colon \overline{K} \rightrightarrows \overline{K}$ is convex-valued, (s,w)-closed, (s,s)-lower semicontinuous and (s,w)-subcontinuous;

(A_3) ϕ is a convex and continuous function on $\overline{K}$;

(A_4) $\langle G(\,\cdot\,,\,\cdot\,),\,\cdot\,\rangle\colon W(X)\times L^2(I,V)\times L^2(I,V)\to\mathbb{R}$ is upper semicontinuous, that is, $u_n\to u_0$, $x_n\to x_0$ and $v_n\to v_0$ imply

$$\limsup_{n\to\infty}\langle G(x_n,u_n),v_n\rangle_{L^2(I,V)}\le\langle G(x_0,u_0),v_0\rangle_{L^2(I,V)};$$

(A_5) $\Gamma\colon X\times X\to X$ is continuous.

Remark 3.1. Notice that each of assumptions (A_1)–(A_5) refers to just one of the data in problem (DMQVI), so there is no interplay between these hypotheses.

Remark 3.2. Assumption (A_2) ensures that $S\colon \overline{K}\rightrightarrows\overline{K}$ is closed, convex set-valued mapping, which will be essential in the forthcoming theorems.

Some of these assumptions can be weakened in certain circumstances as seen in the subsequent sections. Sometimes alternative assumptions can be used. In this respect, we introduce the concept of relaxed α-monotonicity, which is interesting in itself.

Definition 3.3. Given $\alpha\colon L^2(V)\to\mathbb{R}$, the mapping $G\colon\overline{K}\to L^2(I,V)$ is said to be relaxed α-monotone if

$$\langle G(v)-G(u),v-u\rangle\ge\alpha(v-u),\quad\text{for all } u,v\in\overline{K}.$$

In fact, the relaxed α-monotonicity is an abstract concept for monotonicity of a linear or nonlinear operator A on a Hilbert space H. Definition 4.3 is formulated for $A=G$ and $G=L^2(V)$. Certainly, for investigating specific properties it is necessary to assume adequate conditions that the function α should meet. Significant choices for α are as follows:

- if $\alpha(u)=0$, then A is monotone;
- if $\alpha(u)=m_A\|u\|^2$ with $m_A>0$, then A is strongly monotone;
- if $\alpha\colon L^2(V)\to\mathbb{R}_+$ is such that $\alpha(u)>0$ for all $u\in L^2(V)\setminus\{0\}$ and $\alpha(0)=0$, then A is strictly monotone;
- if $\alpha(u)=-m_A\|u\|^2$ with $m_A>0$, then A is relaxed monotone.

Now we list a second group of conditions:

(B_1) $\Phi\colon W(X)\times L^2(I,V)\rightrightarrows L^2(I,X)$ is (s,s)-closed;

(B_2) ϕ is convex, lower semicontinuous $\not\equiv+\infty$ on $\overline{K}$;

(B_3) for every $x\in W(X)$, $G(x,\,\cdot\,)$ is upper hemicontinuous, relaxed α-monotone with $\alpha\colon L^2(I,V)\to\mathbb{R}$ satisfying $\limsup_{n\to\infty}\alpha(u_n)\ge\alpha(u)$ whenever $u_n\to u$, and $\lim_{t\downarrow0}\alpha(tv)/t=0$ for all $v\in L^2(I,V)$, and for every $u\in\overline{K}$, if $x_n\to x_0$ and $v_n\to v_0$, one has

$$\limsup_{n\to\infty}\langle G(x_n,u),v_n\rangle\le\langle G(x_0,u),v_0\rangle.$$

Remark 3.4. Condition (B_1) is weaker than condition (A_1). We make use of it in Theorems 4.2 and 4.3.

Remark 3.5. Condition (B_2) is weaker than condition (A_3). We make use of it in Theorem 4.3.

Remark 3.6. Condition (B_3) is independent of condition (A_4). It is used in Theorem 4.3 as a condition alternative to (A_4) permitting to handle a broad range of situations.

4. Characterization of well-posedness for (DMQVI)

This section is devoted to the metric characterization of well-posedness for (DMQVI).

Theorem 4.1. *Let* $S\colon \overline{K} \rightrightarrows \overline{K}$ *and* $\Phi\colon W(X)\times L^2(I,V) \rightrightarrows L^2(I,V)$ *be set-valued maps. Then* (DMQVI) *is strongly well-posed if and only if the solution set* Σ *of* (DMQVI) *is nonempty and, with the notation in* (2.2)*,*

$$\lim_{\varepsilon\to 0} \operatorname{diam}(\Omega(\varepsilon)) = 0. \tag{4.1}$$

Proof. "⇒" Suppose that (DMQVI) is strongly well-posed. Therefore (DMQVIs) has a unique solution $(x_0,u_0)\in W(X)\times L^2(I,V)$, thus $\Sigma=\{(x_0,u_0)\}$. In order to show that (4.1) holds, we argue by contradiction. We suppose that there exist a constant $\beta>0$ and a sequence $\varepsilon_n\to 0^+$ such that

$$\|(x_n^{(1)},u_n^{(1)})-(x_n^{(2)},u_n^{(2)})\|_{W(X)\times L^2(I,V)}>\beta, \quad \text{for all } n\in\mathbb{N}, \tag{4.2}$$

with $(x_n^{(1)},u_n^{(1)}),(x_n^2,u_n^{(2)})\in\Omega(\varepsilon_n)$. By the strong well-posedness of (DMQVI) we have

$$\lim_{n\to\infty}(x_n^{(1)},u_n^{(1)})=\lim_{n\to\infty}(x_n^{(2)},u_n^{(2)})=(x_0,u_0) \ \text{ in } W(X)\times L^2(I,V). \tag{4.3}$$

From (4.2) and (4.3) we arrive at the contradiction

$$\begin{aligned} 0<\beta&<\|(x_n^{(1)},u_n^1)-(x_n^{(2)},u_n^2)\|_{W(X)\times L^2(I,V)}\\ &\le\|(x_n^{(1)},u_n^1)-(x_0,u_0)\|_{W(X)\times L^2(I,V)}\\ &\quad+\|(x_n^{(2)},u_n^{(2)})-(x_0,u_0)\|_{W(X)\times L^2(I,V)}\to 0.\end{aligned}$$

"⇐" Conversely, assume that (4.1) and $\Sigma\neq\emptyset$ hold. By (4.1) we can easily show that Σ is a singleton $\Sigma=\{(x_0,u_0)\}$. Let $\{(x_n,u_n)\}\subseteq W(X)\times L^2(I,V)$ be an approximating sequence of problem (DMQVI) as introduced in (1.1). Then there exists a sequence $\varepsilon_n \to 0^+$ as $n \to \infty$ such that

$$\begin{cases} d_{L^2(I,X)}(\dot{x}_n, \Phi(x_n, u_n)) \leq \varepsilon_n, \\ d_{L^2(I,V)}(u_n, S(u_n)) \leq \varepsilon_n, \\ \langle G(x_n, u_n), v - u_n \rangle_{L^2(I,V)} + \phi(v) - \phi(u_n) \\ \qquad\qquad \geq -\varepsilon_n \|v - u_n\|_{L^2(I,V)}, \quad \text{for all } v \in S(u_n), \\ \|\Gamma(x_n(0), x_n(T))\|_X \leq \varepsilon_n, \end{cases}$$

which implies $(x_n, u_n) \in \Omega(\varepsilon_n)$ for all n. Since $(x_0, u_0) \in \Omega(\varepsilon_n)$, we infer from (4.1) that

$$\lim_{n\to\infty} \|(x_n, u_n) - (x_0, u_0)\|_{W(X)\times L^2(I,V)} \leq \lim_{n\to\infty} \operatorname{diam}(\Omega(\varepsilon_n)) = 0,$$

so $\{(x_n, u_n)\}$ strongly converges to (x_0, u_0), that is (DMQVIs) is strongly well-posed. □

In the proof of Theorem 4.1, the assumption $\Sigma \neq \emptyset$ plays an important role. In the next theorem, based on other conditions we remove it.

THEOREM 4.2. *If* (B_1), (A_2)–(A_5) *hold, then* (DMQVI) *is strongly well-posed if and only if*

$$\Omega(\varepsilon) \neq \emptyset, \quad \textit{for all } \varepsilon > 0, \quad \text{and} \quad \lim_{\varepsilon\to 0} \operatorname{diam}(\Omega(\varepsilon)) = 0. \tag{4.4}$$

PROOF. The necessity part is obvious from Theorem 4.1. Therefore, we need to prove the sufficiency.

Suppose (4.4). Then there exists $(x_n, u_n) \in \Omega(\varepsilon_n)$ for all $n \in \mathbb{N}$ with $\varepsilon_n \downarrow 0$ as $n \to \infty$. By $\Omega(\varepsilon) \subseteq \Omega(\delta)$ if $\varepsilon \leq \delta$ and condition (4.4), we get that $\{(x_n, u_n)\}$ is a Cauchy sequence, so it converges strongly to some $(x_0, u_0) \in W(X) \times L^2(I, V)$.

Step 1. Feasibility: $u_0 \in S(u_0)$ and $\Gamma(x_0(0), x_0(T)) = 0$.

Since $x_n \to x_0$ in $W(X)$, as noticed before we obtain that $x_n(0) \to x_0(0)$ and $x_n(T) \to x_0(T)$ in X. By virtue of (A_5) and $\|\Gamma(x_n(0), x_n(T)\|_X \leq \varepsilon_n$, it turns out that

$$\Gamma(x_0(0), x_0(T)) = \lim_{n\to\infty} \Gamma(x_n(0), x_n(T)) = 0.$$

Besides, we claim that

$$d_{L^2(I,V)}(u_0, S(u_0)) \leq \liminf_{n\to\infty} d_{L^2(I,V)}(u_n, S(u_n)). \tag{4.5}$$

Assume by contradiction that there exists $\gamma > 0$ such that

$$\liminf_{n\to\infty} d_{L^2(I,V)}(u_n, S(u_n)) < \gamma < d_{L^2(I,V)}(u_0, S(u_0)). \tag{4.6}$$

Then, along subsequences $\{u_{n_k}\}$ and $\{p_{n_k}\}$, it holds

$$\|u_{n_k} - p_{n_k}\|_{L^2(I,V)} < \gamma, \quad \text{for all } k \in \mathbb{N}.$$

Taking into account (A_2), we may admit that $p_{n_k} \rightharpoonup p_0$ in $L^2(I,V)$ and $p_0 \in S(u_0)$, which through (4.6) leads to the contradiction

$$\gamma < d(u_0, S(u_0)) \le \|u_0 - p_0\|_{L^2(V)} \le \liminf_{k\to\infty} \|u_{n_k} - p_{n_k}\|_{L^2(I,V)} \le \gamma.$$

Thus (4.5) ensures the stated feasibility property.

Step 2. $u_0 \in \mathrm{SOL}(G(x_0,\cdot), S(\cdot), \phi)$.

By the (s,s)-lower semicontinuity of S in (A_2), for every $v \in S(u_0)$ one can find a sequence $v_n \in S(u_n)$ with $v_n \to v$ in $L^2(I,V)$. Then the thesis follows from (A_3) and (A_4) because

$$\begin{aligned} 0 &= \limsup_{n\to\infty} [-\varepsilon_n \|v_n - u_n\|_{L^2(I,V)}] \\ &\le \limsup_{n\to\infty} [\langle G(x_n,u_n), v_n - u_n\rangle_{L^2(V)} + \phi(v_n) - \phi(u_n)] \\ &\le \langle G(x_0,u_0), v - u_0\rangle_{L^2(V)} + \phi(v) - \phi(u_0), \end{aligned}$$

for all $v \in S(u_0)$, which implies that $u_0 \in \mathrm{SOL}(G(x_0,\cdot), S(\cdot), \phi)$.

Step 3. $\dot{x}_0 \in \Phi(x_0, u_0)$.

By $d_{L^2(X)}(\dot{x}_n, \Phi(x_n,u_n)) \le \varepsilon_n$, there exists $p_n \in \Phi(x_n,u_n)$ satisfying

$$\|\dot{x}_n - p_n\|_{L^2(X)} \le 2\varepsilon_n.$$

Now it suffices to apply that $\dot{x}_n \to \dot{x_0}$ in $L^2(I,X)$ and Φ is (s,s)-closed according to assumption (B_1).

Step 4. (x_0,u_0) is the only element of the solution set Σ.

If there exists another solution $(x^*,u^*) \in \Sigma$, then $(x_0,u_0),(x^*,u^*) \in \Omega(\varepsilon)$ for all $\varepsilon > 0$. Therefore (4.4) yields

$$\|(x_0,u_0) - (x^*,u^*)\|_{W(X)\times L^2(I,V)} \le \mathrm{diam}(\Omega(\varepsilon)) \to 0 \quad \text{as } \varepsilon \to 0. \qquad \square$$

In the rest of this section, we consider under relaxed conditions the well-posedness of differential mixed variational inequalities ((DMVIs), for short), that is, the special case $S(u) \equiv \overline{K}$ in (1.1).

THEOREM 4.3. *Assume that conditions* (B_1)*,* (B_2)*,* (A_4)*,* (A_5) *hold or conditions* (B_1)*–*(B_3)*,* (A_5) *hold. Then problem* (DMVI) *is strongly well-posed if and only if*

$$\Omega(\varepsilon) \ne \emptyset, \quad \textit{for all } \varepsilon > 0, \quad \textit{and} \quad \lim_{\varepsilon\to 0} \mathrm{diam}(\Omega(\varepsilon)) = 0.$$

PROOF. The necessity part can be proven through the same reasoning as in Theorem 4.1.

For the sufficiency part, in view of Remarks 3.4, 3.5, under assumptions (B_1), (B_3), (A_4), (A_5), the conclusion follows from Theorem 4.1. Now we assume that conditions (B_1)–(B_3), (A_5) hold. Let $\{(x_n,u_n)\}$ be an approximating sequence of problem (DMVI) as defined in Definition 2.6. Since $\mathrm{diam}(\Omega(\varepsilon))$ tends to 0 as

$\varepsilon \to 0$, then $\{(x_n, u_n)\}$ is a Cauchy sequence, so $\{(x_n, u_n)\}$ converges strongly to some point $(x_0, u_0) \in W(X) \times \overline{K} \subseteq W(X) \times L^2(I, V)$. We obtain

$$\Gamma(x_0(0), x_0(T)) = 0.$$

By virtue of conditions (B_2), (B_3), and the fact that $\{(x_n, u_n)\}$ is an approximating sequence of problem (DMVI), we infer that

$$\begin{aligned}\alpha(v-u_0) &\le \limsup_{n\to\infty}\,[-\varepsilon_n\|v-u_n\|_{L^2(I,V)}+\alpha(v-u_n)]\\ &\le \limsup_{n\to\infty}\,[\langle G(x_n,u_n), v-u_n\rangle_{L^2(I,V)}+\phi(v)-\phi(u_n)]\\ &\le \langle G(x_0,v), v-u_0\rangle_{L^2(I,V)}+\phi(v)-\phi(u_0),\end{aligned}$$

for all $v \in \overline{K}$. Since $\overline{K} \subseteq L^2(V)$ is convex, for each $v \in \overline{K}$ and $\lambda \in [0,1]$, we have $v_\lambda := \lambda v + (1-\lambda)u_0 \in \overline{K}$, therefore

$$\begin{aligned}\alpha(\lambda(v-u_0)) = \alpha(v_\lambda-u_0) &\le \langle G(x_0,v_\lambda), v_\lambda-u_0\rangle_{L^2(I,V)}+\phi(v_\lambda)-\phi(u_0)\\ &\le \lambda[\langle G(x_0,v_\lambda), v-u_0\rangle_{L^2(I,V)}+\phi(v)-\phi(u_0)].\end{aligned}$$

Letting $\lambda \to 0$ and using (A_5) and the upper hemicontinuity of $G(x,\,\cdot\,)$ in (B_3) (see Definition 2.2), for any $x \in W(X)$ we get

$$\langle G(x_0,u_0), v-u_0\rangle_{L^2(V)}+\phi(v)-\phi(u_0) \ge 0, \quad \text{for all } v \in \overline{K}.$$

Since $d(\dot{x}_n, \Phi(x_n,u_n)) \le \varepsilon_n$, we can choose $p_n \in \Phi(x_n,u_n)$ such that

$$\|\dot{x}_n - p_n\|_{L^2(X)} \le 2\varepsilon_n.$$

Taking into account that Φ is (s,s)-closed as known from (B_1) and that $\dot{x}_n \to \dot{x_0}$, we get $p_n \to \dot{x}_0 \in \Phi(x_0,u_0)$. We conclude that (x_0,u_0) is a solution of problem (DMVI).

It remains to prove the uniqueness of solution to problem (DMVI). If (x^*,u^*), (x_0,u_0) are solutions of problem (DMVI), it is clear that $(x^*,u^*), (x_0,u_0) \in \Omega(\varepsilon)$, $\varepsilon > 0$, thus

$$\|(x_0,u_0)-(x^*,u^*)\|_{W(X)\times L^2(I,V)} \le \operatorname{diam}(\Omega(\varepsilon)) \to 0 \quad \text{as } \varepsilon \to 0. \qquad \square$$

Remark 4.4. Theorem 4.3 focusses on an important special case of problem (DMVI) in (1.1), namely when $S(u) \equiv \overline{K}$. For obtaining the same conclusion as in Theorem 4.2, it is expected that the assumptions can be weakened, as actually demonstrated in Remark 3.5. Specifically, since the constraint set $\overline{K}$ does not depend on the solution u, assumption (A_2) is automatically satisfied, whereas by the proof of Theorem 4.2 we can observe that the lower semicontinuity is enough for ϕ rather than its continuity (see Remark 3.5). On the other hand, condition (B_3) is an alternative assumption for the operator G in comparison with hypothesis (A_4) covering other relevant situations.

5. Characterization of well-posedness in the generalized sense for (DMQVIs)

We establish the metric characterization of well-posedness in the generalized sense for (DMQVIs). Among other things, we obtain the compactness of the solution set provided the well-posedness in the generalized sense for (DMQVIs) occurs. First, we show that the set $\Omega(\varepsilon)$ introduced in (2.2) is closed for all $\varepsilon > 0$.

Lemma 5.1. *Assume that conditions* (A_1)–(A_5) *hold. Then, for every* $\varepsilon > 0$, $\Omega(\varepsilon)$ *is closed.*

Proof. Let $(x_n, u_n) \to (x_0, u_0)$ in $W(X) \times L^2(I, V)$ with $\{(x_n, u_n)\} \subset \Omega(\varepsilon)$, so according to (2.2),

$$\begin{cases} d_{L^2(I,X)}(\dot{x}_n, \Phi(x_n, u_n)) \le \varepsilon, \\ d_{L^2(I,V)}(u_n, S(u_n)) \le \varepsilon, \\ \langle G(x_n, u_n), v - u_n\rangle_{L^2(I,V)} + \phi(v) - \phi(u_n) \\ \qquad\qquad\qquad \ge -\varepsilon \|v - u_n\|_{L^2(V)}, \quad \text{for all } v \in S(u_n), \\ \|\Gamma(x_n(0), x_n(T))\|_X \le \varepsilon. \end{cases}$$

Since $x_n \to x_0$ in $W(X)$, we obtain that $x_n \to x_0$ in $C(X)$, so $x_n(0) \to x_0(0)$ and $x_n(T) \to x_0(T)$ in X. By (A_5) we conclude that

$$\|\Gamma(x_0(0), x_0(T))\|_X \le \varepsilon. \tag{5.1}$$

Next, we claim that

$$d_{L^2(I,V)}(u_0, S(u_0)) \le \liminf_{n\to\infty} d_{L^2(I,V)}(u_n, S(u_n)). \tag{5.2}$$

If (5.2) were not true, there would exist $\gamma > 0$ such that

$$\liminf_{n\to\infty} d_{L^2(I,V)}(u_n, S(u_n)) < \gamma < d_{L^2(I,V)}(u_0, S(u_0)). \tag{5.3}$$

Hence there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $p_{n_k} \in S(u_{n_k})$ such that

$$\|u_{n_k} - p_{n_k}\|_{L^2(I,V)} < \gamma, \quad \text{for all } k \in \mathbb{N}.$$

Without loss of generality, we may assume that $p_{n_k} \rightharpoonup p_0$ in $L^2(I, V)$ and $p_0 \in S(u_0)$ due to hypothesis (A_2). From (5.3) we reach the contradiction

$$\gamma < d_{L^2(I,V)}(u_0, S(u_0)) \le \|u_0 - p_0\|_{L^2(I,V)} \le \liminf_{k\to\infty} \|u_{n_k} - p_{n_k}\|_{L^2(V)} \le \gamma.$$

Consequently, we can conclude via (5.2) that

$$d_{L^2(I,V)}(u_0, S(u_0)) \le \varepsilon. \tag{5.4}$$

Similarly, by (A_1) we have

$$d_{L^2(I,X)}(\dot{x}_0, \Phi(x_0, u_0)) \le \varepsilon. \tag{5.5}$$

Finally, for any $v \in S(u_0)$, by the (s,s)-lower semicontinuity of S in (A_2), there exists a sequence $\{v_n\}$ with $v_n \in S(u_n)$ such that $v_n \to v$. Then, according to (A_3) and (A_4), we derive

$$(5.6)\qquad \begin{aligned} -\varepsilon\|v_0 - u_0\|_{L^2(I,V)} &\leq \limsup_{n\to\infty} [\langle G(x_n, u_n), v_n - u_n\rangle + \phi(v_n) - \phi(u_n)] \\ &\leq \langle G(x_0, u_0), v - u_0\rangle_{L^2(I,V)} + \phi(v) - \phi(u_0), \end{aligned}$$

for all $v \in S(u_0)$. From (5.1), (5.4)–(5.6), we see that $(x_0, u_0) \in \Omega(\varepsilon)$, which completes the proof. □

Theorem 5.2. *Problem* (DMQVI) *is strongly well-posed in the generalized sense if and only if the solution set* Σ *of* (DMQVIs) *is nonempty, compact and*

$$(5.7)\qquad \lim_{\varepsilon\to 0^+} e(\Omega(\varepsilon), \Sigma) = 0.$$

Proof. "⇒" Suppose that (DMQVI) is strongly well-posed in the generalized sense (see Definition 2.9), thereby $\Sigma \neq \emptyset$ and $\Sigma \subseteq \Omega(\varepsilon) \neq \emptyset, \forall \varepsilon > 0$. Let us show that Σ is compact. Let $\{(x_n, u_n)\} \subset \Sigma$, so $\{(x_n, u_n)\}$ is an approximating sequence for (DMQVI). By virtue of Definition 2.9, $\{(x_n, u_n)\}$ has a subsequence which strongly converges to some point of Σ. Thus, Σ is compact.

In order to prove that (5.7) holds, arguing by contradiction we assume that for every sequence $\varepsilon_n \to 0^+$ there exist $\beta > 0$ and $(x'_n, u'_n) \in \Omega(\varepsilon_n)$ such that

$$d_{W(X)\times L^2(I,V)}((x'_n, u'_n), \Sigma) > \beta, \quad \text{for all } n \in \mathbb{N}.$$

Since $\{(x'_n, u'_n)\}$ is an approximating sequence for the problem (DMQVI), because of the well-posedness in the generalized sense, there exists a subsequence $\{(x'_{n_k}, u'_{n_k})\}$ of $\{(x'_n, u'_n)\}$ strongly converging to some point of Σ. Then we have the contradiction

$$0 < \beta < d_{W(X)\times L^2(I,V)}((x'_{n_k}, u'_{n_k}), \Sigma) \to 0 \quad \text{as } k \to \infty.$$

"⇐" Conversely, assume that (5.7) holds. Let $\{(x_n, u_n)\} \subset W(X) \times L^2(I, V)$ be an approximating sequence of problem (DMQVI). Then there exists $\varepsilon_n \to 0^+$ as $n \to \infty$ such that $(x_n, u_n) \in \Omega(\varepsilon_n)$ for all n. By (5.7) we can find a sequence $\{(x^*_n, u^*_n)\}$ in Σ such that

$$\|(x_n, u_n) - (x^*_n, u^*_n)\|_{W(X)\times L^2(I,V)} \to 0 \quad \text{as } n \to \infty.$$

Since Σ is compact, there exists a subsequence $\{(x^*_{n_k}, u^*_{n_k})\}$ of $\{(x^*_n, u^*_n)\}$ strongly converging to some point $(x_0, u_0) \in \Sigma$, which leads to

$$\begin{aligned} \|(x_{n_k}, u_{n_k}) - (x_0, u_0)\|_{W(X)\times L^2(I,V)} &\leq \|(x_{n_k}, u_{n_k}) - (x^*_{n_k}, u^*_{n_k})\|_{W(X)\times L^2(I,V)} \\ &\quad + \|(x^*_{n_k}, u^*_{n_k}) - (x_0, u_0)\|_{W(X)\times L^2(I,V)} \to 0 \end{aligned}$$

as $k \to \infty$. Therefore (DMQVI) is well-posed in the generalized sense. □

Theorem 5.2 relies on the compactness of Σ. Under certain circumstances we can develop a different approach.

Theorem 5.3. *If* (A_1)–(A_5) *hold and* $S\colon \overline{K} \rightrightarrows \overline{K}$ *is a closed, convex set-valued mapping, then* (DMQVI) *is strongly well-posed in the generalized sense if and only if*

$$\Omega(\varepsilon) \neq \emptyset, \quad \textit{for all } \varepsilon > 0 \quad \textit{and} \quad \lim_{\varepsilon\to 0} \mu(\Omega(\varepsilon)) = 0. \tag{5.8}$$

Proof. "$\Rightarrow$" Suppose that (DMQVI) is well-posed in the generalized sense (see Definition 2.9). Then, for every $\varepsilon > 0$, $\Sigma \subseteq \Omega(\varepsilon) \neq \emptyset$. We find from Theorem 5.2 that

$$\mathcal{H}(\Omega(\varepsilon), \Sigma) = \max\{e(\Omega(\varepsilon), \Sigma), e(\Sigma, \Omega(\varepsilon))\} = e(\Omega(\varepsilon), \Sigma), \tag{5.9}$$

for all $\varepsilon > 0$, and

$$\mu(\Sigma) = 0. \tag{5.10}$$

From (5.9) and (5.10) we deduce

$$\mu(\Omega(\varepsilon)) \leq 2\mathcal{H}(\Omega(\varepsilon), \Sigma) + \mu(\Sigma) = 2e(\Omega(\varepsilon), \Sigma). \tag{5.11}$$

Then condition (5.7) implies that $\lim_{\varepsilon\to 0} \mu(\Omega(\varepsilon)) = 0$.

"$\Leftarrow$" Conversely, assume that (5.8) holds. According to Lemma 5.1, for every $\varepsilon > 0$, $\Omega(\varepsilon)$ is closed. Set $\Omega = \bigcap_{\varepsilon>0} (\Omega(\varepsilon))$. By the generalized Cantor theorem in [16], we have that $\lim_{\varepsilon\to 0} \mathcal{H}(\Omega(\varepsilon), \Omega) = 0$ and Ω is nonempty compact.

In the following, we show that $\Omega = \Sigma$. Obviously, $\Sigma \subseteq \Omega$, so we only need to prove that $\Omega \subseteq \Sigma$. For any $(x_0, u_0) \in \Omega$ and $\varepsilon > 0$ it holds $d_{W(X)\times L^2(I,V)}((x_0, u_0), \Omega(\varepsilon)) = 0$. Then, for each $n \in \mathbb{N}$ there exists $(x_n, u_n) \in \Omega(\varepsilon_n)$ such that

$$\|(x_0, u_0) - (x_n, u_n)\|_{W(X)\times L^2(V)} \leq \varepsilon_n,$$

with $\varepsilon_n \downarrow 0$ as $n \to \infty$. Hence, $x_n \to x_0$ in $W(X)$ and $u_n \to u_0$ in $L^2(I, V)$. As before, we can show that $x_n(0) \to x_0(0)$ and $x_n(T) \to x_0(T)$ in X. Due to the continuity of Γ (see (A_5)), we obtain

$$\Gamma(x_0(0), x_0(T)) = \lim_{n\to\infty} \Gamma(x_n(0), x_n(T)) = 0. \tag{5.12}$$

As noted in Remark 3.4, condition (A_1) is stronger than (B_1). Hence the same arguments as in Theorem 4.2 based on (A_2) and (A_1) provide

$$d_{L^2(V)}(u_0, S(u_0)) \leq \liminf_{n\to\infty} d_{L^2(V)}(u_n, S(u_n)) \leq \lim_{n\to\infty} \varepsilon_n = 0,$$

and

$$d_{L^2(X)}(\dot{x}_0, \Phi(x_0, u_0)) \leq \liminf_{n\to\infty} d_{L^2(X)}(\dot{x}_n, \Phi(x_n, u_n)) \leq \lim_{n\to\infty} \varepsilon_n = 0.$$

These amount to saying that

$$u_0 \in S(u_0) \quad \text{and} \quad \dot{x}_0 \in \Phi(x_0, u_0). \tag{5.13}$$

Let us show that u_0 is a solution of $\mathrm{SOL}(S(\,\cdot\,), G(x_0,\,\cdot\,), \phi)$. Since S is (s,s)-lower semicontinuous (see (A_2)), for any $v \in S(u_0)$ there exists $v_n \in S(u_n)$ such that $v_n \to v$ as $n \to \infty$. According to (A_3) and (A_4), we obtain

$$
\begin{aligned}
(5.14) \qquad 0 &\leq \limsup_{n\to\infty} [-\varepsilon_n \|v_n - u_n\|_{L^2(I,X)}] \\
&\leq \limsup_{n\to\infty} [\langle G(x_n, u_n), v_n - u_n\rangle_{L^2(I,V)} + \phi(v_n) - \phi(u_n)] \\
&\leq \langle G(x_0, u_0), v - u_0\rangle_{L^2(I,V)} + \phi(v) - \phi(u_0),
\end{aligned}
$$

for all $v \in S(u_0)$. Then (5.12)–(5.14) enable us to conclude that $(x_0, u_0) \in \Sigma$, thus $\Sigma = \Omega$.

At this point we know that $\lim\limits_{\varepsilon\to 0} \mathcal{H}(\Omega(\varepsilon), \Sigma) = 0$ and $\lim\limits_{\varepsilon\to 0} e(\Omega(\varepsilon), \Sigma) = 0$. It follows from the compactness of Σ and Theorem 5.2 that (DMQVI) is well-posed in the generalized sense. □

Proceeding as in the proof of Theorem 4.3, we also obtain the well-posedness in the generalized sense for problem (DMVI), which is a special case of (DMQVI), using this time the setting of condition (B_3). In order to avoid repetitions, we omit the proof.

Theorem 5.4. *Assume that conditions* (A_1), (B_2), (A_4), (A_5) *hold or conditions* (A_1), (A_2), (B_3), (A_5) *hold. Then problem* (DMVI) *is strongly well-posed in the generalized sense if and only if*

$$
\Omega(\varepsilon) \neq \emptyset, \quad \text{for all } \varepsilon > 0, \quad \text{and} \quad \lim_{\varepsilon\to 0} \mu(\Omega(\varepsilon)) = 0.
$$

Remark 5.5. As mentioned in Remark 3.4, if Φ is (s,s)-closed, then it is (s,w)-closed. Obviously, the converse is generally not true. In the proof of Theorem 4.2 we have to verify that $\dot{x}_0 \in \Phi(x_0, u_0)$ (see Step 3 therein), which is done by merely assuming (B_1), that is Φ is (s,s)-closed. However, in the proof of Theorem 5.3, to establish the equivalence between the strong well-posedness in generalized sense of (DMQVI) and condition (5.8), we use the generalized Cantor theorem where the closedness of $\Omega(\varepsilon)$ is necessary. In Lemma 5.1, to guarantee the closedness of $\Omega(\varepsilon)$, we need to strengthen the assumption on Φ to be sequentially (s,w)-closed and sequentially (s,w)-subcontinuous, which means to assume condition (A_1).

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