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GROUND STATE SOLUTIONS FOR A CLASS OF SEMILINEAR ELLIPTIC SYSTEMS WITH SUM OF PERIODIC AND VANISHING POTENTIALS

Guofeng Che — Haibo Chen

Abstract. In this paper, we consider the following semilinear elliptic systems:

$$\begin{cases} -\Delta u + V(x)u = F_u(x,u,v) - \Gamma(x)|u|^{q-2}u & \text{in } \mathbb{R}^N, \\ -\Delta v + V(x)v = F_v(x,u,v) - \Gamma(x)|v|^{q-2}v & \text{in } \mathbb{R}^N, \end{cases}$$

where $q \in [2, 2^*)$, $V = V_{\rm per} + V_{\rm loc} \in L^\infty(\mathbb{R}^N)$ is the sum of a periodic potential $V_{\rm per}$ and a localized potential $V_{\rm loc}$ and $\Gamma \in L^\infty(\mathbb{R}^N)$ is periodic and $\Gamma(x) \geq 0$ for almost every $x \in \mathbb{R}^N$. Under some appropriate assumptions on F, we investigate the existence and nonexistence of ground state solutions for the above system. Recent results from the literature are improved and extended.

1. Introduction

In this paper, we consider the existence and nonexistence of ground state solutions to the following semilinear elliptic systems:

$$\begin{cases} -\Delta u + V(x)u = F_u(x,u,v) - \Gamma(x)|u|^{q-2}u & \text{in } \mathbb{R}^N, \\ -\Delta v + V(x)v = F_v(x,u,v) - \Gamma(x)|v|^{q-2}v & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

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where $q \in [2, 2^*)$. We assume that functions Γ, V and F satisfy the following hypotheses:

(Γ) $\Gamma \in L^\infty(\mathbb{R}^N)$ is periodic and $\Gamma(x) \geq 0$ for almost every $x \in \mathbb{R}^N$.

(V) $V = V_{\rm per} + V_{\rm loc} \in L^\infty(\mathbb{R}^N)$, $V_{\rm per} \in L^\infty(\mathbb{R}^N)$ is $\mathbb{Z}^N$-periodic, $V_{\rm loc} \in L^\infty(\mathbb{R}^N)$ and $V_{\rm loc}(x) \to 0$ as $|x| \to \infty$.

(F_1) $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2)$, $F_u(x,u,v)$ and $F_v(x,u,v)$ are measurable, $\mathbb{Z}^N$-periodic in $x \in \mathbb{R}^N$ and continuous in $u, v \in \mathbb{R}$ for almost every $x \in \mathbb{R}^N$ and there exist $c > 0$ and $2 \leq q < p < 2^*$ such that $|F_u(x,u,v)| \leq c(1+|(u,v)|^{p-1})$ and $|F_v(x,u,v)| \leq c(1+|(u,v)|^{p-1})$ for all $(u,v) \in \mathbb{R} \times \mathbb{R}$ and $x \in \mathbb{R}^N$, where $|(u,v)| = (u^2+v^2)^{1/2}$.

(F_2) $F_u(x,u,v) = o(|(u,v)|)$ and $F_v(x,u,v) = o(|(u,v)|)$ uniformly in $x \in \mathbb{R}^N$ as $|(u,v)| \to 0$.

(F_3) $F(x,u,v)/|(u,v)|^q \to \infty$ uniformly in $x \in \mathbb{R}^N$ as $|(u,v)| \to \infty$.

(F_4) For every fixed $v \in \mathbb{R}$, $F_u(x,u,v)/|u|^{q-1}$ is strictly increasing in u on $(-\infty, 0)$ and $(0, +\infty)$ and for every fixed $u \in \mathbb{R}$, $F_v(x,u,v)/|v|^{q-1}$ is strictly increasing in v on $(-\infty, 0)$ and $(0, +\infty)$.

(F_5) $F(x,u,v) \geq 0$, $F_u(x,u,v)u \geq 0$ and $F_v(x,u,v)v \geq 0$ and $F_u(x,u,v)u + F_v(x,u,v)v \geq qF(x,u,v)$ for any $(x,u,v) \in \mathbb{R}^N \times \mathbb{R}^2$.

When Ω is a bounded domain of $\mathbb{R}^N$, the problem

$$(1.2) \qquad \begin{cases} -\Delta u = \lambda(a(x)u + b(x)v) + F_u(x,u,v) & \text{in } \Omega, \\ -\Delta v = \lambda(b(x)u + c(x)v) + F_v(x,u,v) & \text{in } \Omega, \\ u(x) = v(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

which is related to reaction-diffusion systems that appear in chemical and biological phenomena, including the steady and unsteady state situation (see [9], [24]), has been extensively investigated in recent years. For the results on existence, multiple solutions and positive solutions to problem (1.2), we refer the readers to [5], [8], [9], [21], [23], [24], [29] and the references therein. In [21], Qu and Tang obtained the existence and multiplicity of weak solutions to problem (1.2) by using Ekeland's variational principle together with variational methods, some new existence theorems of weak solutions were obtained in Duan et al. [9].

Recently, the problems in the whole space $\mathbb{R}^N$ were considered in some works. For example, see [6], [13]–[15], [17], [27], [28], [30] and the references therein. By applying the theorems of [2], Zhao et al. [30] considered the periodic asymptotically linear elliptic system

$$(1.3) \qquad \begin{cases} -\Delta u + V(x)u = G_u(x,u,v) & \text{for } x \in \mathbb{R}^N, \\ -\Delta v + V(x)v = G_v(x,u,v) & \text{for } x \in \mathbb{R}^N, \end{cases}$$

where V is periodic and 0 lies in a gap of $\sigma(-\Delta + V)$, $G(x,u,v)$ is periodic in x and asymptotically quadratic in (u,v), they obtained infinitely many geometrically distinct solutions. Assuming that the potential V is non-periodic and sign-changing, $G(x,z)$ is non-periodic in x and asymptotically quadratic in $z=(u,v)$, Zhang et al. [28] obtained the existence and multiplicity of solutions of system (1.3) via a variational approach. In [27], the authors obtained the existence of a ground state solution by proving all Cerami sequences for the energy functional are bounded. Moreover, they assumed that 0 lies in a gap of the spectrum $\sigma(-\Delta+V)$, $G_u(x,u,v)=f(x,v)$ and $G_v(x,u,v)=g(x,u)$ are both superlinear at 0 and infinity but they have different increasing rates at infinity. Liao et al. [15] proved system (1.3) has a nontrivial solution under concise super-quadratic conditions. Furthermore, they assumed that V and G are periodic in x, $G(x;z)$ is super-linear in $z=(u,v)$, and these conditions showed that the existence of a nontrivial solution depends mainly on the behavior of $G(x,u,v)$ as $|u+v|\to 0$ and $|au+bv|\to\infty$ for some positive constants a, b.

Recall that in the absence of the localized potential $V_{\mathrm{loc}}=0$, the spectrum $\sigma(-\Delta+V)$ of $-\Delta+V=-\Delta+V_{\mathrm{per}}$ is purely continuous, bounded from below and consists of closed disjoint intervals [22]. In [25], Szulkin and Weth considered the following Schrödinger equation:

$$-\Delta u+V(x)u=f(x,u)\quad \text{in } \mathbb{R}^N. \tag{1.4}$$

When $V(x)$ and $f(x,u)$ are periodic in x and 0 belongs to a spectral gap of $-\Delta+V$, they obtained the existence of ground state solutions for problem (1.4). When 0 is a right boundary point of the essential spectrum of $-\Delta+V$ and $f(x,u)$ is superlinear and subcritical, Mederski [18] obtained the existence of ground state solutions and multiple solutions of system (1.4) with $u(x)\to 0$, as $|x|\to\infty$. Later, Mederski [19] considered the ground state solutions to the system of coupled Schrödinger equations as follows:

$$\triangle u_i+V_i(x)u_i=\partial_{u_i}F(x,u)\quad \text{on } \mathbb{R}^N,\ i=1,\dots,K, \tag{1.5}$$

where F and V_i are periodic in x, $0\notin\sigma(-\Delta+V_i)$, $i=1,\dots,K$. Moreover, they made use of a new linking-type result involving the Nehari–Pankov manifold and assumed that F satisfies the following conditions:

(1) $f_i\colon \mathbb{R}^N\times\mathbb{R}^K\to\mathbb{R}$ is measurable, $\mathbb{Z}^N$-periodic in $x\in\mathbb{Z}^N$ and continuous in $u\in\mathbb{R}^K$ for almost every $x\in\mathbb{R}^N$. Moreover, $f=(f_1,\dots,f_K)=\partial_uF$, where $F\colon\mathbb{R}^N\times\mathbb{R}^K\to\mathbb{R}$ is differentiable with respect to the second variable $u\in\mathbb{R}^K$ and $F(x,0)=0$ for almost every $x\in\mathbb{R}^N$.

(2) There are $a>0$ and $2<p<2^*$ such that

$$|f(x,u)|\le a(1+|u|^{p-1}),\quad \text{for all } u\in\mathbb{R}^K \text{ and a.e. } x\in\mathbb{R}^N.$$

(3) $f(x,u)=o(u)$ uniformly with respect to x as $|u|\to 0$.

In [11], Guo and Mederski considered the existence and nonexistence of ground state solutions of system (1.5) with $K=1$ and $V(x)=V_1(x)-\mu/|x|^2$, where $V_1 \in L^\infty(\mathbb{R}^N)$, V_1 is $\mathbb{Z}^N$-periodic in $x \in \mathbb{R}^N$ and $0 \notin \sigma(-\triangle + V)$. Moreover, they assumed that $f(x,u)$ satisfies (1)–(3) and the following conditions:

(4) $F(x,u)/u^2 \to \infty$ uniformly in x as $|u| \to \infty$, where F is the primitive of f with respect to u, that is, $F(x,u) = \int_0^u f(x,s)\,ds$.
(5) $u \mapsto f(x,u)/|u|$ is non-decreasing on $(-\infty,0)$ and $(0,+\infty)$.

In photonic crystals, the potential V is periodic or close-to-periodic. Namely if the periodic structure has a linear defect, i.e. an additional structure breaking the periodicity, then the photonic crystal can guide light along the defect. In this case the potential has the following form:

$$V(x) = V_{\rm per} + V_{\rm loc}, \tag{1.6}$$

where $V_{\rm per}$ is periodic in $x \in \mathbb{R}^N$ and $V_{\rm loc}$ is a localized potential that vanishes at infinity.

Recently, Bieganowski and Mederski [4] considered system (1.4) with $f(x,u) = g(x,u) - \Gamma(x)|u|^{q-2}u$ sign-changing and $V(x)$ satisfying (1.6). When $f(x,u)$ satisfies (1)–(5), they investigated the existence and nonexistence of ground state solutions of system (1.4) by means of the Nehari techniques.

Inspired by the above facts, more precisely by [4], [25], the aim of this paper is to study the existence and nonexistence of ground state solutions to problem (1.1) via variational methods. To the best of our knowledge, there have been few works concerning this case up to now.

Now, we state our main results.

Theorem 1.1. *Suppose that conditions* (Γ), (V) *and* (F_1)–(F_5) *hold and* $\inf\sigma(-\Delta + V) > 0$. *If* $V_{\rm loc}(x) < 0$ *or* $V_{\rm loc}(x) = 0$, *then* (1.1) *possesses a ground state solution* $(u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, *i.e.* (u,v) *is a critical point of* J *such that* $J(u,v) = \inf_{\mathcal{N}} J$. *Moreover,* u *and* v *are continuous and there exist* $\alpha, C > 0$ *such that*

$$|(u(x),v(x))| \le C\exp(-\alpha|x|), \quad \textit{for any } x \in \mathbb{R}^N.$$

Theorem 1.2. *Suppose that conditions* (Γ), (V) *and* (F_1)–(F_5) *hold and* $\inf\sigma(-\Delta + V_{\rm per}) > 0$. *If* $V_{\rm loc}(x) > 0$ *for almost every* $x \in \mathbb{R}^N$, *then problem* (1.1) *has no ground state solutions.*

Problem 1.3. Note that in our paper if $\Gamma(x) \neq 0$ and $q > 2$, then

$$I(u,v) = \int_{\mathbb{R}^N} F(x,u,v)\,dx - \frac{1}{q}\int_{\mathbb{R}^N} \Gamma(x)(|u|^q + |v|^q)\,dx$$

is sign-changing. Furthermore, for every fixed $v \in \mathbb{R}$,

$$F_u(x,u,v) - \frac{1}{q}\int_{\mathbb{R}^N} \frac{\Gamma(x)|u|^q}{|u|^{q-1}}\,dx$$

is not strictly increasing in u on $(-\infty, 0)$ and $(0, +\infty)$ and for every fixed $u \in \mathbb{R}$, $F_v(x, u, v) - (1/q)\int_{\mathbb{R}^N} \Gamma(x)|v|^q\,dx/|v|^{q-1}$ is not strictly increasing about v on $(-\infty, 0)$ and $(0, +\infty)$. Then the results of [25] do not apply in our case.

PROBLEM 1.4. Under our assumptions, $\mathcal{N}$ is not C^1-manifold, so the classical minimization on the Nehari manifold does not work.

For Problem 1.3, our approach is presented in the abstract setting in Section 2 and we develop a critical point theory which extends the abstract results from [3] for definite functionals and enables us to deal with sign-changing functionals.

For Problem 1.4, we intend to adopt the techniques of [4], [25] based on the observation that $\mathcal{N}$ is a topological manifold homeomorphic with the unit sphere in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, where a minimizing sequence can be found. On the other hand, the abstract results concerning Nehari techniques have been obtained by Szulkin and Weth [25] for positive nonlinear part I or completely continuous I' as well as by Figueiredo and Quoirin in [10] for weak lower semicontinuous $u \mapsto J'(u)u$. In [4], I and J do not satisfy these conditions anymore.

NOTATION 1.5. Throughout this paper, we shall denote by $\|\cdot\|_r$ the L^r-norm and by C various positive generic constants, which may vary from line to line. $2^* = \infty$ for $N = 1, 2$ and $2^* = 2N/(N-2)$ for $N \geq 3$, is the critical Sobolev exponent. Also if we take a subsequence of a sequence $\{(u_n, v_n)\}$ we shall denote it again as $\{(u_n, v_n)\}$.

The paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proofs of main results. The lack of compactness of (PS) sequences requires decomposition of sequences which is provided in Lemma 2.3 that is proved in Section 4.

2. Variational setting and preliminaries

In this section we outline the variational framework for problem (1.1) and give some preliminary lemmas.

Let $H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$ with the norm

$$\|u\|_{H^1} = \bigg(\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)\,dx\bigg)^{1/2}.$$

Let

$$X = \bigg\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \big(|\nabla u|^2 + V(x)u^2\big)\,dx < +\infty\bigg\}$$

with the inner product and norm

$$\langle u, v\rangle_X = \int_{\mathbb{R}^N} \big(\nabla u \nabla v + V(x)uv\big)\,dx, \qquad \|u\|_X = \langle u, u\rangle_X^{1/2}.$$

As usual, for $1 \le p < +\infty$, we let

$$\|u\|_p = \bigg(\int_{\mathbb{R}^N} |u(x)|^p \, dx\bigg)^{1/p}, \quad u \in L^p(\mathbb{R}^N),$$

and

$$\|u\|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} |u(x)|, \quad u \in L^\infty(\mathbb{R}^N).$$

Then $E = X \times X$ is a Hilbert space with the following inner product:

$$\langle (u,v),(\varphi,\psi)\rangle = \langle u,\varphi\rangle_X + \langle v,\psi\rangle_X, \quad (u,v),(\varphi,\psi) \in X \times X$$

and the norm

$$\|(u,v)\|^2 = \langle (u,v),(u,v)\rangle = \|u\|_X^2 + \|v\|_X^2, \quad (u,v) \in X \times X.$$

Since $\inf \sigma(-\triangle + V) > 0$, the norm $\|\cdot\|$ is equivalent to the classic one on $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

Similarly, $\inf \sigma(-\triangle + V_{\rm per}) > 0$ implies that the norm given by

$$\bigg(\int_{\mathbb{R}^N} (|\nabla u|^2 + V_{\rm per} u^2 + |\nabla v|^2 + V_{\rm per} v^2) \, dx\bigg)^{1/2}$$

is equivalent to the classic one on $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

Define a functional J on E by

$$J(u,v) = \frac{1}{2}\,\|(u,v)\|^2 - I(u,v), \tag{2.1}$$

where

$$I(u,v) = \int_{\mathbb{R}^N} F(x,u,v)\, dx - \frac{1}{q}\int_{\mathbb{R}^N} \Gamma(x)(|u|^q + |v|^q)\, dx.$$

It is not difficult to verify that $J \in C^1(E,\mathbb{R})$ under assumptions (V), (Γ), (F_1)–(F_5) and

$$\begin{aligned} \langle J'(u,v),(\varphi,\psi)\rangle &= \int_{\mathbb{R}^N} \nabla u \nabla \varphi \, dx + \int_{\mathbb{R}^N} V(x) u\varphi \, dx \\ &\quad - \int_{\mathbb{R}^N} F_u(x,u,v)\varphi \, dx + \int_{\mathbb{R}^N} \Gamma(x)|u|^{q-2}u\varphi \, dx + \int_{\mathbb{R}^N} \nabla v \nabla \psi \, dx \\ &\quad + \int_{\mathbb{R}^N} V(x) v\psi \, dx - \int_{\mathbb{R}^N} F_v(x,u,v)\psi \, dx + \int_{\mathbb{R}^N} \Gamma(x)|v|^{q-2}v\psi \, dx. \end{aligned} \tag{2.2}$$

In this case the Nehari manifold is given by

$$\begin{aligned} \mathcal{N} &= \big\{(u,v) \in E \setminus \{(0,0)\} : \langle J'(u,v),(u,v)\rangle = 0\big\} \\ &= \big\{(u,v) \in E \setminus \{(0,0)\} : \|(u,v)\|^2 = \langle I'(u,v),(u,v)\rangle\big\}. \end{aligned}$$

Lemma 2.1. *Suppose the following conditions hold*:

(J_1) *There exists $\rho > 0$ such that*

$$\alpha = \inf_{\|(u,v)\|=\rho} J(u,v) > J(0,0) = 0.$$

(J_2) *There exists* $q \geq 2$ *such that* $I(t_nu_n, t_nv_n)/t_n^q \to \infty$ *for* $t_n \to \infty$ *and* $(u_n, v_n) \to (u, v) \neq (0,0)$ *as* $n \to \infty$.

(J_3) *For* $t \in (0,\infty) \setminus \{1\}$ *and* $(u,v) \in \mathcal{N}$, *we have*

$$\frac{t^2-1}{2}\langle I'(u,v),(u,v)\rangle - I(tu,tv) + I(u,v) < 0.$$

(J_4) J *is coercive on* $\mathcal{N}$.

Then $\inf\limits_{\mathcal{N}} J > 0$ *and there exists a bounded minimizing sequence for* J *on* $\mathcal{N}$, *i.e. there is a sequence* $\{u_n, v_n\} \subset \mathcal{N}$ *such that* $J(u_n, v_n) \to \inf\limits_{\mathcal{N}} J$ *and* $J'(u_n, v_n) \to 0$ *as* $n \to \infty$.

Remark 2.2. Observe that condition (J_2) implies that for any $(u,v) \neq (0,0)$, there exists $t > 0$ such that $J(tu,tv) < 0$. Then it follows from (J_1) and (J_2) that J has the mountain pass geometry [1, 26] and we are able to find a Palais–Smale sequence. While, we do not know whether it is a bounded sequence and contained in $\mathcal{N}$. In order to get the boundedness we assume the coercivity in (J_4), which is, in applications, a weaker requirement than the classical Ambrosetti–Rabinowitz condition [25].

Remark 2.3. (a) In order to get (J_3), it is sufficient to check

$$(1-t)(t\langle I'(u,v),(u,v)\rangle - \langle I'(tu,tv),(u,v)\rangle) > 0 \tag{2.3}$$

for any $t \in (0,\infty) \setminus \{1\}$ and (u,v) such that $\langle I'(u,v),(u,v)\rangle > 0$. In fact, let

$$\varphi(t) = \frac{t^2-1}{2}\langle I'(u,v),(u,v)\rangle - I(tu,tv) + I(u,v), \tag{2.4}$$

for $t \in (0,\infty) \setminus \{1\}$ and $(u,v) \in \mathcal{N}$. Therefore, $\langle I'(u,v),(u,v)\rangle = \|(u,v)\|^2$, $\varphi(1) = 0$, $\varphi'(t) = t\langle I'(u,v),(u,v)\rangle - \langle I'(tu,tv),(u,v)\rangle > 0$ for $t < 1$ and $\varphi'(t) < 0$ for $t > 1$. As a result, $\varphi(t) < \varphi(1) = 0,\ t \in (0,\infty) \setminus \{1\}$.

(b) Condition (J_3) is equivalent to the following conditions: $(u,v) \in \mathcal{N}$ is the unique maximum point of $J(tu,tv)$, $t \in (0,+\infty)$. Indeed,

$$\begin{aligned} J(tu,tv) &= J(u,v) + J(tu,tv) - J(u,v) - \frac{t^2-1}{2}\langle J'(u,v),(u,v)\rangle \\ &= J(u,v) + \varphi(t) < J(u,v) \end{aligned} \tag{2.5}$$

if and only if $\varphi(t) < 0$.

Proof of Lemma 2.1. The proof is analogous to the proof of Theorem 2.1 in [4], we omit it here. □

Lemma 2.4. *Suppose that conditions* (Γ), (V), (F_1)–(F_5) *hold and* $\inf\sigma(-\Delta + V) > 0$. *Then* ($J_1$)–($J_4$) *hold.*

Proof. (J_1) For any $\varepsilon > 0$, it follows from (F_1) and (F_2) that there exists $C(\varepsilon) > 0$ such that

$$|F(x,u,v)| \le \varepsilon|(u,v)| + C(\varepsilon)|(u,v)|^p.$$

Then, by the Sobolev embedding theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} F(x,u,v)\,dx - \int_{\mathbb{R}^N} \frac{1}{q}\,T(x)(|u|^q + |v|^q)\,dx \\ \le \int_{\mathbb{R}^N} |F(x,u,v)|\,dx \le C(\varepsilon\|(u,v)\|^2 + C(\varepsilon)\|(u,v)\|^p).\end{aligned}$$

Therefore, there exists $\rho > 0$ such that

$$\int_{\mathbb{R}^N} F(x,u,v)\,dx - \int_{\mathbb{R}^N} \frac{1}{q}\,T(x)(|u|^q + |v|^q)\,dx \le \frac{1}{4}\,\|(u,v)\|^2$$

for $\|(u,v)\| \le \rho$. Hence

$$J(u,v) \ge \frac{1}{4}\,\|(u,v)\|^2 = \frac{1}{4}\,\rho^2 > 0 \quad \text{for } \|(u,v)\| = \rho.$$

(J_2) It follows from (F_3), (F_5) and Fatou's lemma that

$$\frac{I(t_n u_n, t_n v_n)}{t_n^q} = \int_{\mathbb{R}^N} \frac{F(x, t_n u_n, t_n v_n)}{t_n^q}\,dx - \frac{1}{q}\int_{\mathbb{R}^N} \Gamma(x)(|u_n|^q + |v_n|^q)\,dx \to \infty,$$

as $n \to \infty$.

(J_3) Fix (u,v) such that $\langle I'(u,v),(u,v)\rangle > 0$, i.e.

$$\int_{\mathbb{R}^N} (F_u(x,u,v)u + F_v(x,u,v)v)\,dx > \int_{\mathbb{R}^N} \Gamma(x)(|u|^q + |v|^q)\,dx.$$

Observe that

$$\begin{aligned}
&t\langle I'(u,v),(u,v)\rangle - \langle I'(tu,tv),(u,v)\rangle \\
&= \int_{\mathbb{R}^N} (F_u(x,u,v)tu + F_v(x,u,v)tv)\,dx - \int_{\mathbb{R}^N} \Gamma(x)t(|u|^q + |v|^q)\,dx \\
&\quad - \int_{\mathbb{R}^N} (F_u(x,tu,tv)u + F_v(x,tu,tv)v)\,dx + \int_{\mathbb{R}^N} \Gamma(x)t^{q-1}(|u|^q + |v|^q)\,dx \\
&= \int_{\mathbb{R}^N} (F_u(x,u,v)tu - F_u(x,tu,tv)u)\,dx + (t^{q-1} - t)\int_{\mathbb{R}^N} \Gamma(x)|u|^q\,dx \\
&\quad + \int_{\mathbb{R}^N} (F_v(x,u,v)tv - F_v(x,tu,tv)v)\,dx + (t^{q-1} - t)\int_{\mathbb{R}^N} \Gamma(x)|v|^q\,dx.
\end{aligned}$$

Therefore, for $t < 1$, we derive

$$\begin{aligned}
&\int_{\mathbb{R}^N} (F_u(x,u,v)tu - F_u(x,tu,tv)u)\,dx + (t^{q-1} - t)\int_{\mathbb{R}^N} \Gamma(x)|u|^q\,dx \\
&\qquad + \int_{\mathbb{R}^N} (F_v(x,u,v)tv - F_v(x,tu,tv)v)\,dx + (t^{q-1} - t)\int_{\mathbb{R}^N} \Gamma(x)|v|^q\,dx \\
&\quad > \int_{\mathbb{R}^N} (t^{q-1}F_u(x,u,v)u - F_u(x,tu,tv)u)\,dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N}(t^{q-1}F_v(x,u,v)v-F_v(x,tu,tv)v)\,dx\\
&=t^{q-1}\int_{\mathbb{R}^N}\bigg(F_u(x,u,v)u-\frac{F_u(x,tu,tv)u}{t^{q-1}}\bigg)\,dx\\
&+t^{q-1}\int_{\mathbb{R}^N}\bigg(F_v(x,u,v)v-\frac{F_v(x,tu,tv)v}{t^{q-1}}\bigg)\,dx>0
\end{aligned}
$$

by (F_4). Analogously, we get $t\langle I'(u,v),(u,v)\rangle-\langle I'(tu,tv),(u,v)\rangle<0$, for $t>1$. Therefore, it follows from Remark 2.2 that (J_3) holds.

(J_4) Let $(u_n,v_n)\subset\mathcal{N}$ be a sequence such that $\|(u_n,v_n)\|\to\infty$ as $n\to\infty$. If $q=2$, then $\Gamma\equiv 0$. Let $(\omega_n,s_n)=(u_n,v_n)/\|(u_n,v_n)\|$ and $(\omega_n,s_n)\to(0,0)$ in $L^p(\mathbb{R}^N)\times L^p(\mathbb{R}^N)$. Then it follows from (2.5) that

$$
J(u_n,v_n)\geq J(t\omega_n,ts_n)=\frac{t^2}{2}+o(1)\quad\text{for any } t>0,
$$

then $J(\omega_n,s_n)\to\infty$, as $n\to\infty$. If (ω_n,s_n) is bounded away from $(0,0)$ in $L^p(\mathbb{R}^N)\times L^p(\mathbb{R}^N)$, then it follows from Fatou's lemma and Lion's lemma that

$$
\begin{aligned}
&\frac{J(u_n,v_n)}{\|(u_n,v_n)\|^2}\\
&=\frac{1}{2}-\int_{\mathbb{R}^N}\frac{F(x,u_n(x+y_n),v_n(x+y_n))}{|(u_n(x+y_n),v_n(x+y_n))|^2}\,|(\omega_n(x+y_n),s_n(x+y_n))|^2\,dx\to-\infty
\end{aligned}
$$

for some sequence $(y_n)\subset\mathbb{R}^N$ such that $(\omega_n(x+y_n),s_n(x+y_n))\rightharpoonup(\omega,s)\neq(0,0)$ for some $(\omega,s)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ and $(\omega_n(x+y_n),s_n(x+y_n))\to(\omega(x),s(x))$ for almost every $x\in\mathbb{R}^N$. Then we get a contradiction with $J(u_n,v_n)/\|(u_n,v_n)\|\geq 0$. If $q>2$, then it follows from (F_5) that

$$
\begin{aligned}
J(u_n,v_n)&=J(u_n,v_n)-\frac{1}{q}\,\langle J'(u_n,v_n),(u_n,v_n)\rangle\\
&=\bigg(\frac{1}{2}-\frac{1}{q}\bigg)\|(u_n,v_n)\|^2\\
&\quad+\int_{\mathbb{R}^N}\bigg[\frac{1}{q}\,(F_u(x,u_n,v_n)u+F_v(x,u_n,v_n)v)-F(x,u_n,v_n)\bigg]\,dx\\
&\geq\bigg(\frac{1}{2}-\frac{1}{q}\bigg)\|(u_n,v_n)\|^2\to\infty,
\end{aligned}
$$

as $\|(u_n,v_n)\|\to\infty$, which implies that J is coercive on $\mathcal{N}$. □

Lemma 2.5. *Suppose that* (V) *and* (F_1)–(F_4) *hold. Let* (u_n,v_n) *be a bounded Palais–Smale sequence for* J. *Then passing to a subsequence of* (u_n,v_n), *there exist* $l\geq 0$ *and sequences* $(y_n^k)\subset\mathbb{Z}^N$, $(\omega_1^k,\omega_2^k)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, $k=1,\ldots,l$, *such that*:

(a) $(u_n,v_n)\rightharpoonup(u_0,v_0)$ *and* $J'(u_0,v_0)=0$;
(b) $|y_n^k|\to\infty$ *and* $|y_n^k-y_n^{k'}|\to\infty$, *as* $n\to\infty$ *for* $k\neq k'$;

(c) $(\omega_1^k,\omega_2^k)\neq(0,0)$ *and* $J'_{\rm per}(\omega_1^k,\omega_2^k)=0$ *for each* $k\neq k'$, *where*

$$J_{\rm per}(u,v)=J(u,v)-\frac{1}{2}\int_{\mathbb{R}^N}V_{\rm loc}(x)(u^2+v^2)\,dx;$$

(d) $(u_n,v_n)-(u_0,v_0)-\sum_{k=1}^{l}(\omega_1^k(\cdot\,-y_n^k),\omega_2^k(\cdot\,-y_n^k))\to 0$, *as* $n\to\infty$, *in* $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$;

(e) $J(u_n,v_n)\to J(u_0,v_0)+\sum_{k=1}^{l}J_{\rm per}(\omega_1^k,\omega_2^k)$.

Since the proof of Lemma 2.5 is technical, we postpone it to Section 4.

3. Proofs of main results

Proof of Theorem 1.1. It follows from Lemma 2.1 that there exists a bounded sequence $(u_n,v_n)\subset\mathcal{N}$ for J . Let $c=\inf_{(u,v)\in\mathcal{N}}J(u,v)$ and

$$c_{\rm per}=\inf\big\{J_{\rm per}(u,v):(u,v)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)\setminus\{(0,0)\},\ J'_{\rm per}(u,v)=0\big\}.$$

If $V_{\rm loc}=0$, we have $J=J_{\rm per}$. Thus, it follows from Lemma 2.5 that there exists an integer $l\geq 0$ and sequences $(y_n^k)\subset\mathbb{Z}^N$, $(\omega_1^k,\omega_2^k)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, $k=1,\ldots,l$, such that $(u_n,v_n)\rightharpoonup(u_0,v_0)$, $J'(u_0,v_0)=0$, $(\omega_1^k,\omega_2^k)\neq(0,0)$, $J'_{\rm per}(\omega_1^k,\omega_2^k)=0$ for each $1\leq k\leq l$ and

$$J_{\rm per}(u_n,v_n)\to J_{\rm per}(u_0,v_0)+\sum_{k=1}^{l}J_{\rm per}(\omega_1^k,\omega_2^k),\quad n\to\infty.$$

Therefore

$$c=c_{\rm per}=J_{\rm per}(u_0,v_0)+\sum_{k=1}^{l}J_{\rm per}(\omega_1^k,\omega_2^k)\geq J_{\rm per}(u_0,v_0)+lc_{\rm per}.$$

If $(u_0,v_0)\neq(0,0)$, then $l=0$ and $J_{\rm per}(u_0,v_0)=c_{\rm per}$ and (u_0,v_0) is a ground state solution. If $(u_0,v_0)=(0,0)$, we have $c_{\rm per}\geq lc_{\rm per}$. Since $c_{\rm per}>0$, then $l=1$ and (ω_1^1,ω_2^1) is a ground state solution.

Suppose that $V_{\rm loc}(x)<0$ for almost every $x\in\mathbb{R}^N$. Again, it follows from Lemma 2.5 that there is $(u_{\rm per},v_{\rm per})\neq(0,0)$ such that $J'_{\rm per}(u_{\rm per},v_{\rm per})=0$ and $J_{\rm per}(u_{\rm per},v_{\rm per})=c_{\rm per}$. Let $t>0$ be such that $(tu_{\rm per},tv_{\rm per})\in\mathcal{N}$. Since $V(x)<V_{\rm per}(x)$, we have

$$c_{\rm per}=J_{\rm per}(u_{\rm per},v_{\rm per})\geq J_{\rm per}(tu_{\rm per},tv_{\rm per})>J(tu_{\rm per},tv_{\rm per})\geq c>0.$$

It follows from Lemma 2.5 that there exists (u_0,v_0) such that $J'(u_0,v_0)=0$. Furthermore,

$$J(u_n,v_n)\to J(u_0,v_0)+\sum_{k=1}^{l}J_{\rm per}(\omega_1^k,\omega_2^k),\quad n\to\infty.$$

Hence

$$c = J(u_0, v_0) + \sum_{k=1}^{l} J_{\mathrm{per}}(\omega_1^k, \omega_2^k) \geq J(u_0, v_0) + l c_{\mathrm{per}}.$$

Since $c_{\mathrm{per}} > c$, we obtain that $l = 0$ and (u_0, v_0) is a ground state solution. It follows from Theorem 2 in [20] that there exist $\lambda, C > 0$ such that

$$|(u(x), v(x))| \leq C \exp(-\lambda|x|), \quad x \in \mathbb{R}^N. \qquad \square$$

Proof of Theorem 1.2. Let

$$\mathcal{N}_{\mathrm{per}} = \big\{(u, v) \in E \setminus \{0, 0\} : \langle J'_{\mathrm{per}}(u, v), (u, v)\rangle = 0\big\},$$

where

$$J_{\mathrm{per}}(u, v) = J(u, v) - \int_{\mathbb{R}^N} V_{\mathrm{loc}}(x) u^2\, dx - \int_{\mathbb{R}^N} V_{\mathrm{loc}}(x) v^2\, dx.$$

Argue by contradiction, suppose that there exists a ground state solution (u_0, v_0) in $\mathcal{N}$ of J. Let $t_{\mathrm{per}} > 0$ be such that $(t_{\mathrm{per}} u_0, t_{\mathrm{per}} v_0) \in \mathcal{N}_{\mathrm{per}}$. Since $V_{\mathrm{loc}}(x) > 0$ for almost every $x \in \mathbb{R}^N$, then we get

$$\int_{\mathbb{R}^N} V_{\mathrm{loc}}(x) u_0^2\, dx + \int_{\mathbb{R}^N} V_{\mathrm{loc}}(x) v_0^2\, dx > 0.$$

Then

$$\begin{aligned} c_{\mathrm{per}} = \inf_{(u,v)\in\mathcal{N}_{\mathrm{per}}} J_{\mathrm{per}}(u, v) &\leq J_{\mathrm{per}}(t_{\mathrm{per}} u_0, t_{\mathrm{per}} v_0) \\ &< J(t_{\mathrm{per}} u_0, t_{\mathrm{per}} v_0) \leq J(u_0, v_0) = c. \end{aligned} \tag{3.1}$$

On the other hand, let $(u, v) \in \mathcal{N}_{\mathrm{per}}$ and we denote $(u_y, v_y) = (u(\,\cdot - y), v(\,\cdot - y))$ for $y \in \mathbb{Z}^N$. For each $y \in \mathbb{Z}^N$, let t_y be a number such that $(t_y u_y, t_y v_y) \in \mathcal{N}$. Now observe that

$$\begin{aligned} J_{\mathrm{per}}(u, v) &= J_{\mathrm{per}}(u_y, v_y) \geq J_{\mathrm{per}}(t_y u_y, t_y v_y) \\ &= J(t_y u_y, t_y v_y) - \int_{\mathbb{R}^N} V_{\mathrm{loc}}(x)(t_y u_y)^2\, dx - \int_{\mathbb{R}^N} V_{\mathrm{loc}}(x)(t_y v_y)^2\, dx \\ &\geq c - \int_{\mathbb{R}^N} V_{\mathrm{loc}}(x)(t_y u_y)^2\, dx - \int_{\mathbb{R}^N} V_{\mathrm{loc}}(x)(t_y v_y)^2\, dx. \end{aligned}$$

Next, we will prove that

$$\int_{\mathbb{R}^N} V_{\mathrm{loc}}(x)(t_y u_y)^2\, dx + \int_{\mathbb{R}^N} V_{\mathrm{loc}}(x)(t_y v_y)^2\, dx \to 0. \tag{3.2}$$

In fact

$$\int_{\mathbb{R}^N} V_{\mathrm{loc}}(x)(t_y u_y)^2\, dx = t_y^2 \int_{\mathbb{R}^N} V_{\mathrm{loc}}(x + y) u_y^2\, dx$$

and

$$\int_{\mathbb{R}^N} V_{\mathrm{loc}}(x)(t_y v_y)^2\, dx = t_y^2 \int_{\mathbb{R}^N} V_{\mathrm{loc}}(x + y) v_y^2\, dx.$$

It follows from condition (V) that

$$\int_{\mathbb{R}^N} V_{\rm loc}(x+y)u_y^2\,dx \to 0, \quad \text{as } |y|\to\infty,$$

and

$$\int_{\mathbb{R}^N} V_{\rm loc}(x+y)u_y^2\,dx \to 0, \quad \text{as } |y|\to\infty.$$

Since $J_{\rm per}$ is coercive on $\mathcal{N}_{\rm per}$, we have

$$J_{\rm per}(t_y u_y, t_y v_y) = J_{\rm per}(t_y u, t_y v) \le c_{\rm per},$$

which implies that (t_y) is bounded. Then

$$\int_{\mathbb{R}^N} V_{\rm loc}(x)(t_y u_y)^2\,dx = t_y^2 \int_{\mathbb{R}^N} V_{\rm loc}(x+y)u_y^2\,dx \to 0, \quad \text{as } |y|\to\infty, \tag{3.3}$$

and

$$\int_{\mathbb{R}^N} V_{\rm loc}(x)(t_y v_y)^2\,dx = t_y^2 \int_{\mathbb{R}^N} V_{\rm loc}(x+y)v_y^2\,dx \to 0, \text{as } |y|\to\infty. \tag{3.4}$$

Then it follows from (3.3) and (3.4) that (3.2) holds. Therefore

$$J_{\rm per}(u,v) \ge c - \int_{\mathbb{R}^N} V_{\rm loc}(x)(t_y u_y)^2\,dx - \int_{\mathbb{R}^N} V_{\rm loc}(x)(t_y v_y)^2\,dx \to c, \quad \text{as } |y|\to\infty.$$

Taking infimum over all $(u,v)\in\mathcal{N}_{\rm per}$, we have $c_{\rm per}\ge c$, which is a contradiction with (3.1). □

4. Decomposition of bounded Palais–Smale sequences

In this section we obtain a decomposition result of bounded Palais–Smale sequences in the spirit of [4], [7], [11], [16], which is a key step in the proofs of Theorems 1.1 and 1.2 and generalizes Theorem 5.1 in [12].

Proof of Lemma 2.5. Let $G(x,u,v) = F(x,u,v) - \Gamma(x)|u|^q - \Gamma(x)|v|^q$, then it follows from (Γ), (F_1) and (F_2) that for any $\varepsilon>0$, there exists $C(\varepsilon)>0$ such that

$$|G(x,u,v)| \le \varepsilon|(u,v)| + C(\varepsilon)|(u,v)|^p + C|(u,v)|^q. \tag{4.1}$$

Step 1. We may find a subsequence of (u_n,v_n) such that $(u_n,v_n)\rightharpoonup(u_0,v_0)$, where (u_0,v_0) is a critical point of J. Indeed, since (u_n,v_n) is a bounded Palais–Smale sequence, then there exists $(u_0,v_0)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ such that $(u_n,v_n)\rightharpoonup(u_0,v_0)$ and $(u_n(x),v_n(x))\to(u_0(x),v_0(x))$ for almost every $x\in\mathbb{R}^N$. Let $\varphi,\psi\in C_0^\infty(\mathbb{R}^N)$, then

$$\begin{aligned}
&\langle J'(u_n,v_n),(\varphi,\psi)\rangle - \langle J'(u_0,v_0),(\varphi,\psi)\rangle \\
&\quad= \int_{\mathbb{R}^N} \nabla(u_n-u_0)\nabla\varphi\,dx + \int_{\mathbb{R}^N} V(x)(u_n-u_0)\varphi\,dx \\
&\qquad+ \int_{\mathbb{R}^N} \nabla(v_n-v_0)\nabla\psi\,dx + \int_{\mathbb{R}^N} V(x)(v_n-v_0)\varphi\,dx
\end{aligned} \tag{4.2}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^N} (G_u(x, u_n, v_n) - G_u(x, u_0, v_0))\varphi\, dx \\
&- \int_{\mathbb{R}^N} (G_v(x, u_n, v_n) - G_v(x, u_0, v_0))\psi\, dx \\
= &\int_{\operatorname{supp}\varphi} \nabla(u_n - u_0)\nabla\varphi\, dx + \int_{\operatorname{supp}\varphi} V(x)(u_n - u_0)\varphi\, dx \\
&+ \int_{\operatorname{supp}\psi} \nabla(v_n - v_0)\nabla\psi\, dx + \int_{\operatorname{supp}\psi} V(x)(v_n - v_0)\psi\, dx \\
&- \int_{\operatorname{supp}\varphi} (G_u(x, u_n, v_n) - G_u(x, u_0, v_0))\varphi\, dx \\
&- \int_{\operatorname{supp}\psi} (G_v(x, u_n, v_n) - G_v(x, u_0, v_0))\psi\, dx.
\end{aligned}
$$

Since $(u_n, v_n) \rightharpoonup (u_0, v_0)$, we have

$$
\int_{\operatorname{supp}\varphi} \nabla(u_n - u_0)\nabla\varphi\, dx + \int_{\operatorname{supp}\psi} \nabla(v_n - v_0)\nabla\psi\, dx \to 0, \quad \text{as } n \to \infty. \tag{4.3}
$$

It follows from the Vitali convergence theorem that

$$
\int_{\operatorname{supp}\varphi} V(x)(u_n - u_0)\varphi\, dx + \int_{\operatorname{supp}\psi} V(x)(v_n - v_0)\varphi\, dx, \quad \text{as } n \to \infty. \tag{4.4}
$$

On the other hand, from (4.1) and the Hölder inequality, we obtain

$$
\begin{aligned}
&\int_{\operatorname{supp}\varphi} G_u(x, u_n, v_n)\varphi\, dx \\
&\qquad \le C\big(\|(u_n, v_n)\|_2\|\chi_E\|_2 + \|(u_n, v_n)\|_p^{p-1}\|\chi_E\|_p + \|(u_n, v_n)\|_q^{q-1}\|\chi_E\|_q\big)
\end{aligned}
$$

for a measurable set $E \subset \operatorname{supp}\varphi$. Therefore

$$
\int_{\operatorname{supp}\varphi} (G_u(x, u_n, v_n) - G_u(x, u_0, v_0))\varphi\, dx \to 0, \quad \text{as } n \to \infty, \tag{4.5}
$$

and

$$
\int_{\operatorname{supp}\psi} (G_v(x, u_n, v_n) - G_v(x, u_0, v_0))\psi\, dx \to 0, \quad \text{as } n \to \infty. \tag{4.4}
$$

Thus, it follows from (4.3)–(4.6) that (4.2) holds. Then

$$
\langle J'(u_n, v_n), (\varphi, \psi)\rangle - \langle J'(u_0, v_0), (\varphi, \psi)\rangle, \quad \text{as } n \to \infty.
$$

Since (u_n, v_n) is a Palais–Smale sequence, we have $\langle J'(u_0, v_0), (\varphi, \psi)\rangle = 0$ for any $\varphi, \psi \in C_0^\infty(\mathbb{R}^N)$, that is, $J'(u_0, v_0) = 0$.

Step 2. Let $(u_n^1, v_n^1) = (u_n - u_0, v_n - v_0)$. Suppose that

$$
\sup_{z \in \mathbb{R}^N} \int_{B(z,1)} |(u_n^1, v_n^1)|^2\, dx \to 0, \quad \text{as } n \to \infty. \tag{4.7}
$$

Then $(u_n, v_n) \to (u_0, v_0)$ and (a)–(e) hold for $l = 0$. In fact

$$\begin{aligned}\langle J'(u_n, v_n),(u_n^1, v_n^1)\rangle = &\int_{\mathbb{R}^N} |\nabla u_n^1|^2\,dx + \int_{\mathbb{R}^N} \nabla u_0 \nabla u_n^1\,dx \\ &+ \int_{\mathbb{R}^N} V(x)|u_n^1|^2\,dx + \int_{\mathbb{R}^N} V(x) u_0 u_n^1\,dx + \int_{\mathbb{R}^N} |\nabla v_n^1|^2\,dx \\ &+ \int_{\mathbb{R}^N} \nabla v_0 \nabla v_n^1\,dx + \int_{\mathbb{R}^N} V(x)|v_n^1|^2\,dx + \int_{\mathbb{R}^N} V(x) v_0 v_n^1\,dx \\ &- \int_{\mathbb{R}^N} G_u(x, u_n, v_n) u_n^1\,dx - \int_{\mathbb{R}^N} G_v(x, u_n, v_n) v_n^1\,dx.\end{aligned}$$

Hence

$$\begin{aligned}\|(u_n^1, v_n^1)\|^2 = &\langle J'(u_n, v_n), (u_n^1, v_n^1)\rangle - \int_{\mathbb{R}^N} \nabla u_0 \nabla u_n^1\,dx \\ &- \int_{\mathbb{R}^N} V(x) u_0 u_n^1\,dx - \int_{\mathbb{R}^N} \nabla v_0 \nabla v_n^1\,dx - \int_{\mathbb{R}^N} V(x) v_0 v_n^1\,dx \\ &+ \int_{\mathbb{R}^N} G_u(x, u_n, v_n) u_n^1\,dx + \int_{\mathbb{R}^N} G_v(x, u_n, v_n) v_n^1\,dx.\end{aligned}$$

On the other hand, since $\langle J'(u_0, v_0), (u_n^1, v_n^1)\rangle = 0$, we have

$$\begin{aligned}\|(u_n^1, v_n^1)\|^2 = &\langle J'(u_n, v_n), (u_n^1, v_n^1)\rangle \\ &+ \int_{\mathbb{R}^N} (G_u(x, u_n, v_n) - G_u(x, u_n^1, v_n^1)) u_n^1\,dx \\ &+ \int_{\mathbb{R}^N} (G_v(x, u_n, v_n) - G_v(x, u_n^1, v_n^1)) v_n^1\,dx.\end{aligned}$$

It follows from Lions' lemma [16], [26] and (4.7) that $u_n^1 \to 0$ in $L^r(\mathbb{R}^N)$ and $v_n^1 \to 0$ in $L^r(\mathbb{R}^N)$, $r \in (2, 2^*)$. Since (u_n^1, v_n^1) is bounded, we obtain

$$\|\langle J'(u_n, v_n), (u_n^1, v_n^1)\rangle\| \le \|J'(u_n, v_n)\| \|(u_n^1, v_n^1)\| \to 0, \quad \text{as } n \to \infty.$$

Furthermore, from (4.1) and the Hölder inequality, we derive

$$\begin{aligned}\bigg| \int_{\mathbb{R}^N} G_u(x, u_n, v_n) u_n^1\,dx \bigg| \le &\,\varepsilon \|(u_n, v_n)\|_2 \|u_n^1\|_2 \\ &+ C(\varepsilon) \|(u_n, v_n)\|_p^{p-1} \|u_n^1\|_p + C \|(u_n, v_n)\|_q^{q-1} \|u_n^1\|_q.\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^N} G_u(x, u_n, v_n) u_n^1\,dx \to 0, \quad \text{as } n \to \infty,$$

since $p, q \in (2, 2^*)$. In a similar way, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} G_u(x, u_0, v_0) u_n^1\,dx \to 0, \quad \text{as } n \to \infty, \\ &\int_{\mathbb{R}^N} G_v(x, u_n, v_n) v_n^1\,dx \to 0, \quad \text{as } n \to \infty, \\ &\int_{\mathbb{R}^N} G_v(x, u_0, v_0) v_n^1\,dx \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Thus, $(u_n^1, v_n^1) \to (0,0)$, as $n \to \infty$, that is, $(u_n, v_n) \to (u_0, v_0)$, as $n \to \infty$.

Step 3. Suppose that there exists a sequence $(z_n) \subset \mathbb{Z}^N$ such that

$$\liminf_{n\to\infty} \int_{B(z_n, 1+\sqrt{N})} |(u_n^1, v_n^1)|^2 \, dx > 0.$$

Then there exists (ω_1, ω_2) such that

(1) $|z_n| \to \infty$,
(2) $(u_n(\,\cdot\, + z_n), v_n(\,\cdot\, + z_n)) \rightharpoonup (\omega_1, \omega_2) \neq (0,0)$,
(3) $J'_{\rm per}(\omega_1, \omega_2) = 0$.

Indeed, conditions (1) and (2) are standard, so let us concentrate on (3). Let $(x_n^1, x_n^2) = (u_n(\,\cdot\, + z_n), v_n(\,\cdot\, + z_n))$, then similarly as in Step 1, we have

$$\langle J'_{\rm per}(x_n^1, x_n^2), (\varphi, \psi)\rangle - \langle J'_{\rm per}(\omega^1, \omega^2), (\varphi, \psi)\rangle \to 0, \quad \text{as } n \to \infty,$$

for each $\varphi, \psi \in C_0^\infty(\mathbb{R}^N)$. Next, we will prove that $\langle J'_{\rm per}(x_n^1, x_n^2), (\varphi, \psi)\rangle \to 0$, as $n \to \infty$. Observe that

$$\begin{aligned}
\langle J'(u_n, v_n), &(\varphi(\,\cdot\, - z_n), \psi(\,\cdot\, - z_n))\rangle \\
&= \int_{\mathbb{R}^N} \nabla u_n(\,\cdot\, + z_n)\nabla\varphi \, dx + \int_{\mathbb{R}^N} V(x+z_n) u_n(\,\cdot\, + z_n)\varphi \, dx \\
&\quad + \int_{\mathbb{R}^N} \nabla v_n(\,\cdot\, + z_n)\nabla\psi \, dx + \int_{\mathbb{R}^N} V(x+z_n) v_n(\,\cdot\, + z_n)\psi \, dx \\
&\quad - \int_{\mathbb{R}^N} G_u(u_n(\,\cdot\, + z_n), v_n(\,\cdot\, + z_n))\varphi \, dx \\
&\quad - \int_{\mathbb{R}^N} G_v(u_n(\,\cdot\, + z_n), v_n(\,\cdot\, + z_n))\psi \, dx.
\end{aligned}$$

Furthermore, there exists $M > 0$ such that

$$\begin{aligned}
\|\langle J'(u_n, v_n), (\varphi(\,\cdot\, - z_n), \psi(\,\cdot\, - z_n))\rangle\| &\leq \|J'(u_n, v_n)\| \|(\varphi(\,\cdot\, - z_n), \psi(\,\cdot\, - z_n))\| \\
&\leq M\|J'(u_n, v_n)\| \|(\varphi(\,\cdot\, - z_n), \psi(\,\cdot\, - z_n))\|_{H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)} \to 0,
\end{aligned}$$

as $n \to \infty$. Hence

$$\begin{aligned}
o(1) &= \int_{\mathbb{R}^N} \nabla u_n(\,\cdot\, + z_n)\nabla\varphi \, dx \, dx + \int_{\mathbb{R}^N} V(x+z_n) u_n(\,\cdot\, + z_n)\varphi \, dx \\
&\quad + \int_{\mathbb{R}^N} \nabla v_n(\,\cdot\, + z_n)\nabla\psi \, dx \, dx + \int_{\mathbb{R}^N} V(x+z_n) v_n(\,\cdot\, + z_n)\psi \, dx \\
&\quad - \int_{\mathbb{R}^N} G_u(u_n(\,\cdot\, + z_n), v_n(\,\cdot\, + z_n))\varphi \, dx \\
&\quad - \int_{\mathbb{R}^N} G_v(u_n(\,\cdot\, + z_n), v_n(\,\cdot\, + z_n))\psi \, dx,
\end{aligned} \tag{4.8}$$

as $n \to \infty$. It follows from $V_{\rm loc}(x+z_n) \to 0$ as $|z_n| \to \infty$ for almost every $x \in \mathbb{R}^N$, then we have

$$\int_{\mathbb{R}^N} (V(x+z_n) - V_{\rm per}(x+z_n))x_n^1 \varphi\, dx + \int_{\mathbb{R}^N} (V(x+z_n) - V_{\rm per}(x+z_n))x_n^2 \psi\, dx \to 0, \tag{4.9}$$

as $|z_n| \to \infty$.

It follows from (4.8) and (4.9) that

$$\begin{aligned}\langle J'_{\rm per}(x_n^1,x_n^2),(\varphi,\psi)\rangle &= \int_{\mathbb{R}^N} \nabla x_n^1 \nabla \varphi\, dx + \int_{\mathbb{R}^N} V_{\rm per}(x) x_n^1 \varphi\, dx + \int_{\mathbb{R}^N} \nabla x_n^2 \nabla \psi\, dx \\ &\quad + \int_{\mathbb{R}^N} V_{\rm per}(x) x_n^2 \psi\, dx - \int_{\mathbb{R}^N} G_u(x,x_n^1,x_n^2)\varphi\, dx - \int_{\mathbb{R}^N} G_v(x,x_n^1,x_n^2)\psi\, dx \\ &= \int_{\mathbb{R}^N} \nabla x_n^1 \nabla \varphi\, dx + \int_{\mathbb{R}^N} V(x) x_n^1 \varphi\, dx + \int_{\mathbb{R}^N} \nabla x_n^2 \nabla \psi\, dx \\ &\quad + \int_{\mathbb{R}^N} V(x) x_n^2 \psi\, dx - \int_{\mathbb{R}^N} G_u(x,x_n^1,x_n^2)\varphi\, dx - \int_{\mathbb{R}^N} G_v(x,x_n^1,x_n^2)\psi\, dx \\ &\quad - \int_{\mathbb{R}^N} V_{\rm loc}(x) x_n^1 \varphi\, dx - \int_{\mathbb{R}^N} V_{\rm loc}(x) x_n^2 \psi\, dx \to 0,\end{aligned}$$

as $n \to \infty$.

Step 4. Suppose that $m \geq 1$, $(y_n^k) \subset \mathbb{Z}^N$, $(\omega_1^k, \omega_2^k) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ for $1 \leq k \leq m$ such that

$$\begin{gathered}|y_n^k| \to \infty, \quad |y_n^k - y_n^{k'}| \to \infty, \quad \text{as } n \to \infty, \quad \text{for } k \neq k', \\ (u_n(\,\cdot\, + y_n^k), v_n(\,\cdot\, + y_n^k)) \to (\omega_1^k, \omega_2^k) \neq (0,0), \quad \text{for each } 1 \leq k \leq m, \\ J'_{\rm per}(\omega_1^k, \omega_2^k) = 0, \quad \text{for each } 1 \leq k \leq m.\end{gathered}$$

Then: (i) If

$$\sup_{z \in \mathbb{R}^N} \int_{B(z,1)} \bigg| \bigg(u_n - u_0 - \sum_{k=1}^m \omega_1^k(\,\cdot\, - y_n^k), v_n - v_0 - \sum_{k=1}^m \omega_2^k(\,\cdot\, - y_n^k) \bigg) \bigg|^2 dx \to 0,$$

as $n \to \infty$, then

$$\bigg\| \bigg(u_n - u_0 - \sum_{k=1}^m \omega_1^k(\,\cdot\, - y_n^k), v_n - v_0 - \sum_{k=1}^m \omega_2^k(\,\cdot\, - y_n^k) \bigg) \bigg\| \to 0, \quad \text{as } n \to \infty;$$

(ii) If there exists $(z_n) \subset \mathbb{Z}^N$ such that

$$\liminf_{n\to\infty} \int_{B(z_n,1+\sqrt{N})} \bigg| \bigg(u_n - u_0 - \sum_{k=1}^m \omega_1^k(\,\cdot\, - y_n^k), v_n - v_0 - \sum_{k=1}^m \omega_2^k(\,\cdot\, - y_n^k) \bigg) \bigg|^2 dx > 0,$$

then there exists $(\omega_1^{m+1}, \omega_2^{m+1}) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ such that

(a) $|z_n| \to \infty$, $|z_n - y_n^k| \to \infty$, for $1 \leq k \leq m$;
(b) $(u_n(\,\cdot\, + z_n), v_n(\,\cdot\, + z_n)) \rightharpoonup (\omega_1^{m+1}, \omega_2^{m+1}) \neq (0,0)$;

(c) $J'_{\text{per}}(\omega_1^{m+1}, \omega_2^{m+1}) = 0$.

Suppose that

$$\sup_{z\in\mathbb{R}^N} \int_{B(z,1)} \bigg| \bigg(u_n - u_0 - \sum_{k=1}^m \omega_1^k(\,\cdot\, - y_n^k), v_n - v_0 - \sum_{k=1}^m \omega_2^k(\,\cdot\, - y_n^k) \bigg) \bigg|^2 \,dx \to 0,$$

as $n \to \infty$.

Let

$$\xi_n = u_n - u_0 - \sum_{k=1}^m \omega_1^k(\,\cdot\, - y_n^k) \quad \text{and} \quad \eta_n = v_n - v_0 - \sum_{k=1}^m \omega_2^k(\,\cdot\, - y_n^k).$$

Then it follows from Lions' lemma that $\xi_n \to 0$ in $L^r(\mathbb{R}^N)$ and $\eta_n \to 0$ in $L^r(\mathbb{R}^N)$ for $r \in (2, 2^*)$. By a direct computation, we have

$$\begin{aligned}
\langle J'(u_n,v_n), (\xi_n, \eta_n) \rangle &= \int_{\mathbb{R}^N} \nabla\xi_n \nabla\xi_n \,dx + \int_{\mathbb{R}^N} \nabla\xi_n \nabla u_0 \,dx \\
&+ \int_{\mathbb{R}^N} \nabla \bigg(\sum_{k=1}^m \omega_1^k(\,\cdot\, - y_n^k) \bigg) \nabla\xi_n \,dx + \int_{\mathbb{R}^N} V(x)\xi_n^2 \,dx + \int_{\mathbb{R}^N} V(x)\xi_n u_0 \,dx \\
&+ \int_{\mathbb{R}^N} V(x) \bigg(\sum_{k=1}^m \omega_1^k(\,\cdot\, - y_n^k) \bigg) \xi_n \,dx + \int_{\mathbb{R}^N} \nabla\eta_n \nabla\eta_n \,dx + \int_{\mathbb{R}^N} \nabla\eta_n \nabla v_0 \,dx \\
&+ \int_{\mathbb{R}^N} \nabla \bigg(\sum_{k=1}^m \omega_2^k(\,\cdot\, - y_n^k) \bigg) \nabla\eta_n \,dx + \int_{\mathbb{R}^N} V(x)\eta_n^2 \,dx + \int_{\mathbb{R}^N} V(x)\eta_n v_0 \,dx \\
&+ \int_{\mathbb{R}^N} V(x) \bigg(\sum_{k=1}^m \omega_2^k(\,\cdot\, - y_n^k) \bigg) \eta_n \,dx \\
&- \int_{\mathbb{R}^N} G_u(x, u_n, v_n)\xi_n \,dx - \int_{\mathbb{R}^N} G_v(x, u_n, v_n)\eta_n \,dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|(\xi_n, \eta_n)\|^2 = \langle J'(u_n, v_n), (\xi_n, \eta_n) \rangle &- \int_{\mathbb{R}^N} \nabla\xi_n \nabla u_0 \,dx \\
&- \int_{\mathbb{R}^N} \nabla \bigg(\sum_{k=1}^m \omega_1^k(\,\cdot\, - y_n^k) \bigg) \nabla\xi_n \,dx - \int_{\mathbb{R}^N} V(x)\xi_n u_0 \,dx \\
&- \int_{\mathbb{R}^N} V(x) \bigg(\sum_{k=1}^m \omega_1^k(\,\cdot\, - y_n^k) \bigg) \xi_n \,dx - \int_{\mathbb{R}^N} \nabla\eta_n \nabla v_0 \,dx \\
&- \int_{\mathbb{R}^N} \nabla \bigg(\sum_{k=1}^m \omega_2^k(\,\cdot\, - y_n^k) \bigg) \nabla\eta_n \,dx - \int_{\mathbb{R}^N} V(x)\eta_n v_0 \,dx \\
&- \int_{\mathbb{R}^N} V(x) \bigg(\sum_{k=1}^m \omega_2^k(\,\cdot\, - y_n^k) \bigg) \eta_n \,dx \\
&+ \int_{\mathbb{R}^N} G_u(x, u_n, v_n)\xi_n \,dx + \int_{\mathbb{R}^N} G_v(x, u_n, v_n)\eta_n \,dx.
\end{aligned}$$

Since $\langle J'(u_0,v_0),(\xi_n,\eta_n)\rangle = 0$, we obtain

$$\begin{aligned}\|(\xi_n,\eta_n)\|^2 &= \langle J'(u_n,v_n),(\xi_n,\eta_n)\rangle \\ &\quad - \int_{\mathbb{R}^N} G_u(x,u_0,v_0)\xi_n\,dx - \int_{\mathbb{R}^N} G_v(x,u_0,v_0)\eta_n\,dx \\ &\quad - \sum_{k=1}^m \int_{\mathbb{R}^N} \nabla(\omega_1^k(\,\cdot\, - y_n^k))\nabla\xi_n\,dx - \sum_{k=1}^m \int_{\mathbb{R}^N} V_{\rm per}(x)(\omega_1^k(\,\cdot\, - y_n^k))\xi_n\,dx \\ &\quad - \sum_{k=1}^m \int_{\mathbb{R}^N} V_{\rm loc}(x)(\omega_1^k(\,\cdot\, - y_n^k))\xi_n\,dx - \sum_{k=1}^m \int_{\mathbb{R}^N} \nabla(\omega_2^k(\,\cdot\, - y_n^k))\nabla\eta_n\,dx \\ &\quad - \sum_{k=1}^m \int_{\mathbb{R}^N} V_{\rm per}(x)(\omega_2^k(\,\cdot\, - y_n^k))\eta_n\,dx - \sum_{k=1}^m \int_{\mathbb{R}^N} V_{\rm loc}(x)(\omega_2^k(\,\cdot\, - y_n^k))\eta_n\,dx \\ &\quad + \int_{\mathbb{R}^N} G_u(x,u_n,v_n)\xi_n\,dx + \int_{\mathbb{R}^N} G_v(x,u_n,v_n)\eta_n\,dx.\end{aligned}$$

On the other hand, since $J'_{\rm per}(\omega_1^k,\omega_2^k) = 0$, we obtain

$$\begin{aligned}\|(\xi_n,\eta_n)\|^2 &= \langle J'(u_n,v_n),(\xi_n,\eta_n)\rangle \\ &\quad - \sum_{k=1}^m \int_{\mathbb{R}^N} G_u(x,\omega_1^k,\omega_2^k)\xi_n(\,\cdot\, + y_n^k)\,dx \\ &\quad - \sum_{k=1}^m \int_{\mathbb{R}^N} G_v(x,\omega_1^k,\omega_2^k)\eta_n(\,\cdot\, + y_n^k)\,dx \\ &\quad - \sum_{k=1}^m \int_{\mathbb{R}^N} V_{\rm loc}(x)(\omega_1^k(\,\cdot\, - y_n^k))\xi_n\,dx \\ &\quad - \sum_{k=1}^m \int_{\mathbb{R}^N} V_{\rm loc}(x)(\omega_2^k(\,\cdot\, - y_n^k))\eta_n\,dx \\ &\quad + \int_{\mathbb{R}^N} (G_u(x,u_n,v_n) - G_u(x,u_0,v_0))\xi_n\,dx \\ &\quad + \int_{\mathbb{R}^N} (G_v(x,u_n,v_n) - G_v(x,u_0,v_0))\eta_n\,dx.\end{aligned} \tag{4.10}$$

Observe that

$$\|\langle J'(u_n,v_n),(\xi_n,\eta_n)\rangle\| \le \|J'(u_n,v_n)\|\|(\xi_n,\eta_n)\| \to 0, \quad \text{as } n\to\infty.$$

Moreover, it follows from $\|\xi_n\|_r \to 0$ and $\|\eta_n\|_r \to 0$, $r\in(2,2^*)$, that

$$\begin{aligned}\bigg|\int_{\mathbb{R}^N} G_u(x,u_0,v_0)\xi_n\bigg| &\le \varepsilon \int_{\mathbb{R}^N} |(u_0,v_0)||\xi_n|\,dx \\ &\quad + C(\varepsilon)\int_{\mathbb{R}^N} |(u_0,v_0)|^{p-1}|\xi_n|\,dx + C\int_{\mathbb{R}^N} |(u_0,v_0)|^{q-1}|\xi_n|\,dx.\end{aligned}$$

Analogously, we can prove that all the integrals in (4.10) tend to 0. Therefore, $\|(\xi_n,\eta_n)\| \to 0,\ n\to\infty$. That is

$$\bigg\|\bigg(u_n-u_0-\sum_{k=1}^m \omega_1^k(\,\cdot\,-y_n^k), v_n-v_0-\sum_{k=1}^m \omega_2^k(\,\cdot\,-y_n^k)\bigg)\bigg\|\to 0,\quad \text{as } n\to\infty.$$

Suppose that

$$\liminf_{n\to\infty}\int_{B(z_n,1+\sqrt{N})}\bigg|\bigg(u_n-u_0-\sum_{k=1}^m \omega_1^k(\,\cdot\,-y_n^k), v_n-v_0-\sum_{k=1}^m \omega_2^k(\,\cdot\,-y_n^k)\bigg)\bigg|^2\,dx>0,$$

for some $(z_n)\subset\mathbb{Z}^N$. Then (a) and (b) hold similarly as in Step 3. To prove (c), let $(x_n^1,x_n^2)=(u_n(\,\cdot\,+z_n),v_n(\,\cdot\,+z_n))$,

$$\big\langle J'_{\rm per}(x_n^1,x_n^2),(\varphi,\psi)\big\rangle-\langle J'_{\rm per}(\omega^1,\omega^2),(\varphi,\psi)\big\rangle\to 0,\quad \text{as } n\to\infty,$$

for each $\varphi,\psi\in C_0^\infty(\mathbb{R}^N)$, and $\langle J'_{\rm per}(x_n^1,x_n^2),(\varphi,\psi)\rangle\to 0$, as $n\to\infty$, which completes the proof of (c).

Step 5. Conclusion. It follows from Step 1 that $(u_n,v_n)\rightharpoonup(u_0,v_0)$ and $J'(u_0,v_0)=0$, which complete the proof of (1). If (4.7) in Step 2 holds, then $(u_n,v_n)\to(u_0,v_0)$ and Lemma 2.5 holds for $l=0$. Moreover, one has

$$\liminf_{n\to\infty}\int_{B(y_n,1)}|(u_n^1,v_n^1)|^2\,dx>0$$

for some $(y_n)\subset\mathbb{R}^N$. For each $(y_n)\subset\mathbb{R}^N$, there exists $(z_n)\subset\mathbb{R}^N$ such that

$$B(y_n,1)\subset B(z_n,1+\sqrt{N}).$$

Therefore

$$\liminf_{n\to\infty}\int_{B(z_n,1+\sqrt{N})}|(u_n^1,v_n^1)|^2\,dx\ge\liminf_{n\to\infty}\int_{B(y_n,1)}|(u_n^1,v_n^1)|^2\,dx>0.$$

Then it follows from Step 3 that there exists (ω_1,ω_2) such that (1)–(3) hold. Let $y_n^1=z_n$ and $(\omega_1^1,\omega_2^1)=(\omega_1,\omega_2)$. If (i) in Step 4 holds with $m=1$, then (2)–(4) hold. Otherwise (ii) holds and we put $y_n^2=z_n$ and $(\omega_1^2,\omega_2^2)=(\omega_1,\omega_2)$. Then we iterate the Step 4. To complete the proof of (2)–(4), it is sufficient to prove that this procedure will finish after a finite number of steps. Indeed, observe that

$$\begin{aligned}\lim_{n\to\infty}\|(u_n,v_n)\|^2-\|(u_0,v_0)\|^2-\sum_{k=1}^m\|(\omega_1^k,\omega_2^k)\|^2\\=\lim_{n\to\infty}\bigg\|\bigg(u_n-u_0-\sum_{k=1}^m \omega_1^k(\,\cdot\,-y_n^k), v_n-v_0-\sum_{k=1}^m \omega_2^k(\,\cdot\,-y_n^k)\bigg)\bigg\|^2\ge 0\end{aligned}$$

for each $m\ge 1$. Since (ω_1^k,ω_2^k) are critical points of $J_{\rm per}$, then there exists $\eta>0$ such that $\|(\omega_1^k,\omega_2^k)\|\ge\eta>0$, then after a finite number of steps, say l steps, condition (i) from Step 4 will hold.

Step 6. We will show that (5) holds:

$$J(u_n,v_n)\to J(u_0,v_0)+\sum_{k=1}^{l}J_{\rm per}(\omega_1^k,\omega_2^k).$$

Observe that

$$\begin{aligned}J(u_n,v_n)&=\frac12\int_{\mathbb{R}^N}|\nabla u_0|^2\,dx+\frac12\int_{\mathbb{R}^N}|\nabla(u_n-u_0)|^2\,dx\\&\quad+\int_{\mathbb{R}^N}\nabla u_0\nabla(u_n-u_0)\,dx+\frac12\int_{\mathbb{R}^N}V(x)u_0^2\,dx\\&\quad+\frac12\int_{\mathbb{R}^N}V(x)(u_n-u_0)^2\,dx+\int_{\mathbb{R}^N}V(x)u_0(u_n-u_0)\,dx\\&\quad+\frac12\int_{\mathbb{R}^N}|\nabla v_0|^2\,dx+\frac12\int_{\mathbb{R}^N}|\nabla(v_n-v_0)|^2\,dx\\&\quad+\int_{\mathbb{R}^N}\nabla v_0\nabla(v_n-v_0)\,dx+\frac12\int_{\mathbb{R}^N}V(x)v_0^2\,dx\\&\quad+\frac12\int_{\mathbb{R}^N}V(x)(v_n-v_0)^2\,dx+\int_{\mathbb{R}^N}V(x)v_0(v_n-v_0)\,dx\\&\quad-\int_{\mathbb{R}^N}G(x,u_n,v_n)\,dx.\end{aligned}$$

Hence

$$\begin{aligned}J(u_n,v_n)&=J(u_0,v_0)+J_{\rm per}(u_n-u_0,v_n-v_0)+\int_{\mathbb{R}^N}\nabla u_0\nabla(u_n-u_0)\,dx\\&\quad+\frac12\int_{\mathbb{R}^N}V_{\rm loc}(x)(u_n-u_0)^2\,dx+\int_{\mathbb{R}^N}V(x)u_0(u_n-u_0)\,dx\\&\quad+\int_{\mathbb{R}^N}\nabla v_0\nabla(v_n-v_0)\,dx+\frac12\int_{\mathbb{R}^N}V_{\rm loc}(x)(v_n-v_0)^2\,dx\\&\quad+\int_{\mathbb{R}^N}V(x)v_0(v_n-v_0)\,dx+\int_{\mathbb{R}^N}G(x,u_n-u_0,v_n-v_0)\,dx\\&\quad+\int_{\mathbb{R}^N}G(x,u_0,v_0)\,dx,-\int_{\mathbb{R}^N}G(x,u_n,v_n)\,dx.\end{aligned}$$

Therefore, it is sufficient to prove that

$$\int_{\mathbb{R}^N}[G(x,u_n-u_0,v_n-v_0)+G(x,u_0,v_0)-G(x,u_n,v_n)]\,dx\to 0, \tag{4.11}$$

as $n\to\infty$, and

$$J_{\rm per}(u_n-u_0,v_n-v_0)\to\sum_{k=1}^{l}J_{\rm per}(\omega_1^k,\omega_2^k). \tag{4.12}$$

Let us consider the function $L\colon\mathbb{R}^N\times[0,1]\to\mathbb{R}$ given by

$$L(x,t)=G(x,u_n-tu_0,v_n-tv_0).$$

Then

$$G(x,u_n-u_0,v_n-v_0)-G(x,u_n,v_n)=L(x,1)-L(x,0)=\int_0^1\frac{\partial L}{\partial s}(x,s)\,ds.$$

Furthermore,

$$\begin{aligned}(4.13)\qquad &\int_{\mathbb{R}^N}[G(x,u_n-u_0,v_n-v_0)+G(x,u_0,v_0)-G(x,u_n,v_n)]\,dx\\ &=\int_{\mathbb{R}^N}\bigg[\int_0^1\frac{\partial L}{\partial s}(x,s)\,ds+G(x,u_0,v_0)\bigg]\,dx\\ &=\int_{\mathbb{R}^N}\int_0^1\frac{\partial L}{\partial s}(x,s)\,ds\,dx+\int_{\mathbb{R}^N}G(x,u_0,v_0)\,dx\\ &=\int_0^1\int_{\mathbb{R}^N}[-G_u(x,u_n-su_0,v_n-sv_0)u_0\\ &\quad-G_v(x,u_n-su_0,v_n-sv_0)v_0]\,dx\,ds\\ &\quad+\int_{\mathbb{R}^N}G(x,u_0,v_0)\,dx.\end{aligned}$$

Let $E\subset\mathbb{R}^N$ be a measurable set, then it follows from the Hölder inequality that

$$\begin{aligned}\int_E|G_u(x,u_n-su_0,v_n-sv_0)u_0|\,dx\le{}&\varepsilon\int_E|(u_n-su_0,v_n-sv_0)||u_0|\,dx\\ &+C(\varepsilon)\int_E|(u_n-su_0,v_n-sv_0)|^{p-1}|u_0|\,dx\\ &+C\int_E|(u_n-su_0,v_n-sv_0)|^{q-1}|u_0|\,dx\\ \le{}&\varepsilon\|(u_n-su_0,v_n-sv_0)\chi_E\|_2\|u_0\chi_E\|_2\\ &+C(\varepsilon)\|(u_n-su_0,v_n-sv_0)\chi_E\|_p^{p-1}\|u_0\chi_E\|_p\\ &+C\|(u_n-su_0,v_n-sv_0)\chi_E\|_q^{q-1}\|u_0\chi_E\|_q.\end{aligned}$$

Therefore, $G_u(x,u_n-su_0,v_n-sv_0)u_0$ is uniformly integrable and by the Vitali convergence theorem, we derive

$$\begin{aligned}&\int_0^1\int_{\mathbb{R}^N}-G_u(x,u_n-su_0,v_n-sv_0)u_0\,dx\,ds\\ &\qquad\to\int_0^1\int_{\mathbb{R}^N}-G_u(x,u_0-su_0,v_0-sv_0)u_0\,dx\,ds\end{aligned}$$

as $n\to\infty$. Analogously

$$\begin{aligned}&\int_0^1\int_{\mathbb{R}^N}-G_v(x,u_n-su_0,v_n-sv_0)v_0\,dx\,ds\\ &\qquad\to\int_0^1\int_{\mathbb{R}^N}-G_v(x,u_0-su_0,v_0-sv_0)v_0\,dx\,ds\end{aligned}$$

as $n \to \infty$. Furthermore,

$$\begin{aligned}
(4.14)\qquad \int_0^1 \int_{\mathbb{R}^N} &[-G_u(x, u_n - su_0, v_n - sv_0)u_0 \\
&- G_v(x, u_n - su_0, v_n - sv_0)v_0]\, dx\, ds \\
&= \int_{\mathbb{R}^N} \int_0^1 [-G_u(x, u_n - su_0, v_n - sv_0)u_0 \\
&\quad - G_v(x, u_n - su_0, v_n - sv_0)v_0]\, ds\, dx \\
&= \int_{\mathbb{R}^N} \int_0^1 -\frac{\partial}{\partial s}[G(x, u_n - su_0, v_n - sv_0)]\, ds\, dx \\
&= \int_{\mathbb{R}^N} (G(x, 0, 0) - G(x, u_0, v_0))\, dx \\
&= \int_{\mathbb{R}^N} -G(x, u_0, v_0)\, dx.
\end{aligned}$$

Then it follows from (4.13) and (4.14) that (4.11) holds. Next, we prove that (4.12) holds. Observe that

$$\begin{aligned}
J_{\rm per}(u_n - u_0, v_n - v_0) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla(u_n - u_0)|^2\, dx + \int_{\mathbb{R}^N} V_{\rm per}(x)(u_n - u_0)^2\, dx \\
+ \frac{1}{2}\int_{\mathbb{R}^N} |\nabla(v_n - v_0)|^2\, dx + \int_{\mathbb{R}^N} V_{\rm per}(x)(v_n - v_0)^2\, dx \\
- \int_{\mathbb{R}^N} G(x, u_n - u_0, v_n - v_0)\, dx.
\end{aligned}$$

Now we show that

$$\int_{\mathbb{R}^N} G(x, u_n - u_0, v_n - v_0)\, dx \to \sum_{k=1}^{l} \int_{\mathbb{R}^N} G(x, \omega_1^k, \omega_2^k)\, dx, \quad \text{as } n \to \infty.$$

Put $a_m^n = u_n - u_0 - \sum_{k=1}^{m} \omega_1(\,\cdot\, - y_n^k)$ and $b_m^n = v_n - v_0 - \sum_{k=1}^{m} \omega_2(\,\cdot\, - y_n^k)$. Observe that in (4.11) we have proved that

$$\int_{\mathbb{R}^N} [G(x, a_0^n, b_0^n) + G(x, u_0, v_0) - G(x, u_n, v_n)]\, dx \to 0, \quad \text{as } n \to \infty.$$

Taking (a_0^n, b_0^n) instead of (u_n, v_n) and (ω_1^1, ω_2^1) instead of (u_0, v_0), then we obtain

$$(4.15)\qquad \int_{\mathbb{R}^N} [G(x, a_1^n, b_1^n) + G(x, \omega_1^1, \omega_2^1) - G(x, a_0^n, b_0^n)]\, dx \to 0, \quad \text{as } n \to \infty.$$

Now taking (a_1^n, b_1^n) instead of (u_n, v_n) and (ω_1^1, ω_2^2) instead of (u_0, v_0), we obtain

$$(4.16)\qquad \int_{\mathbb{R}^N} [G(x, a_2^n, b_2^n) + G(x, \omega_1^2, \omega_2^2) - G(x, a_1^n, b_1^n)]\, dx \to 0, \quad \text{as } n \to \infty.$$

Then it follows from (4.15) and (4.16) that

$$\int_{\mathbb{R}^N} [G(x, a_2^n, b_2^n) + G(x, \omega_1^2, \omega_2^2) + G(x, \omega_1^1, \omega_2^1) - G(x, a_0^n, b_0^n)]\, dx \to 0,$$

as $n \to \infty$. Repeating this reasoning, putting $(a^n_{l-1}(\,\cdot+y^l_n), b^n_{l-1}(\,\cdot+y^l_n))$ instead of (u_n, v_n) and (ω^l_1, ω^l_2) instead of (u_0, v_0), we obtain

$$\int_{\mathbb{R}^N} [G(x, a^n_l, b^n_l) + G(x, \omega^l_1, \omega^l_2) - G(x, a^n_{l-1}, b^n_{l-1})]\, dx \to 0, \quad \text{as } n \to \infty.$$

Using, already proved convergence, we get, respectively,

$$\int_{\mathbb{R}^N} [G(x, a^n_l, b^n_l) + G(x, \omega^l_1, \omega^l_2) - G(x, a^n_{l-1}, b^n_{l-1})]\, dx \to 0,$$

$$\int_{\mathbb{R}^N} [G(x, a^n_l, b^n_l) + G(x, \omega^l_1, \omega^l_2) + G(x, \omega^{l-1}_1, \omega^{l-1}_2) - G(x, a^n_{l-2}, b^n_{l-2})]\, dx \to 0,$$

. . . .

$$\int_{\mathbb{R}^N} \bigg[G(x, a^n_l, b^n_l) + \sum_{k=1}^{l} G(x, \omega^k_1, \omega^k_2) - G(x, a^n_0, b^n_0)\bigg]\, dx \to 0,$$

as $n \to \infty$. Observe that $(a^n_l, b^n_l) \to (0,0)$, as $n \to \infty$, then

$$\int_{\mathbb{R}^N} G(x, a^n_l, b^n_l)\, dx \to 0, \quad \text{as } n \to \infty.$$

Therefore

$$\int_{\mathbb{R}^N} G(x, u_n - u_0, v_n - v_0)\, dx \to \sum_{k=1}^{l} G(x, \omega^k_1, \omega^k_2)\, dx, \quad \text{as } n \to \infty.$$

Observe that

$$\begin{aligned}\int_{\mathbb{R}^N} &\bigg|\nabla\bigg(u_n - u_0 - \sum_{k=1}^{l} \omega^k_1(\,\cdot - y^k_n)\bigg)\bigg|^2 \\ &+ V(x)\bigg(u_n - u_0 - \sum_{k=1}^{l} \omega^k_1(\,\cdot - y^k_n)\bigg)^2 dx + \int_{\mathbb{R}^N} \bigg|\nabla\bigg(v_n - v_0 - \sum_{k=1}^{l} \omega^k_2(\,\cdot - y^k_n)\bigg)\bigg|^2 \\ &\qquad + V(x)\bigg(v_n - v_0 - \sum_{k=1}^{l} \omega^k_2(\,\cdot - y^k_n)\bigg)^2 dx \to 0,\end{aligned}$$

as $n \to \infty$, which is equivalent to

$$\begin{aligned}\int_{\mathbb{R}^N} &|\nabla(u_n - u_0)|^2\, dx + \sum_{k=1}^{l} \int_{\mathbb{R}^N} |\nabla \omega^k_1(\,\cdot - y^k_n)|^2\, dx \\ &- 2\int_{\mathbb{R}^N} \sum_{k=1}^{l} \nabla(u_n - u_0)\nabla\omega^k_1(\,\cdot - y^k_n)\, dx \\ &+ 2\sum_{k\neq k'} \int_{\mathbb{R}^N} \nabla\omega^k_1(\,\cdot - y^k_n)\nabla\omega^{k'}_1(\,\cdot - y^k_n)\, dx \\ &+ \int_{\mathbb{R}^N} V_{\mathrm{loc}}(x)\bigg(\nabla\bigg(u_n - u_0 - \sum_{k=1}^{l} \omega^k_1(\,\cdot - y^k_n)\bigg)\bigg)^2 dx\end{aligned}$$

$$+\int_{\mathbb{R}^N} V_{\rm per}(x)\bigg(\nabla\bigg(u_n-u_0-\sum_{k=1}^{l}\omega_1^k(\,\cdot\,-y_n^k)\bigg)\bigg)^2\,dx$$

$$+\int_{\mathbb{R}^N}|\nabla(v_n-v_0)|^2\,dx+\sum_{k=1}^{l}\int_{\mathbb{R}^N}|\nabla\omega_2^k(\,\cdot\,-y_n^k)|^2\,dx$$

$$-2\int_{\mathbb{R}^N}\sum_{k=1}^{l}\nabla(v_n-v_0)\nabla\omega_2^k(\,\cdot\,-y_n^k)\,dx$$

$$+2\sum_{k\neq k'}\int_{\mathbb{R}^N}\nabla\omega_2^k(\,\cdot\,-y_n^k)\nabla\omega_2^{k'}(\,\cdot\,-y_n^k)\,dx$$

$$+\int_{\mathbb{R}^N} V_{\rm loc}(x)\bigg(\nabla\bigg(v_n-v_0-\sum_{k=1}^{l}\omega_2^k(\,\cdot\,-y_n^k)\bigg)\bigg)^2\,dx$$

$$+\int_{\mathbb{R}^N} V_{\rm per}(x)\bigg(\nabla\bigg(v_n-v_0-\sum_{k=1}^{l}\omega_2^k(\,\cdot\,-y_n^k)\bigg)\bigg)^2\,dx\to 0,$$

as $n\to\infty$. Note that

$$\int_{\mathbb{R}^N} V_{\rm loc}(x)\bigg(u_n-u_0-\sum_{k=1}^{l}\omega_1^k(\,\cdot\,-y_n^k)\bigg)^2\,dx\ \to 0,\quad \text{as } n\to\infty,$$

$$\int_{\mathbb{R}^N} V_{\rm loc}(x)\bigg(v_n-v_0-\sum_{k=1}^{l}\omega_2^k(\,\cdot\,-y_n^k)\bigg)^2\,dx\ \to 0,\quad \text{as } n\to\infty.$$

On the other hand

$$\bigg|\int_{\mathbb{R}^N} V_{\rm loc}(x)\bigg(u_n-u_0-\sum_{k=1}^{l}\omega_1^k(\,\cdot\,-y_n^k)\bigg)^2\,dx\bigg| \le \|V_{\rm loc}\|_\infty\bigg\|u_n-u_0-\sum_{k=1}^{l}\omega_1^k(\,\cdot\,-y_n^k)\bigg\|_2^2.$$

Furthermore,

$$-2\int_{\mathbb{R}^N}\sum_{k=1}^{l}\nabla(u_n-u_0)\nabla\omega_1^k(\,\cdot\,-y_n^k)\,dx\to-2\int_{\mathbb{R}^N}\sum_{k=1}^{l}|\nabla\omega_1^k(\,\cdot\,-y_n^k)|^2\,dx,$$

as $n\to\infty$. Since

$$\int_{\mathbb{R}^N}\sum_{k=1}^{l}\nabla(u_n-u_0)\nabla\omega_1^k(\,\cdot\,-y_n^k)\,dx=\int_{\mathbb{R}^N}\sum_{k=1}^{l}\nabla(u_n(\,\cdot\,+y_n^k)-u_0(\,\cdot\,+y_n^k))\nabla\omega_1^k\,dx$$

and $u_n(\,\cdot\,+y_n^k)\rightharpoonup\omega_1^k$. Analogously

$$-2\int_{\mathbb{R}^N}\sum_{k=1}^{l}\nabla(v_n-v_0)\nabla\omega_2^k(\,\cdot\,-y_n^k)\,dx\to-2\int_{\mathbb{R}^N}\sum_{k=1}^{l}|\nabla\omega_2^k(\,\cdot\,-y_n^k)|^2dx, n\to\infty.$$

Hence

$$\int_{\mathbb{R}^N} |\nabla(u_n-u_0)|^2\,dx - \sum_{k=1}^{l}\int_{\mathbb{R}^N} |\nabla\omega_1^k(\,\cdot\,-y_n^k)|^2\,dx$$
$$+\,2\sum_{k\neq k'}\int_{\mathbb{R}^N} \nabla\omega_1^k(\,\cdot\,-y_n^k)\nabla\omega_1^{k'}(\,\cdot\,-y_n^k)\,dx$$
$$+\int_{\mathbb{R}^N} V_{\rm per}(x)\bigg(\nabla\bigg(u_n-u_0-\sum_{k=1}^{l}\omega_1^k(\,\cdot\,-y_n^k)\bigg)\bigg)^2\,dx$$
$$+\int_{\mathbb{R}^N} |\nabla(v_n-v_0)|^2\,dx - \sum_{k=1}^{l}\int_{\mathbb{R}^N} |\nabla\omega_2^k(\,\cdot\,-y_n^k)|^2\,dx$$
$$+\,2\sum_{k\neq k'}\int_{\mathbb{R}^N} \nabla\omega_2^k(\,\cdot\,-y_n^k)\nabla\omega_2^{k'}(\,\cdot\,-y_n^k)\,dx$$
$$+\int_{\mathbb{R}^N} V_{\rm per}(x)\bigg(\nabla\bigg(v_n-v_0-\sum_{k=1}^{l}\omega_2^k(\,\cdot\,-y_n^k)\bigg)\bigg)^2\,dx \to 0,$$

as $n\to\infty$. Observe that

$$\int_{\mathbb{R}^N} \nabla\omega_1^k(\,\cdot\,-y_n^k)\nabla\omega_1^{k'}(\,\cdot\,-y_n^k)\,dx = \int_{\mathbb{R}^N} \nabla\omega_1^k\nabla\omega_1^{k'}(\,\cdot\,+y_n^k-y_n^{k'})\,dx \to 0,$$
$$\int_{\mathbb{R}^N} \nabla\omega_2^k(\,\cdot\,-y_n^k)\nabla\omega_2^{k'}(\,\cdot\,-y_n^k)\,dx = \int_{\mathbb{R}^N} \nabla\omega_2^k\nabla\omega_2^{k'}(\,\cdot\,+y_n^k-y_n^{k'})dx \to 0,$$

as $n\to\infty$, since $|y_n^k-y_n^{k'}|\to\infty$ for $k\neq k'$. Then

$$\int_{\mathbb{R}^N} |\nabla(u_n-u_0)|^2\,dx - \sum_{k=1}^{l}\int_{\mathbb{R}^N} |\nabla\omega_1^k(\,\cdot\,-y_n^k)|^2\,dx$$
$$+\int_{\mathbb{R}^N} V_{\rm per}(x)\bigg(u_n-u_0-\sum_{k=1}^{l}\omega_1^k(\,\cdot\,-y_n^k)\bigg)^2\,dx$$
$$+\int_{\mathbb{R}^N} |\nabla(v_n-v_0)|^2\,dx - \sum_{k=1}^{l}\int_{\mathbb{R}^N} |\nabla\omega_2^k(\,\cdot\,-y_n^k)|^2\,dx$$
$$+\int_{\mathbb{R}^N} V_{\rm per}(x)\bigg(v_n-v_0-\sum_{k=1}^{l}\omega_2^k(\,\cdot\,-y_n^k)\bigg)^2\,dx \to 0,$$

as $n\to\infty$, which is equivalent to

$$\int_{\mathbb{R}^N} |\nabla(u_n-u_0)|^2\,dx - \sum_{k=1}^{l}\int_{\mathbb{R}^N} |\nabla\omega_1^k(\,\cdot\,-y_n^k)|^2\,dx + \int_{\mathbb{R}^N} V_{\rm per}(x)(u_n-u_0)^2\,dx$$
$$-\,2\sum_{k=1}^{l}\int_{\mathbb{R}^N} V_{\rm per}(x)(u_n-u_0)\omega_1^k(\,\cdot\,-y_n^k)\,dx$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N} V_{\rm per}(x)\bigg(\sum_{k=1}^{l}\omega_1^k(\,\cdot\,-y_n^k)\bigg)^2\,dx+\int_{\mathbb{R}^N}|\nabla(v_n-v_0)|^2\,dx\\
&-\sum_{k=1}^{l}\int_{\mathbb{R}^N}|\nabla\omega_2^k(\,\cdot\,-y_n^k)|^2\,dx+\int_{\mathbb{R}^N}V_{\rm per}(x)(v_n-v_0)^2\,dx\\
&-2\sum_{k=1}^{l}\int_{\mathbb{R}^N}V_{\rm per}(x)(v_n-v_0)\omega_2^k(\,\cdot\,-y_n^k)\,dx\\
&+\int_{\mathbb{R}^N} V_{\rm per}(x)\bigg(\sum_{k=1}^{l}\omega_2^k(\,\cdot\,-y_n^k)\bigg)^2\,dx\to 0,
\end{aligned}
$$

as $n\to\infty$. In a similar way, we obtain

$$
\begin{aligned}
2\sum_{k=1}^{l}\int_{\mathbb{R}^N}V_{\rm per}(x)(u_n-u_0)\omega_1^k(\,\cdot\,-y_n^k)\,dx&=-2\int_{\mathbb{R}^N}V_{\rm per}(x)(u_n-u_0)^2\,dx+o(1),\\
2\sum_{k=1}^{l}\int_{\mathbb{R}^N}V_{\rm per}(x)(v_n-v_0)\omega_2^k(\,\cdot\,-y_n^k)\,dx&=-2\int_{\mathbb{R}^N}V_{\rm per}(x)(v_n-v_0)^2\,dx+o(1).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\int_{\mathbb{R}^N}|\nabla(u_n-u_0)|^2\,dx-\sum_{k=1}^{l}\int_{\mathbb{R}^N}|\nabla\omega_1^k(\,\cdot\,-y_n^k)|^2\,dx\\
&\quad-\int_{\mathbb{R}^N}V_{\rm per}(x)(u_n-u_0)^2\,dx+\int_{\mathbb{R}^N} V_{\rm per}(x)\bigg(\sum_{k=1}^{l}\omega_1^k(\,\cdot\,-y_n^k)\bigg)^2\,dx\\
&\quad+\int_{\mathbb{R}^N}|\nabla(v_n-v_0)|^2\,dx-\sum_{k=1}^{l}\int_{\mathbb{R}^N}|\nabla\omega_2^k(\,\cdot\,-y_n^k)|^2\,dx\\
&\quad-\int_{\mathbb{R}^N}V_{\rm per}(x)(v_n-v_0)^2\,dx+\int_{\mathbb{R}^N} V_{\rm per}(x)\bigg(\sum_{k=1}^{l}\omega_2^k(\,\cdot\,-y_n^k)\bigg)^2\,dx\to 0,
\end{aligned}
$$

as $n\to\infty$, and

$$
\begin{aligned}
&\int_{\mathbb{R}^N}|\nabla(u_n-u_0)|^2\,dx-\sum_{k=1}^{l}\int_{\mathbb{R}^N}|\nabla\omega_1^k(\,\cdot\,-y_n^k)|^2\,dx\\
&\quad-\int_{\mathbb{R}^N}V_{\rm per}(x)(u_n-u_0)^2\,dx+\sum_{k=1}^{l}\int_{\mathbb{R}^N}V_{\rm per}(x)(\omega_1^k(\,\cdot\,-y_n^k))^2\,dx\\
&\quad+\int_{\mathbb{R}^N}|\nabla(v_n-v_0)|^2dx-\sum_{k=1}^{l}\int_{\mathbb{R}^N}|\nabla\omega_2^k(\,\cdot\,-y_n^k)|^2\,dx\\
&\quad-\int_{\mathbb{R}^N}V_{\rm per}(x)(v_n-v_0)^2\,dx+\sum_{k=1}^{l}\int_{\mathbb{R}^N}V_{\rm per}(x)(\omega_2^k(\,\cdot\,-y_n^k))^2\,dx
\end{aligned}
$$

$$+ 2\sum_{k\neq k'}\int_{\mathbb{R}^N} V_{\rm per}(x)\omega_1^k(\,\cdot\,-y_n^k)\omega_1^k(\,\cdot\,-y_n^{k'})\,dx$$
$$+ 2\sum_{k\neq k'}\int_{\mathbb{R}^N} V_{\rm per}(x)\omega_2^k(\,\cdot\,-y_n^k)\omega_2^k(\,\cdot\,-y_n^{k'})\,dx \to 0,$$

as $n\to\infty$. Note that

$$\int_{\mathbb{R}^N} V_{\rm per}(x)\omega_1^k(\,\cdot\,-y_n^k)\omega_1^k(\,\cdot\,-y_n^{k'})\,dx \to 0,$$
$$\int_{\mathbb{R}^N} V_{\rm per}(x)\omega_2^k(\,\cdot\,-y_n^k)\omega_2^k(\,\cdot\,-y_n^{k'})\,dx \to 0,$$

as $n\to\infty$. Therefore

$$J_{\rm per}(u_n-u_0,v_n-v_0)-\sum_{k=1}^{l}J_{\rm per}(\omega_1^k,\omega_2^k)$$
$$= J_{\rm per}(u_n-u_0,v_n-v_0)-\sum_{k=1}^{l}J_{\rm per}(\omega_1^k(\,\cdot\,-y_n^{k'}),\omega_2^k(\,\cdot\,-y_n^{k'}))\to 0,$$

as $n\to\infty$. Then

$$J_{\rm per}(u_n-u_0,v_n-v_0)\to\sum_{k=1}^{l}J_{\rm per}(\omega_1^k,\omega_2^k),$$

which completes the proof of (4.12). □

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Guofeng Che and Haibo Chen (corresponding author)
School of Mathematics and Statistics
Central South University
Changsha, 410083 Hunan, P.R. CHINA

E-mail address: cheguofeng222@163.com, math_chb@163.com