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NONHOMOGENEOUS DIRICHLET PROBLEMS WITHOUT THE AMBROSETTI–RABINOWITZ CONDITION

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Abstract. We consider the existence of solutions of the following $p(x)$-Laplacian Dirichlet problem without the Ambrosetti–Rabinowitz condition:

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We give a new growth condition and we point out its importance for checking the Cerami compactness condition. We prove the existence of solutions of the above problem via the critical point theory, and also provide some multiplicity properties. The present paper extend previous results of Q. Zhang and C. Zhao (Existence of strong solutions of a $p(x)$-Laplacian Dirichlet problem without the Ambrosetti–Rabinowitz condition, *Computers and Mathematics with Applications*, 2015) and we establish the existence of solutions under weaker hypotheses on the nonlinear term.

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1. Introduction

In recent years, the study of differential equations and variational problems with variable exponent growth conditions has been a topic of great interest. This type of problems has very strong background, for instance in image processing, nonlinear electrorheological fluids and elastic mechanics. Some of these phenomena are related to the *Winslow effect*, which describes the behavior of certain fluids that become solids or quasi-solids when subjected to an electric field. The result was named after American engineer Willis M. Winslow.

There are many papers dealing with problems with variable exponents, see [1]–[8], [10]–[25], [28], [33], [34], [37], [38], [40]–[46], [48]–[49]. On results concerning the existence of solutions of these kinds of problems, we refer to [8], [14], [15], [18], [21], [33], [36], [45]. We also refer to the recent monograph [35] which treats variational methods in the framework of nonlinear problems with variable exponent.

In this paper, we consider the existence of solutions of the following class of Dirichlet problems:

$$\text{(P)}\qquad \begin{cases} -\Delta_{p(x)}u := -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^{1,\alpha}$ smooth boundary, and $p(\,\cdot\,) > 1$ is of class $C^1(\overline{\Omega})$.

Since the elliptic operator with variable exponent is not homogeneous, new methods and techniques are needed to study these types of problems. We point out that commonly known methods and techniques for studying constant exponent equations fail in the setting of problems involving variable exponents. For instance, the eigenvalues of the $p(x)$-Laplacian Dirichlet problem were studied in [16]. In this case, if $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, then the Rayleigh quotient

$$\lambda_{p(\,\cdot\,)} = \inf_{u \in W_0^{1,p(\,\cdot\,)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \frac{1}{p(x)}\,|\nabla u|^{p(x)}\,dx}{\displaystyle\int_\Omega \frac{1}{p(x)}\,|u|^{p(x)}\,dx} \tag{1.1}$$

is in general zero, and $\lambda_{p(\,\cdot\,)} > 0$ holds only under some special conditions.

In [41], the author generalized the Picone identities for half-linear elliptic operators with $p(x)$-Laplacian. In the same paper some applications to Sturmian comparison theory are also presented, but the formula is different from the constant exponent case. In a related setting, we point out that the formula

$$\int_\Omega |u(x)|^p\,dx = p\int_0^\infty t^{p-1}\,|\{x\in\Omega;\ |u(x)|>t\}|\,dt$$

has no variable exponent analogue.

In [23] and [46], the authors deal with the local boundedness and the Harnack inequality for the $p(x)$-Laplace equation. However, it was shown in [23] that even in the case of a very nice exponent, for example,

$$p(x) := \begin{cases} 3 & \text{for } 0 < x \le \dfrac{1}{2}, \\ 3 - 2\left(x - \dfrac{1}{2}\right) & \text{for } \dfrac{1}{2} < x < 1, \end{cases}$$

the constant in the Harnack inequality depends on the minimizer, that is, the inequality $\sup u \le c \inf u$ does not hold for any absolute constant c.

The standard norm in variable exponent Sobolev spaces is the so-called Luxemburg norm $|u|_{p(\,\cdot\,)}$ (see Section 2) and the integral $\int_\Omega |u(x)|^{p(x)}\,dx$ does not satisfy the constant power relation.

On several occasions, it is difficult to judge whether or not results about p-Laplacian can be generalized to $p(x)$-Laplacian, and even if this can be done, it is still difficult to figure out the form in which the results should be.

Our main goal is to obtain a couple of existence results for problem (P) without the Ambrosetti–Rabinowitz condition via critical point theory. For this purpose, we use a new method for checking the Cerami compactness condition under a new growth condition. Our results can be regarded as extensions of the corresponding results for the p-Laplacian problems, but the growth condition and the methods for checking the Cerami compactness condition are different with respect to quasilinear equations with constant exponent.

Next, we give a review of some results related to our work. Since the Ambrosetti–Rabinowitz type condition is quite restrictive and excludes many cases of nonlinearity, there are many papers dealing with the problem without the Ambrosetti–Rabinowitz type growth condition. For the constant exponent case $p(\,\cdot\,) \equiv p$, we refer to [26], [27], [31], [39].

In [26], the authors considered problem (P) for $p(\,\cdot\,) \equiv p$, and proved the existence of weak solutions under the following assumptions:

$$\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^p} = +\infty, \quad \text{where } F(x,t) = \int_0^t f(x,s)\,ds;$$

and there exists a constant $C_* > 0$ such that $H(x,t) \le H(x,s) + C_*$ for each $x \in \Omega$, $0 < t < s$ or $s < t < 0$, where $H(x,t) = tf(x,t) - pF(x,t)$.

In [27], the author studied problem (P) for $p(\,\cdot\,) \equiv p$. Under the assumption that $f(x,s)/|s|^{p-2}s$ is increasing when $s \ge s_0$ and decreasing when $s \le -s_0$, for all $x \in \Omega$, the existence of weak solutions was obtained.

In [31], the authors studied problem (P) for $p(\,\cdot\,) \equiv 2$, which becomes a Laplacian problem. The main result in [31] establishes the existence of weak solutions by assuming that $f(x,s)/s$ is increasing when $s \ge s_0$ and decreasing when $s \le -s_0$, for all $x \in \Omega$.

In [39], the author also studied problem (P) for $p(\,\cdot\,) \equiv 2$ and proved the existence of weak solutions under the assumption

$$sf(x,u) \geq C_0|s|^{\mu}, \quad \text{where } \mu > 2 \text{ and } C_0 > 0.$$

If $p(\,\cdot\,)$ is a general function, results on variable exponent problem without the Ambrosetti–Rabinowitz type growth condition are rare due to the complexity of $p(x)$-Laplacian (see [3], [5], [20], [19], [42]). However, their assumptions imply $G_{p^+}(x,t) = f(x,t)t - p^+F(x,t) \geq 0$ and $F(x,t) > 0$ as $t \to +\infty$, so we can see that $F(x,t) \geq Ct^{p^+}$ as $t \to +\infty$. This is too strong and unnatural for the $p(x)$-Laplacian problems.

In [45], the author considered problem (P) under the following growth condition:

- there exist constants $M, C_1, C_2 > 0, a > p$ on $\overline{\Omega}$ such that, for all $x \in \Omega$ and all $|t| \geq M$,

$$C_1|t|^{p(x)}[\ln(e+|t|)]^{a(x)-1} \leq C_2\,\frac{tf(x,t)}{\ln(e+|t|)} \leq tf(x,t) - p(x)F(x,t). \tag{1.2}$$

A typical example is $f(x,t) = |t|^{p(x)-2}t\,[\ln(1+|t|)]^{a(x)}$. This function satisfies the above condition (1.2), but does not satisfy the Ambrosetti–Rabinowitz condition.

Our paper was motivated by [45]. We further weaken condition (1.2). To begin, we point out that the assumption $a > p$ on $\overline{\Omega}$ is unnecessary in the present paper.

Before stating our main results, we make the following assumptions:

(f_0) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and

$$|f(x,t)| \leq C(1+|t|^{\alpha(x)-1}), \quad \text{for all } (x,t) \in \Omega \times \mathbb{R},$$

where $\alpha \in C(\overline{\Omega})$ and $p(x) < \alpha(x) < p^*(x)$ on $\overline{\Omega}$.

(f_1) There exist constants $M, C > 0$, such that

$$C\,\frac{tf(x,t)}{K(t)} \leq tf(x,t) - p(x)F(x,t), \quad \text{for all } |t| \geq M \text{ and all } x \in \overline{\Omega}, \tag{1.3}$$

and

$$\frac{tf(x,t)}{|t|^{p(x)}[K(t)]^{p(x)}} \to +\infty \quad \text{uniformly as } |t| \to +\infty \text{ for } x \in \overline{\Omega}, \tag{1.4}$$

where K satisfies the following hypotheses:

(K) $1 \leq K(\,\cdot\,) \in C^1([0,+\infty),[1,+\infty))$ is increasing and $[\ln(e+t)]^2 \geq K(t) \to +\infty$ as $|t| \to +\infty$, which satisfies $tK'(t)/K(t) \leq \sigma_0 \in (0,1)$, where σ_0 is a constant.

(f_2) $f(x,t) = o(|t|^{p(x)-1})$ uniformly for $x \in \Omega$ as $t \to 0$.

(f_3) $f(x,-t) = -f(x,t)$, for all $x \in \overline{\Omega}$, for all $t \in \mathbb{R}$.

(f_4) F satisfies

$$\frac{F(x,t)}{|t|^{p(x)}[\ln(e+|t|)]^{p(x)}} \to +\infty \quad \text{uniformly as } |t| \to +\infty \text{ for } x \in \overline{\Omega}.$$

(p_1) There is a vector $l \in \mathbb{R}^N \setminus \{0\}$ such that for any $x \in \Omega$, $\rho(t) = p(x+tl)$ is monotone for $t \in I_x(l) = \{t \mid x + tl \in \Omega\}$.

(p_2) p has a local maximum point, that is, there exist $x_0 \in \Omega$ and $\delta > 0$ such that $\overline{B(x_0, 3\delta)} \subset \Omega$ and

$$\min_{|x-x_0|\le\delta} p(x) > \max_{2\delta\le|x-x_0|\le3\delta} p(x).$$

(p_3) p has a sequence of local maximum points, that is, there exist a sequence of points $x_n \in \Omega$ and $\delta_n > 0$ such that $\overline{B(x_0, 3\delta_n)}$ are mutually disjoint and

$$\min_{|x-x_n|\le\delta_n} p(x) > \max_{2\delta_n\le|x-x_n|\le3\delta_n} p(x).$$

We state our main results in what follows.

Theorem 1.1. *Assume that hypotheses* (f_0)–(f_2), (p_1) *and* (f_4) *or* (p_2) *are fulfilled. Then problem* (P) *has a nontrivial solution.*

Theorem 1.2. *Assume that hypotheses* (f_0), (f_1), (f_3) *and* (f_4) *or* (p_3) *are fulfilled. Then problem* (P) *has infinitely many pairs of solutions.*

Remark 1.3. (a) The following functions satisfy hypothesis (K):

$$\begin{aligned} K_1(t) &= \ln(e+|t|), \\ K_2(t) &= \ln(e+\ln(e+|t|)), \\ K_3(t) &= [\ln(e+\ln(e+|t|))]\ln(e+|t|). \end{aligned}$$

Let $K = K_1$, and $f(x,t) = |t|^{p(x)-2}t[\ln(1+|t|)]^{p(x)}\rho(|t|)$, where $1 \le \rho(|t|) \le [\ln(e+|tt|)]^2$, $\rho' \ge 0$ and $\rho(|t|) \to +\infty$ as $|t| \to +\infty$, for example $\rho(|t|) = \ln(e+\ln(e+|t|))$. Then f satisfies conditions (f_0)–(f_4), but it does not satisfy the Ambrosetti–Rabinowitz condition, and does not satisfy (1.2).

(b) We do not need any monotonicity assumption on $f(x,\cdot)$.

This paper is organized as follows. In Section 2, we do some preparatory work including some basic properties of the variable exponent Sobolev spaces, which can be regarded as a special class of generalized Orlicz–Sobolev spaces. In Section 3, we give proofs of the results stated above.

2. Preliminary results

Throughout this paper, we use letters c, c_i, C, C_i, $i = 1, 2, \ldots$, to denote generic positive constants which may vary from line to line, and we will specify them whenever necessary.

One of the reasons for the huge development of the theory of classical Lebesgue and Sobolev spaces L^p and $W^{1,p}$ (where $1 \leq p \leq \infty$) is its usefulness for the description of many phenomena arising in applied sciences. For instance, many materials can be modeled with sufficient accuracy by using the function spaces L^p and $W^{1,p}$, where p is a fixed constant. For some materials with nonhomogeneities, for instance electrorheological fluids (sometimes referred to as "smart fluids"), this approach is not adequate, but rather the exponent p should be allowed to vary. This leads us to the study of variable exponent Lebesgue and Sobolev spaces, $L^{p(\,\cdot\,)}$ and $W^{1,p(\,\cdot\,)}$, where p is a real-valued function.

In order to discuss problem (P), we need some results about the space $W_0^{1,p(\,\cdot\,)}(\Omega)$, which we call the *variable exponent Sobolev space*. We first state some basic properties of $W_0^{1,p(\cdot)}(\Omega)$ (for details, see [12], [17], [15], [25], [35] and [38]). Denote

$$
\begin{aligned}
C_+(\overline{\Omega}) &= \{h \mid h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for } x \in \overline{\Omega}\},\\
h^+ &= \max_{\overline{\Omega}} h(x), \quad h^- = \min_{\overline{\Omega}} h(x), \text{ for any } h \in C(\overline{\Omega}),\\
L^{p(\,\cdot\,)}(\Omega) &= \bigg\{u \ \bigg|\ u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\bigg\}.
\end{aligned}
$$

We introduce the norm on $L^{p(\,\cdot\,)}(\Omega)$ by

$$
|u|_{p(\,\cdot\,)} = \inf\bigg\{\lambda > 0 \ \bigg|\ \int_\Omega \bigg|\frac{u(x)}{\lambda}\bigg|^{p(x)}\,dx \leq 1\bigg\}.
$$

Then $(L^{p(\,\cdot\,)}(\Omega),\ |\cdot|_{p(\,\cdot\,)})$ becomes a Banach space and it is called the *variable exponent Lebesgue space*.

Proposition 2.1 (see [12], [35]). (a) *The space $(L^{p(\,\cdot\,)}(\Omega), |\cdot|_{p(\,\cdot\,)})$ is a separable, uniform convex Banach space, and its conjugate space is $L^{q(\,\cdot\,)}(\Omega)$, where $1/q(\,\cdot\,) + 1/p(\,\cdot\,) \equiv 1$. For any $u \in L^{p(\,\cdot\,)}(\Omega)$ and $v \in L^{q(\,\cdot\,)}(\Omega)$, we have*

$$
\bigg|\int_\Omega uv\,dx\bigg| \leq \bigg(\frac{1}{p^-} + \frac{1}{q^-}\bigg)|u|_{p(\,\cdot\,)}|v|_{q(\,\cdot\,)}.
$$

(b) *If $p_1, p_2 \in C_+(\overline{\Omega})$, $p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then $L^{p_2(\,\cdot\,)}(\Omega) \subset L^{p_1(\,\cdot\,)}(\Omega)$, and this imbedding is continuous.*

Proposition 2.2 (see [15], [35]). *If $f\colon \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies*

$$
|f(x,s)| \leq a(x) + b|s|^{p_1(x)/p_2(x)} \quad \textit{for any } x \in \Omega,\ s \in \mathbb{R},
$$

where $p_1, p_2 \in C_+(\overline{\Omega})$, $a \in L^{p_2(\,\cdot\,)}(\Omega)$, $a(x) \geq 0$, $b \geq 0$, then the Nemytskiĭ operator from $L^{p_1(\,\cdot\,)}(\Omega)$ to $L^{p_2(\,\cdot\,)}(\Omega)$ defined by $(N_f u)(x) = f(x,u(x))$, is a continuous and bounded operator.

PROPOSITION 2.3 (see [15], [35]). *If we denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}\,dx, \quad \textit{for all } u \in L^{p(\,\cdot\,)}(\Omega),$$

then there exists $\xi \in \overline{\Omega}$ *such that* $|u|_{p(\,\cdot\,)}^{p(\xi)} = \int_\Omega |u|^{p(x)}\,dx$ *and*

(a) $|u|_{p(\,\cdot\,)} < 1$ $(= 1;\ > 1)$ *if and only if* $\rho(u) < 1$ $(= 1;\ > 1)$;

(b) *if* $|u|_{p(\,\cdot\,)} > 1$ *then* $|u|_{p(\,\cdot\,)}^{p^-} \le \rho(u) \le |u|_{p(\,\cdot\,)}^{p^+}$;
if $|u|_{p(\,\cdot\,)} < 1$ *then* $|u|_{p(\,\cdot\,)}^{p^-} \ge \rho(u) \ge |u|_{p(\,\cdot\,)}^{p^+}$;

(c) $|u|_{p(\,\cdot\,)} \to 0$ *if and only if* $\rho(u) \to 0$;
$|u|_{p(\,\cdot\,)} \to \infty$ *if and only if* $\rho(u) \to \infty$.

PROPOSITION 2.4 (see [15], [35]). *If* $u, u_n \in L^{p(\,\cdot\,)}(\Omega)$, $n = 1, 2, \ldots$, *then the following statements are equivalent*:

(a) $\lim_{k\to\infty} |u_k - u|_{p(\,\cdot\,)} = 0$;

(b) $\lim_{k\to\infty} \rho(u_k - u) = 0$;

(c) $u_k \to u$ *in measure in* Ω *and* $\lim_{k\to\infty} \rho(u_k) = \rho(u)$.

The space $W^{1,p(\,\cdot\,)}(\Omega)$ is defined by

$$W^{1,p(\,\cdot\,)}(\Omega) = \{u \in L^{p(\,\cdot\,)}(\Omega) \mid \nabla u \in (L^{p(\,\cdot\,)}(\Omega))^N\},$$

and it can be equipped with the norm

$$\|u\| = |u|_{p(\,\cdot\,)} + |\nabla u|_{p(\,\cdot\,)}, \quad \text{for all } u \in W^{1,p(\,\cdot\,)}(\Omega).$$

We denote by $W_0^{1,p(\,\cdot\,)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\,\cdot\,)}(\Omega)$ and set

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \ge N. \end{cases}$$

Then we have the following properties.

PROPOSITION 2.5 (see [12], [15], [35]).

(a) $W^{1,p(\,\cdot\,)}(\Omega)$ *and* $W_0^{1,p(\,\cdot\,)}(\Omega)$ *are separable reflexive Banach spaces*;

(b) *if* $q \in C_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$, *then the imbedding from* $W^{1,p(\,\cdot\,)}(\Omega)$ *to* $L^{q(\,\cdot\,)}(\Omega)$ *is compact*;

(c) *there is a constant* $C > 0$ *such that*

$$|u|_{p(\,\cdot\,)} \le C|\nabla u|_{p(\,\cdot\,)}, \quad \textit{for all } u \in W_0^{1,p(\,\cdot\,)}(\Omega).$$

It follows from (a) of Proposition 2.5 that $|\nabla u|_{p(\,\cdot\,)}$ and $\|u\|$ are equivalent norms on $W_0^{1,p(\,\cdot\,)}(\Omega)$. From now on, we will use $|\nabla u|_{p(\,\cdot\,)}$ instead of $\|u\|$ as the norm on $W_0^{1,p(\,\cdot\,)}(\Omega)$.

The Lebesgue and Sobolev spaces with variable exponents coincide with the usual Lebesgue and Sobolev spaces provided that p is constant. These function spaces $L^{p(x)}$ and $W^{1,p(x)}$ have some unusual properties, see [35, p. 8–9]. Some of these properties are the following:

(i) Assuming that $1 < p^- \le p^+ < \infty$ and $p\colon \overline{\Omega} \to [1,\infty)$ is a smooth function, the following co-area formula

$$\int_\Omega |u(x)|^p \, dx = p \int_0^\infty t^{p-1} \, |\{x \in \Omega;\ |u(x)| > t\}| \, dt$$

has no analogue in the framework of variable exponents.

(ii) Spaces $L^{p(x)}$ do *not* satisfy the *mean continuity property.* More exactly, if p is nonconstant and continuous in an open ball B, then there is some $u \in L^{p(x)}(B)$ such that $u(x+h) \notin L^{p(x)}(B)$ for every $h \in \mathbb{R}^N$ with arbitrary small norm.

(iii) Function spaces with variable exponent are *never* invariant with respect to translations. The convolution is also limited. For instance, the classical Young inequality

$$|f * g|_{p(x)} \le C |f|_{p(x)} \|g\|_{L^1}$$

holds if and only if p is constant.

Proposition 2.6 (see [16]). *If the assumption* (p_1) *is satisfied, then* $\lambda_{p(\,\cdot\,)}$ *defined in* (1.1) *is positive.*

Next, we prove some results related to the $p(x)$-Laplace operator $-\Delta_{p(x)}$ as defined at the beginning of Section 1. Consider the following functional:

$$J(u) = \int_\Omega \frac{1}{p(x)} \, |\nabla u|^{p(x)} \, dx, \quad u \in X := W_0^{1,p(\,\cdot\,)}(\Omega).$$

Then (see [9]) $J \in C^1(X,\mathbb{R})$ and the $p(x)$-Laplace operator is the derivative operator of J in the weak sense. We denote $L = J'\colon X \to X^*$, then

$$(L(u), v) = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx, \quad \text{for all } v, u \in X.$$

Theorem 2.7 (see [15], [21]).

(a) *$L\colon X \to X^*$ is a continuous, bounded and strictly monotone operator*;

(b) *L is a mapping of type* (S_+)*, that is, if $u_n \rightharpoonup u$ in X and $\overline{\lim}_{n\to+\infty} (L(u_n) - L(u), u_n - u) \le 0$, then $u_n \to u$ in X;*

(c) *$L\colon X \to X^*$ is a homeomorphism.*

Denote

$$B(x_0, \varepsilon, \delta, \theta) = \left\{ x \in \mathbb{R}^N \;\middle|\; \delta \le |x - x_0| \le \varepsilon,\ \frac{x - x_0}{|x - x_0|} \cdot \frac{\nabla p(x_0)}{|\nabla p(x_0)|} \ge \cos\theta \right\},$$

where $\theta \in (0, \pi/2)$. Then we obtain the following.

LEMMA 2.8. *If $p \in C^1(\overline{\Omega})$, $x_0 \in \Omega$ satisfy $\nabla p(x_0) \neq 0$, then there exists a small enough $\varepsilon > 0$ such that*

$$(x - x_0) \cdot \nabla p(x) > 0, \quad \text{for all } x \in B(x_0, \varepsilon, \delta, \theta), \tag{2.1}$$

$$\max\{p(x) \mid x \in \overline{B(x_0, \varepsilon)}\} = \max\{p(x) \mid x \in B(x_0, \varepsilon, \delta, \theta),\ |x - x_0| = \varepsilon\}. \tag{2.2}$$

PROOF. A proof of this lemma can be found in [45]. For readers' convenience, we include it here.

Since $p \in C^1(\overline{\Omega})$, for any $x \in B(x_0, \varepsilon, \delta, \theta)$, when $\varepsilon > 0$ is small enough, we have

$$\begin{aligned}\nabla p(x) \cdot (x - x_0) &= (\nabla p(x_0) + o(1)) \cdot (x - x_0) \\ &= \nabla p(x_0) \cdot (x - x_0) + o(|x - x_0|) \\ &\geq |\nabla p(x_0)|\, |x - x_0| \cos\theta + o(|x - x_0|) > 0,\end{aligned}$$

where $o(1) \in \mathbb{R}^N$ is a function and $o(1) \to 0$ uniformly as $|x - x_0| \to 0$.

When ε is small enough, condition (2.1) is valid. Since $p \in C^1(\overline{\Omega})$, there exists a small enough positive ε such that

$$p(x) - p(x_0) = \nabla p(y) \cdot (x - x_0) = (\nabla p(x_0) + o(1)) \cdot (x - x_0),$$

where $y = x_0 + \tau(x - x_0)$ and $\tau \in (0, 1)$, $o(1) \in \mathbb{R}^N$ is a function and $o(1) \to 0$ uniformly as $|x - x_0| \to 0$.

Suppose that $x \in \overline{B(x_0, \varepsilon)} \setminus B(x_0, \varepsilon, \delta, \theta)$. Let $x^* = x_0 + \varepsilon \nabla p(x_0)/|\nabla p(x_0)|$. Suppose that

$$\frac{x - x_0}{|x - x_0|} \cdot \frac{\nabla p(x_0)}{|\nabla p(x_0)|} < \cos\theta.$$

When ε is small enough, we have

$$\begin{aligned}p(x) - p(x_0) &= (\nabla p(x_0) + o(1)) \cdot (x - x_0) < |\nabla p(x_0)|\, |x - x_0| \cos\theta + \varepsilon \cdot o(1) \\ &\leq (\nabla p(x_0) + o(1)) \cdot \varepsilon \nabla p(x_0)/|\nabla p(x_0)| = p(x^*) - p(x_0),\end{aligned}$$

where $o(1) \in \mathbb{R}^N$ is a function and $o(1) \to 0$ as $\varepsilon \to 0$.

Suppose that $|x - x_0| < \delta$. When ε is small enough, we have

$$\begin{aligned}p(x) - p(x_0) &= (\nabla p(x_0) + o(1)) \cdot (x - x_0) \leq |\nabla p(x_0)|\, |x - x_0| + \varepsilon \cdot o(1) \\ &< (\nabla p(x_0) + o(1)) \cdot \varepsilon \nabla p(x_0)/|\nabla p(x_0)| = p(x^*) - p(x_0),\end{aligned}$$

where $o(1) \in \mathbb{R}^N$ is a function and $o(1) \to 0$ as $\varepsilon \to 0$. Thus

$$\max\{p(x) \mid x \in \overline{B(x_0, \varepsilon)}\} = \max\{p(x) \mid x \in B(x_0, \varepsilon, \delta, \theta)\}. \tag{2.3}$$

It follows from (2.1) and (2.3) that relation (2.2) holds. □

LEMMA 2.9. *Suppose that $F(x, u)$ satisfies* (f_4). *Let*

$$h(x) = \begin{cases} 0 & \text{if } |x - x_0| > \varepsilon, \\ \varepsilon - |x - x_0| & \text{if } |x - x_0| \leq \varepsilon, \end{cases}$$

where ε is defined as in Lemma 2.8. *Then*

$$\int_\Omega |\nabla th|^{p(x)}\,dx - \int_\Omega F(x,th)\,dx \to -\infty \quad \text{as } t \to +\infty.$$

Proof. Obviously,

$$\int_\Omega \frac{1}{p(x)}\,|\nabla th|^{p(x)}\,dx \le C_2 \int_{B(x_0,\varepsilon,\delta,\theta)} |\nabla th|^{p(x)}\,dx.$$

We make a spherical coordinate transformation. Denote $r = |x - x_0|$. Since $p \in C^1(\overline{\Omega})$, it follows from (2.1) that there exist positive constants c_1 and c_2 such that

$$p(\varepsilon,\omega) - c_2(\varepsilon - r) \le p(r,\omega) \le p(\varepsilon,\omega) - c_1(\varepsilon - r), \quad \text{for all } (r,\omega) \in B(x_0,\varepsilon,\delta,\theta).$$

Therefore

$$\begin{aligned}\int_{B(x_0,\varepsilon,\delta,\theta)} |\nabla th|^{p(x)}\,dx &= \int_{B(x_0,\varepsilon,\delta,\theta)} |t|^{p(r,\omega)} r^{N-1}\,dr\,d\omega \\ &\le \int_{B(x_0,\varepsilon,\delta,\theta)} |t|^{p(\varepsilon,\omega)-c_1(\varepsilon-r)} r^{N-1}\,dr\,d\omega \\ &\le \varepsilon^{N-1} \int_{B(x_0,\varepsilon,\delta,\theta)} t^{p(\varepsilon,\omega)-c_1(\varepsilon-r)}\,dr\,d\omega \\ &\le \varepsilon^{N-1} \int_{B(x_0,1,1,\theta)} \frac{t^{p(\varepsilon,\omega)}}{c_1 \ln t}\,d\omega.\end{aligned} \tag{2.4}$$

Denote

$$G(x,u) = \frac{F(x,u)}{|u|^{p(x)}[\ln(e+|u|)]^{p(x)}}.$$

Then

$$G(x,u) \to +\infty \quad \text{uniformly as } |u| \to +\infty \text{ for } x \in \overline{\Omega}. \tag{2.5}$$

Thus there exists a positive constant M such that $G(x,u) \ge 1$, for all $|u| \ge M$ and for all $x \in \overline{\Omega}$. Denote

$$\begin{aligned}&E_1 = \{x \in B(x_0,\varepsilon) \mid th \ge M\} = \{x \in B(x_0,\varepsilon) \mid |x - x_0| \le \varepsilon - M/t\},\\ &E_2 = B(x_0,\varepsilon) \setminus E_1.\end{aligned}$$

Then we have

$$\begin{aligned}\int_\Omega F(x,th)\,dx &= \int_{B(x_0,\varepsilon)} F(x,th)\,dx \\ &= \int_{E_1} F(x,th)\,dx + \int_{E_2} F(x,th)\,dx \ge \int_{E_1} F(x,th)\,dx - C_1.\end{aligned}$$

When t is large enough, we have

$$\begin{aligned}
\int_{E_1} F(x,th)\,dx &= \int_{E_1} |th|^{p(x)}[\ln(e+|th|)]^{p(x)} G(x,th)\,dx \\
&= \int_{B(x_0,\varepsilon-M/t,\delta,\theta)} C_1 |th|^{p(x)}[\ln(e+|th|)]^{p(x)} G(x,th)\,dx \\
&= \int_{B(x_0,\varepsilon-M/t,\delta,\theta)} C_1 |t(\varepsilon-r)|^{p(r,\omega)} r^{N-1} \\
&\quad \cdot [\ln(e+|t(\varepsilon-r)|)]^{p(r,\omega)} G(r,\omega,t(\varepsilon-r))\,dr\,d\omega \\
&\geq C_1\delta^{N-1} \int_{B(x_0,\varepsilon-M/t,\delta,\theta)} |t|^{p(\varepsilon,\omega)-c_2(\varepsilon-r)} |\varepsilon-r|^{p(\varepsilon,\omega)-c_1(\varepsilon-r)} \\
&\quad \cdot [\ln(e+|t(\varepsilon-r)|)]^{p(r,\omega)} G(r,\omega,t(\varepsilon-r))\,dr\,d\omega \\
&= C_1\delta^{N-1} \int_{B(x_0,1,1,\theta)} d\omega \int_{\delta}^{\varepsilon-M/t} |t|^{p(\varepsilon,\omega)-c_2(\varepsilon-r)} |\varepsilon-r|^{p(\varepsilon,\omega)-c_1(\varepsilon-r)} \\
&\quad \cdot [\ln(e+|t(\varepsilon-r)|)]^{p(r,\omega)} G(r,\omega,t(\varepsilon-r))\,dr \\
&\geq C_1\delta^{N-1} \int_{B(x_0,1,1,\theta)} d\omega \int_{\delta}^{\varepsilon-1/\ln t} |t|^{p(\varepsilon,\omega)-c_2(\varepsilon-r)} |\varepsilon-r|^{p(\varepsilon,\omega)} \\
&\quad \cdot [\ln(e+|t(\varepsilon-r)|)]^{p(r,\omega)} G(r,\omega,t(\varepsilon-r))\,dr \\
&\geq C_2\delta^{N-1} G(r_t,\omega_t,t(\varepsilon-r_t)) \int_{B(x_0,1,1,\theta)} \left(\frac{1}{\ln t}\right)^{p(\varepsilon,\omega)} \left[\ln\left(e+\frac{t}{\ln t}\right)\right]^{p(\varepsilon,\omega)} \\
&\quad \cdot \int_{\delta}^{\varepsilon-1/\ln t} |t|^{p(\varepsilon,\omega)-c_2(\varepsilon-r)}\,dr\,d\omega \\
&\geq C_3\delta^{N-1} G(r_t,\omega_t,t(\varepsilon-r_t)) \int_{B(x_0,1,1,\theta)} \frac{|t|^{p(\varepsilon,\omega)-c_2/\ln t}}{c_2\ln t}\,d\omega \\
&\geq C_4\delta^{N-1} G(r_t,\omega_t,t(\varepsilon-r_t)) \int_{B(x_0,1,1,\theta)} \frac{|t|^{p(\varepsilon,\omega)}}{c_2\ln t}\,d\omega,
\end{aligned}$$

where $(r_t,\omega_t)\in E_1$ is such that

$$G(r_t,\omega_t,t(\varepsilon-r_t)) = \min\left\{ G(r,\omega,t(\varepsilon-r)) \;\middle|\; (r,\omega)\in B\left(x_0,\varepsilon-\frac{1}{\ln t},\delta,\theta\right)\right\}.$$

Note that $t(\varepsilon-r_t)\geq t/\ln t\to+\infty$ as $t\to+\infty$. Thus

$$\int_\Omega F(x,th)\,dx \geq G(r_t,\omega_t,t(\varepsilon-r_t))C_5 \int_{B(x_0,1,1,\theta)} \frac{|t|^{p(\varepsilon,\omega)}}{\ln t}\,d\omega - C_1 \tag{2.6}$$

as $t\to+\infty$. It follows from (2.4), (2.5) and (2.6) that $\Psi(th)\to-\infty$. $\square$

Lemma 2.10. *The following* K_i $(i = 1, 2, 3)$ *satisfy hypothesis* (K)

$$\begin{aligned} K_1(t) &= \ln(e + |t|); \\ K_2(t) &= \ln(e + \ln(e + |t|)); \\ K_3(t) &= [\ln(e + \ln(e + |t|))] \ln(e + |t|). \end{aligned}$$

Proof. We only need to check that $K_3(t)$ satisfies hypothesis (K). The proofs for the other functions are similar.

We observe that $1 \le K(\,\cdot\,) \in C^1([0, +\infty), [1, +\infty))$ is increasing and $K(t) \to +\infty$ as $t \to +\infty$. So we only need to prove that $tK'(t)/K(t) \le \sigma \in (0, 1)$, where σ is a constant. By computation we obtain

$$\begin{aligned} \frac{tK'}{K} &= \frac{t}{K} \bigg\{ \frac{[\ln(e + |t|)] \operatorname{sgn} t}{[e + \ln(e + |t|)](e + |t|)} + \frac{[\ln(e + \ln(e + |t|))] \operatorname{sgn} t}{(e + |t|)} \bigg\} \\ &= \frac{|t|}{[\ln(e + \ln(e + |t|))][e + \ln(e + |t|)](e + |t|)} \\ &\quad + \frac{|t|}{[\ln(e + |t|)](e + |t|)}. \end{aligned}$$

We have

$$\begin{aligned} |t| &\le \frac{1}{3} [\ln(e + \ln(e + |t|))][e + \ln(e + |t|)](e + |t|), \\ |t| &\le \frac{1}{2} [\ln(e + |t|)](e + |t|) \end{aligned}$$

and we complete the proof by observing that $tK'/K \le 5/6$, for all $t \in \mathbb{R}$. $\square$

3. Proofs of main results

In this section we give the proofs of our main results.

Definition 3.1. We say that $u \in W_0^{1,p(\,\cdot\,)}(\Omega)$ is a weak solution of (P) if

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v \, dx = \int_\Omega f(x, u) v \, dx, \quad \text{for all } v \in X := W_0^{1,p(\,\cdot\,)}(\Omega).$$

The corresponding functional of (P) is

$$\varphi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx - \int_\Omega F(x, u) \, dx, \quad u \in X,$$

where $F(x, t) = \int_0^t f(x, s) \, ds$.

Definition 3.2. We say that φ satisfies the Cerami condition in X, if any sequence $\{u_n\} \subset X$ such that $\{\varphi(u_n)\}$ is bounded and $\|\varphi'(u_n)\|(1 + \|u_n\|) \to 0$ as $n \to +\infty$ has a convergent subsequence.

Lemma 3.3. *If f satisfies* (f_0) *and* (f_1), *then φ satisfies the Cerami condition.*

PROOF. Let $\{u_n\} \subset X$ be a Cerami sequence, that is $\varphi(u_n) \to c$ and $\|\varphi'(u_n)\|(1+\|u_n\|) \to 0$. Therefore $\varphi'(u_n) = L(u_n) - f(x,u_n) \to 0$ in X^*, so we have $L(u_n) = f(x,u_n) + o_n(1)$, where $o_n(1) \to 0$ in X^* as $n \to \infty$. Suppose that $\{u_n\}$ is bounded. Then $\{u_n\}$ has a weakly convergent subsequence in X. Without loss of generality, we may assume that $u_n \rightharpoonup u$. Then by Propositions 2.2 and 2.5, we have $f(x,u_n) \to f(x,u)$ in X^*. Thus $L(u_n) = f(x,u_n) + o_n(1) \to f(x,u)$ in X^*. Since L is a homeomorphism, we have $u_n \to L^{-1}(f(x,u))$ in X, and so φ satisfies the Cerami condition. Therefore $u = L^{-1}(f(x,u))$, so $L(u) = f(x,u)$, which means that u is a solution of (P). Thus we only need to prove the boundedness of the Cerami sequence $\{u_n\}$.

We argue by contradiction. Then there exist $c \in \mathbb{R}$ and $\{u_n\} \subset X$ satisfying:

$$\varphi(u_n) \to c, \qquad \|\varphi'(u_n)\|(1+\|u_n\|) \to 0, \qquad \|u_n\| \to +\infty.$$

Obviously,

$$\left|\frac{1}{p(x)}\,u_n\right|_{p(\,\cdot\,)} \le \frac{1}{p^-}\,|u_n|_{p(\,\cdot\,)}, \qquad \left|\nabla\frac{1}{p(x)}\,u_n\right|_{p(\,\cdot\,)} \le \frac{1}{p^-}\,|\nabla u_n|_{p(\,\cdot\,)} + C|u_n|_{p(\,\cdot\,)}.$$

Thus $\|u_n/p(x)\| \le C\|u_n\|$. Therefore $(\varphi'(u_n), u_n/p(x)) \to 0$. We may assume that

$$\begin{aligned}
c+1 &\ge \varphi(u_n) - \left(\varphi'(u_n), \frac{1}{p(x)}\,u_n\right)\\
&= \int_\Omega \frac{1}{p(x)}\,|\nabla u_n|^{p(x)}\,dx - \int_\Omega F(x,u_n)\,dx - \bigg\{\int_\Omega \frac{1}{p(x)}\,|\nabla u_n|^{p(x)}\,dx\\
&\quad - \int_\Omega \frac{1}{p(x)}\,f(x,u_n)u_n\,dx - \int_\Omega \frac{1}{p^2(x)}\,u_n|\nabla u_n|^{p(x)-2}\nabla u_n\nabla p\,dx\bigg\}\\
&\ge \int_\Omega \frac{1}{p^2(x)}\,u_n|\nabla u_n|^{p(x)-2}\nabla u_n\nabla p\,dx + \int_\Omega\bigg\{\frac{1}{p(x)}\,f(x,u_n)u_n - F(x,u_n)\bigg\}\,dx.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_\Omega \bigg\{\frac{f(x,u_n)u_n}{p(x)} - F(x,u_n)\bigg\}\,dx &\le C_0\bigg(\int_\Omega |u_n||\nabla u_n|^{p(x)-1}\,dx + 1\bigg)\\
&\le \sigma\int_\Omega \frac{|\nabla u_n|^{p(x)}}{K(|u_n|)}\,dx + C_1 + C(\sigma)\int_\Omega |u_n|^{p(x)}[K(|u_n|)]^{p(x)-1}\,dx,
\end{aligned} \tag{3.1}$$

where σ is a small enough positive constant. Due to hypothesis (K), it is easy to check that $u_n/K(|u_n|) \in X$, and $\|u_n/K(|u_n|)\| \le C_2\|u_n\|$. Let $u_n/K(|u_n|)$ be a test function. We have

$$\begin{aligned}
\int_\Omega f(x,u_n)\,\frac{u_n}{K(|u_n|)}\,dx &= \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\nabla\,\frac{u_n}{K|u_n|)}\,dx + o(1)\\
&= \int_\Omega \frac{|\nabla u_n|^{p(x)}}{K(|u_n|)}\,dx - \int_\Omega u_n|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\,\frac{1}{K(|u_n|)}\,dx + o(1).
\end{aligned}$$

By computation, we obtain

$$\begin{aligned}\bigg|\int_\Omega u_n|\nabla u_n|^{p(x)-2}\nabla u_n\nabla \frac{1}{K(|u_n|)}\,dx\bigg| &\le \int_\Omega |u_n||\nabla u_n|^{p(x)-1}\,\frac{|\nabla K(|u_n|)|}{K^2(|u_n|)}\,dx\\ &\le \int_\Omega \frac{|\nabla u_n|^{p(x)}}{K(|u_n|)}\,\frac{|u_n|K'|u_n|)}{K(|u_n|)}\,dx.\end{aligned}$$

Note that $|u_n|K'(|u_n|)/K(|u_n|) \le \sigma_0 \in (0,1)$. Thus

$$C_3\int_\Omega \frac{|\nabla u_n|^{p(x)}}{K(|u_n|)}\,dx - C_4 \le \int_\Omega \frac{f(x,u_n)u_n}{K(|u_n|)}\,dx \le C_5\int_\Omega \frac{|\nabla u_n|^{p(x)}}{K(|u_n|)}\,dx + C_6. \tag{3.2}$$

By (3.1), (3.2) and conditions (f_0) and (f_1), we have

$$\begin{aligned}\int_\Omega f(x,u_n)\,\frac{u_n}{K(|u_n|)}\,dx &\overset{(\mathrm{f}_1)}{\le} C_7\int_\Omega\bigg\{\frac{f(x,u_n)u_n}{p(x)} - F(x,u_n)\bigg\}\,dx + C_7\\ &\le\ C_7\bigg\{\sigma\int_\Omega \frac{|\nabla u_n|^{p(x)}}{K(|u_n|)}\,dx + C_8 + C(\sigma)\int_\Omega |u_n|^{p(x)}[K(|u_n|)]^{p(x)-1}\,dx\bigg\}\\ &\le\ C_7\sigma\int_\Omega \frac{|\nabla u_n|^{p(x)}}{K(|u_n|)}\,dx + C_7C(\sigma)\int_\Omega |u_n|^{p(x)}[K(|u_n|)]^{p(x)-1}dx + C_9\\ &\overset{(3.2)}{\le} \frac{1}{2}\int_\Omega \frac{f(x,u_n)u_n}{K(|u_n|)}\,dx + C_7C(\sigma)\int_\Omega |u_n|^{p(x)}[K(|u_n|)]^{p(x)-1}\,dx + C_{10}.\end{aligned}$$

Thus, by condition (f_1) and the above inequality, we can see that

$$\int_\Omega f(x,u_n)\,\frac{u_n}{K(|u_n|)}\,dx \le C_{11}\int_\Omega |u_n|^{p(x)}[K(|u_n|)]^{p(x)-1}\,dx + C_{12}. \tag{3.3}$$

Note that $tf(x,t)/(|t|^{p(x)}[K(t)]^{p(x)}) \to +\infty$ uniformly as $|t| \to +\infty$ for $x \in \overline{\Omega}$. We claim that

$$\int_\Omega |u_n|^{p(x)}[K(|u_n|)]^{p(x)-1}\,dx \quad \text{is bounded.}$$

This means that

$$\int_\Omega f(x,u_n)\,\frac{u_n}{K(|u_n|)}\,dx \quad \text{is bounded.}$$

In fact, by (K), we observe that there exists $M > 0$ large enough such that

$$\frac{tf(x,t)}{K(t)} > 2C_{11}|t|^{p(x)}[K(t)]^{p(x)-1}, \quad \text{for all } |t| \ge M. \tag{3.4}$$

Denote $\Omega_n = \{x \in \Omega \mid |u_n| \ge M\}$. We have

$$\int_\Omega f(x,u_n)\,\frac{u_n}{K(|u_n|)}\,dx \ge \int_{\Omega_n} 2C_{11}|u_n|^{p(x)}[K(|u_n|)]^{p(x)-1}\,dx - C_{12}. \tag{3.5}$$

Combining (3.3)–(3.5), we obtain

$$\int_{\Omega_n} C_{11}|u_n|^{p(x)}[K(|u_n|)]^{p(x)-1}\,dx \le C_{13},$$

and hence

$$\int_\Omega C_{11}|u_n|^{p(x)}[K(|u_n|)]^{p(x)-1}\,dx \le C_{14}.$$

Thus

$$\int_\Omega f(x,u_n)\,\frac{u_n}{K(|u_n|)}\,dx \le C_{14}, \quad \text{for any } n = 1,2,\ldots$$

This combined with (f_0) implies that

$$\left\{\int_\Omega \frac{|f(x,u_n)u_n|}{K(|u_n|)}\,dx\right\} \quad \text{is bounded.} \tag{3.6}$$

Let $\varepsilon > 0$ satisfy $\varepsilon < \min\{1, p^- - 1, 1/p^{*+}, (p^*/\alpha)^- - 1\}$. Since $\|\varphi'(u_n)\|\|u_n\| \to 0$, we get

$$\begin{aligned}\int_\Omega |\nabla u_n|^{p(x)}\,dx &= \int_\Omega f(x,u_n)u_n\,dx + o(1)\\ &\le \int_\Omega^\varepsilon |f(x,u_n)u_n|^\varepsilon [K(|u_n|)]^{1-\varepsilon}\left[\frac{|f(x,u_n)u_n|}{K(|u_n|)}\right]^{1-\varepsilon} dx + o(1).\end{aligned}$$

By condition (f_1), we have $|f(x,u_n)u_n| \ge |u_n|^{p(x)}$ for large enough $vertu_n|$, and $[K(|u_n|)]^{1-\varepsilon} \le [\ln(e+|u_n|)]^{2(1-\varepsilon)}$ for large enough $|u_n|$, so we have

$$|f(x,u_n)u_n|^\varepsilon [K(|u_n|)]^{1-\varepsilon} \le C_{15}(|f(x,u_n)u_n|^{\varepsilon(1+\varepsilon)} + 1).$$

Therefore

$$\begin{aligned}&\int_\Omega |\nabla u_n|^{p(x)}\,dx = \int_\Omega f(x,u_n)u_n\,dx + o(1)\\ &\le C_{15}(1+\|u_n\|)^{1+\varepsilon}\int_\Omega \left[\frac{|f(x,u_n)u_n|^{1+\varepsilon}+1}{(1+\|u_n\|)^{(1+\varepsilon)/\varepsilon}}\right]^\varepsilon \left[\frac{|f(x,u_n)u_n|}{K(|u_n|)}\right]^{1-\varepsilon} dx + o(1).\end{aligned}$$

By Young's inequality, we have

$$\begin{aligned}&\int_\Omega |\nabla u_n|^{p(x)}\,dx\\ &\quad\le C_{15}(1+\|u_n\|)^{1+\varepsilon}\int_\Omega \frac{|f(x,u_n)u_n|^{1+\varepsilon}+1}{(1+\|u_n\|)^{(1+\varepsilon)/\varepsilon}} + \frac{|f(x,u_n)u_n|}{K(|u_n|)}\,dx + o(1).\end{aligned} \tag{3.7}$$

According to the definition of ε, we have

$$|f(x,u_n)u_n|^{1+\varepsilon} + 1 \le C(|u_n|^{p^*(x)} + 1)$$

and

$$(1+\|u_n\|)^{(1+\varepsilon)/\varepsilon} \ge (1+\|u_n\|)^{(1+\varepsilon)(p^*)^+}.$$

Therefore

$$\begin{aligned}\int_\Omega \frac{|f(x,u_n)u_n|^{1+\varepsilon}+1}{(1+\|u_n\|)^{(1+\varepsilon)/\varepsilon}}\,dx &\le \int_\Omega \frac{C(|u_n|^{p^*(x)}+1)}{(1+\|u_n\|)^{(1+\varepsilon)/\varepsilon}}\,dx\\ &\le \frac{C(|u_n|^{(p^*)^+}+1)}{(1+\|u_n\|)^{(1+\varepsilon)/\varepsilon}} \le \frac{C_\#(\|u_n\|^{(p^*)^+}+1)}{(1+\|u_n\|)^{(1+\varepsilon)/\varepsilon}}.\end{aligned}$$

Thus, the sequence

$$\left\{\int_\Omega \frac{|f(x,u_n)u_n|^{1+\varepsilon}+1}{(1+\|u_n\|)^{(1+\varepsilon)/\varepsilon}}\,dx\right\}$$

is bounded. This combined with (3.6) and (3.7) implies

$$\int_\Omega |\nabla u_n|^{p(x)}\,dx \le C_{16}(1+\|u_n\|)^{1+\varepsilon} + C_{17}.$$

Note that $\varepsilon < p^- - 1$. This is a contradiction, hence $\{u_n\}$ is bounded in X, as claimed. $\square$

Proof of Theorem 1.1. We first establish the existence of a nontrivial weak solution. We show that φ satisfies conditions of the mountain pass lemma. By Lemma 3.3, φ satisfies the Cerami condition. Since $p(x) < \alpha(x) < p^*(x)$, the embedding $X \hookrightarrow L^{\alpha(\,\cdot\,)}(\Omega)$ is compact. Hence there exists $C_0 > 0$ such that

$$|u|_{p(\,\cdot\,)} \le C_0\|u\|, \quad \text{for all } u \in X.$$

Let $\sigma > 0$ be small enough such that $\sigma \le \frac{1}{4}\,\lambda_{p(\,\cdot\,)}$. By assumptions ($\mathrm{f}_0$) and ($\mathrm{f}_2$), we obtain

$$F(x,t) \le \sigma\,\frac{1}{p(x)}\,|t|^{p(x)} + C(\sigma)|t|^{\alpha(x)}, \quad \text{for all } (x,t) \in \Omega\times\mathbb{R}.$$

By (p_1) and Lemma 2.6, we have $\lambda_{p(\,\cdot\,)} > 0$ and

$$\int_\Omega \frac{1}{p(x)}\,|\nabla u|^{p(x)}\,dx - \sigma\int_\Omega \frac{1}{p(x)}\,|u|^{p(x)}\,dx \ge \frac{3}{4}\int_\Omega \frac{1}{p(x)}\,|\nabla u|^{p(x)}.$$

Since $\alpha \in C(\overline{\Omega})$ and $p(x) < \alpha(x) < p^*(x)$, we can divide the domain Ω into n_0 disjoint small subdomains Ω_i $(i = 1,\ldots,n_0)$ such that $\overline{\Omega} = \bigcup_{i=1}^{n_0}\overline{\Omega_i}$ and

$$\sup_{\Omega_i} p(x) < \inf_{\Omega_i}\alpha(x) \le \sup_{\Omega_i}\alpha(x) < \inf_{\Omega_i} p^*(x).$$

Let $\varepsilon = \min_{1\le i\le n_0}\Big\{\inf_{\Omega_i}\alpha(x) - \sup_{\Omega_i} p(x)\Big\}$ and denote by $\|u\|_{\Omega_i}$ the norm of u on Ω_i, that is

$$\int_{\Omega_i} \frac{1}{p(x)}\left|\nabla\,\frac{u}{\|u\|_{\Omega_i}}\right|^{p(x)} dx + \int_{\Omega_i}\frac{1}{p(x)}\left|\frac{u}{\|u\|_{\Omega_i}}\right|^{p(x)} dx = 1.$$

Then $\|u\|_{\Omega_i} \le C\|u\|$ and there exist $\xi_i, \eta_i \in \overline{\Omega_i}$ such that

$$\begin{aligned}
|u|_{\alpha(\,\cdot\,)}^{\alpha(\xi_i)} &= \int_{\Omega_i} |u|^{\alpha(x)}\,dx,\\
\|u\|_{\Omega_i}^{p(\eta_i)} &= \int_{\Omega_i}\left(\frac{1}{p(x)}\,|\nabla u|^{p(x)} + \frac{1}{p(x)}\,|u|^{p(x)}\right)dx.
\end{aligned}$$

When $\|u\|$ is small enough, we have

$$\begin{aligned}
C(\sigma)\int_\Omega |u|^{\alpha(x)}\,dx &= C(\sigma)\sum_{i=1}^{n_0}\int_{\Omega_i}|u|^{\alpha(x)}\,dx\\
&= C(\sigma)\sum_{i=1}^{n_0}|u|_{\alpha(\,\cdot\,)}^{\alpha(\xi_i)} \qquad (\text{where } \xi_i \in \overline{\Omega_i})
\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{i=1}^{n_0}\|u\|_{\Omega_i}^{\alpha(\xi_i)} \qquad\text{(by Proposition 2.5)}\\
&\le C\|u\|^{\varepsilon}\sum_{i=1}^{n_0}\|u\|_{\Omega_i}^{p(\eta_i)} \quad\text{(where } \eta_i\in\overline{\Omega_i})\\
&= C\|u\|^{\varepsilon}\sum_{i=1}^{n_0}\int_{\Omega_i}\bigg(\frac{1}{p(x)}\,|\nabla u|^{p(x)}+\frac{1}{p(x)}\,|u|^{p(x)}\bigg)\,dx\\
&= C\|u\|^{\varepsilon}\int_{\Omega}\bigg(\frac{1}{p(x)}\,|\nabla u|^{p(x)}+\frac{1}{p(x)}\,|u|^{p(x)}\bigg)\,dx\\
&\le \frac{1}{4}\int_{\Omega}\frac{1}{p(x)}\,|\nabla u|^{p(x)}\,dx.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\varphi(u)\ge\int_{\Omega}\frac{1}{p(x)}\,|\nabla u|^{p(x)}-\sigma\int_{\Omega}\frac{1}{p(x)}\,|u|^{p(x)}\,dx-C(\sigma)\int_{\Omega}|u|^{\alpha(x)}\,dx\\
\ge\frac{1}{2}\int_{\Omega}\frac{1}{p(x)}\,|\nabla u|^{p(x)}
\end{aligned}
$$

when $\|u\|$ is small enough. Therefore, there exist $r>0$ and $\delta>0$ such that $\varphi(u)\ge\delta>0$ for every $u\in X$ and $\|u\|=r$.

Suppose (p_2) is satisfied. Define $h\in C_0\overline{(B(x_0,3\delta))}$ as follows:

$$
h(x)=\begin{cases}0 & \text{if } |x-x_0|\ge 3\delta,\\ 3\delta-|x-x_0| & \text{if } 2\delta\le|x-x_0|<3\delta,\\ \delta & \text{if } |x-x_0|<2\delta.\end{cases}
$$

Note that $\min\limits_{|x-x_0|\le\delta}p(x)>\max\limits_{2\delta\le|x-x_0|\le3\delta}p(x)$. It is now easy to check that

$$
\begin{aligned}
\varphi(th)&=\int_{\Omega}\frac{1}{p(x)}\,|\nabla th|^{p(x)}-\int_{\Omega}F(x,th)\,dx\\
&\le\int_{\overline{B(x_0,3\delta)}\setminus(\overline{B(x_0,2\delta)})}\frac{1}{p(x)}\,|\nabla th|^{p(x)}-\int_{(\overline{B(x_0,\delta)})}C_1|th|^{p(x)}\,dx+C_2\to-\infty
\end{aligned}
$$

as $t\to+\infty$. Since $\varphi(0)=0$, the functional φ satisfies the conditions of the mountain pass lemma. So φ admits at least one nontrivial critical point, which implies that problem (P) has a nontrivial weak solution u.

Suppose (f_4) is satisfied. We may assume that there exists $x_0\in\Omega$ such that $\nabla p(x_0)\ne0$. Define $h\in C_0(\overline{B(x_0,\varepsilon)})$ as follows:

$$
h(x)=\begin{cases}0 & \text{if } |x-x_0|\ge\varepsilon,\\ \varepsilon-|x-x_0| & \text{if } |x-x_0|<\varepsilon.\end{cases}
$$

By (f_4) and Lemma 2.9, there exists $\varepsilon>0$ small enough such that

$$
\varphi(th)=\int_{\Omega}\frac{1}{p(x)}\,|\nabla th|^{p(x)}-\int_{\Omega}F(x,th)\,dx\to-\infty \text{ as } t\to+\infty.
$$

Since functional $\varphi(0)=0$, φ satisfies the conditions of the mountain pass lemma. So φ admits at least one nontrivial critical point, which implies that problem (P) has a nontrivial weak solution u. $\square$

In order to prove Theorem 1.2, we need to make some preparations. Note that $X := W_0^{1,p(\,\cdot\,)}(\Omega)$ is a reflexive and separable Banach space (see [47], Section 17, Theorems 2 and 3). Therefore there exist $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\operatorname{span}}\{e_j,\ j=1,2,\ldots\}, \qquad X^* = \overline{\operatorname{span}}^{W^*}\{e_j^*,\ j=1,2,\ldots\},$$

and

$$\langle e_j^*, e_j\rangle = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i\neq j. \end{cases}$$

For convenience, we write

$$X_j = \operatorname{span}\{e_j\}, \quad Y_k = \bigoplus_{j=1}^{k} X_j \quad \text{and} \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}.$$

Lemma 3.4. *Assume that $\alpha \in C_+(\overline{\Omega})$, $\alpha(x) < p^*(x)$ for any $x \in \overline{\Omega}$. If*

$$\beta_k = \sup\{|u|_{\alpha(\,\cdot\,)} \mid \|u\| = 1,\ u \in Z_k\},$$

then $\lim\limits_{k\to\infty} \beta_k = 0$.

Proof. Obviously, $0 < \beta_{k+1} \le \beta_k$, so $\beta_k \to \beta \ge 0$. Let $u_k \in Z_k$ satisfy

$$\|u_k\| = 1, \qquad 0 \le \beta_k - |u_k|_{\alpha(\,\cdot\,)} < \frac{1}{k}.$$

Then there exists a subsequence of $\{u_k\}$ (which we still denote by u_k) such that $u_k \rightharpoonup u$, and

$$\langle e_j^*, u\rangle = \lim_{k\to\infty} \langle e_j^*, u_k\rangle = 0, \quad \text{for all } e_j^*.$$

This implies that $u=0$, and so $u_k \rightharpoonup 0$. Since the embedding from $W_0^{1,p(\,\cdot\,)}(\Omega)$ into $L^{\alpha(\,\cdot\,)}(\Omega)$ is compact, we can conclude that $u_k \to 0$ in $L^{\alpha(\,\cdot\,)}(\Omega)$. Hence we get $\beta_k \to 0$ as $k \to \infty$. $\square$

In order to prove Theorem 1.2, we need the following auxiliary result, see [50, Theorem 4.7]. If the Cerami condition is replaced by the PS condition, we can use the following property, see [9, Theorem 3.6].

Lemma 3.5. *Suppose that $\varphi \in C^1(X,\mathbb{R})$ is even and satisfies the Cerami condition. Let $V^+, V^- \subset X$ be closed subspaces of X with* $\operatorname{codim} V^+ + 1 = \dim V^-$. *Suppose that*:

(1) $\varphi(0) = 0$;
(2) *there exist $\tau > 0$, $\gamma > 0$ such that, for all $u \in V^+$, if $\|u\| = \gamma$ then $\varphi(u) \ge \tau$; and*
(3) *there exists $\rho > 0$ such that, for all $u \in V^-$, if $\|u\| \ge \rho$ then $\varphi(u) \le 0$.*

Consider the set:

$$\Gamma = \{g \in C^0(X, X) \mid g \text{ is odd, } g(u) = u \text{ if } u \in V^- \text{ and } \|u\| \geq \rho\}.$$

Then

(a) *for all* $\delta > 0$, $g \in \Gamma$, $S_\delta^+ \cap g(V^-) \neq \emptyset$, *and it satisfies* $S_\delta^+ = \{u \in V^+ \mid \|u\| = \delta\}$; *and*
(b) *the number* $\varpi := \inf_{g \in \Gamma} \sup_{u \in V^-} \varphi(g(u)) \geq \tau > 0$ *is a critical value for* φ.

PROOF OF THEOREM 1.2. We first establish the existence of infinitely many pairs of weak solutions. According to (f_0), (f_1) and (f_3), the functional φ is an even functional and it satisfies the Cerami condition. Let $V_k^+ = Z_k$ be a closed linear subspace of X and $V_k^+ \oplus Y_{k-1} = X$.

Suppose that (f_4) is satisfied. We may assume that there exists $x_n \in \Omega$ such that $\nabla p(x_n) \neq 0$. Define $h_n \in C_0(\overline{B(x_n, \varepsilon_n)})$ by

$$h_n(x) = \begin{cases} 0 & \text{if } |x - x_n| \geq \varepsilon_n, \\ \varepsilon_n - |x - x_n| & \text{if } |x - x_n| < \varepsilon_n. \end{cases}$$

Without loss of generality, we may assume that $\operatorname{supp} h_i \cap \operatorname{supp} h_j = \emptyset$, for all $i \neq j$. By Lemma 2.9, we can let $\varepsilon_n > 0$ be small enough, so that

$$\varphi(th_n) = \int_\Omega \frac{1}{p(x)} |\nabla th_n|^{p(x)} - \int_\Omega F(x, th_n)\, dx \to -\infty \quad \text{as } t \to +\infty.$$

Suppose that (p_3) is satisfied. Define $h_n \in C_0(\overline{B(x_n, \varepsilon_n)})$ by

$$h_n(x) = \begin{cases} 0 & \text{if } |x - x_n| \geq 3\delta_n, \\ 3\delta_n - |x - x_n| & \text{if } 2\delta_n \leq |x - x_n| < 3\delta_n, \\ \delta_n & \text{if } |x - x_n| < 2\delta_n. \end{cases}$$

Note that $\min_{|x - x_n| \leq \delta_n} p(x) > \max_{2\delta_n \leq |x - x_n| \leq 3\delta_n} p(x)$. It follows that

$$\begin{aligned} \varphi(th_n) &= \int_\Omega \frac{1}{p(x)} |\nabla th_n|^{p(x)} - \int_\Omega F(x, th_n)\, dx \\ &\leq \int_{2\delta_n \leq |x - x_n| \leq 3\delta_n} \frac{1}{p(x)} |\nabla th_n|^{p(x)} - \int_{|x - x_n| \leq \delta_n} C_1 |th_n|^{p(x)}\, dx + C_2 \to -\infty \end{aligned}$$

as $t \to +\infty$.

Set $V_k^- = \operatorname{span}\{h_1, \ldots, h_k\}$. We will prove that there exist infinitely many pairs of V_k^+ and V_k^-, such that φ satisfies the conditions of Lemma 3.5 and the corresponding critical value satisfies

$$\varpi_k := \inf_{g \in \Gamma} \sup_{u \in V_k^-} \varphi(g(u)) \to +\infty$$

when $k \to +\infty$. This shows that there are infinitely many pairs of solutions of problem (P). For any $m = 1, 2, \ldots$, we will prove that there exist $\rho_m > \gamma_m > 0$ and large enough k_m such that

(A_1) $b_{k_m} := \inf\{\varphi(u) \mid u \in V_{k_m}^+,\ \|u\| = \gamma_m\} \to +\infty$ ($m \to +\infty$); and
(A_2) $a_{k_m} := \max\{\varphi(u) \mid u \in V_{k_m}^-,\ \|u\| = \rho_m\} \le 0$.

First, we prove (A_1) as follows. By computation, for any $u \in Z_{k_m}$ with $\|u\| = \gamma_m = m$, we have

$$\begin{aligned}
\varphi(u) &= \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\,dx - \int_\Omega F(x,u)\,dx \\
&\ge \frac{1}{p^+} \int_\Omega |\nabla u|^{p(x)}\,dx - C\int_\Omega |u|^{\alpha(x)}\,dx - C_1 \int_\Omega |u|\,dx \\
&\ge \frac{1}{p^+} \|u\|^{p^-} - C|u|_{\alpha(\,\cdot\,)}^{\alpha(\xi)} - C_2|u|_{\alpha(\,\cdot\,)} \quad (\text{where } \xi \in \Omega) \\
&\ge \begin{cases} \dfrac{1}{p^+} \|u\|^{p^-} - C\beta_{k_m}^{\alpha^-}\|u\|^{\alpha^-} - C_2\beta_{k_m}\|u\| & \text{if } |u|_{\alpha(\,\cdot\,)} \le 1, \\ \dfrac{1}{p^+} \|u\|^{p^-} - C\beta_{k_m}^{\alpha^+}\|u\|^{\alpha^+} - C_2\beta_{k_m}\|u\| & \text{if } |u|_{\alpha(\,\cdot\,)} > 1, \end{cases} \\
&\ge \frac{1}{p^+} \|u\|^{p^-} - C\beta_{k_m}^{\alpha^-}(\|u\|^{\alpha^+} + 1) - C_2\beta_{k_m}\|u\|.
\end{aligned}$$

Obviously, there exists a large enough k_m such that, for all $u \in Z_{k_m}$ with $\|u\| = \gamma_m = m$,

$$\frac{1}{p^+} \|u\|^{p^-} - C\beta_{k_m}^{\alpha^-}(\|u\|^{\alpha^+} + 1) - C_2\beta_{k_m}\|u\| \ge \frac{1}{2p^+} \|u\|^{p^-}.$$

Therefore $\varphi(u) \ge \|u\|^{p^-}/2p^+$, for all $u \in Z_{k_m}$ with $\|u\| = \gamma_m = m$. Hence $b_{k_m} \to +\infty$ as $m \to \infty$.

Next, we give a proof of (A_2). According to the above discussion, it is easy to see that

$$\Psi(th_{k_m}) \to -\infty \quad \text{as } t \to +\infty.$$

We conclude that, for all $h \in V_{k_m}^- = \operatorname{span}\{h_1, \ldots, h_{k_m}\}$ with $\|h\| = 1$,

$$\varphi(th) \to -\infty \quad \text{as } t \to +\infty. \qquad \square$$

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