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SOME PROPERTIES OF SETS, FIXED POINT THEOREMS IN ORDERED PRODUCT SPACES AND APPLICATIONS TO A NONLINEAR SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We study a partial order in product spaces and then present some new properties of sets via the partial order. Based on these properties and monotone iterative technique, we establish some new fixed point theorems in product spaces. As an application, we utilize the main fixed point theorem to study a nonlinear system of fractional differential equations. We get the existence-uniqueness of positive solutions for this system, which complements the existing results of positive solutions for this nonlinear problem in the literature.

1. Introduction

During the past several decades, nonlinear functional analysis has been an active area of research. As an important content of nonlinear functional analysis, nonlinear operator theory has attracted much attention and has been widely studied (see for example [1]–[3], [6], [7], [10], [16], [18], [27], [31]). As we know, nonlinear operator theory is an important theoretical foundation and basic tool of nonlinear sciences, and it is a research field of modern mathematics which

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has profound theories and extensive applications. Nonlinear operator theory has been extensively used to study nonlinear differential equations, integral equations, matrix equations and boundary value problems, etc. In most cases, we deal with these nonlinear problems by means of some fixed point theorems for single operator. However, multiple operators in product spaces can be regarded as an integration, this motivates to consider multiple operators and establish some nonlinear operator theories in product spaces. Let us mention the following papers concerned with fixed point theorems in product spaces: Fora [9], Kuczumow [14], Tan and Xu [21], Ding et al. [8], Wiśnicki [23], Kohlenbach and Leustean [13]. These operators' results have not been widely utilized to study equation problems. The reason is that the conditions are difficult to verify for particular operators.

In this article, we first study one partial order in product spaces, and then present some properties of sets. Using these properties and monotone iterative technique, we establish some new fixed point theorems in product spaces. Here we mainly consider the following operator equation:

$$(x,y) = (A(x,y), B(x,y)). \tag{1.1}$$

Motivated by our works [27], [31], we will establish some existence and uniqueness results of positive solutions for operator equation (1.1), which extend the results of [31] to some degree.

In the last section of this paper, we study a nonlinear system of fractional differential equations. We give the existence-uniqueness of positive solutions for this system, which complements the existence results of positive solutions for this nonlinear problem. Moreover, we note that our main fixed point theorems can be applied easily to many nonlinear problems.

2. Preliminaries and one partial order

Let E be a linear space on a scalar field K. Then the product space $E\times E$ is a linear space with

$$(x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2),$$
$$\lambda(x,y) = (\lambda x, \lambda y), \quad \lambda\in K.$$

If E is a Banach space with the norm $\|\cdot\|_E$, then the product space $E\times E$ is also a Banach space with the norm

$$\|(u,v)\|_{E\times E} = \|u\|_E + \|v\|_E, \quad (u,v)\in E\times E; \tag{2.1}$$

or

$$\|(u,v)\|_{E\times E} = \max\{\|u\|_E, \|v\|_E\}, \quad (u,v)\in E\times E. \tag{2.2}$$

Moreover, the norms (2.1) and (2.2) are equivalent.

For convenience, we recall some definitions, notations and known results which can be found in [1], [7], [10], [27], [31].

Let $(E, \|\cdot\|_E)$ be a real Banach space, by θ we denote the zero element of E. A non-empty closed convex set $P \subset E$ is a cone if it satisfies

(i) if $x \in P$, $r \geq 0$ then $rx \in P$;
(ii) if $x \in P$, $-x \in P$ then $x = \theta$.

Then E is partially ordered by P, i.e. $x \leq y$ if and only if $y - x \in P$. $x < y$ or $y > x$ means that $x \leq y$ and $x \neq y$.

Put $\mathrm{int}(P) = \{x \in P : x \text{ is an interior point of } P\}$. If $\mathrm{int}(P)$ is non-empty, then the cone P is said to be solid. If there is a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\|_E \leq N\|y\|_E$, then P is called *normal*, in this case N is the infimum of such constants and it is called the *normality constant* of P. If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E : x_1 \leq x \leq x_2\}$ is called the order interval between x_1 and x_2.

For all $x, y \in E$, the notation $x \sim y$ means that there are $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Evidently, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e. $h \geq \theta$ and $h \neq \theta$), we define the set $P_h = \{x \in E : x \sim h\}$. It is easy to see that $P_h \subset P$.

2.1. One partial order. In this subsection we consider the product set $P \times P = \{(x, y) : x \geq \theta, y \geq \theta\}$. First, $P \times P$ is a closed convex set in $E \times E$.

Lemma 2.1. *$P \times P$ is a cone in $E \times E$.*

Proof. We only need prove that

(i) if $(x, y) \in P \times P$, $r \geq 0$ then $r(x, y) \in P \times P$;
(ii) if $(x, y), -(x, y) \in P \times P$ then $(x, y) = (\theta, \theta)$.

On the one hand, evidently, $r(x, y) = (rx, ry)$. Since $x, y \geq \theta$, we know that $rx, ry \geq \theta$. So $r(x, y) = (rx, ry) \in P \times P$. On the other hand, if $(x, y) \in P \times P$ then $x, y \in P$ and $-(x, y) = (-x, -y) \in P \times P$ and then $-x, -y \in P$. Since P is a cone in E, we get $x = \theta$, $y = \theta$. That is, $(x, y) = (\theta, \theta)$. Consequently, $P \times P$ is a cone in $E \times E$. □

Remark 2.2. From Lemma 2.1, we obtain that $E \times E$ is partially ordered by $P \times P$. That is,

$$\begin{aligned}(x_1, y_1) \mathrel{\dot{\leq}} (x_2, y_2) \;&\Leftrightarrow\; (x_2, y_2) - (x_1, y_1) = (x_2 - x_1, y_2 - y_1) \in P \times P \\ &\Leftrightarrow\; x_2 \geq x_1,\; y_2 \geq y_1.\end{aligned}$$

We call this partial order *partial-order-I*. Further, we can define monotone operators on the product space $E \times E$, we say that an operator $A\colon E \times E \to E$ is increasing (decreasing) if $(x_1, y_1) \mathrel{\dot{\leq}} (x_2, y_2)$ implies $A(x_1, y_1) \leq A(x_2, y_2)$ (if $(x_1, y_1) \mathrel{\dot{\geq}} (x_2, y_2)$ implies $A(x_1, y_1) \geq A(x_2, y_2)$).

Lemma 2.3. *P is normal if and only if $P\times P$ is normal; and their normality constants coincide.*

Proof. Suppose that P is normal and N is the normality constant of P. For $(x_1,y_1)\mathrel{\dot\le}(x_2,y_2)$, we have $x_2\ge x_1$, $y_2\ge y_1$. Then $\|x_1\|_E\le N\|x_2\|_E$, $\|y_1\|_E\le N\|y_2\|_E$. Thus

$$\|(x_1,y_1)\|_{E\times E}=\|x_1\|_E+\|y_1\|_E\le N(\|x_2\|_E+\|y_2\|_E)=N\|(x_2,y_2)\|_{E\times E};$$

or

$$\begin{aligned}\|(x_1,y_1)\|_{E\times E}&=\max\{\|x_1\|_E,\|y_1\|_E\}\le\max\{N\|x_2\|_E,N\|y_2\|_E\}\\&=N\max\{\|x_2\|_E,\|y_2\|_E\}=N\|(x_2,y_2)\|_{E\times E}.\end{aligned}$$

So, $P\times P$ is normal.

Conversely, suppose that $P\times P$ is normal and N_1 is the normality constant. From the above argument and the definition of normality, we have $N\ge N_1$. For any $x_1\le x_2$, we have $(x_1,\theta)\mathrel{\dot\le}(x_2,\theta)$, and then $\|(x_1,\theta)\|_{E\times E}\le N_1\|(x_2,\theta)\|_{E\times E}$. That is,

$$\|x_1\|_E+\|\theta\|_E\le N_1(\|x_2\|_E+\|\theta\|_E).$$

Thus, $\|x_1\|_E\le N_1\|x_2\|_E$; or $\max\{\|x_1\|_E,\|\theta\|_E\}\le N_1\max\{\|x_2\|_E,\|\theta\|_E\}$. Thus, $\|x_1\|_E\le N_1\|x_2\|_E$. So P is normal and $N_1\ge N$. Moreover, we have $N_1=N$.□

2.2. Some properties of sets by partial-order-I. Given $h_0^{(1)},h_0^{(2)}\ge\theta$ with $h_0^{(1)}\ne\theta$, $h_0^{(2)}\ne\theta$. Let $h_0=(h_0^{(1)},h_0^{(2)})$, then $h_0\in P\times P$. Define the set $\widetilde{P_{h_0}}=\{(x,y):x\in P_{h_0^{(1)}},\ y\in P_{h_0^{(2)}}\}=P_{h_0^{(1)}}\times P_{h_0^{(2)}}$.

Lemma 2.4.

$$\widetilde{P_{h_0}}=\big\{(x,y):\text{there exist }\lambda,\mu>0\text{ such that }\lambda(h_0^{(1)},h_0^{(2)})\mathrel{\dot\le}(x,y)\mathrel{\dot\le}\mu(h_0^{(1)},h_0^{(2)})\big\},$$

where λ and μ depend on x and y.

Proof. Set

$$D=\big\{(x,y):\text{there exist }\lambda,\mu>0\text{ such that }\lambda(h_0^{(1)},h_0^{(2)})\mathrel{\dot\le}(x,y)\mathrel{\dot\le}\mu(h_0^{(1)},h_0^{(2)})\big\}.$$

For $(x,y)\in\widetilde{P_{h_0}}$, we know that $x\in P_{h_0^{(1)}}$, $y\in P_{h_0^{(2)}}$. Then there exist $\lambda_1,\lambda_2,\mu_1,\mu_2>0$ such that $\lambda_1h_0^{(1)}\le x\le\mu_1h_0^{(1)}$ and $\lambda_2h_0^{(2)}\le y\le\mu_2h_0^{(2)}$.

Let $\lambda=\min\{\lambda_1,\lambda_2\}$, $\mu=\max\{\mu_1,\mu_2\}$. Then

$$\begin{aligned}\lambda(h_0^{(1)},h_0^{(2)})&=(\lambda h_0^{(1)},\lambda h_0^{(2)})\mathrel{\dot\le}(x,y)\\&\mathrel{\dot\le}(\mu_1h_0^{(1)},\mu_2h_0^{(2)})\mathrel{\dot\le}(\mu h_0^{(1)},\mu h_0^{(2)})=\mu(h_0^{(1)},h_0^{(2)}).\end{aligned}$$

That is, $(x,y)\in D$ and thus $\widetilde{P_{h_0}}\subset D$. Conversely, if $(x,y)\in D$, then

$$\lambda(h_0^{(1)},h_0^{(2)})=(\lambda h_0^{(1)},\lambda h_0^{(2)})\dot{\leq}(x,y)\dot{\leq}\mu(h_0^{(1)},h_0^{(2)})=(\mu h_0^{(1)},\mu h_0^{(2)}).$$

So we have $\lambda h_0^{(1)}\leq x\leq \mu h_0^{(1)}$, $\lambda h_0^{(2)}\leq y\leq \mu h_0^{(2)}$. That is, $x\in P_{h_0^{(1)}}$, $y\in P_{h_0^{(2)}}$. Hence, $(x,y)\in \widetilde{P_{h_0}}$. Consequently, $D\subset \widetilde{P_{h_0}}$. Therefore, $\widetilde{P_{h_0}}=D=\big\{(x,y):$ there exist $\lambda,\mu>0$ such that $\lambda(h_0^{(1)},h_0^{(2)})\dot{\leq}(x,y)\dot{\leq}\mu(h_0^{(1)},h_0^{(2)})\big\}$. □

Remark 2.5. $\widetilde{P_{h_0}}\subset P\times P$ and $\lambda\widetilde{P_{h_0}}\subset\widetilde{P_{h_0}}$ for all $\lambda\geq 0$.

Lemma 2.6. *Let* $\operatorname{int}(P\times P)$ *be the set of all interior points of* $P\times P$. *Then* $\operatorname{int}(P\times P)=\operatorname{int}(P)\times\operatorname{int}(P)$.

Proof. Suppose $M_0(x_0,y_0)\in\operatorname{int}(P\times P)$, then there exists $r>0$ such that $B(M_0,r)=\{(x,y):\|(x,y)-(x_0,y_0)\|_{E\times E}\leq r\}\subset P\times P$. For any $(x,y)\in B(M_0,r)$, we have $\|(x,y)-(x_0,y_0)\|_{E\times E}\leq r$. That is,

$$\|(x-x_0,y-y_0)\|_{E\times E}=\|x-x_0\|_E+\|y-y_0\|_E\leq r;$$

or

$$\|(x-x_0,y-y_0)\|_{E\times E}=\max\{\|x-x_0\|_E,\|y-y_0\|_E\}\leq r.$$

Thus, $\|x-x_0\|_E\leq r$, $\|y-y_0\|_E\leq r$. Hence, $B(x_0,r)\subset P$, $B(y_0,r)\subset P$. So, $x_0\in\operatorname{int}(P)$, $y_0\in\operatorname{int}(P)$ and thus $(x_0,y_0)\in\operatorname{int}(P)\times\operatorname{int}(P)$.

Conversely, for any $M_0(x_0,y_0)\in\operatorname{int}(P)\times\operatorname{int}(P)$, we know that $x_0\in\operatorname{int}(P)$, $y_0\in\operatorname{int}(P)$. Then there exist $B(x_0,r_1),B(y_0,r_2)\subset P$. Let $r=\min\{r_1,r_2\}$, for any $(x,y)\in B(M_0,r)$, we obtain $\|(x,y)-(x_0,y_0)\|_{E\times E}\leq r$, that is, $\|x-x_0\|_E+\|y-y_0\|_E\leq r$ or $\max\{\|x-x_0\|_E,\|y-y_0\|_E\}\leq r$. So $\|x-x_0\|_E\leq r\leq r_1$, $\|y-y_0\|_E\leq r\leq r_2$. Hence, $x\in B(x_0,r_1)\subset P$, $y\in B(x_0,r_2)\subset P$. Then we get $B(M_0,r)\subset P\times P$. Consequently, (x_0,y_0) is an interior point of $P\times P$, that is, $(x_0,y_0)\in\operatorname{int}(P\times P)$. □

Remark 2.7. If $h_0\in\operatorname{int}(P\times P)$, then $\widetilde{P_{h_0}}=\operatorname{int}(P\times P)$.

3. Fixed point theorems by partial-order-I

In this section we present some new results for operator equation (1.1).

Theorem 3.1. *Let* P *be a normal cone in a Banach space* E *and* $h_0=(h_0^{(1)},h_0^{(2)})\in P\times P$ *with* $h_0^{(1)},h_0^{(2)}\neq\theta$. *Let operators* $A,B\colon P\times P\to P$ *be increasing and satisfy the following conditions:*

(H$_1$) *for* $x,y\in P$, *there exist* $\varphi_1,\varphi_2\colon(0,1)\to(0,1)$ *such that*

$$A(tx,ty)\geq\varphi_1(t)A(x,y),\qquad B(tx,ty)\geq\varphi_2(t)B(x,y),$$

where $\varphi_i(t)>t$, $t\in(0,1)$, $i=1,2$;

(H$_2$) *there exists* $(e_1,e_2)\in\widetilde{P_{h_0}}$ *such that* $A(e_1,e_2)\in P_{h_0^{(1)}}$, $B(e_1,e_2)\in P_{h_0^{(2)}}$.

Then:

(a) $A\colon \widetilde{P_{h_0}} \to P_{h_0^{(1)}}$, $B\colon \widetilde{P_{h_0}} \to P_{h_0^{(2)}}$ *and there exist* $u_0^{(1)}, v_0^{(1)} \in P_{h_0^{(1)}}$, $u_0^{(2)}, v_0^{(2)} \in P_{h_0^{(2)}}$, $r \in (0,1)$ *such that*

$$\begin{gathered} r(v_0^{(1)}, v_0^{(2)}) \dot{\le} (u_0^{(1)}, u_0^{(2)}) \dot{\le} (v_0^{(1)}, v_0^{(2)}), \\ u_0^{(1)} \le A(u_0^{(1)}, u_0^{(2)}) \le v_0^{(1)}, \quad u_0^{(2)} \le B(u_0^{(1)}, u_0^{(2)}) \le v_0^{(2)}; \end{gathered}$$

(b) *operator equation* (1.1) *has a unique solution* (x^*, y^*) *in* $\widetilde{P_{h_0}}$. *In addition, for any given point* $(x_0, y_0) \in \widetilde{P_{h_0}}$, *if*

$$(x_n, y_n) = (A(x_{n-1}, y_{n-1}), B(x_{n-1}, y_{n-1})), \quad n = 1, 2, \ldots,$$

then $\|x_n - x^*\|_E \to 0$, $\|y_n - y^*\|_E \to 0$ *as* $n \to \infty$.

Proof. From (H_1), for any $x, y \in P$, $t \in (0,1)$, we have

$$\begin{aligned} A(x,y) &= A\bigg(t \cdot \frac{1}{t}\,x, t \cdot \frac{1}{t}\,y\bigg) \ge \varphi_1(t) A\bigg(\frac{1}{t}\,x, \frac{1}{t}\,y\bigg), \\ B(x,y) &= B\bigg(t \cdot \frac{1}{t}\,x, t \cdot \frac{1}{t}\,y\bigg) \ge \varphi_2(t) B\bigg(\frac{1}{t}\,x, \frac{1}{t}\,y\bigg), \end{aligned}$$

and thus

$$A\bigg(\frac{1}{t}\,x, \frac{1}{t}\,y\bigg) \le \frac{1}{\varphi_1(t)}\,A(x,y), \qquad B\bigg(\frac{1}{t}\,x, \frac{1}{t}\,y\bigg) \le \frac{1}{\varphi_2(t)}\,B(x,y). \tag{3.1}$$

Since $e_1 \in P_{h_0^{(1)}}$, $e_2 \in P_{h_0^{(2)}}$, $A(e_1, e_2) \in P_{h_0^{(1)}}$, $B(e_1, e_2) \in P_{h_0^{(2)}}$, we can choose sufficiently small numbers $t_i \in (0,1)$ ($i = 1, 2, 3, 4$) such that

$$\begin{aligned} t_1 h_0^{(1)} &\le e_1 \le \frac{1}{t_1}\,h_0^{(1)}, & t_2 h_0^{(2)} &\le e_2 \le \frac{1}{t_2}\,h_0^{(2)}, \\ t_3 h_0^{(1)} &\le A(e_1, e_2) \le \frac{1}{t_3}\,h_0^{(1)}, & t_4 h_0^{(2)} &\le B(e_1, e_2) \le \frac{1}{t_4}\,h_0^{(2)}. \end{aligned} \tag{3.2}$$

Also, for $u \in P_{h_0^{(1)}}$, $v \in P_{h_0^{(2)}}$, there exist $\mu_1, \mu_2 \in (0,1)$ such that

$$\mu_1 h_0^{(1)} \le u \le \frac{1}{\mu_1}\,h_0^{(1)}, \qquad \mu_2 h_0^{(2)} \le v \le \frac{1}{\mu_2}\,h_0^{(2)}. \tag{3.3}$$

Let $\mu = \min\{\mu_1, \mu_2\}$, $t_0 = \min\{t_1, t_2\}$, then $\mu, t_0 \in (0,1)$. From (H_1) and (3.1)–(3.3), we have

$$\begin{aligned} A(u,v) &\ge A(\mu_1 h_0^{(1)}, \mu_2 h_0^{(2)}) \ge A(\mu h_0^{(1)}, \mu h_0^{(2)}) \\ &\ge \varphi_1(\mu) A(h_0^{(1)}, h_0^{(2)}) \ge \varphi_1(\mu) A(t_1 e_1, t_2 e_2) \\ &\ge \varphi_1(\mu) A(t_0 e_1, t_0 e_2) \ge \varphi_1(\mu)\varphi_1(t_0) A(e_1, e_2) \ge \varphi_1(\mu)\varphi_1(t_0) t_0 h_0^{(1)}, \end{aligned}$$

$$\begin{aligned}A(u,v) &\le A\left(\frac{1}{\mu_1}\,h_0^{(1)},\frac{1}{\mu_2}\,h_0^{(2)}\right)\le A\left(\frac{1}{\mu}\,h_0^{(1)},\frac{1}{\mu}\,h_0^{(2)}\right)\\&\le \frac{1}{\varphi_1(\mu)}\,A(h_0^{(1)},h_0^{(2)})\le \frac{1}{\varphi_1(\mu)}\,A\left(\frac{1}{t_1}\,e_1,\frac{1}{t_2}\,e_2\right)\\&\le \frac{1}{\varphi_1(\mu)}\,A\left(\frac{1}{t_0}\,e_1,\frac{1}{t_0}\,e_2\right)\le \frac{1}{\varphi_1(\mu)\varphi_1(t_0)}\,A(e_1,e_2)\le \frac{1}{\varphi_1(\mu)\varphi_1(t_0)t_3}\,h_0^{(1)}.\end{aligned}$$

From (H_1), $\varphi_1(\mu)\varphi_1(t_0)t_3>\mu t_0t_3>0$ and thus $A(u,v)\in P_{h_0^{(1)}}$. Hence, we have $A\colon P_{h_0^{(1)}}\times P_{h_0^{(2)}}\to P_{h_0^{(1)}}$. That is, $A\colon \widetilde{P_{h_0}}\to P_{h_0^{(1)}}$. Similarly, we can prove that $B\colon \widetilde{P_{h_0}}\to P_{h_0^{(2)}}$. Since $h_0=(h_0^{(1)},h_0^{(2)})\in\widetilde{P_{h_0}}$, we obtain

$$Ah_0=A(h_0^{(1)},h_0^{(2)})\in P_{h_0^{(1)}},\qquad Bh_0=B(h_0^{(1)},h_0^{(2)})\in P_{h_0^{(2)}}.$$

So there are $t_5,t_6\in(0,1)$ such that

$$t_5h_0^{(1)}\le A(h_0^{(1)},h_0^{(2)})\le\frac{1}{t_5}\,h_0^{(1)},\qquad t_6h_0^{(2)}\le B(h_0^{(1)},h_0^{(2)})\le\frac{1}{t_6}\,h_0^{(2)}.\tag{3.4}$$

Set $\tau_0=\min\{t_5,t_6\}$, then $\tau_0\in(0,1)$. Note $\tau_0<\varphi_i(\tau_0)<1$, $i=1,2$, so we can choose a positive integer k such that

$$\left(\frac{\varphi_1(\tau_0)}{\tau_0}\right)^k\ge\frac{1}{\tau_0},\qquad\left(\frac{\varphi_2(\tau_0)}{\tau_0}\right)^k\ge\frac{1}{\tau_0}.\tag{3.5}$$

Let $u_0^{(1)}=\tau_0^kh_0^{(1)}$, $u_0^{(2)}=\tau_0^kh_0^{(2)}$, $v_0^{(1)}=\tau_0^kh_0^{(1)}/\tau_0^k$, $v_0^{(2)}=\tau_0^kh_0^{(2)}/\tau_0^k$. Clearly, $u_0^{(1)},v_0^{(1)}\in P_{h_0^{(1)}}$, $u_0^{(2)},v_0^{(2)}\in P_{h_0^{(2)}}$ and

$$u_0^{(1)}=\tau_0^{2k}v_0^{(1)}<v_0^{(1)},\qquad u_0^{(2)}=\tau_0^{2k}v_0^{(2)}<v_0^{(2)}.$$

Take any $r\in(0,\tau_0^{2k}]$, then $r\in(0,1)$ and $u_0^{(1)}\ge rv_0^{(1)}$, $u_0^{(2)}\ge rv_0^{(2)}$. By (H_1), (3.1), (3.4), (3.5), we have

$$\begin{aligned}A(u_0^{(1)},u_0^{(2)})&=A(\tau_0^kh_0^{(1)},\tau_0^kh_0^{(2)})=A(\tau_0\tau_0^{k-1}h_0^{(1)},\tau_0\tau_0^{k-1}h_0^{(2)})\\&\ge\varphi_1(\tau_0)A(\tau_0^{k-1}h_0^{(1)},\tau_0^{k-1}h_0^{(2)})\ge\ldots\ge(\varphi_1(\tau_0))^kA(h_0^{(1)},h_0^{(2)})\\&\ge(\varphi_1(\tau_0))^kt_5h_0^{(1)}\ge(\varphi_1(\tau_0))^k\tau_0h_0^{(1)}\ge\tau_0^kh_0^{(1)}=u_0^{(1)},\end{aligned}$$

$$\begin{aligned}A(v_0^{(1)},v_0^{(2)})&=A\left(\frac{1}{\tau_0^k}\,h_0^{(1)},\frac{1}{\tau_0^k}\,h_0^{(2)}\right)=A\left(\frac{1}{\tau_0}\,\frac{1}{\tau_0^{k-1}}\,h_0^{(1)},\frac{1}{\tau_0}\,\frac{1}{\tau_0^{k-1}}\,h_0^{(2)}\right)\\&\le\frac{1}{\varphi_1(\tau_0)}\,A\left(\frac{1}{\tau_0^{k-1}}\,h_0^{(1)},\frac{1}{\tau_0^{k-1}}\,h_0^{(2)}\right)\le\ldots\le\frac{1}{(\varphi_1(\tau_0))^k}\,A(h_0^{(1)},h_0^{(2)})\\&\le\frac{1}{(\varphi_1(\tau_0))^kt_5}\,h_0^{(1)}\le\frac{1}{(\varphi_1(\tau_0))^k\tau_0}\,h_0^{(1)}\le\frac{1}{\tau_0^k}\,h_0^{(1)}=v_0^{(1)}.\end{aligned}$$

Similarly, we can prove that

$$B(u_0^{(1)},u_0^{(2)})\ge u_0^{(2)},\qquad B(v_0^{(1)},v_0^{(2)})\le v_0^{(2)}.$$

Let $u_0 = (u_0^{(1)}, u_0^{(2)})$, $v_0 = (v_0^{(1)}, v_0^{(2)})$, $\varphi(t) = \min\{\varphi_1(t), \varphi_2(t)\}$. From the above, we have

$$u_0, v_0 \in \widetilde{P_{h_0}}, \quad \varphi(t) \in (t,1), \quad u_0 \le v_0, \quad Au_0 \ge u_0^{(1)}, \quad Av_0 \le v_0^{(1)},$$
$$Bu_0 \ge u_0^{(2)}, \qquad Bv_0 \le v_0^{(2)},$$
$$u_0 = (u_0^{(1)}, u_0^{(2)}) \dot{\ge} (rv_0^{(1)}, rv_0^{(2)}) = r(v_0^{(1)}, v_0^{(2)}) = rv_0.$$

Now, let us define an operator $T\colon E \times E \to E \times E$ by

$$T(x,y) = (A(x,y), B(x,y)).$$

Due to monotonicity of operators A, B, we know that $T\colon P \times P \to P \times P$ is increasing. Further, for any $(x,y) \in P \times P$, $t \in (0,1)$, we have

$$\begin{aligned} T(tx,ty) &= (A(tx,ty), B(tx,ty)) \dot{\ge} (\varphi_1(t)A(x,y), \varphi_2(t)B(x,y)) \\ &\dot{\ge} (\varphi(t)A(x,y), \varphi(t)B(x,y)) = \varphi(t)T(x,y). \end{aligned} \tag{3.6}$$

Moreover,

$$\begin{aligned} Tu_0 &= (Au_0, Bu_0) \dot{\ge} (u_0^{(1)}, u_0^{(2)}) = u_0, \\ Tv_0 &= (Av_0, Bv_0) \dot{\le} (v_0^{(1)}, v_0^{(2)}) = v_0, \\ Tu_0 &= (Au_0, Bu_0) \dot{\le} (Av_0, Bv_0) = Tv_0, \\ Tu_0 &= (Au_0, Bu_0) \dot{\ge} (A(rv_0), B(rv_0)) \\ &\dot{\ge} (\varphi_1(r)Av_0, \varphi_2(r)Bv_0) \dot{\ge} (\varphi(r)Av_0, \varphi(r)Bv_0) = \varphi(r)Tv_0. \end{aligned}$$

Consider the following sequences:

$$u_1 = Tu_0, \ \ldots, \ u_n = Tu_{n-1}, \ldots, \qquad v_1 = Tv_0, \ \ldots, \ v_n = Tv_{n-1}, \ldots$$

Then, $\{u_n\}, \{v_n\} \subset P \times P$. In a usual way, we obtain

$$u_0 \dot{\le} u_1 \dot{\le} \ldots \dot{\le} u_n \dot{\le} \ldots \dot{\le} v_n \dot{\le} \ldots \dot{\le} v_1 \dot{\le} v_0. \tag{3.7}$$

Note that $u_0 \dot{\ge} rv_0$, thus $u_n \dot{\ge} u_0 \dot{\ge} rv_0 \dot{\ge} rv_n$, $n = 1, 2, \ldots$ Define

$$t_n = \sup\{t > 0 : u_n \dot{\ge} tv_n\}, \quad n = 1, 2, \ldots$$

It is clear that $u_n \dot{\ge} t_n v_n$, $n = 1, 2, \ldots$, and thus

$$u_{n+1} \dot{\ge} u_n \dot{\ge} t_n v_n \dot{\ge} t_n v_{n+1}, \quad n = 1, 2, \ldots$$

So, $t_{n+1} \ge t_n$, that is, $\{t_n\}$ is increasing with $\{t_n\} \subset (0,1]$. Let $t_n \to t^*$ as $n \to \infty$, then $t^* = 1$. Otherwise, $0 < t^* < 1$, then from (H_1), (3.6) and $t_n/t^* \le 1$, we easily obtain

$$u_{n+1} = Tu_n \dot{\ge} T(t_n v_n) = T\left(\frac{t_n}{t^*}\, t^* v_n\right) \dot{\ge} \frac{t_n}{t^*}\, T(t^* v_n) \dot{\ge} \frac{t_n}{t^*}\, \varphi(t^*) T v_n.$$

It follows from the definition of t_n that $t_{n+1} \ge t_n \varphi(t^*)/t^*$. Letting $n \to \infty$, we get $t^* \ge \varphi(t^*) > t^*$, this is a contradiction. So, $\lim_{n\to\infty} t_n = 1$.

Next we prove that $\{u_n\}, \{v_n\}$ are Cauchy sequences. For any given natural number p, we have

$$(\theta,\theta)\,\dot{\leq}\,u_{n+p}-u_n\,\dot{\leq}\,v_n-u_n\,\dot{\leq}\,v_n-t_nv_n=(1-t_n)v_n\,\dot{\leq}\,(1-t_n)v_0,$$
$$(\theta,\theta)\,\dot{\leq}\,v_n-v_{n+p}\,\dot{\leq}\,v_n-u_n\,\dot{\leq}\,(1-t_n)v_0.$$

As P is normal, from Lemma 2.3, we know that $P\times P$ is normal. Let $n\to\infty$, then

$$\|u_{n+p}-u_n\|_{E\times E}\ \leq N(1-t_n)\|v_0\|_{E\times E}\to 0,$$
$$\|v_n-v_{n+p}\|_{E\times E}\ \leq N(1-t_n)\|v_0\|_{E\times E}\to 0,$$

where N is the normality constant of P. So, $\{u_n\}, \{v_n\}$ are Cauchy sequences in $E\times E$. Since $E\times E$ is complete, there exist $u^*, v^*\in E\times E$ such that $u_n\to u^*$, $v_n\to v^*$ as $n\to\infty$. By (3.7), we have $u_n\,\dot{\leq}\,u^*\,\dot{\leq}\,v^*\,\dot{\leq}\,v_n$ with $u^*, v^*\in \widetilde{P_{h_0}}$ and $(\theta,\theta)\,\dot{\leq}\,v^*-u^*\,\dot{\leq}\,v_n-u_n\,\dot{\leq}\,(1-t_n)v_0$. Further, we get

$$\|v^*-u^*\|_{E\times E}\leq N(1-t_n)\|v_0\|_{E\times E}\to 0,\quad n\to\infty,$$

which shows that $u^*=v^*$. So we obtain

$$u_{n+1}=Tu_n\,\dot{\leq}\,Tu^*\,\dot{\leq}\,Tv_n=v_{n+1}.$$

Also, we get $u^*=Tu^*$ as $n\to\infty$. That is, u^* is a fixed point of T in $\widetilde{P_{h_0}}$. Let $u^*=(x^*,y^*)$, then

$$(x^*,y^*)=T(x^*,y^*)=(A(x^*,y^*),B(x^*,y^*)).$$

This implies that (x^*,y^*) is a positive solution of operator equation (1.1).

In the sequel, we show that $u^*=(x^*,y^*)$ is the unique positive solution of operator equation (1.1) in $\widetilde{P_{h_0}}$. Suppose $\widetilde{u}$ is any positive solution of operator equation (1.1) in $\widetilde{P_{h_0}}$. It is easy to see that $u^*,\widetilde{u}$ are fixed points of the operator T. Since $u^*,\widetilde{u}\in\widetilde{P_{h_0}}$, there exist positive numbers $\mu_1,\mu_2,\lambda_1,\lambda_2>0$ such that

$$\mu_1h_0\,\dot{\leq}\,u^*\,\dot{\leq}\,\lambda_1h_0,\qquad \mu_2h_0\,\dot{\leq}\,\widetilde{u}\,\dot{\leq}\,\lambda_2h_0.$$

Then we obtain $\widetilde{u}\,\dot{\geq}\,\mu_2h_0\,\dot{\geq}\,\mu_2u^*/\lambda_1$. Let $\widetilde{t}=\sup\{t>0:\widetilde{u}\geq tu^*\}$. We get $0<\widetilde{t}<\infty$, $\widetilde{u}\,\dot{\geq}\,\widetilde{t}u^*$. Next we prove that $\widetilde{t}\geq 1$. If $0<\widetilde{t}<1$, then

$$\widetilde{u}=T\widetilde{u}\,\dot{\geq}\,T(\widetilde{t}u^*)\,\dot{\geq}\,\varphi(\widetilde{t})Tu^*=\varphi(\widetilde{t})u^*.$$

Note that $\varphi(\widetilde{t})>\widetilde{t}$, this contradicts the definition of $\widetilde{t}$. Hence, $\widetilde{t}\geq 1$ and we get $\widetilde{u}\,\dot{\geq}\,\widetilde{t}u^*\,\dot{\geq}\,u^*$. Similarly, we can show that $u^*\geq\widetilde{u}$, thus $\widetilde{u}=u^*$. Therefore, T has a unique fixed point u^* in $\widetilde{P_{h_0}}$. That is to say, operator equation (1.1) has a unique solution in $\widetilde{P_{h_0}}$.

Now we construct successively the sequence

$$(x_n,y_n)=(A(x_{n-1},y_{n-1}),B(x_{n-1},y_{n-1})),\quad n=1,2,\ldots,$$

for any given point $(x_0, y_0) \in \widetilde{P_{h_0}}$. Since $(x_0, y_0) \in \widetilde{P_{h_0}}$, we can choose a sufficiently small number $\lambda_0 \in (0,1)$ such that

$$\lambda_0(h_0^{(1)}, h_0^{(2)}) \dot{\leq} (A(x_0, y_0), B(x_0, y_0)) \dot{\leq} \frac{1}{\lambda_0}(h_0^{(1)}, h_0^{(2)}). \tag{3.8}$$

Since $\lambda_0 < \varphi(\lambda_0) < 1$, there is a positive integer m such that

$$\left(\frac{\varphi(\lambda_0)}{\lambda_0}\right)^m > \frac{1}{\lambda_0}.$$

Let $\overline{u}_0 = \lambda_0^m(x_0, y_0) = (\lambda_0^m x_0, \lambda_0^m y_0)$, $\overline{v}_0 = (x_0, y_0)/\lambda_0^m = (x_0/\lambda_0^m, y_0/\lambda_0^m)$. Clearly, $\overline{u}_0, \overline{v}_0 \in \widetilde{P_{h_0}}$ and $\overline{u}_0 \dot{\leq} (x_0, y_0) \dot{\leq} \overline{v}_0$. Put $\overline{u}_n = T\overline{u}_{n-1}$, $\overline{v}_n = T\overline{v}_{n-1}$, $n = 1, 2, \ldots$ Similarly to the above proof, it follows that there exists $\overline{u}^* \in \widetilde{P_{h_0}}$ such that $T\overline{u}^* = \overline{u}^*$. Due to uniqueness of fixed points of the operator T in $\widetilde{P_{h_0}}$, we get $\overline{u}^* = u^*$. By induction, $\overline{u}_n \dot{\leq} T^n(x_0, y_0) \dot{\leq} \overline{v}_n$, $n = 1, 2, \ldots$ Since the cone $P \times P$ is normal, we have $T^n(x_0, y_0) \to u^*$ as $n \to \infty$. That is,

$$(x_n, y_n) = (A(x_{n-1}, y_{n-1}), B(x_{n-1}, y_{n-1})) \to (x^*, y^*) \quad \text{as } n \to \infty.$$

Evidently, $\|x_n - x^*\|_E \to 0$, $\|y_n - y^*\|_E \to 0$ as $n \to \infty$. □

From the proof of Theorem 3.1, it is ease to obtain the following conclusion.

Corollary 3.2. *Let P be a normal cone in a Banach space E and $h_0^{(1)} \in P$ with $h_0^{(1)} \neq \theta$. Let the operator $A\colon P \times P \to P$ be increasing and satisfy* (H$_1$). *Let there exist $e_0 \in P_{h_0^{(1)}}$ such that $A(e_0, e_0) \in P_{h_0^{(1)}}$. Then*:

(a) *$A\colon P_{h_0^{(1)}} \times P_{h_0^{(1)}} \to P_{h_0^{(1)}}$ and there are $u_0^{(1)}, v_0^{(1)} \in P_{h_0^{(1)}}$ and $r_1 \in (0,1)$ such that*

$$r_1 v_0^{(1)} \leq u_0^{(1)} \leq v_0^{(1)}, \qquad u_0^{(1)} \leq A(u_0^{(1)}, u_0^{(1)}) \leq A(v_0^{(1)}, v_0^{(1)}) \leq v_0^{(1)};$$

(b) *the operator equation $A(x, x) = x$ has a unique solution x^* in $P_{h_0^{(1)}}$. In addition, for any given point $x_0 \in P_{h_0^{(1)}}$, if $x_n = A(x_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$, then $\|x_n - x^*\|_E \to 0$ as $n \to \infty$.*

Theorem 3.3. *Let P be a normal cone in a Banach space E and $h_0 = (h_0^{(1)}, h_0^{(2)}) \in P \times P$ with $h_0^{(1)}, h_0^{(2)} \neq \theta$. Let operators $A, B, C\colon P \times P \to P$ be increasing and satisfy the following conditions*:

(H$_3$) *there exist $\sigma \in (0,1)$, $\delta > 0$ such that, for $x, y \in P$, $t \in (0,1)$,*

$$A(tx, ty) \geq t^\sigma A(x, y), \qquad B(tx, ty) \geq tB(x, y), \qquad A(x, y) \geq \delta B(x, y);$$

(H$_4$) *for $x, y \in P$, $t \in (0,1)$, there exists $\varphi_3(t) \in (t, 1)$ such that*

$$C(tx, ty) \geq \varphi_3(t) C(x, y);$$

(H$_5$) *there exists $(e_1, e_2) \in \widetilde{P_{h_0}}$ such that*

$$A(e_1, e_2) \in P_{h_0^{(1)}}, \qquad B(e_1, e_2) \in P_{h_0^{(1)}}, \qquad C(e_1, e_2) \in P_{h_0^{(2)}}.$$

Then:

(a) *there exist* $u_0^{(1)}, v_0^{(1)} \in P_{h_0^{(1)}}$, $u_0^{(2)}, v_0^{(2)} \in P_{h_0^{(2)}}$, $r \in (0,1)$ *such that*

$$r(v_0^{(1)}, v_0^{(2)}) \dot{\leq} (u_0^{(1)}, u_0^{(2)}) \dot{\leq} (v_0^{(1)}, v_0^{(2)}),$$
$$u_0^{(1)} \leq A(u_0^{(1)}, u_0^{(2)}) + B(u_0^{(1)}, u_0^{(2)}) \leq v_0^{(1)},$$
$$u_0^{(2)} \leq C(u_0^{(1)}, u_0^{(2)}) \leq v_0^{(2)};$$

(b) *the operator equation*

$$(x,y) = (A(x,y) + B(x,y), C(x,y)) \tag{3.9}$$

has a unique solution (x^*, y^*) *in* $\widetilde{P_{h_0}}$. *In addition, for any given point* $(x_0, y_0) \in \widetilde{P_{h_0}}$, *if*

$$(x_n, y_n) = (A(x_{n-1}, y_{n-1}) + B(x_{n-1}, y_{n-1}), C(x_{n-1}, y_{n-1})), \quad n = 1, 2, \ldots,$$

then $\|x_n - x^*\|_E \to 0$, $\|y_n - y^*\|_E \to 0$ *as* $n \to \infty$.

PROOF. Define an operator $F\colon P \times P \to P$ by $F(x,y) = A(x,y) + B(x,y)$. From Theorem 3.1, we only need to prove that F satisfies conditions (H_1), (H_2). As A, B are increasing, we know that F is increasing. From (H_5), $F(e_1, e_2) = A(e_1,e_2)+B(e_1,e_2) \in P_{h_0^{(1)}}$. In the sequel, we show that there exists $\varphi(t) \in (t,1)$, $t \in (0,1)$, such that

$$F(tx, ty) \geq \varphi(t) F(x,y), \quad \text{for all } x, y \in P.$$

Consider the following function:

$$f(t,\beta) = \frac{t^\beta - t}{t^\sigma - t^\beta}, \quad t \in (0,1), \ \beta \in (\sigma, 1).$$

One can easily prove that $f(t,\beta)$ is increasing in $t \in (0,1)$ for fixed $\beta \in (\sigma,1)$, and

$$\lim_{t\to 0^+} f(t,\beta) = 0, \qquad \lim_{t \to 1^-} f(t,\beta) = \frac{1-\beta}{\beta - \sigma}.$$

Further, for fixed $t \in (0,1)$, we get

$$\lim_{\beta \to 1^-} f(t,\beta) = \lim_{\beta\to 1^-} \frac{t^\beta - t}{t^\sigma - t^\beta} = 0.$$

So there is $\beta_0(t) \in (\sigma, 1)$ depending on t such that

$$\frac{t^{\beta_0(t)} - t}{t^\sigma - t^{\beta_0(t)}} \leq \delta, \quad t \in (0,1).$$

Hence, from (H_3) we get

$$A(x,y) \geq \delta B(x,y) \geq \frac{t^{\beta_0(t)} - t}{t^\sigma - t^{\beta_0(t)}} B(x,y), \quad \text{for all } t \in (0,1), \ x, y \in P.$$

Then

$$t^\sigma A(x,y) + tB(x,y) \geq t^{\beta_0(t)}[A(x,y) + B(x,y)], \quad \text{for all } t \in (0,1), \ x, y \in P.$$

Consequently, for any $t \in (0,1)$ and $x, y \in P$,

$$\begin{aligned} F(tx,ty) &= A(tx,ty) + B(tx,ty) \geq t^{\sigma} A(x,y) + tB(x,y) \\ &\geq t^{\beta_0(t)}[A(x,y) + B(x,y)] = t^{\beta_0(t)} F(x,y). \end{aligned}$$

Set $\varphi(t) = t^{\beta_0(t)}$, $t \in (0,1)$. Then $\varphi(t) \in (t,1)$ and $F(tx,ty) \geq \varphi(t)F(x,y)$ for any $t \in (0,1)$ and $x, y \in P$. Hence condition (H_1) in Theorem 3.1 is satisfied. By Theorem 3.1, we have the following conclusions:

(i) there exist $u_0^{(1)}, v_0^{(1)} \in P_{h_0^{(1)}}$, $u_0^{(2)}, v_0^{(2)} \in P_{h_0^{(2)}}$, $r \in (0,1)$ such that

$$r(v_0^{(1)}, v_0^{(2)}) \mathrel{\dot{\leq}} (u_0^{(1)}, u_0^{(2)}) \mathrel{\dot{\leq}} (v_0^{(1)}, v_0^{(2)}),$$
$$u_0^{(1)} \leq F(u_0^{(1)}, u_0^{(2)}) \leq v_0^{(1)}, \qquad u_0^{(2)} \leq C(u_0^{(1)}, u_0^{(2)}) \leq v_0^{(2)};$$

(ii) the operator equation $(x,y) = (F(x,y), C(x,y))$ has a unique solution (x^*, y^*) in $\widetilde{P_{h_0}}$. Moreover, for any $(x_0, y_0) \in \widetilde{P_{h_0}}$, if

$$(x_n, y_n) = (F(x_{n-1}, y_{n-1}), C(x_{n-1}, y_{n-1})), \quad n = 1, 2, \ldots,$$

then $\|x_n - x^*\|_E \to 0$, $\|y_n - y^*\|_E \to 0$ as $n \to \infty$. That is, the conclusion of Theorem 3.3 holds. □

Theorem 3.4. *Assume that all conditions of Theorem* 3.1 *hold. Then, for any given* $\lambda, \mu > 0$, *the operator equation*

$$(x,y) = (\lambda A(x,y), \mu B(x,y)) \tag{3.10}$$

has a unique solution $(x^*_{\lambda,\mu}, y^*_{\lambda,\mu})$ *in* $\widetilde{P_{h_0}}$. *In addition, for any given point* $(x_0, y_0) \in \widetilde{P_{h_0}}$, *if*

$$(x_n, y_n) = (\lambda A(x_{n-1}, y_{n-1}), \mu B(x_{n-1}, y_{n-1})), \quad n = 1, 2, \ldots,$$

then $\|x_n - x^*_{\lambda,\mu}\|_E \to 0, \|y_n - y^*_{\lambda,\mu}\|_E \to 0$ *as* $n \to \infty$.

Proof. Let $A_\lambda = \lambda A$, $B_\mu = \mu B$ for $\lambda, \mu > 0$. Then operators A_λ, B_μ satisfy (H_1), (H_2). From Theorem 3.1, the operator equation $(x,y) = (A_\lambda(x,y), B_\mu(x,y))$ has a unique solution $(x^*_{\lambda,\mu}, y^*_{\lambda,\mu})$ in $\widetilde{P_{h_0}}$. Moreover, for any initial point $(x_0, y_0) \in \widetilde{P_{h_0}}$, constructing successively the sequence $(x_n, y_n) = (A_\lambda(x_{n-1}, y_{n-1}), B_\mu(x_{n-1}, y_{n-1}))$, $n = 1, 2, \ldots$, we have $\|x_n - x^*_{\lambda,\mu}\|_E \to 0$, $\|y_n - y^*_{\lambda,\mu}\|_E \to 0$ as $n \to \infty$. That is, the operator equation $(x,y) = (\lambda A(x,y), \mu B(x,y))$ has a unique solution $(x^*_{\lambda,\mu}, y^*_{\lambda,\mu})$ in $\widetilde{P_{h_0}}$. Further, for any $(x_0, y_0) \in \widetilde{P_{h_0}}$, constructing the sequence $(x_n, y_n) = (\lambda A(x_{n-1}, y_{n-1}), \mu B(x_{n-1}, y_{n-1}))$, $n = 1, 2, \ldots$, we have $\|x_n - x^*_{\lambda,\mu}\|_E \to 0$, $\|y_n - y^*_{\lambda,\mu}\|_E \to 0$ as $n \to \infty$. □

From Remark 2.7 and Lemma 2.6, we the following conclusions which are similar to Theorem 3.1, Corollary 3.2, Theorems 3.3 and 3.4 can be established.

THEOREM 3.5. *Let P be a normal cone in a Banach space E. Let operators $A, B\colon \operatorname{int}(P \times P) \to \operatorname{int}(P)$ be increasing and satisfy* (H$_1$) *for $x, y \in \operatorname{int}(P)$, $t \in (0,1)$. Then*:

(a) *there exist $u_0^{(1)}, v_0^{(1)}, u_0^{(2)}, v_0^{(2)} \in \operatorname{int}(P)$, $r \in (0,1)$ such that*

$$r(v_0^{(1)}, v_0^{(2)}) \dot{\leq} (u_0^{(1)}, u_0^{(2)}) \dot{\leq} (v_0^{(1)}, v_0^{(2)}),$$
$$u_0^{(1)} \leq A(u_0^{(1)}, u_0^{(2)}) \leq v_0^{(1)}, \qquad u_0^{(2)} \leq B(u_0^{(1)}, u_0^{(2)}) \leq v_0^{(2)};$$

(b) *operator equation* (1.1) *has a unique solution (x^*, y^*) in $\operatorname{int}(P \times P)$. In addition, for any given point $(x_0, y_0) \in \operatorname{int}(P \times P)$, if*

$$(x_n, y_n) = (A(x_{n-1}, y_{n-1}), B(x_{n-1}, y_{n-1})), \quad n = 1, 2, \ldots,$$

then $\|x_n - x^*\|_E \to 0, \|y_n - y^*\|_E \to 0$ as $n \to \infty$.

THEOREM 3.6. *Let P be a normal cone in a Banach space E. Let the operator $A\colon \operatorname{int}(P \times P) \to \operatorname{int}(P)$ be increasing and satisfy* (H$_1$). *Then*:

(a) *there are $u_0^{(1)}, v_0^{(1)} \in \operatorname{int}(P)$ and $r_1 \in (0,1)$ such that*

$$r_1 v_0^{(1)} \leq u_0^{(1)} \leq v_0^{(1)}, \qquad u_0^{(1)} \leq A(u_0^{(1)}, u_0^{(1)}) \leq A(v_0^{(1)}, v_0^{(1)}) \leq v_0^{(1)};$$

(b) *the operator equation $A(x,x) = x$ has a unique solution x^* in $\operatorname{int}(P)$. In addition, for any given point $x_0 \in \operatorname{int}(P)$, if $x_n = A(x_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$, then $\|x_n - x^*\|_E \to 0$ as $n \to \infty$.*

THEOREM 3.7. *Let P be a normal cone in a Banach space E. Let operators $A, B, C\colon \operatorname{int}(P \times P) \to \operatorname{int}(P)$ be increasing and satisfy* (H$_3$), (H$_4$) *for $x, y \in \operatorname{int}(P)$, $t \in (0,1)$. Then*:

(a) *there exist $u_0^{(1)}, v_0^{(1)}, u_0^{(2)}, v_0^{(2)} \in \operatorname{int}(P)$, $r \in (0,1)$ such that*

$$r(v_0^{(1)}, v_0^{(2)}) \dot{\leq} (u_0^{(1)}, u_0^{(2)}) \dot{\leq} (v_0^{(1)}, v_0^{(2)}),$$
$$u_0^{(1)} \leq A(u_0^{(1)}, u_0^{(2)}) + B(u_0^{(1)}, u_0^{(2)}) \leq v_0^{(1)}, \qquad u_0^{(2)} \leq C(u_0^{(1)}, u_0^{(2)}) \leq v_0^{(2)};$$

(b) *the operator equation*

$$(x, y) = (A(x,y) + B(x,y), C(x,y))$$

has a unique solution (x^*, y^*) in $\operatorname{int}(P \times P)$. In addition, for any given point $(x_0, y_0) \in \operatorname{int}(P \times P)$, if

$$(x_n, y_n) = (A(x_{n-1}, y_{n-1}) + B(x_{n-1}, y_{n-1}), C(x_{n-1}, y_{n-1})), \quad n = 1, 2, \ldots,$$

then $\|x_n - x^*\|_E \to 0$, $\|y_n - y^*\|_E \to 0$ as $n \to \infty$.

THEOREM 3.8. *Assume that all conditions of Theorem* 3.5 *hold. Then, for any given $\lambda, \mu > 0$, the operator equation $(x, y) = (\lambda A(x,y), \mu B(x,y))$ has a* *unique solution* $(x^*_{\lambda,\mu}, y^*_{\lambda,\mu})$ *in* $\mathrm{int}(P \times P)$. *In addition, for any given point* $(x_0, y_0) \in \mathrm{int}(P \times P)$, *if*

$$(x_n, y_n) = (\lambda A(x_{n-1}, y_{n-1}), \mu B(x_{n-1}, y_{n-1})), \quad n = 1, 2, \ldots,$$

then $\|x_n - x^*_{\lambda,\mu}\|_E \to 0$, $\|y_n - y^*_{\lambda,\mu}\|_E \to 0$ *as* $n \to \infty$.

Remark 3.9. As we know, the condition of upper-lower solutions is difficult to verify for particular operators. So the condition was required directly in many known results. Here we do not suppose the condition and we give the iterative forms. Moreover, the existence of a unique solution for (1.1) has not been studied in literature.

4. Applications

Many problems that arise from differential equations, integral equations, nonlinear matrix equations and boundary value problems, etc., have been studied via various operator equations but none results were obtained via operator equation (1.1). In this section, we apply the main result of Section 3 to study a nonlinear system of fractional differential equations. Let D^α_{0+} be the Riemann–Liouville fractional derivative of order $\alpha > 0$, defined by

$$D^\alpha_{0+} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-\alpha-1} y(\tau)\, d\tau,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, see [20].

We study the existence-uniqueness of positive solutions for the following system of nonlinear fractional differential equations:

$$\begin{cases} -D^{\nu_1}_{0+} y_1(t) = f(t, y_1(t), y_2(t)), \\ -D^{\nu_2}_{0+} y_2(t) = g(t, y_1(t), y_2(t)), \end{cases} \tag{4.1}$$

where $t \in (0, 1)$, $\nu_1, \nu_2 \in (n-1, n]$ for $n > 3$ and $n \in N$, subject to a couple of boundary conditions

$$y_1^{(i)}(0) = 0 = y_2^{(i)}(0), \qquad 0 \le i \le n-2, \tag{4.2}$$

$$[D^\alpha_{0+} y_1(t)]_{t=1} = 0 = [D^\alpha_{0+} y_2(t)]_{t=1}, \quad 1 \le \alpha \le n-2. \tag{4.3}$$

In the recent years, many fractional differential equations that arise from physics, mechanics, chemistry, engineering and biological sciences (see [12], [17], [19]) have been studied (see the papers [5], [11], [15], [20], [22], [24] and the references therein). In many papers, the authors have investigated the existence of positive solutions for nonlinear fractional differential equation boundary value problems. In addition, the uniqueness of positive solutions for nonlinear fractional differential equation boundary value problems has been considered by several authors, see [25]–[30] for example. However, there are few papers concerned with the uniqueness of positive solutions for systems of nonlinear fractional differential equations. In this section, we apply Theorem 3.1 to study the system of nonlinear fractional differential equations (4.1) with boundary conditions (4.2) and (4.3).

LEMMA 4.1 (see [11]). *If* $g \in C[0,1]$, *then the solution for the problem* $-D^{\nu}_{0+}y(t) = g(t)$ *with boundary conditions* $y^{(i)}(0) = 0 = [D^{\alpha}_{0+}y(t)]_{t=1}$, *where* $1 \le \alpha \le n-2$ *and* $0 \le i \le n-2$, *is*

$$y(t) = \int_0^1 G(t,s)g(s)\,ds,$$

where

$$G(t,s) = \begin{cases} \dfrac{t^{\nu-1}(1-s)^{\nu-\alpha-1} - (t-s)^{\nu-1}}{\Gamma(\nu)}, & 0 \le s \le t \le 1, \\ \dfrac{t^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)}, & 0 \le t \le s, \le 1, \end{cases}$$

is the Green function for this problem.

LEMMA 4.2 (see [11], [28]). *Let* G *be as given in the statement of Lemma* 4.1. *Then*:

(a) G *is a continuous function on the unit square* $[0,1]\times[0,1]$;
(b) $G(t,s) \ge 0$ *for every* $(t,s) \in [0,1]\times[0,1]$ *and, for* $t,s \in [0,1]$,

$$[1-(1-s)^{\alpha}](1-s)^{\nu-\alpha-1}t^{\nu-1} \le \Gamma(\nu)G(t,s) \le (1-s)^{\nu-\alpha-1}t^{\nu-1}.$$

In the following, set $E = C[0,1]$, the Banach space of continuous functions on $[0,1]$ with the norm $\|y\| = \max\{|y(t)| : t \in [0,1]\}$. Let $P = \{y \in C[0,1] : y(t) \ge 0,\ t \in [0,1]\}$. Then P is a normal cone with the normality constant 1. The partial ordering defined by P is given by $x \le y \Leftrightarrow x(t) \le y(t)$ for all $t \in [0,1]$.

THEOREM 4.3. *Assume that*:

(D_1) $f,g \in C([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty))$ *and* $f(t,0,0), g(t,0,0) \not\equiv 0$;
(D_2) $f(t,u_1,v_1) \le f(t,u_2,v_2)$, $g(t,u_1,v_1) \le g(t,u_2,v_2)$, *for any* $t \in [0,1]$, $u_2 \ge u_1 \ge 0$, $v_2 \ge v_1 \ge 0$;
(D_3) *for any* $\lambda \in (0,1)$, *there exist* $\varphi_i(\lambda) \in (\lambda,1)$, $i=1,2$, *such that*

$$f(t,\lambda u,\lambda v) \ge \varphi_1(\lambda)f(t,u,v), \qquad g(t,\lambda u,\lambda v) \ge \varphi_2(\lambda)g(t,u,v)$$

for $t \in [0,1]$, $u,v \in [0,+\infty)$.

Then:

(a) *there exist* $x_0^{(1)}, y_0^{(1)} \in P_{t^{\nu_1-1}}$, $x_0^{(2)}, y_0^{(2)} \in P_{t^{\nu_2-1}}$ *and* $r \in (0,1)$ *such that* $r(y_0^{(1)},y_0^{(2)}) \dot{\le} (x_0^{(1)},x_0^{(2)}) \dot{\le} (y_0^{(1)},y_0^{(2)})$ *and*

$$x_0^{(1)}(t) \le \int_0^1 G_1(t,s)f(s,x_0^{(1)}(s),x_0^{(2)}(s))\,ds \le y_0^{(1)}(s), \quad t \in [0,1],$$

$$x_0^{(2)}(t) \le \int_0^1 G_2(t,s) g(s, x_0^{(1)}(s), x_0^{(2)}(s))\, ds \le y_0^{(2)}(s), \quad t \in [0,1],$$

where G_i, $i = 1,2$, are the Green functions of Lemma 4.1 *with ν replaced by ν_i, $i = 1,2$;*

(b) *problem* (4.1)–(4.3) *has a unique positive solution (x^*, y^*) in $\widetilde{P_{h_0}}$, where $h_0(t) = (t^{\nu_1 - 1}, t^{\nu_2 - 1})$, $t \in [0,1]$;*

(c) *for any given point $(x_0, y_0) \in \widetilde{P_{h_0}}$, if*

$$\begin{aligned} x_{n+1}(t) &= \int_0^1 G_1(t,s) f(s, x_n(s), y_n(s))\, ds, \quad n = 1,2,\ldots, \\ y_{n+1}(t) &= \int_0^1 G_2(t,s) g(s, x_n(s), y_n(s))\, ds, \quad n = 1,2,\ldots, \end{aligned}$$

then $x_n(t) \to x^*(t)$, $y_n(t) \to y^*(t)$ as $n \to \infty$.

Proof. We work in the product space $E \times E = C[0,1] \times C[0,1]$ with the partial-order-I. Define two operators $A, B \colon P \times P \to E$ by

$$\begin{aligned} A(x,y)(t) &= \int_0^1 G_1(t,s) f(s, x(s), y(s))\, ds, \\ B(x,y)(t) &= \int_0^1 G_2(t,s) g(s, x(s), y(s))\, ds, \end{aligned}$$

where G_i, $i = 1,2$, are the Green functions of Lemma 4.1 with ν replaced by ν_i, $i = 1,2$. From [11], we know that a pair of functions $(x,y) \in E \times E$ is a solution of problem (4.1)–(4.3) if and only if (x,y) is a solution of the operator equation $(x,y) = (A(x,y), B(x,y))$. From Lemma 4.2 and (D_1), we know that $A, B \colon P \times P \to P$. In the sequel, we check that A, B satisfy all assumptions of Theorem 3.1.

Firstly, we prove that A, B are increasing. Indeed, for $x_i, y_i \in P$, $i = 1,2$, with $x_1 \le x_2$, $y_1 \le y_2$, we know that $x_1(t) \le x_2(t)$, $y_1(t) \le y_2(t)$, $t \in [0,1]$, and by (D_2) and Lemma 4.2,

$$\begin{aligned} A(x_1,y_1)(t) &= \int_0^1 G_1(t,s) f(s, x_1(s), y_1(s))\, ds \\ &\le \int_0^1 G_1(t,s) f(s, x_2(s), y_2(s))\, ds = A(x_2,y_2)(t), \\ B(x_1,y_1)(t) &= \int_0^1 G_2(t,s) g(s, x_1(s), y_1(s))\, ds \\ &\le \int_0^1 G_2(t,s) g(s, x_2(s), y_2(s))\, ds = B(x_2,y_2)(t). \end{aligned}$$

That is, $A(x_1,y_1) \le A(x_2,y_2)$ and $B(x_1,y_1) \le B(x_2,y_2)$.

Further, we prove that A, B satisfy condition (H_1) of Theorem 3.1. For any $\lambda \in (0,1)$ and $x, y \in P$, by (D_3) we have

$$\begin{aligned} A(\lambda x, \lambda y)(t) &= \int_0^1 G_1(t,s) f(s, \lambda x(s), \lambda y(s))\, ds \\ &\geq \varphi_1(\lambda) \int_0^1 G_1(t,s) f(s, x(s), y(s))\, ds = \varphi_1(\lambda) A(x,y)(t), \\ B(\lambda x, \lambda y)(t) &= \int_0^1 G_2(t,s) g(s, \lambda x(s), \lambda y(s))\, ds \\ &\geq \varphi_2(\lambda) \int_0^1 G_2(t,s) g(s, x(s), y(s))\, ds = \varphi_2(\lambda) B(x,y)(t). \end{aligned}$$

That is, $A(\lambda x, \lambda y) \geq \varphi_1(\lambda) A(x,y)$, $B(\lambda x, \lambda y) \geq \varphi_2(\lambda) B(x,y)$ for any $\lambda \in (0,1)$, $x, y \in P$.

Let $h_0 = (h_0^{(1)}, h_0^{(2)})$, where $h_0^{(1)}(t) = t^{\nu_1 - 1}$, $h_0^{(2)}(t) = t^{\nu_2 - 1}$, $t \in [0,1]$. Then $(h_0^{(1)}, h_0^{(2)}) \in \widetilde{P_{h_0}}$. Next we show that $A(h_0^{(1)}, h_0^{(2)}) \in P_{h_0^{(1)}}$, $B(h_0^{(1)}, h_0^{(2)}) \in P_{h_0^{(2)}}$. On the one hand, from (D_2) and Lemma 4.2, for any $t \in [0,1]$, we have

$$\begin{aligned} A(h_0^{(1)}, h_0^{(2)})(t) &= \int_0^1 G_1(t,s) f(s, s^{\nu_1 - 1}, s^{\nu_2 - 1})\, ds \\ &\geq \frac{1}{\Gamma(\nu_1)}\, h_0^{(1)}(t) \int_0^1 [1-(1-s)^{\alpha}](1-s)^{\nu_1 - \alpha - 1} f(s,0,0)\, ds. \end{aligned}$$

On the other hand, also from (D_2) and Lemma 4.2, for any $t \in [0,1]$, we obtain

$$\begin{aligned} A(h_0^{(1)}, h_0^{(2)})(t) &= \int_0^1 G_1(t,s) f(s, s^{\nu_1 - 1}, s^{\nu_2 - 1})\, ds \\ &\leq \frac{1}{\Gamma(\nu_1)}\, h_0^{(1)}(t) \int_0^1 (1-s)^{\nu_1 - \alpha - 1} f(s,1,1)\, ds. \end{aligned}$$

From (D_2), we have $f(s,1,1) \geq f(s,0,0) \geq 0$. Since $f(t,0,0) \not\equiv 0$, we get

$$[1-(1-t)^{\alpha}](1-t)^{\nu_1 - \alpha - 1} f(t,0,0) \not\equiv 0, \qquad (1-t)^{\nu_1 - \alpha - 1} f(t,1,1) \not\equiv 0.$$

Note that $\nu_1 - \alpha - 1 > 0$, so we have

$$\begin{aligned} l_1 &:= \frac{1}{\Gamma(\nu_1)} \int_0^1 [1-(1-s)^{\alpha}](1-s)^{\nu_1 - \alpha - 1} f(s,0,0)\, ds > 0, \\ l_2 &:= \frac{1}{\Gamma(\nu_1)} \int_0^1 (1-s)^{\nu_1 - \alpha - 1} f(s,1,1)\, ds > 0. \end{aligned}$$

So $l_1 h_0^{(1)}(t) \leq A(h_0^{(1)}, h_0^{(2)})(t) \leq l_2 h_0^{(1)}(t)$, $t \in [0,1]$; and hence $A(h_0^{(1)}, h_0^{(2)}) \in P_{h_0^{(1)}}$. Similarly, we can prove that $B(h_0^{(1)}, h_0^{(2)}) \in P_{h_0^{(2)}}$.

Finally, by Theorem 3.1, we have the following conclusions:

(1) there exist $x_0^{(1)}, y_0^{(1)} \in P_{t^{\nu_1 - 1}}$, $x_0^{(2)}, y_0^{(2)} \in P_{t^{\nu_2 - 1}}$ and $r \in (0,1)$ such that

$$r(y_0^{(1)}, y_0^{(2)}) \dot{\leq} (x_0^{(1)}, x_0^{(2)}) \dot{\leq} (y_0^{(1)}, y_0^{(2)}),$$
$$x_0^{(1)} \leq A(x_0^{(1)}, x_0^{(2)}) \leq y_0^{(1)}, \qquad x_0^{(2)} \leq B(x_0^{(1)}, x_0^{(2)}) \leq y_0^{(2)},$$

that is

$$\begin{aligned} x_0^{(1)}(t) &\leq \int_0^1 G_1(t,s) f(s, x_0^{(1)}(s), x_0^{(2)}(s))\, ds \leq y_0^{(1)}(s), \quad t \in [0,1], \\ x_0^{(2)}(t) &\leq \int_0^1 G_2(t,s) g(s, x_0^{(1)}(s), x_0^{(2)}(s))\, ds \leq y_0^{(2)}(s), \quad t \in [0,1]; \end{aligned}$$

(2) the operator equation $(x,y) = (A(x,y), B(x,y))$ has a unique solution (x^*, y^*) in $\widetilde{P_{h_0}}$, that is, problem (4.1)–(4.3) has a unique positive solution (x^*, y^*) in $\widetilde{P_{h_0}}$;

(3) for any $(x_0, y_0) \in \widetilde{P_{h_0}}$, if $(x_n, y_n) = (A(x_{n-1}, y_{n-1}),\ B(x_{n-1}, y_{n-1}))$, $n = 1, 2, \ldots$, then $\|x_n - x^*\|_E \to 0$, $\|y_n - y^*\|_E \to 0$ as $n \to \infty$. That is, for

$$\begin{aligned} x_{n+1}(t) &= \int_0^1 G_1(t,s) f(s, x_n(s), y_n(s))\, ds, \quad n = 1, 2, \ldots, \\ y_{n+1}(t) &= \int_0^1 G_2(t,s) g(s, x_n(s), y_n(s))\, ds, \quad n = 1, 2, \ldots, \end{aligned}$$

we have $x_n(t) \to x^*(t)$, $y_n(t) \to y^*(t)$ as $n \to \infty$. □

Example 4.4. Let us consider the following system:

$$\begin{cases} -D_{0+}^{7/2} y_1(t) = [y_1(t)]^{\tau_1} + [y_2(t)]^{\tau_1} + \psi_1(t), & t \in (0,1), \\ -D_{0+}^{10/3} y_2(t) = [y_1(t)]^{\tau_2} + [y_2(t)]^{\tau_2} + \psi_2(t), & t \in (0,1), \end{cases} \tag{4.4}$$

subject to a couple of boundary conditions

$$y_1^{(i)}(0) = 0 = y_2^{(i)}(0), \quad 0 \leq i \leq 2,, \tag{4.5}$$

$$[D_{0+}^{3/2} y_1(t)]_{t=1} = 0 = [D_{0+}^{3/2} y_2(t)]_{t=1}, \tag{4.6}$$

where $\tau_1, \tau_2 \in (0,1)$, $\psi_1, \psi_2 \colon [0,1] \to [0, +\infty)$ are continuous with $\psi_i \not\equiv 0$. Let

$$f(t,u,v) = u^{\tau_1} + v^{\tau_1} + \psi_1(t), \qquad g(t,u,v) = u^{\tau_2} + v^{\tau_2} + \psi_2(t).$$

Take $n = 4$, $\nu_1 = 7/2$, $\nu_2 = 10/3$, $\alpha = 3/2$. Then $\nu_1, \nu_2 \in (3,4]$, $\alpha \in [0,2]$. Obviously, $f, g \in C([0,1] \times [0,+\infty) \times [0,+\infty), [0,+\infty))$ and

$$f(t,0,0) = \psi_1(t), \qquad g(t,0,0) = \psi_2(t) \not\equiv 0.$$

Note that x^{τ_i}, $i = 1, 2$, are increasing in $[0,+\infty)$, $f(t,u,v), g(t,u,v)$ are increasing in u, v for any $t \in [0,1]$. Moreover, set $\varphi_1(\lambda) = \lambda^{\tau_1}$, $\varphi_2(\lambda) = \lambda^{\tau_2}$, $\lambda \in (0,1)$. Then $\varphi_1(\lambda), \varphi_2(\lambda) \in (\lambda, 1)$ and

$$f(t, \lambda u, \lambda v) = \lambda^{\tau_1}[u^{\tau_1} + v^{\tau_1}] + \psi_1(t) \geq \lambda^{\tau_1}[u^{\tau_1} + v^{\tau_1}] + \lambda^{\tau_1}\psi_1(t) = \lambda^{\tau_1} f(t,u,v).$$

Similarly, $g(t,\lambda u,\lambda v) \geq \lambda^{\tau_2} g(t,u,v)$ for $t \in [0,1]$, $u,v \in [0,+\infty)$. Hence, all conditions of Theorem 4.3 are satisfied. Application of Theorem 4.3 implies that problem (4.4)–(4.6) has a unique positive solution (x^*,y^*) in $\widetilde{P_{h_0}}$, where $h_0(t) = (t^{5/2}, t^{7/3})$, $t \in [0,1]$, and for any given point $(x_0,y_0) \in \widetilde{P_{h_0}}$, if

$$x_{n+1}(t) = \int_0^1 G_1(t,s)\{[x_n(s)]^{\tau_1} + [y_n(s)]^{\tau_1} + \psi_1(s)\}\,ds, \quad n = 1,2,\ldots,$$

$$y_{n+1}(t) = \int_0^1 G_2(t,s)\{[x_n(s)]^{\tau_2} + [y_n(s)]^{\tau_2} + \psi_2(s)\}\,ds, \quad n = 1,2,\ldots,$$

then $x_n(t) \to x^*(t)$, $y_n(t) \to y^*(t)$ as $n \to \infty$, where

$$G_1(t,s) = \begin{cases} \dfrac{t^{5/2}(1-s) - (t-s)^{5/2}}{\Gamma(7/2)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{t^{5/2}(1-s)}{\Gamma(7/2)}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$G_2(t,s) = \begin{cases} \dfrac{t^{7/3}(1-s)^{5/6} - (t-s)^{7/3}}{\Gamma(10/3)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{t^{7/3}(1-s)^{5/6}}{\Gamma(10/3)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

REMARK 4.5. If in Example 4.4 we replace f,g by $f \equiv \Gamma(7/2)$, $g \equiv \Gamma(10/3)$, then problem (4.4)–(4.6) has a unique solution (x^*,y^*), where $x^*(t) = (1/2 - 2t/7)t^{5/2}$, $y^*(t) = (6/11 - 3t/10)t^{7/3}$, $t \in [0,1]$. We can obtain that

$$\frac{3}{14}\,t^{5/2} \leq x^*(t) \leq \frac{1}{2}\,t^{5/2}, \quad \frac{27}{110}\,t^{7/3} \leq y^*(t) \leq \frac{6}{11}\,t^{7/3}, \quad t \in [0,1].$$

So the unique solution is a positive solution and $(x^*,y^*) \in \widetilde{P}_{(t^{5/2},t^{7/3})}$.

REFERENCES

[1] H. AMANN, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), no. 4, 620–709.

[2] A.B. AMAR, S. CHOUAYEKH AND A. JERIBI, *New fixed point theorems in Banach algebras under weak topology features and applications to nonlinear integral equations*, J. Funct. Anal. **259** (2010), 2215–2237.

[3] R.I. AVERY AND A.C. PETERSON, *Three positive fixed points of nonlinear operators on ordered Banach spaces*, Comput. Math. Appl. **42** (2001), 313–322.

[4] Z.B. BAI AND H.S. LÜ, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl. **311** (2005), 495–505.

[5] A. BAKAKHAI AND V.D. GEJJI, *Existence of positive solutions of nonlinear fractional differential equations*, J. Math. Anal. Appl. **278** (2003), 434–442.

[6] Y.Z. CHEN, *Stability of positive fixed points of nonlinear operators*, Positivity **6** (2002), 47–57.

[7] K. DEIMLING, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.

[8] X.P. DING, J.Y. PARK AND I.H. JUNG, *Fixed point theorems on product topological spaces and applications*, Positivity **8** (2004), 315–326.

[9] A. Fora, *A fixed point theorem for product spaces*, Pacific J. Math. **99** (1982), no. 2, 327–335.

[10] D.J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Notes and Reports in Mathematics in Science and Engineering, vol. 5. Academic Press, Boston, 1988.

[11] C.S. Goodrich, *Existence of a positive solution to systems of differential equations of fractional order*, Comput. Math. Appl. **62** (2011), 1251–1268.

[12] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204, Elsevier, Amsterdam, 2006.

[13] U. Kohlenbach and L. Leustean, *The approximate fixed point property in product spaces*, Nonlinear Anal. **66** (2007), 806–818.

[14] T. Kuczumow, *Fixed point theorems in product spaces*, Proc. Amer. Math. Soc. **108** (1990), no. 3, 727–729.

[15] K.X. Li, J.G. Peng and J.X. Jia, *Cauchy problems for fractional differential equations with Riemann–Liouville fractional derivatives*, J. Funct. Anal. **263** (2012), 476–510.

[16] W.R. Melvin, *Some extensions of the Krasnoselskii fixed point theorems*, J. Differential Equations **11** (1972), 335–348.

[17] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[18] J.J. Nieto, R.L. Pouso and R. Rodriguez-Lopez, *Fixed point theorems in ordered abstract spaces*, Proc. Amer. Math. Soc. **135** (2007), 2505–2517.

[19] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering, vol. 198, Academic Press, San Diego, 1999.

[20] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integral and Derivatives*, Gordon and Breach, Yverdon, 1993.

[21] K.K. Tan and H.K. Xu, *On fixed point theorems of nonexpansive mappings in product spaces*, Proc. Amer. Math. Soc. **113** (1991), no. 4, 983–989.

[22] Y.Q. Wang, L.S. Liu and Y.H. Wu, *Positive solutions for a nonlocal fractional differential equation*, Nonlinear Anal. **74** (2011), 3599–3605.

[23] A. Wiśnicki, *On the super fixed point property in product spaces*, J. Funct. Anal. **236** (2006), 447–456.

[24] X. Xu, D. Jiang and C. Yuan, *Multiple positive solutions for boundary value problem of nonlinear fractional differential equation*, Nonlinear Anal. **71** (2009), 4676–4688.

[25] L. Yang and H. Chen, *Unique positive solutions for fractional differential equation boundary value problems*, Appl. Math. Lett. **23** (2010), 1095–1098.

[26] C. Yang and C.B. Zhai, *Uniqueness of positive solutions for a fractional differential equation via a fixed point theorem of a sum operator*, Electron. J. Differential Equations **70** (2012), 1–8.

[27] C.B. Zhai and M.R. Hao, *Fixed point theorems for mixed monotone operators with perturbation and applications to fractional differential equation boundary value problems*, Nonlinear Anal. **75** (2012), 2542–2551.

[28] ______, *Mixed monotone operator methods for the existence and uniqueness of positive solutions to Riemann–Liouville fractional differential equation boundary value problems*, Boundary Value Problems, 2013, 2013:85.

[29] C.B. Zhai and L. Xu, *Properties of positive solutions to a class of four-point boundary value problem of Caputo fractional differential equations with a parameter*, Commun. Nonlinear Sci. Numer. Simul. **19** (2014), 2820–2827.

[30] C.B. Zhai, W.P. Yan and C. Yang, *A sum operator method for the existence and uniqueness of positive solutions to Riemann–Liouville fractional differential equation boundary value problems*, Commun. Nonlinear Sci. Numer. Simul. **18** (2013), 858–866.

[31] C.B. Zhai, C. Yang and X.Q. Zhang, *Positive solutions for nonlinear operator equations and several classes of applications*, Math. Z. **266** (2010), 43–63.

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