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GLOBAL WELL-POSEDNESS AND ATTRACTOR FOR DAMPED WAVE EQUATION WITH SUP-CUBIC NONLINEARITY AND LOWER REGULAR FORCING ON $\mathbb{R}^3$

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Abstract. The dissipative wave equation with sup-cubic nonlinearity and lower regular forcing term which belongs to $H^{-1}(\mathbb{R}^3)$ in the whole space $\mathbb{R}^3$ is considered. Well-posedness of a translational regular solution is achieved by establishing extra space-time translational regularity of an energy solution. Furthermore, a global attractor in the naturally defined energy space $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ is built.

1. Introduction

In this paper we study the following weakly damped wave equation defined on $\mathbb{R}^3$:

$$\begin{cases} u_{tt} + \gamma u_t - \Delta u + mu + f(x,u) = g(x) & \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ u(x,0) = u_0, \quad u_t(x,0) = u_1 & \text{for } x \in \mathbb{R}^3. \end{cases} \tag{1.1}$$

Here $\gamma, m > 0$ and the initial data (u_0, u_1) belongs to the energy space $\mathcal{H} = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, $g \in H^{-1}(\mathbb{R}^3)$ is independent of time, $f \in C^1(\mathbb{R}^4)$, $f(0) = 0$

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and satisfies the following conditions:

$$|\partial_s f(x,s)| \le C(1+|s|^{p-1}), \tag{1.2}$$

$$\liminf_{|x|+|s|\to\infty} \frac{f(x,s)}{s} \ge 0, \tag{1.3}$$

where $3 < p < 4$. Here (and throughout) the symbol C stands for a generic constant indexed occasionally for the sake of clarity, the different positive constants C_i, $i \in \mathbb{N}$, are also used for special differentiation, and for the convenience, we denote $u(t) = u(x,t)$ and $u_t(t) = u_t(t,x)$ throughout the paper.

When the spacial variable x belongs to a bounded domain, the existence of global attractors for weakly damped wave equations (1.1) has been studied extensively, see [1], [2], [4], [7]–[9], [17]–[19], [23], [26] and references therein. In the case that the spatial domain is unbounded, the typical Sobolev embedding is continuous but no longer compact, and the spaces $L^p(\mathbb{R}^{\mathbb{N}})$ are not nested, so there is a substantial difficulty in dealing with the compactness of the corresponding operator semigroup, which is a key point to obtain the existence of global attractor. To overcome this difficulty, several classical methods have been established. For example, in [14], Karachalios and Stavrakakis have employed weighted Sobolev spaces to deal with (1.1). However, when working in weighted spaces we have to impose an additional condition that the initial data and forcing term also belong to the corresponding weighted spaces. Combining with the idea of "tail estimate" in [27], Djiby and You proved that solutions of (1.1) are uniformly small for large spacial and time variables, and then obtained the asymptotic compactness in [10].

On the other hand, mathematical properties of (1.1) including well-posedness and asymptotic behavior also depend strongly on the growth rate of the nonlinearity f. For a long time, exactly the cubic growth rate of the nonlinearity f has been considered as critical in bounded domains, see for instance [2], [3], [23] and the literature cited therein. Based on the recent progress in Strichartz estimates for bounded domains (for the dimension $N = 3$), Burq et al. [6] proved the global well-posedness of weak solutions with quintic nonlinearity growth of hyperbolic equation, Kalantarov et al. [16] established the corresponding attractor theory for the dissipative wave equation in bounded domains. For the case of unbounded domains, since the Strichartz estimates have been known a little earlier, based on that, Kapitanski and Feireisl constructed global attractors in [15] and [11], [12], respectively.

Concerning regularity of the external force term g, all papers mentioned above require $g \in L^2$ or specially $g = 0$. Still, a larger range of g, i.e. $g \in H^{-1}$, can also be considered. We refer the reader to [22], [25], [28], in which the existence and asymptotic regularity of global attractor have been discussed for the strong damping wave equations. As mentioned in [28], the strong damping term $-\Delta u_t$ brings many advantages which can counteract the unfavorable effect of $g \in H^{-1}$. However, for the weakly damped wave equation (1.1), it seems difficult to apply the corresponding method to verify asymptotic compactness of the solution semigroup. Just recently, in [20], we presented a new method about decomposition of the solution to deal with the case $g \in H^{-1}(\Omega)$ in bounded domains. In [21], by proper decomposition of the solution and using finite speed of propagation for the wave equation, we built a global attractor for (1.1) in the unbounded domain with cubic nonlinearity and lower regular forcing.

The case of sup-cubic growth rate with forcing term belonging to $H^{-1}(\mathbb{R}^3)$ is indeed much more delicate, in contrast to [12], the crucial difficulty here is related to the fact that multiplying formally (1.1) by u_t directly to obtain the necessary energy equality fails, since $g \in H^{-1}$, $u_t \in L^2$, $\int_\Omega g u_t \, dx$ may be meaningless (in the cubic or sub-cubic case, we overcame the difficulty by rewriting the equation as $u_{tt} + \gamma u_t - \Delta u + mu - g(x) = -f(x,u)$ and observing that $f \in L^2$, see [20], [21] for more details. Unfortunately, this argument does not work in the sup-cubic case. It appears that to get well-posedness one needs some extra regularity (see Definition 3.2).

In the present work, we give a positive answer to the well-posedness and asymptotic behavior of equation (1.1) in the above case. In detail, the aim of this work is two fold. The first one is to establish and prove existence and uniqueness of a translational regular solution (see the rigorous definition in Section 3) for weakly damped wave equation (1.1) in $\mathbb{R}^3$ with super-cubic nonlinearity and $g \in H^{-1}(\mathbb{R}^3)$. The second one is to build the corresponding global attractor for (1.1).

The rest of the paper is organized as follows. The preliminary things, including the key Strichartz estimates for the damped wave equation and properties of the elliptic equation are discussed in Section 2. The global well-posedness for the translational regular solution is verified in Section 3. The existence of a global attractor is established in Section 4.

2. Preliminaries

We begin with the Strichartz estimate for the Klein–Gordon equation [5] and [13]:

PROPOSITION 2.1. *Suppose $2 \le r < \infty$, if u is a finite energy solution of the Klein–Gordon equation*

$$u_{tt} - \Delta u + mu = G(t), \quad u(0) = u_0, \quad u_t(0) = u_1,$$

then $u \in L^r(0,T; L^q(\mathbb{R}^3))$ and the following estimate holds:

$$\|u\|_{L^r(0,T;L^q(\mathbb{R}^3))} \le C\big(\|u_0\|_{H^1(\mathbb{R}^3)} + \|u_1\|_{L^2(\mathbb{R}^3)} + \|G\|_{L^1(0,T;L^2(\mathbb{R}^3))}\big), \tag{2.1}$$

where C is independent of T, u_0, u_1 and G, provided (r,q) satisfy $1/r + 3/q = 1/2$.

For $(u_0, u_1) \in \mathcal{H}$, since the weak solution $u(t)$ of the damped wave equation

$$u_{tt} + \gamma u_t - \Delta u + mu = G(t), \quad u(0) = u_0, \quad u_t(0) = u_1, \tag{2.2}$$

satisfies that $(u, u_t) \in L^\infty(0, T; \mathcal{H})$, the damping term u_t can be dominated via energy estimate, i.e. $u_t \in L^\infty(0, T; L^2(\mathbb{R}^3))$. Therefore we obtain the following Strichartz estimate.

Corollary 2.2. *For $(u_0, u_1) \in \mathcal{H}$, suppose that $u(t)$ is a solution of* (2.2) *satisfying $(u, u_t) \in L^\infty(0, T; \mathcal{H})$. Then $u \in L^{4/(p-2)}(0, T; L^{12/(4-p)}(\mathbb{R}^3))$ and we have the following estimate*:

$$\begin{aligned} \|u\|_{L^{4/(p-2)}(0,T;L^{12/(4-p)}(\mathbb{R}^3))} \\ \leq C\big(\|u_0\|_{H^1(\mathbb{R}^3)} + \|u_1\|_{L^2(\mathbb{R}^3)} + \|G\|_{L^1(0,T;L^2(\mathbb{R}^3))}\big), \end{aligned} \tag{2.3}$$

where C is independent of T, u_0, u_1 and G.

The next elementary fact will be used to establish the well-posedness of (1.1) in some sense.

Lemma 2.3. *Assume that f satisfies* (1.2), (1.3)*, then there exists a solution $h \in H^1(\mathbb{R}^3)$ for the following elliptic equation*:

$$-\Delta u + mu + f(x, u) = g(x) \in H^{-1}. \tag{2.4}$$

On the other hand, if h_1 and h_2 are two solutions of (2.4)*, then $h_1 - h_2 \in L^q(\mathbb{R}^3)$, for all $q < \infty$.*

Proof. The energy functional corresponding to elliptic equation (2.4) is weakly lower semi-continuous and bounded from below on $H^1(\mathbb{R}^3)$, thus existence of h can be guaranteed. Suppose that h_1, h_2 are two solutions of (2.4), denote

$$a(x) = \int_0^1 \partial_s f(x, (1-\tau)h_1 + \tau h_2)\, d\tau,$$

then $h_1 - h_2$ satisfies $-\Delta(h_1 - h_2) + m(h_1 - h_2) = a(x)(h_1 - h_2)$. Due to (1.2), $a \in L^2(\mathbb{R}^3)$, and by the de Giorgi–Moser iteration, we have $h_1 - h_2 \in L^q(\mathbb{R}^3)$, for all $q < \infty$. □

3. Well-posedness

The aim of this section is to introduce and study translational regular solutions of problem (1.1), i.e. weak solutions satisfying some translational regularity. We begin with recalling the definition of weak solutions.

Definition 3.1. For any $T > 0$, a function $u(t)$, $t \in [0, T]$, is a *weak solution* of (1.1) if $(u, u_t) \in L^\infty(0, T; \mathcal{H})$, and equation (1.1) is satisfied in the sense of distributions, i.e.

$$-\int_0^\infty \langle u_t, \phi_t\rangle\, dt + \gamma \int_0^\infty \langle u_t, \phi\rangle\, dt + \int_0^\infty \langle \nabla u, \nabla \phi\rangle\, dt \\ + m\int_0^\infty \langle u, \phi\rangle\, dt + \int_0^\infty \langle f(x,u), \phi\rangle\, dt = \int_0^\infty \langle g, \phi\rangle\, dt$$

for any $\phi \in C_0^\infty((0,T)\times\mathbb{R}^3)$.

Definition 3.2. A weak solution $u(t)$, $t\in[0,T]$, is a *translational regular solution* of problem (1.1) if the following additional regularity holds:

$$u(t) - h \in L^{4/(p-2)}(0,T; L^{12/(4-p)}(\mathbb{R}^3)),$$

where h is any solution of elliptic equation (2.4) .

Remark 3.3. By Lemma 2.3, the above definition is independent of the choice of h, hence it is well-defined.

Define the energy functional

$$E(u) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{m}{2}\|u\|^2 + \int_{\mathbb{R}^3} F(x,u)\, dx - \langle g, u\rangle,$$

where $F(x,s) = \int_0^s f(x,\tau)\, d\tau$. Here and below, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ stand for the inner product and norm of L^2, respectively.

Due to condition (1.3),

$$E(u) \ge C_1(\|u\|^2 + \|\nabla u\|^2) - C_2(\|g\|^2_{H^{-1}} + 1).$$

Theorem 3.4. *Let $g\in H^{-1}(\mathbb{R}^3)$ and the nonlinearity f satisfy conditions* (1.2) *and* (1.3). *Then for any initial data $(u_0,u_1)\in\mathcal{H}$, problem* (1.1) *possesses a global translational regular solution $u(t)$ satisfying the following energy estimate:*

$$\|(u(t),u_t(t))\|_{\mathcal{H}} + \|u-h\|_{L^{4/(p-2)}(t,t+1;L^{12/(4-p)}(\mathbb{R}^3))} \\ \le Q(\|(u_0,u_1)\|_{\mathcal{H}} + \|g\|_{H^{-1}}), \tag{3.1}$$

where the positive monotone increasing function Q is independent of u and t.

Proof. In order to construct the solution, we adopt an approximating scheme. Choose a regular approximating sequence $\{(u_0^n,u_1^n)\}\subset H^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$ and $\{g^n\}\subset L^2(\mathbb{R}^3)$, satisfying $(u_0^n,u_1^n)\to(u_0,u_1)$ in $\mathcal{H}$ and $g^n\to g$ in $H^{-1}(\mathbb{R}^3)$, as $n\to\infty$. According to Proposition 1.1 of [12], there exists a unique strong solution u^n of the following approximating problem:

$$\begin{cases} u_{tt}^n + \gamma u_t^n - \Delta u^n + mu^n + f(x,u^n) = g^n(x), \\ u^n(0) = u_0^n, \quad u_t^n(0) = u_1^n, \end{cases} \tag{3.2}$$

with $u^n \in L^\infty_{\rm loc}(\mathbb{R}^+; H^2(\mathbb{R}^3))$. Furthermore, multiplying (3.2) by $\partial_t u^n(t)$, we obtain the following energy estimate:

$$\|(u^n(t), u^n_t(t))\|_{\mathcal{H}} \le Q_0(\|(u_0, u_1)\|_{\mathcal{H}} + \|g\|_{H^{-1}}), \tag{3.3}$$

where the positive monotone increasing function Q_0 is independent of n and t.

Due to $u^n \in L^\infty_{\rm loc}(\mathbb{R}^+; H^2(\mathbb{R}^3))$, $u^n(t) \in L^{4/(p-2)}_{\rm loc}(\mathbb{R}^+; L^{12/(4-p)}(\mathbb{R}^3))$, and we will make a more precise estimate for it in what follows. Denote $v^n(t) = u^n(t) - h^n$, where h^n satisfies

$$-\Delta h^n + \lambda h^n + f(x, h^n) = g^n(x).$$

Then $v^n(t)$ solves the equation

$$\begin{cases} v^n_{tt} + \gamma v^n_t - \Delta v^n + m v^n + f(x, u^n) - f(x, h^n) = 0, \\ v^n(0) = u^n_0 - h^n, \quad v^n_t(0) = u^n_1. \end{cases} \tag{3.4}$$

According to the Strichartz inequality (2.3) on $[t, t+\tau]$ for some $\tau > 0$ to be determined later, we get

$$\begin{aligned} \|v^n\|_{L^{4/(p-2)}(t,t+\tau;L^{12/(4-p)}(\mathbb{R}^3))} \le C_3\big(&\|u^n(t) - h^n\|_{H^1(\mathbb{R}^3)} \\ &+ \|u^n_t(t)\| + \|f(x, u^n) - f(x, h^n)\|_{L^1(t,t+\tau;L^2(\mathbb{R}^3))}\big). \end{aligned} \tag{3.5}$$

For the last term, by the mean value theorem, we obtain

$$\begin{aligned} |f(x, u^n) - f(x, h^n)| &= \bigg| \int_0^1 \partial_s f(x, \eta u^n + (1-\eta) h^n) v^n \, d\eta \bigg| \\ &\le C(1 + |u^n|^{p-1} + |h^n|^{p-1})|v^n|. \end{aligned}$$

Hence

$$\begin{aligned} \|f(x, u^n) &- f(x, h^n)\|_{L^1(t,t+\tau;L^2)} \\ &\le C\|(1 + |u^n|^{p-1} + |h^n|^{p-1}) v^n\|_{L^1(t,t+\tau;L^2(\mathbb{R}^3))} \\ &\le C\big(1 + \|u^n\|^{p-1}_{L^\infty(t,t+\tau;L^{12(p-1)/(2+p)}(\mathbb{R}^3))} \\ &\quad + \|h^n\|^{p-1}_{L^{12(p-1)/(2+p)}(\mathbb{R}^3)}\big)\|v^n\|_{L^1(t,t+\tau;L^{12/(4-p)}(\mathbb{R}^3))} \\ &\le C\tau^{(6-p)/4}\big(1 + \|u^n\|^{p-1}_{L^\infty(t,t+\tau;H^1(\mathbb{R}^3))} \\ &\quad + \|h^n\|^{p-1}_{H^1(\mathbb{R}^3)}\big)\|v^n\|_{L^{4/(p-2)}(t,t+\tau;L^{12/(4-p)}(\mathbb{R}^3))}. \end{aligned} \tag{3.6}$$

Combining with the energy estimate (3.3), we derive

$$\|f(x, u^n) - f(x, h^n)\|_{L^1(t,t+\tau;L^2(\mathbb{R}^3))} \le C_4 \tau^{(6-p)/4} \|v^n\|_{L^{4/(p-2)}(t,t+\tau;L^{12/(4-p)}(\mathbb{R}^3))},$$

where $C_4 = C_4(\|u_0\|_{H^1}, \|u_1\|, \|g\|_{H^{-1}})$.

Choose τ such that $\tau^{(6-p)/4} C_3 C_4 = 1/2$, then (3.5) implies

$$\|v^n\|_{L^{4/(p-2)}(t,t+\tau;L^{12/(4-p)}(\mathbb{R}^3))} \le 2C_3\big(\|u^n(t) - h^n\|_{H^1(\mathbb{R}^3)} + \|u^n_t(t)\|\big). \tag{3.7}$$

Combining with (3.3), there exists a positive monotone increasing function Q which is independent of u^n and t such that

$$\|(u^n(t),u_t^n(t))\|_{\mathcal{H}}+\|u^n-h^n\|_{L^{4/(p-2)}(t,t+\tau;L^{12/(4-p)}(\mathbb{R}^3))} \le Q(\|(u_0,u_1)\|_{\mathcal{H}}+\|g\|_{H^{-1}}).$$

Notice that $\{h^n\}$ is bounded in $H^1(\mathbb{R}^3)$, hence, without loss of generality we may assume that

$$h^n \rightharpoonup h \quad \text{in } H^1(\mathbb{R}^3), \qquad u^n \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(\mathbb{R}^+;H^1(\mathbb{R}^3)),$$
$$u^n-h^n \rightharpoonup u-h \quad \text{in } L^{4/(p-2)}_{\mathrm{loc}}(\mathbb{R}^+;L^{12/(4-p)}(\mathbb{R}^3)).$$

It is easy to verify that h and u solve

$$-\Delta h+mh+f(x,h)=g(x)$$

and

$$u_{tt}+\gamma u_t-\Delta(u-h)+m(u-h)+f(x,u)-f(x,h)=0,$$

respectively. Therefore, u is a solution of equation (1.1), and satisfies the energy estimate (3.1). The proof is completed. □

Next, we will verify the uniqueness of the translational regular solution.

Theorem 3.5. *Under the conditions of Theorem* 3.4, *the translational regular solution of* (1.1) *is unique and for the difference* $\overline{u}$ *of two translational regular solutions* u^1 *and* u^2 *with the corresponding initial data* $(u_0^1,u_1^1),(u_0^2,u_1^2)\in\mathcal{H}$ *the following estimate is valid*:

$$\|(\overline{u}(t),\overline{u}_t(t))\|_{\mathcal{H}} \le e^{KT}\|(\overline{u}(0),\overline{u}_t(0))\|_{\mathcal{H}}, \quad t\in[0,T], \tag{3.8}$$

where K is a positive constant depending on the $\mathcal{H}$-norm of $u^1(t)$ and $u^2(t)$, $t\in[0,T]$.

Proof. Define $R=\sup_{[0,T]}\{\|(u^1(t),u_t^1(t))\|_{\mathcal{H}},\|(u^2(t),u_t^2(t))\|_{\mathcal{H}}\}$. The function $\overline{u}(t)$ as the difference of two solutions $u^1(t)$ and $u^2(t)$ solves

$$\begin{cases}\overline{u}_{tt}+\gamma\overline{u}_t-\Delta\overline{u}+m\overline{u}+f(x,u^1)-f(x,u^2)=0,\\ \overline{u}(0)=u_0^1-u_0^2, \quad \overline{u}_t(0)=u_1^1-u_1^2.\end{cases} \tag{3.9}$$

For $0\le S\le T$, similarly to (3.6), we have

$$\begin{aligned}\|f(x,u^1)-f(x,u^2)\|_{L^1(0,S;L^2(\mathbb{R}^3))} &\le CS^{(6-p)/4}\big(1+\|u^1\|^{p-1}_{L^\infty(0,S;H^1(\mathbb{R}^3))}+\|u^2\|^{p-1}_{L^\infty(0,S;H^1(\mathbb{R}^3))}\big)\\ &\quad\times\|\overline{u}\|_{L^{4/(p-2)}(0,T;L^{12/(4-p)}(\mathbb{R}^3))}\\ &\le C_5R^{p-1}S^{(6-p)/4}\|\overline{u}\|_{L^{4/(p-2)}(0,S;L^{12/(4-p)}(\mathbb{R}^3))}.\end{aligned} \tag{3.10}$$

Applying the Strichartz inequality to (3.9), we gain

$$\|\overline{u}\|_{L^{4/(p-2)}(0,S;L^{12/(4-p)}(\mathbb{R}^3))} \le C_6\big(\|\overline{u}(0)\|_{H^1(\mathbb{R}^3)} + \|\overline{u}_t(0)\| + \|f(x,u^1)-f(x,u^2)\|_{L^1(0,S;L^2(\mathbb{R}^3))}\big).$$

Combining the above two inequalities, we have

$$\|f(x,u^1)-f(x,u^2)\|_{L^1(0,S;L^2(\mathbb{R}^3))} \le C_6C_5R^{p-1}S^{(6-p)/4}\big(\|\overline{u}(0)\|_{H^1(\mathbb{R}^3)} + \|\overline{u}_t(0)\| + \|f(x,u^1)-f(x,u^2)\|_{L^1(0,S;L^2(\mathbb{R}^3))}\big).$$

Choose an appropriate S such that $C_6C_5R^{p-1}S^{(6-p)/4} \le 1/2$, then we immediately get

$$\|f(x,u^1)-f(x,u^2)\|_{L^1(0,S;L^2(\mathbb{R}^3))} \le \|\overline{u}(0)\|_{H^1(\mathbb{R}^3)} + \|\overline{u}_t(0)\|.$$

Multiplying equation (3.9) by $\overline{u}_t$, we obtain the following estimate:

$$\begin{aligned}\|\overline{u}_t(t)\| + \|\overline{u}(t)\|_{H^1(\mathbb{R}^3)} &\le 2\big(\|\overline{u}(0)\|_{H^1(\mathbb{R}^3)} + \|\overline{u}_t(0)\| + \|f(x,u^1)-f(x,u^2)\|_{L^1(0,S;L^2(\mathbb{R}^3))}\big) \\ &\le 4(\|\overline{u}(0)\|_{H^1(\mathbb{R}^3)} + \|\overline{u}_t(0)\|),\end{aligned}$$

for any $0 \le t \le S$. Iterating the above process on $[s, s+S]$, we have

$$\|\overline{u}_t(t)\| + \|\overline{u}(t)\|_{H^1(\mathbb{R}^3)} \le 4(\|\overline{u}(s)\|_{H^1(\mathbb{R}^3)} + \|\overline{u}_t(s)\|), \quad s \le t \le s+S.$$

Therefore, we end up with the estimate

$$\|\overline{u}_t(t)\| + \|\overline{u}(t)\|_{H^1(\mathbb{R}^3)} \le 4^{T/S}(\|\overline{u}(0)\|_{H^1(\mathbb{R}^3)} + \|\overline{u}_t(0)\|), \quad 0 \le t \le T,$$

that completes the proof. □

Basing on the above discussion about well-posedness of (1.1), we are now ready to define the solution semigroup $S(t)\colon \mathcal{H} \to \mathcal{H}$ associated with equation (1.1):

$$S(t)(u_0, u_1) = (u(t), u_t(t)).$$

where $u(t)$ is a unique translational regular solution of (1.1). Then, according to Theorems 3.3 and 3.4, the semigroup is well-posed and continuous.

4. Existence of global attractor

In this section, we prove the existence of a global attractor for problem (1.1) under conditions (1.2) and (1.3) on the nonlinearity and forcing term $g \in H^{-1}(\mathbb{R}^3)$. We start with stating the result about the dissipativity of the semigroup.

Multiplying the approximating equation (3.2) by $u_t^n + \alpha u^n$, where α is a properly chosen positive constant, and integrating in x, the desired estimates are obtained by passing to the limit from the analogous estimates for the approximating solutions. By using of the Gronwall Lemma, the existence of the bounded absorbing set for the semigroup $S(t)$ can be obtained. We only state the result as follows:

LEMMA 4.1. *Suppose $g \in H^{-1}(\mathbb{R}^3)$ and f satisfies (1.2) and (1.3), then $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set $\mathscr{B}_0$ in $\mathcal{H}$, that is, for any bounded subset $B \subset \mathcal{H}$, there exists T_0 which depends on $\mathcal{H}$-bounds of B such that*

$$S(t)B \subset \mathscr{B}_0 \quad \text{for all } t \geq T_0. \tag{4.1}$$

For the convenience of the reader, we now recall the definition of the global attractor, see [3], [26] for more details.

DEFINITION 4.2. Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a metric space (E, d). A subset $\mathscr{A}$ of E is called a *global attractor* for the semigroup, if $\mathscr{A}$ is compact and enjoys the following properties:

(a) $\mathscr{A}$ is invariant , i.e. $S(t)\mathscr{A} = \mathscr{A}$ for all $t \geq 0$;
(b) $\mathscr{A}$ attracts all bounded sets of E, that is, for any bounded subset B of E, $d(S(t)B, \mathscr{A}) \to 0$ as $t \to 0$, where $d(B, A)$ is the semidistance between two sets A and B,

$$d(B, A) = \sup_{x\in B} \inf_{y\in A} d(x, y).$$

Now we are ready to state our main result in this section.

THEOREM 4.3. *Let $g \in H^{-1}(\mathbb{R}^3)$ and $f \in C^1(\mathbb{R}^4)$ satisfy (1.2) and (1.3), $\{S(t)\}_{t\geq 0}$ be the semigroup generated by a solution of problem (1.1) in $\mathcal{H}$. Then $\{S(t)\}_{t\geq 0}$ possesses a compact global attractor $\mathscr{A}$ in $\mathcal{H}$.*

To prove Theorem 4.3, it is crucial to verify the following result.

PROPOSITION 4.4. *For any positive invariant set $\mathcal{B}$, i.e. $\mathcal{B} \subset E$ is bounded, satisfying $S(t)\mathcal{B} \subset \mathcal{B}$ for all $t \geq 0$, the ω-limit set $\omega(\mathcal{B}) = \bigcap_{t\geq 0} \overline{S(t)\mathcal{B}}$ is compact in E.*

The remaining part will be concerned with the proof of Proposition 4.4. Decompose a solution $u(t)$ as follows:

$$u(t) = h + v(t) + w(t), \tag{4.2}$$

where h solves the elliptic equation

$$\begin{cases} -\Delta h + mh + f(x, h) = g(x), \\ h \in H^1(\mathbb{R}^3), \end{cases} \tag{4.3}$$

$v(t)$ solves the cutting-off equation

$$\begin{cases} v_{tt}+\gamma v_t-\Delta v+mv+\mu(|x|-\delta)(f(x,v+h)-f(x,h))=0, \\ v(0)=u_0-h, \quad v_t(0)=u_1, \end{cases} \tag{4.4}$$

with the cutting-off function $\mu\in C^\infty(\mathbb{R})$ defined by

$$\mu\in C^\infty(\mathbb{R}), \quad 0\le\mu\le 1, \quad \begin{cases} \mu(s)=0 & \text{for } s<0, \\ \mu\in[0,1] & \text{for } s\in[0,1], \\ \mu(s)=1 & \text{for } s>1, \end{cases} \tag{4.5}$$

and the remainder $w(t)$ satisfies

$$\begin{cases} w_{tt}+\gamma w_t-\Delta w+mw \\ \qquad =f(x,h)-f(x,u)+\mu(|x|-\delta)(f(x,v+h)-f(x,h)), \\ w(0)=0, \quad w_t(0)=0, \end{cases} \tag{4.6}$$

where the parameter $\delta>0$ will be determined later.

Then due to assumptions (1.2) and (1.3), the energy functional corresponding to elliptic equation (4.3) is weakly lower semi-continuous and bounded from below on $H^1(\mathbb{R}^3)$, thus the existence of h can be guaranteed.

In terms of $v(t)$, in [21], by the dissipative condition on the nonlinearity f, we have proved that for δ large enough, there exists $\beta>0$ such that the following decay estimate holds:

$$\|v_t\|+\|v\|_{H^1}\le C_7e^{-\beta t}(\|u_0\|_{H^1}+\|u_1\|+\|g\|_{H^{-1}}). \tag{4.7}$$

Note that the dissipative condition (1.3) on the nonlinearity f is the same to that in [21], hence (4.7) also holds for $v(t)$ here.

Next, we will discuss the w-component, which is strongly depending on the growth rate of the nonlinearity, since the nonlinearity in this paper grows faster than that in [21], although the proceeding steps are similar, for the convenience of the reader, we also give its proof here.

Lemma 4.5. *Given $T>0$, there is a compact set $\mathfrak{C}_T\subset\mathcal{H}$ such that for any $0\le t\le T$, it holds $(w(t),w_t(t))\in\mathfrak{C}_T$, where $w=u-h-v$ and h,v solve* (4.3) *and* (4.4) *respectively, with the initial data $(u_0,u_1)\in\mathscr{B}_0$.*

Proof. Let us write $u^1=u-h$, then u^1 satisfies the following equation:

$$\begin{cases} u^1_{tt}+\gamma u^1_t-\Delta u^1+mu^1+f(x,u^1+h)-f(x,h)=0, \\ u^1(0)=u_0-h, \quad u^1_t(0)=u_1. \end{cases} \tag{4.8}$$

Note that $\mu(|x|-\delta)=1$, $|x|>\delta$, hence equation (4.4) coincides with (4.8) on $\mathbb{R}^3\setminus B(\delta)$. Based on the finite speed of propagation property, observe that

$$v=u-h \quad \text{on } (\mathbb{R}^3\setminus B_0(\delta+T+1))\times[0,T]. \tag{4.9}$$

Consequently, the difference $w = u - h - v$ satisfies

$$w \equiv 0 \quad \text{on } (\mathbb{R}^3 \setminus B_0(\delta + T + 1)) \times [0, T]. \tag{4.10}$$

According to the continuous dependence of the solution on the right-hand side for equation (4.6), it suffices to prove that $f(\,\cdot\,,u) - f(\,\cdot\,,h)$ and $f(\,\cdot\,,v+h) - f(\,\cdot\,,h)$ belong to a compact subset of $L^1(0,T;L^2(B_0(\delta+T+1)))$.

By the energy estimate (3.1), u, v are bounded, both in $L^\infty(0,T;H^1(\mathbb{R}^3))$ and $H^1([0,T]\times\mathbb{R}^3)$. Then u, v belong to a compact set of $C([0,T];L^3(B_0(\delta+T+1)))$ via the Aubin–Lions lemma [24]. By the growth condition on f, we have

$$\begin{aligned}
&\|f(x,u) - f(x,h)\|_{L^1(t,t+T;L^2(B_0(\delta+T+1)))}\\
&\leq C\|(1+|u|^{p-1}+|h|^{p-1})(u-h)\|_{L^1(t,t+T;L^2(B_0(\delta+T+1)))}\\
&\leq C\big(1+\|u\|^{p-1}_{L^\infty(t,t+T;L^6(\mathbb{R}^3))}+\|h\|^{p-1}_{L^6(\mathbb{R}^3)}\big)\|u-h\|_{L^1(t,t+T;L^{6/(4-p)}(B_0(\delta+T+1)))},
\end{aligned}$$

and

$$\begin{aligned}
&\|f(x,v+h) - f(x,h)\|_{L^1(t,t+T;L^2(B_0(\delta+T+1)))}\\
&\leq C\|(1+|v+h|^{p-1}+|h|^{p-1})v\|_{L^1(t,t+T;L^2(B_0(\delta+T+1)))}\\
&\leq C\big(1+\|v+h\|^{p-1}_{L^\infty(t,t+T;L^6(\mathbb{R}^3))}+\|h\|^{p-1}_{L^6(\mathbb{R}^3)}\big)\|v\|_{L^1(t,t+T;L^{6/(4-p)}(B_0(\delta+T+1)))}.
\end{aligned}$$

Due to the continuous embedding

$$L^\infty(0,T;L^3) \cap L^{4/(p-2)}(0,T;L^{12/(4-p)}) \hookrightarrow L^1(0,T;L^{6/(4-p)}),$$

it is easy to see that $f(\,\cdot\,,u) - f(\,\cdot\,,h)$ and $f(\,\cdot\,,v+h) - f(\,\cdot\,q,h)$ are compact in $L^1(0,T;L^2(B_0(\delta+T+1)))$. □

Consider the ω-limit set of the absorbing set $\mathscr{B}_0$ (obtained in Lemma 4.1), without loss of generality, assume that $\mathscr{B}_0$ is positive invariant, then $\omega(\mathscr{B}_0) = \bigcap_{t\geq 0} \overline{S(t)\mathscr{B}_0}$. From (4.7) and Lemma 4.1, for any $\varepsilon > 0$, there is $t_\varepsilon > 0$ such that the set $S(t)\mathscr{B}_0$ admits a finite covering of radius $\varepsilon > 0$ as $t \geq t_\varepsilon$. Hence $\omega(\mathscr{B}_0)$ is compact. Proposition 4.2 has been proved. Consequently, the existence of a global attractor for (1.1) is guaranteed, i.e. the proof of Theorem 4.2 is completed. □

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