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ASYMPTOTICALLY ALMOST PERIODIC MOTIONS IN IMPULSIVE SEMIDYNAMICAL SYSTEMS

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Abstract. Recursive properties on impulsive semidynamical systems are considered. We obtain results about almost periodic motions and asymptotically almost periodic motions in the context of impulsive systems. The concept of asymptotic almost periodic motions is introduced via time reparametrizations. We also present asymptotic properties for impulsive systems and for their associated discrete systems.

1. Introduction

The theory of impulsive differential equations is an important tool to describe the evolution of systems where the continuous development of a process is interrupted by abrupt changes of state. An impulsive differential equation is modeled by a system that encompasses a differential equation, which describes the period of continuous variation of state, and additional conditions, which describe the discontinuities of the solutions of the differential equation or of their derivatives at the moments of impulses.

One of the branches of the theory of impulsive differential equations is the theory of impulsive dynamical systems. In recent years, a significant progress has been made in the study of discontinuous dynamical systems. Moreover,

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this theory found application in many fields such as physics, pharmacokinetics, biotechnology, economics, chemical technology, population dynamics and others. The reader may consult, for instance, [5], [6], [9]–[11], [17].

The concepts of almost periodic and asymptotically almost periodic functions were introduced by Bohr in [2] and Fréchet in [18], respectively. Later, these concepts were developed in the context of dynamical systems by Bhatia and Szego in [1] and by Cheban in [12]. The existence of almost periodic and asymptotically almost periodic solutions is one of the most attractive topics in the qualitative theory of differential equations, see [12] and [19], for instance.

The goal of this paper is to consider almost periodic motions in the context of impulsive semidynamical systems. We shall give sufficient conditions to obtain the existence of asymptotic almost periodic motions in impulsive semidynamical systems. Since almost periodicity of motions is deeply connected with stability, we also investigate this connection on impulsive systems. The reader may consult some results about almost periodic motions on impulsive systems in [11].

In the next lines, we describe the organization of this paper. Section 2 deals with the basis of the theory of semidynamical systems with impulses. Section 3 concerns with the main results. This section is divided in four parts. In Subsection 3.1, we study some results about almost periodic motions. We show that all almost periodic points are positively Poisson stable in impulsive systems, see Theorem 3.9. In Subsection 3.2, we present the concepts of asymptotic almost periodic, stationary, periodic, recurrent and Poisson stable motions using time reparametrizations. Some topological properties for these motions are considered. We also use the concept of quasi stability of Zhukovskiĭ for impulsive systems to get asymptotic properties. In Subsection 3.3, we consider discrete systems in the sense of Kaul, [21], which are naturally associated to impulsive semidynamical systems. We study the concepts of almost periodicity and asymptotic almost periodicity for these systems. Asymptotic properties are obtained relating impulsive systems and their associated discrete systems. Finally, in Subsection 3.4, we present sufficient conditions to obtain Zhukovskiĭ quasi stability via Lyapunov stability.

2. Preliminaries

Let (X,d) be a metric space, $\mathbb{R}_+$ be the set of non-negative real numbers, $\mathbb{Z}_+$ be the set of non-negative integers and $\mathbb{N} = \{1,2,\ldots\}$ be the set of natural numbers. The triple $(X,\pi,\mathbb{R}_+)$ is called a *continuous semidynamical system* on X if the mapping $\pi\colon X \times \mathbb{R}_+ \to X$ is continuous with $\pi(x,0) = x$ and $\pi(\pi(x,t),s) = \pi(x,t+s)$, for all $x \in X$ and $t,s \geq 0$.

Along to this text, we shall denote the system $(X,\pi,\mathbb{R}_+)$ simply by (X,π) and we will call it as a semidynamical system, that is, dropping the word continuous.

For every $x \in X$, we consider the continuous mapping $\pi_x\colon \mathbb{R} \to X$ given by $\pi_x(t) = \pi(x,t)$ and we call it the *motion* of x. The *positive orbit* of a point $x \in X$ is given by $\pi^+(x) = \{\pi(x,t) : t \in \mathbb{R}_+\}$. We define

$$\pi(A,\Delta) = \bigcup \{\pi(x,t) : x \in A \text{ and } t \in \Delta\} \quad \text{and} \quad \pi^+(A) = \bigcup \{\pi^+(x) : x \in A\},$$

where $A \subset X$ and $\Delta \subset [0,+\infty)$.

For $t \geq 0$ and $x \in X$, we write $F(x,t) = \{y \in X : \pi(y,t) = x\}$ and, for $\Delta \subset [0,+\infty)$ and $D \subset X$, we define

$$F(D,\Delta) = \bigcup \{F(x,t) : x \in D \text{ and } t \in \Delta\}.$$

A point $x \in X$ is called an *initial point*, if $F(x,t) = \emptyset$ for all $t > 0$.

An *impulsive semidynamical system* $(X,\pi;M,I)$ consists of a semidynamical system (X,π), a nonempty closed subset M of X such that for every $x \in M$, there exists $\varepsilon_x > 0$ such that

$$F(x,(0,\varepsilon_x)) \cap M = \emptyset \quad \text{and} \quad \pi(x,(0,\varepsilon_x)) \cap M = \emptyset,$$

and a continuous function $I\colon M \to X$ whose action we explain below in the description of the impulsive semitrajectory. The set M is called the *impulsive set* and the function I is called the *impulse function*. Denote by $M^+(x)$ the set $\pi(x,(0,+\infty)) \cap M$, $x \in X$.

Consider the function $\phi\colon X \to (0,+\infty]$ given by

$$\phi(x) = \begin{cases} s & \text{if } \pi(x,s) \in M \text{ and } \pi(x,t) \notin M \text{ for } 0 < t < s, \\ +\infty & \text{if } M^+(x) = \emptyset. \end{cases} \tag{2.1}$$

This function is well-defined as presented in [14]. Note that if $M^+(x) \neq \emptyset$ then $\phi(x)$ represents the least positive time for which the trajectory of $x \in X$ meets M. Thus for each $x \in X$, we call $\pi(x,\phi(x))$ the *impulsive point* of x.

The *impulsive semitrajectory* of x in $(X,\pi;M,I)$ is an X-valued function $\widetilde{\pi}_x$ defined on the subset $[0,s)$ of $\mathbb{R}_+$ (s may be $+\infty$). The description of such trajectory follows inductively as described in the following lines.

If $M^+(x) = \emptyset$, then $\widetilde{\pi}_x(t) = \pi(x,t)$ for all $t \in \mathbb{R}_+$ and $\phi(x) = +\infty$. However, if $M^+(x) \neq \emptyset$, then $\phi(x) < +\infty$, $\pi(x,\phi(x)) = x_1 \in M$ and $\pi(x,t) \notin M$ for $0 < t < \phi(x)$. Then we define $\widetilde{\pi}_x$ on $[0,\phi(x)]$ by

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x,t) & \text{if } 0 \leq t < \phi(x), \\ x_1^+ & \text{if } t = \phi(x), \end{cases}$$

where $x_1^+ = I(x_1)$. We denote x by x_0^+.

Since $\phi(x) < +\infty$, the process now continues from x_1^+ onwards. If $M^+(x_1^+) = \emptyset$, then we define $\widetilde{\pi}_x(t) = \pi(x_1^+, t - \phi(x))$, for $\phi(x) \leq t < +\infty$, and $\phi(x_1^+) = +\infty$. When $M^+(x_1^+) \neq \emptyset$, it follows that $\phi(x_1^+) < \infty$, $\pi(x_1^+, \phi(x_1^+)) = x_2 \in M$ and

$\pi(x_1^+, t - \phi(x)) \notin M$, for $\phi(x) < t < \phi(x) + \phi(x_1^+)$. Then we define $\widetilde{\pi}_x$ on $[\phi(x), \phi(x) + \phi(x_1^+)]$ by

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x_1^+, t - \phi(x)) & \text{if } \phi(x) \leq t < \phi(x) + \phi(x_1^+), \\ x_2^+ & \text{if } t = \phi(x) + \phi(x_1^+), \end{cases}$$

where $x_2^+ = I(x_2)$, and so on. Notice that $\widetilde{\pi}_x$ is defined on each interval $[t_n, t_{n+1}]$, where $t_0 = 0$ and $t_{n+1} = \sum_{i=0}^{n} \phi(x_i^+)$, $n = 0, 1, \ldots$ Hence, $\widetilde{\pi}_x$ is defined on $[0, t_{n+1}]$.

The process above ends after a finite number of steps, whenever $M^+(x_n^+) = \emptyset$ for some n. However, it continues infinitely if $M^+(x_n^+) \neq \emptyset$, $n = 1, 2, \ldots$, and in this case $\widetilde{\pi}_x$ is defined on the interval $[0, T(x))$, where $T(x) = \sum_{i=0}^{+\infty} \phi(x_i^+)$.

The *impulsive positive orbit* of a point $x \in X$ is given by

$$\widetilde{\pi}^+(x) = \{\widetilde{\pi}(x, t) : t \in [0, T(x))\}.$$

Analogously to the non-impulsive case, an impulsive semidynamical system satisfies the following standard properties: $\widetilde{\pi}(x, 0) = x$ for all $x \in X$ and $\widetilde{\pi}(\widetilde{\pi}(x, t), s) = \widetilde{\pi}(x, t + s)$, for all $t, s \in [0, T(x))$ such that $t + s \in [0, T(x))$. See [3] for a proof of it.

For details about the structure of these types of impulsive systems, the reader may consult [3]–[7], [10], [14]–[17], [20]–[22].

2.1. Continuity of function ϕ. In the previous section, we defined the function ϕ, see (2.1), which describes the times of meeting the impulsive set M. In this section we discuss the continuity of this function. The reader may consult [13] and [14] for more details.

Let (X, π) be a semidynamical system. Any closed set $S \subset X$ containing x ($x \in X$) is called a *section* or a λ-*section* through x, with $\lambda > 0$, if there exists a closed set $L \subset X$ such that:

(a) $F(L, \lambda) = S$;
(b) $F(L, [0, 2\lambda])$ is a neighbourhood of x;
(c) $F(L, \mu) \cap F(L, \nu) = \emptyset$, for $0 \leq \mu < \nu \leq 2\lambda$.

The set $F(L, [0, 2\lambda])$ is called a *tube* or a λ-*tube* and the set L is called a *bar*.

Any tube $F(L, [0, 2\lambda])$ given by a section S through $x \in X$ such that $S \subset M \cap F(L, [0, 2\lambda])$ is called a *TC-tube* on x. We say that a point $x \in M$ fulfills the *tube condition* and we write TC, if there exists a TC-tube $F(L, [0, 2\lambda])$ through x. In particular, if $S = M \cap F(L, [0, 2\lambda])$ we have a *STC-tube* on x and we say that a point $x \in M$ fulfills the *strong tube condition* (we write STC), if there exists a STC-tube $F(L, [0, 2\lambda])$ through x.

The following result concerns the continuity of ϕ which is accomplished outside of M.

THEOREM 2.1 ([14, Theorem 3.8]). *Consider an impulsive semidynamical system $(X, \pi; M, I)$. Assume that no initial point in (X, π) belongs to the impulsive set M and that each element of M satisfies TC. Then ϕ is continuous in x if and only if $x \notin M$.*

2.2. Additional definitions and auxiliary results. Throughout this paper, we shall assume the following conditions:

(H1) No initial point in (X, π) belongs to the impulsive set M and each element of M satisfies STC, consequently ϕ is continuous on $X \setminus M$.
(H2) $M \cap I(M) = \emptyset$.
(H3) For each $x \in X$, the motion $\widetilde{\pi}(x,t)$ is defined for every $t \geq 0$.

For each $x \in X$, we denote the nth jump instant of x by $t_n(x)$, where

$$t_0(x) = 0 \quad \text{and} \quad t_n(x) = \sum_{j=0}^{n-1} \phi(x_j^+), \quad n = 1, 2, \ldots,$$

where $x_0^+ = x$.

Given $A \subset X$ and $\Delta \subset [0, +\infty)$, we set $\widetilde{\pi}(A, \Delta) = \bigcup\{\widetilde{\pi}(x,t) : x \in A,\ t \in \Delta\}$. If $\widetilde{\pi}(A,t) \subset A$ for every $t \geq 0$, we say that A is *positively $\widetilde{\pi}$-invariant*.

The *positive limit set* of a point $x \in X$ in $(X, \pi; M, I)$ is given by

$$\widetilde{L}^+(x) = \Big\{ y \in X : \text{there is a sequence } \{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+ \\ \text{such that } \lambda_n \xrightarrow{n\to+\infty} +\infty \text{ and } \widetilde{\pi}(x, \lambda_n) \xrightarrow{n\to+\infty} y \Big\}.$$

It is well-known that $\widetilde{L}^+(x) \setminus M$ is positively $\widetilde{\pi}$-invariant, see [10, Proposition 4.1].

A point x is called a *stationary* or *rest point* with respect to $(X, \pi; M, I)$, if $\widetilde{\pi}(x,t) = x$ for all $t \geq 0$. If $\widetilde{\pi}(x,\tau) = x$ for some $\tau > 0$, then x will be called *$\widetilde{\pi}$-periodic* with respect to $(X, \pi; M, I)$.

Next, we mention some important results that will be very useful later on.

LEMMA 2.2 ([4, Corollary 3.9]). *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $x \in X \setminus M$. Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X which converges to x. Then, given $t \geq 0$, there is a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $\varepsilon_n \xrightarrow{n\to+\infty} 0$ and $\widetilde{\pi}(x_n, t+\varepsilon_n) \xrightarrow{n\to+\infty} \widetilde{\pi}(x,t)$.*

LEMMA 2.3 ([4, Lemma 3.8]). *Let $(X, \pi; M, I)$ be an impulsive semidynamical system, $z \notin M$ and $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $X \setminus M$ such that $z_n \xrightarrow{n\to+\infty} z$. Then if $\alpha_n \xrightarrow{n\to+\infty} 0$ and $\alpha_n \geq 0$, for all $n \in \mathbb{N}$, we have $\widetilde{\pi}(z_n, \alpha_n) \xrightarrow{n\to+\infty} z$.*

We note that if $t \geq 0$ is not a jump instant of a point x, that is, $\widetilde{\pi}(x,t) \neq x_j^+$ for every $j = 1, 2, \ldots$, then the convergence in Lemma 2.2 does not depend on the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$. Thus, $\widetilde{\pi}(x_n, t) \xrightarrow{n\to+\infty} \widetilde{\pi}(x,t)$ whenever $t \neq t_k(x)$, for every $k = 1, 2, \ldots$ We formalize this fact in the next lemma.

Lemma 2.4. *Let $(X,\pi;M,I)$ be an impulsive semidynamical system, $x \in X \setminus M$ and $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a sequence in X which converges to x. Given $t \geq 0$ such that $t \neq t_k(x)$, $k = 1,2,\ldots$, and $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ is a sequence with $\lambda_n \xrightarrow{n\to+\infty} t$, then $\widetilde{\pi}(x_n,\lambda_n) \xrightarrow{n\to+\infty} \widetilde{\pi}(x,t)$.*

Proof. If $t = 0$ the result follows by Lemma 2.3. Let $k \in \{0,1,\ldots\}$ be such that $t_k(x) < t < t_{k+1}(x)$. Since $\phi((x_n)_j^+) \xrightarrow{n\to+\infty} \phi(x_j^+)$ for each $j = 0,1,\ldots$, and $\lambda_n \xrightarrow{n\to+\infty} t$, we may assume that $\lambda_n \in (t_k(x_n), t_{k+1}(x_n))$ for all $n \in \mathbb{N}$. Then

$$\widetilde{\pi}(x_n,\lambda_n) = \pi((x_n)_k^+, \lambda_n - t_k(x_n)) \xrightarrow{n\to+\infty} \pi(x_k^+, t - t_k(x)) = \widetilde{\pi}(x,t). \qquad \square$$

Lemma 2.5. *Let $(X,\pi;M,I)$ be an impulsive semidynamical system, $x \in X \setminus M$ and $t \geq 0$. Suppose that the sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ is such that $\lambda_n \geq t$, for every $n \in \mathbb{N}$, and $\lambda_n \xrightarrow{n\to+\infty} t$. If $\{x_n\}_{n\in\mathbb{N}} \subset X$ is a sequence which converges to x then there is a sequence $\{\beta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $\beta_n \xrightarrow{n\to+\infty} 0$ and*

$$\widetilde{\pi}(x_n, \lambda_n + \beta_n) \xrightarrow{n\to+\infty} \widetilde{\pi}(x,t).$$

Proof. If $t \neq t_k(x)$, for every $k = 1,2,\ldots$, then in virtue of Lemma 2.4, we get the result. However, if $t = t_k(x)$ for some $k \in \{1,2,\ldots\}$, then since $\lambda_n \geq t$ for every $n \in \mathbb{N}$, we can write $\lambda_n = t + s_n$ with $s_n \geq 0$ and $s_n \xrightarrow{n\to+\infty} 0$. Now, since $\phi((x_n)_j^+) \xrightarrow{n\to+\infty} \phi(x_j^+)$ for all $j = 0,1,\ldots$, we have

$$t_k(x_n) \xrightarrow{n\to+\infty} t_k(x).$$

Define $T_n = t_k(x_n) - t_k(x)$, $n = 1,2,\ldots$ Thus,

$$\lambda_n = t_k(x_n) - T_n + s_n, \quad n = 1,2,\ldots$$

Taking $\beta_n = |T_n|$, $n = 1,2,\ldots$, and using Lemma 2.3, we obtain

$$\begin{aligned}
\widetilde{\pi}(x_n, \lambda_n + \beta_n) &= \widetilde{\pi}(x_n, t_k(x_n) - T_n + |T_n| + s_n) \\
&= \widetilde{\pi}(\widetilde{\pi}(x_n, t_k(x_n)), |T_n| - T_n + s_n) \\
&= \widetilde{\pi}((x_n)_k^+, |T_n| - T_n + s_n) \xrightarrow{n\to+\infty} \pi(x_k^+, 0) = \widetilde{\pi}(x,t). \qquad \square
\end{aligned}$$

Lemma 2.6. *Let $(X,\pi;M,I)$ be an impulsive semidynamical system, $x \in X \setminus M$ and $t = t_k(x)$, for some $k \in \mathbb{N}$. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be a sequence which converges to t and $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a sequence which converges to x.*

(a) *If $\lambda_n < t$ for all $n \in \mathbb{N}$ then $\widetilde{\pi}(x_n,\lambda_n) \xrightarrow{n\to+\infty} x_k$.*
(b) *If $\lambda_n \geq t$ for all $n \in \mathbb{N}$ then $\{\widetilde{\pi}(x_n,\lambda_n)\}_{n\in\mathbb{N}}$ possesses a subsequence which converges in $\overline{\widetilde{\pi}^+(x)}$.*

Proof. (a) Since $\lambda_n < t$ for every $n \in \mathbb{N}$, there are a sequence $\{s_n\}_{n\in\mathbb{N}} \subset [0, \phi((x_n)^+_{k-1}))$ with $s_n \xrightarrow{n\to+\infty} \phi(x^+_{k-1})$ and $n_0 \in \mathbb{N}$ such that $\lambda_n = t_{k-1}(x_n)+s_n$ for all $n \geq n_0$. Then,

$$\widetilde{\pi}(x_n, \lambda_n) = \pi((x_n)^+_{k-1}, s_n) \xrightarrow{n\to+\infty} \pi(x^+_{k-1}, \phi(x^+_{k-1})) = x_k.$$

(b) Following the proof of Lemma 2.5 for $t = t_k(x)$, if $\{s_n - T_n\}_{n\in\mathbb{N}}$ admits a subsequence $\{s_{n_\ell} - T_{n_\ell}\}_{\ell\in\mathbb{N}}$ such that $s_{n_\ell} - T_{n_\ell} \geq 0$ for all $\ell \in \mathbb{N}$ then, using Lemma 2.3, we have

$$\widetilde{\pi}(x_{n_\ell}, \lambda_{n_\ell}) = \widetilde{\pi}(\widetilde{\pi}(x_{n_\ell}, t_k(x_{n_\ell})), -T_{n_\ell} + s_{n_\ell}) \xrightarrow{\ell\to+\infty} x^+_k \in \overline{\widetilde{\pi}^+(x)}.$$

However, if $\{s_n - T_n\}_{n\in\mathbb{N}}$ admits a subsequence $\{s_{m_\ell} - T_{m_\ell}\}_{\ell\in\mathbb{N}}$ such that $s_{m_\ell} - T_{m_\ell} < 0$ for all $\ell \in \mathbb{N}$ then we write $\lambda_{m_\ell} = t_{k-1}(x_{m_\ell}) + \phi((x_{m_\ell})^+_{k-1}) - T_{m_\ell} + s_{m_\ell}$, $\ell = 1, 2, \ldots$ Note that $\phi((x_{m_\ell})^+_{k-1}) - T_{m_\ell} + s_{m_\ell} > 0$ for ℓ sufficiently large. Then

$$\begin{aligned}\widetilde{\pi}(x_{m_\ell}, \lambda_{m_\ell}) &= \widetilde{\pi}(\widetilde{\pi}(x_{m_\ell}, t_{k-1}(x_{m_\ell})), \phi((x_{m_\ell})^+_{k-1}) - T_{m_\ell} + s_{m_\ell}) \\ &= \widetilde{\pi}((x_{m_\ell})^+_{k-1}, \phi((x_{m_\ell})^+_{k-1}) - T_{m_\ell} + s_{m_\ell}) \\ &\xrightarrow{\ell\to+\infty} \pi(x^+_{k-1}, \phi(x^+_{k-1})) = x_k \in \overline{\widetilde{\pi}^+(x)}. \end{aligned}$$ □

Lemma 2.7 ([5, Lemma 3.6]). *Let $A \subset X$ be non-empty and relatively compact. Then the set $\widetilde{\pi}(A, [0, \ell])$ is relatively compact in X for each $\ell > 0$.*

Remark 2.8. Let $y \in M$. By hypothesis (H1), M satisfies STC. Then there is a STC-tube $F(L_y, [0, 2\lambda_y])$ through y given by a section S_y. Moreover, since the tube is a neighbourhood of y, there is $\eta_y > 0$ such that

$$B(y, \eta_y) \subset F(L_y, [0, 2\lambda_y]).$$

From now on, we shall denote $H^{(y)}_1 = F(L_y, (\lambda_y, 2\lambda_y]) \cap B(y, \eta_y)$ and $H^{(y)}_2 = F(L_y, [0, \lambda_y]) \cap B(y, \eta_y)$.

3. Main results

This section concerns with the main results and it is divided in four subsections. In Subsection 3.1, we present results about almost periodic motions. In Subsection 3.2, we deal with asymptotically almost periodic motions. In Subsection 3.3, we study recursive properties for impulsive systems and for discrete systems in the sense of Kaul. Finally, Subsection 3.4 deals with Lyapunov stability and Zhukovskiĭ quasi stability.

3.1. Almost periodic motions. In [10], Bonotto and Jimenez developed a study about almost periodic motions in impulsive systems.

Definition 3.1. A point $x \in X$ is said to be *almost $\widetilde{\pi}$-periodic* if for every $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ such that for every $\alpha \geq 0$, the interval $[\alpha, \alpha + T]$ contains a number $\tau_\alpha > 0$ such that

$$d(\widetilde{\pi}(x, t + \tau_\alpha), \widetilde{\pi}(x, t)) < \varepsilon, \quad \text{for all } t \geq 0. \tag{3.1}$$

The set $\{\tau_\alpha : \alpha \geq 0\}$ is called a *family of almost period* of x.

In [10, Lemma 4.22] it is proved that if $x \in X$ is almost $\widetilde{\pi}$-periodic, then every point $y \in \widetilde{\pi}^+(x)$ is also almost $\widetilde{\pi}$-periodic. We shall proof a more general result.

Theorem 3.2. *If $x \in X$ is almost $\widetilde{\pi}$-periodic, then every point $y \in \overline{\widetilde{\pi}^+(x)} \setminus M$ is almost $\widetilde{\pi}$-periodic. Moreover, if $\{\tau_\alpha : \alpha \geq 0\}$ is a family of almost period of x then $\{\tau_\alpha : \alpha \geq 0\}$ is also a family of almost period for each $y \in \overline{\widetilde{\pi}^+(x)} \setminus M$.*

Proof. Let $\varepsilon > 0$ be given. Since $x \in X$ is almost $\widetilde{\pi}$-periodic, there is $T = T(\varepsilon/3) > 0$ such that for every $\alpha \geq 0$, the interval $[\alpha, \alpha + T]$ contains a number $\tau_\alpha > 0$ such that

$$d(\widetilde{\pi}(x, t), \widetilde{\pi}(x, t + \tau_\alpha)) < \frac{\varepsilon}{3}, \quad \text{for all } t \geq 0. \tag{3.2}$$

Let $y \in \overline{\widetilde{\pi}^+(x)} \setminus M$. Then there exists a sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ in $\mathbb{R}_+$ such that

$$y_n = \widetilde{\pi}(x, \lambda_n) \xrightarrow{n \to +\infty} y.$$

For each $\alpha \geq 0$, we consider $\tau_\alpha \in [\alpha, \alpha + T]$ which satisfies (3.2). Let $t \geq 0$ be fixed and arbitrary. By Lemma 2.2 there is a sequence $\{\varepsilon_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\varepsilon_n \xrightarrow{n \to +\infty} 0$ and $\widetilde{\pi}(y_n, t + \varepsilon_n) \xrightarrow{n \to +\infty} \widetilde{\pi}(y, t)$. Since $t + \tau_\alpha + \varepsilon_n \geq t + \tau_\alpha$, for every $n \in \mathbb{N}$, and $t + \tau_\alpha + \varepsilon_n \xrightarrow{n \to +\infty} t + \tau_\alpha$, it follows by Lemma 2.5 that there exists a sequence $\{\beta_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\beta_n \xrightarrow{n \to +\infty} 0$ and

$$\widetilde{\pi}(y_n, t + \tau_\alpha + \varepsilon_n + \beta_n) \xrightarrow{n \to +\infty} \widetilde{\pi}(y, t + \tau_\alpha).$$

Moreover, since $\widetilde{\pi}(y, t) \notin M$, it follows by Lemma 2.3, that

$$\widetilde{\pi}(y_n, t + \varepsilon_n + \beta_n) \xrightarrow{n \to +\infty} \widetilde{\pi}(y, t).$$

Thus, there is $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$d(\widetilde{\pi}(y_n, t + \varepsilon_n + \beta_n), \widetilde{\pi}(y, t)) < \frac{\varepsilon}{3} \quad \text{and} \quad d(\widetilde{\pi}(y_n, t + \tau_\alpha + \varepsilon_n + \beta_n), \widetilde{\pi}(y, t + \tau_\alpha)) < \frac{\varepsilon}{3}.$$

Define $\eta_n = \varepsilon_n + \beta_n$, $n \in \mathbb{N}$. Then by the above inequalities and (3.2), we have

$$\begin{aligned} &d(\widetilde{\pi}(y, t), \widetilde{\pi}(y, t + \tau_\alpha)) \leq d(\widetilde{\pi}(y, t), \widetilde{\pi}(y_{n_0}, t + \eta_{n_0})) \\ &+ d(\widetilde{\pi}(y_{n_0}, t + \eta_{n_0}), \widetilde{\pi}(y_{n_0}, t + \tau_\alpha + \eta_{n_0})) + d(\widetilde{\pi}(y_{n_0}, t + \tau_\alpha + \eta_{n_0}), \widetilde{\pi}(y, t + \tau_\alpha)) < \varepsilon. \end{aligned}$$

Since t was taken arbitrary, it shows that $y \in \overline{\widetilde{\pi}^+(x)} \setminus M$ is almost $\widetilde{\pi}$-periodic with the same family of almost period $\{\tau_\alpha : \alpha \geq 0\}$. □

Theorem 3.2 shows that all the points from $\overline{\widetilde{\pi}^+(x)} \setminus M$ are almost $\widetilde{\pi}$-periodic provided that x is almost $\widetilde{\pi}$-periodic. In Theorem 3.3, we characterize the points from $\overline{\widetilde{\pi}^+(x)} \cap M$.

THEOREM 3.3. *Let $x \in X$ be almost $\widetilde{\pi}$-periodic, $y \in \overline{\widetilde{\pi}^+(x)} \cap M$ and $\{y_n\}_{n\in\mathbb{N}} \subset \widetilde{\pi}^+(x)$ be a sequence such that $y_n \xrightarrow{n\to+\infty} y$. In notations of Remark 2.8 we have*:

(a) *If $\{y_n\}_{n\in\mathbb{N}}$ admits a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}} \subset H_2^{(y)}$, then y is almost $\widetilde{\pi}$-periodic. Moreover, y admits the same family of almost period of x.*
(b) *If $\{y_n\}_{n\in\mathbb{N}}$ admits a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}} \subset H_1^{(y)}$, then $I(y)$ is almost $\widetilde{\pi}$-periodic. Moreover, $I(y)$ admits the same family of almost period of x.*

PROOF. Let $y \in \overline{\widetilde{\pi}^+(x)} \cap M$. The proof of item (a) is similar to the proof of Theorem 3.2.

Let us show that item (b) holds. Suppose without loss of generality that $\{y_n\}_{n\in\mathbb{N}} \subset H_1^{(y)}$. Then $\phi(y_n) \xrightarrow{n\to+\infty} 0$ and by continuity of I and π, we obtain

$$z_n = \widetilde{\pi}(y_n, \phi(y_n)) \xrightarrow{n\to+\infty} I(y).$$

Since $\{z_n\}_{n\in\mathbb{N}} \subset \widetilde{\pi}^+(x) \setminus M$ (see hypothesis (H2)), it follows by Theorem 3.2 that z_n is almost $\widetilde{\pi}$-periodic, for each $n \in \mathbb{N}$, with the same family of almost period of x. Consequently, given $\varepsilon > 0$, there is $T = T(\varepsilon/3) > 0$ such that for every $\alpha \geq 0$, we can find $\tau_\alpha \in [\alpha, \alpha + T]$ which satisfies

$$d(\widetilde{\pi}(z_n, t), \widetilde{\pi}(z_n, t + \tau_\alpha)) < \frac{\varepsilon}{3}, \quad \text{for all } t \geq 0 \quad \text{and for all } n \in \mathbb{N}. \tag{3.3}$$

Fix $\alpha \geq 0$ and take $\tau_\alpha \in [\alpha, \alpha + T]$. Now let $t \geq 0$ be fixed and arbitrary. Since $I(y) \notin M$ by hypothesis (H2), it follows from Lemma 2.2 that there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $\varepsilon_n \xrightarrow{n\to+\infty} 0$ and $\widetilde{\pi}(z_n, t + \varepsilon_n) \xrightarrow{n\to+\infty} \widetilde{\pi}(I(y), t)$. Note that $t + \tau_\alpha + \varepsilon_n \geq t + \tau_\alpha$, for every $n \in \mathbb{N}$, and $t + \tau_\alpha + \varepsilon_n \xrightarrow{n\to+\infty} t + \tau_\alpha$. Thus, by Lemma 2.5, there exists a sequence $\{\beta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $\beta_n \xrightarrow{n\to+\infty} 0$ and

$$\widetilde{\pi}(z_n, t + \tau_\alpha + \varepsilon_n + \beta_n) \xrightarrow{n\to+\infty} \widetilde{\pi}(I(y), t + \tau_\alpha).$$

Also, since $\widetilde{\pi}(I(y), t) \notin M$, it follows by Lemma 2.3, that

$$\widetilde{\pi}(z_n, t + \varepsilon_n + \beta_n) \xrightarrow{n\to+\infty} \widetilde{\pi}(I(y), t).$$

Set $\eta_n = \varepsilon_n + \beta_n$, $n \in \mathbb{N}$. We can choose $n_0 \in \mathbb{N}$ such that

$$d(\widetilde{\pi}(z_n, t + \eta_n), \widetilde{\pi}(I(y), t)) < \frac{\varepsilon}{3} \quad \text{and} \quad d(\widetilde{\pi}(z_n, t + \tau_\alpha + \eta_n), \widetilde{\pi}(I(y), t + \tau_\alpha)) < \frac{\varepsilon}{3},$$

for all $n \geq n_0$. In virtue of the above inequalities and (3.3), we have

$$\begin{aligned} d(\widetilde{\pi}(I(y),t),\widetilde{\pi}(I(y),t+\tau_\alpha)) &\leq d(\widetilde{\pi}(I(y),t),\widetilde{\pi}(z_{n_0},t+\eta_{n_0})) \\ &\quad + d(\widetilde{\pi}(z_{n_0},t+\eta_{n_0}),\widetilde{\pi}(z_{n_0},t+\tau_\alpha+\eta_{n_0})) \\ &\quad + d(\widetilde{\pi}(z_{n_0},t+\tau_\alpha+\eta_{n_0}),\widetilde{\pi}(I(y),t+\tau_\alpha)) < \varepsilon. \end{aligned}$$

Since the above inequality holds for each $\tau_\alpha \in [\alpha, \alpha+T]$ $(\alpha \geq 0)$ and $t \geq 0$ was taken arbitrary, $I(y)$ is almost $\widetilde{\pi}$-periodic. □

Remark 3.4. If $x \in X$ is $\widetilde{\pi}$-periodic, then using the same arguments of the proof of Theorem 3.2, we can prove that each point $y \in \overline{\widetilde{\pi}^+(x)} \setminus M$ is $\widetilde{\pi}$-periodic. If $y \in \overline{\widetilde{\pi}^+(x)} \cap M$ then y is $\widetilde{\pi}$-periodic or $I(y)$ is $\widetilde{\pi}$-periodic, its proof is analogous to the proof of Theorem 3.3.

Definition 3.5 deals with the concept of relatively dense sets. This concept is presented in Definition 3.11, Chapter III of [1].

Definition 3.5. A set $D \subset \mathbb{R}_+$ is said to be *relatively dense* in $\mathbb{R}_+$ if there is a number $L > 0$ such that $D \cap [t, t+L] \neq \emptyset$ for all $t \in \mathbb{R}_+$.

Next, we present a result which relates the concept of almost $\widetilde{\pi}$-periodic motions with relatively dense sets.

Lemma 3.6. *A point $x \in X$ is almost $\widetilde{\pi}$-periodic if and only if for every $\varepsilon > 0$ the set*

$$D(\varepsilon) = \Big\{\tau \in \mathbb{R}_+ : \sup_{t \in \mathbb{R}_+} d(\widetilde{\pi}(x,t+\tau),\widetilde{\pi}(x,t)) < \varepsilon\Big\}$$

is relatively dense in $\mathbb{R}_+$.

Proof. Let $x \in X$ be almost $\widetilde{\pi}$-periodic and $\varepsilon > 0$ be given. Then there is $T = T(\varepsilon/2) > 0$ such that for all $\alpha \geq 0$, one can find $0 \neq \tau_\alpha \in [\alpha, \alpha+T]$ such that $\sup_{t \in \mathbb{R}_+} d(\widetilde{\pi}(x,t+\tau_\alpha),\widetilde{\pi}(x,t)) \leq \varepsilon/2 < \varepsilon$. Thus, $D(\varepsilon) \cap [\alpha, \alpha+T] \neq \emptyset$ and the set $D(\varepsilon)$ is relatively dense in $\mathbb{R}_+$. The converse is straightforward. □

A point $x \in X$ is called $\widetilde{\pi}$-*recurrent* if for every $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that for every $t, s \geq 0$, the interval $[0, T]$ contains a number $\tau > 0$ such that $d(\widetilde{\pi}(x,t),\widetilde{\pi}(x,s+\tau)) < \varepsilon$. In [10, Theorem 4.23], the authors show that if a point $x \in X \setminus M$ is almost $\widetilde{\pi}$-periodic then it is $\widetilde{\pi}$-recurrent provided that $\overline{\widetilde{\pi}^+(x)}$ is compact. However, if X is a complete metric space then $\overline{\widetilde{\pi}^+(x)}$ is compact provided $x \in X$ is almost $\widetilde{\pi}$-periodic, see Theorem 3.7 below. In conclusion, if X is complete and $x \in X \setminus M$ is almost $\widetilde{\pi}$-periodic then x is $\widetilde{\pi}$-recurrent, see Corollary 3.8.

Theorem 3.7. *Let $(X, \pi; M, I)$ be an impulsive system and X be a complete metric space. If $x \in X$ is almost $\widetilde{\pi}$-periodic then the set $\overline{\widetilde{\pi}^+(x)}$ is compact.*

PROOF. Let $\varepsilon > 0$ and $x \in X$ be almost $\widetilde{\pi}$-periodic. From Lemma 3.6, the set $D(\varepsilon/4)$ is relatively dense in $\mathbb{R}_+$. Thus

$$d(\widetilde{\pi}(x,t+\tau),\widetilde{\pi}(x,t)) < \frac{\varepsilon}{4}, \tag{3.4}$$

for all $t \in \mathbb{R}_+$ and $\tau \in D(\varepsilon/4)$. Besides, we obtain

$$\begin{aligned} &d(\widetilde{\pi}(x,t+\tau_1),\widetilde{\pi}(x,t+\tau_2)) \\ &\qquad \le d(\widetilde{\pi}(x,t+\tau_1),\widetilde{\pi}(x,t)) + d(\widetilde{\pi}(x,t+\tau_2),\widetilde{\pi}(x,t)) < \frac{\varepsilon}{2}, \end{aligned} \tag{3.5}$$

for all $t \in \mathbb{R}_+$ and for all $\tau_1,\tau_2 \in D(\varepsilon/4)$.

Define $\alpha = \inf\{\tau : \tau \in D(\varepsilon/4)\}$. Then there is a sequence $\{\tau_n\}_{n\in\mathbb{N}} \subset D(\varepsilon/4)$ such that $\tau_n \xrightarrow{n\to+\infty} \alpha$ and $\tau_n \ge \alpha$ for all $n \in \mathbb{N}$. Since $\widetilde{\pi}(x,\,\cdot\,)$ is continuous from the right, we get

$$\widetilde{\pi}(x,t+\tau_n) \xrightarrow{n\to+\infty} \widetilde{\pi}(x,t+\alpha).$$

Then, using (3.5), we obtain

$$\begin{aligned} &d(\widetilde{\pi}(x,t+\alpha),\widetilde{\pi}(x,t+\tau)) \le d(\widetilde{\pi}(x,t+\alpha),\widetilde{\pi}(x,t+\tau_n)) \\ &\qquad + d(\widetilde{\pi}(x,t+\tau_n),\widetilde{\pi}(x,t+\tau)) \le d(\widetilde{\pi}(x,t+\alpha),\widetilde{\pi}(x,t+\tau_n)) + \frac{\varepsilon}{2} \end{aligned} \tag{3.6}$$

for all $t \in \mathbb{R}_+$ and for all $\tau \in D(\varepsilon/4)$. When n approaches to $+\infty$ in (3.6), we obtain

$$d(\widetilde{\pi}(x,t+\alpha),\widetilde{\pi}(x,t+\tau)) \le \frac{\varepsilon}{2}, \quad \text{for all } t \in \mathbb{R}_+ \text{ and for all } \tau \in D\bigg(\frac{\varepsilon}{4}\bigg). \tag{3.7}$$

On the other hand, as $D(\varepsilon/4)$ is relatively dense in $\mathbb{R}_+$, there is $L > 0$ such that $D(\varepsilon/4) \cap [t,t+L] \ne \emptyset$ for all $t \in \mathbb{R}_+$. Let $s > L$, then one can choose $\tau_s \in D(\varepsilon/4) \cap [s-L,s]$. By (3.7) we have

$$d(\widetilde{\pi}(x,s),\widetilde{\pi}(x,s-\tau_s+\alpha)) = d(\widetilde{\pi}(x,(s-\tau_s)+\tau_s),\widetilde{\pi}(x,(s-\tau_s)+\alpha)) < \varepsilon,$$

which implies that $\widetilde{\pi}(x,s) \in B(\widetilde{\pi}(x,[\alpha,L+\alpha]),\varepsilon)$ for all $s > L$. Thus,

$$\widetilde{\pi}(x,t) \in B(\widetilde{\pi}(x,[0,L+\alpha]),\varepsilon) \subset B(Q_\alpha,\varepsilon), \quad \text{for all } t \ge 0,$$

where $Q_\alpha = \overline{\widetilde{\pi}(x,[0,L+\alpha])}$. Since Q_α is compact (Lemma 2.7), we get that $\overline{\widetilde{\pi}^+(x)}$ is totally bounded and, as X is complete, we conclude that $\overline{\widetilde{\pi}^+(x)}$ is compact. □

COROLLARY 3.8. *If X is complete and $x \in X \setminus M$ is almost $\widetilde{\pi}$-periodic then x is $\widetilde{\pi}$-recurrent.*

PROOF. The proof follows from Theorem 3.7 and [10, Theorem 4.23]. □

Let us recall that a point $x \in X$ is said to be *positively Poisson $\widetilde{\pi}$-stable* if $x \in \widetilde{L}^+(x)$. For details, the reader may consult [8]. The next result says that if $x \in X$ is almost $\widetilde{\pi}$-periodic then the positive orbit of x is dense in its positive limit set. As a consequence of Theorem 3.9, we conclude that x is positively Poisson $\widetilde{\pi}$-stable.

Theorem 3.9. *If $x \in X$ is almost $\widetilde{\pi}$-periodic, then $\widetilde{L}^+(x) = \overline{\widetilde{\pi}^+(x)}$. Moreover, x is positively Poisson $\widetilde{\pi}$-stable.*

Proof. It is clear that $\widetilde{L}^+(x) \subset \overline{\widetilde{\pi}^+(x)}$. Let us show that $\overline{\widetilde{\pi}^+(x)} \subset \widetilde{L}^+(x)$. Let $\varepsilon > 0$ and $p \in \overline{\widetilde{\pi}^+(x)}$, then there is a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $\widetilde{\pi}(x,\lambda_n) \xrightarrow{n\to+\infty} p$. Take $n_0 \in \mathbb{N}$ such that

$$d(\widetilde{\pi}(x,\lambda_n),p) < \frac{\varepsilon}{2}, \quad n \geq n_0.$$

Since x is almost $\widetilde{\pi}$-periodic, one can conclude that $\widetilde{\pi}(x,\lambda_n)$ is almost $\widetilde{\pi}$-periodic for each $n \in \mathbb{N}$ with the same family of almost period of x, see Theorem 3.2. Then, there are $T > 0$ and $\tau_n \in [n, n+T]$, $n \in \mathbb{N}$, such that

$$d(\widetilde{\pi}(x,\lambda_n+\tau_n),\widetilde{\pi}(x,\lambda_n)) < \frac{\varepsilon}{2}, \quad \text{for all } n \in \mathbb{N}.$$

Thus, for all $n \geq n_0$, we get

$$d(\widetilde{\pi}(x,\lambda_n+\tau_n),p) \leq d(\widetilde{\pi}(x,\lambda_n+\tau_n),\widetilde{\pi}(x,\lambda_n)) + d(\widetilde{\pi}(x,\lambda_n),p) < \varepsilon.$$

As $\lambda_n + \tau_n \xrightarrow{n\to+\infty} +\infty$, we have $p \in \widetilde{L}^+(x)$ and the proof is complete. □

3.2. Asymptotically almost periodic motions. In [12], the author presents a study of asymptotic stability in the sense of Poisson for dynamical systems (in particular, asymptotic almost periodic motions). He shows various results of asymptotic periodicity and asymptotic almost periodicity using Lyapunov stability. In this section, we present some generalizations for these results to the impulsive case. In order to do that we use the concept of Zhukovskiĭ stability once that it permits lapse of time. For this purpose, we introduce the notion of time reparametrization.

Definition 3.10. A function $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ is called a *time reparametrization* if h is a homeomorphism and $h(0) = 0$.

Definition 3.11. A point $x \in X$ is called *asymptotically $\widetilde{\pi}$-stationary* (resp. *asymptotically $\widetilde{\pi}$-periodic, asymptotically almost $\widetilde{\pi}$-periodic, asymptotically $\widetilde{\pi}$-recurrent, asymptotically Poisson $\widetilde{\pi}$-stable*) if there exist a stationary (resp. $\widetilde{\pi}$-periodic, almost $\widetilde{\pi}$-periodic, $\widetilde{\pi}$-recurrent, positively Poisson $\widetilde{\pi}$-stable) point $p \in X$ and a reparametrization h_p such that

$$\lim_{t\to+\infty} d(\widetilde{\pi}(x,t),\widetilde{\pi}(p,h_p(t))) = 0. \tag{3.8}$$

Remark 3.12. If h is a time reparametrization, then the inverse h^{-1} is also a time reparametrization. Choosing $s = h(t)$, $t \in \mathbb{R}_+$, one can write

$$d(\widetilde{\pi}(x,t), \widetilde{\pi}(y,h(t))) = d(\widetilde{\pi}(x,h^{-1}(s)), \widetilde{\pi}(y,s)).$$

The next result says that if x possesses an asymptotic property then each point from its positive orbit also possesses this property.

Lemma 3.13. *If $x \in X$ is asymptotically almost $\widetilde{\pi}$-periodic (resp. asymptotically $\widetilde{\pi}$-stationary, asymptotically $\widetilde{\pi}$-periodic), then every point $y \in \widetilde{\pi}^+(x)$ is also asymptotically almost $\widetilde{\pi}$-periodic (resp., asymptotically $\widetilde{\pi}$-stationary, asymptotically $\widetilde{\pi}$-periodic).*

Proof. Suppose that $x \in X$ is asymptotically almost $\widetilde{\pi}$-periodic. Then, there exist an almost $\widetilde{\pi}$-periodic point $p \in X$ and a time reparametrization $h_p\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\lim_{t\to+\infty} d(\widetilde{\pi}(x,t), \widetilde{\pi}(p,h_p(t))) = 0. \tag{3.9}$$

Let $y \in \widetilde{\pi}^+(x)$, then $y = \widetilde{\pi}(x,s)$ for some $s \in \mathbb{R}_+$. Note that $q = \widetilde{\pi}(p,h_p(s))$ is almost $\widetilde{\pi}$-periodic, see Theorem 3.2. Consider the function $g_y\colon \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$g_y(t) = h_p(t+s) - h_p(s), \quad \text{for all } t \in \mathbb{R}_+.$$

It is clear that $g_y(0) = 0$ and g_y is a continuous function possessing a continuous inverse function given by $g_y^{-1}(t) = h_p^{-1}(t + h_p(s)) - s$. Then g_y is a time reparametrization. Moreover,

$$\begin{aligned} d(\widetilde{\pi}(y,t), \widetilde{\pi}(q,g_y(t))) &= d(\widetilde{\pi}(x,t+s), \widetilde{\pi}(p,h_p(s) + h_p(t+s) - h_p(s))) \\ &= d(\widetilde{\pi}(x,t+s), \widetilde{\pi}(p,h_p(t+s))) \xrightarrow{t\to+\infty} 0, \end{aligned}$$

where the convergence follows by (3.9). Therefore, $y \in \widetilde{\pi}^+(x)$ is asymptotically almost $\widetilde{\pi}$-periodic. The other cases are analogous. $\square$

Theorem 3.14. *Let $(X,\pi;M,I)$ be an impulsive system and X be a complete metric space. If $x \in X$ is asymptotically almost $\widetilde{\pi}$-periodic, then:*

(a) *$\overline{\widetilde{\pi}^+(x)}$ is compact;*
(b) *$\widetilde{L}^+(x)$ coincides with the closure of an almost $\widetilde{\pi}$-periodic orbit.*

Proof. First, we show that (a) holds. Let $x \in X$ be asymptotically almost $\widetilde{\pi}$-periodic and $\varepsilon > 0$. Then there are an almost $\widetilde{\pi}$-periodic point $p \in X$, a time reparametrization $h_p\colon \mathbb{R}_+ \to \mathbb{R}_+$ and $t_0 > 0$ such that

$$d(\widetilde{\pi}(x,t), \widetilde{\pi}(p,h_p(t))) < \frac{\varepsilon}{4}, \quad \text{for all } t \geq t_0. \tag{3.10}$$

It shows that $\widetilde{\pi}(x,[t_0,+\infty)) \subset B(\overline{\widetilde{\pi}^+(p)}, \varepsilon/4)$.

According to Theorem 3.7, we have that $\overline{\widetilde{\pi}^+(p)}$ is compact because X is complete. Thus, there are $p_1, \ldots, p_k \in \overline{\widetilde{\pi}^+(p)}$ such that

$$\overline{\widetilde{\pi}^+(p)} \subset B\left(p_1, \frac{\varepsilon}{4}\right) \cup \ldots \cup B\left(p_k, \frac{\varepsilon}{4}\right). \tag{3.11}$$

Consequently,

$$B\left(\overline{\widetilde{\pi}^+(p)}, \frac{\varepsilon}{4}\right) \subset B\left(p_1, \frac{\varepsilon}{2}\right) \cup \ldots \cup B\left(p_k, \frac{\varepsilon}{2}\right).$$

Let $h_p(t_0) = \eta$. By Theorem 3.9, there is $\lambda_j > \eta$ such that

$$d(\widetilde{\pi}(p, \lambda_j), p_j) < \frac{\varepsilon}{4}, \quad \text{for each } j = 1, \ldots, k.$$

Then, using (3.11), we get

$$\overline{\widetilde{\pi}^+(p)} \subset B\left(\widetilde{\pi}(p, \lambda_1), \frac{\varepsilon}{2}\right) \cup \ldots \cup B\left(\widetilde{\pi}(p, \lambda_k), \frac{\varepsilon}{2}\right).$$

For each $j = 1, \ldots, k$, let $s_j > t_0$ (because $\lambda_j > \eta$) be such that $h_p(s_j) = \lambda_j$. From (3.10) we have

$$d(\widetilde{\pi}(x, s_j), \widetilde{\pi}(p, \lambda_j)) = d(\widetilde{\pi}(x, s_j), \widetilde{\pi}(p, h_p(s_j))) < \frac{\varepsilon}{4}.$$

We claim that $\widetilde{\pi}(x, [t_0, +\infty)) \subset B(\widetilde{\pi}(x, s_1), \varepsilon) \cup \ldots \cup B(\widetilde{\pi}(x, s_k), \varepsilon)$. Indeed, let $a \in \widetilde{\pi}(x, [t_0, +\infty))$, then there is $j_0 \in \{1, \ldots, k\}$ such that $d(a, p_{j_0}) < \varepsilon/2$. Thus,

$$d(a, \widetilde{\pi}(x, s_{j_0})) \leq d(a, p_{j_0}) + d(p_{j_0}, \widetilde{\pi}(p, \lambda_{j_0})) + d(\widetilde{\pi}(p, \lambda_{j_0}), \widetilde{\pi}(x, s_{j_0})) < \varepsilon.$$

This shows that $\overline{\widetilde{\pi}(x, [t_0, +\infty))}$ is totally bounded and therefore, it is compact since X is complete. By Lemma 2.7, the set $\overline{\widetilde{\pi}(x, [0, t_0])}$ is compact. Hence, $\overline{\widetilde{\pi}^+(x)}$ is compact.

Now, let us show that assertion (b) holds. It is enough to show that $\widetilde{L}^+(x) = \overline{\widetilde{\pi}^+(p)}$, where p is the almost $\widetilde{\pi}$-periodic point found above. Let $q \in \overline{\widetilde{\pi}^+(p)}$. By Theorem 3.9, we have $q \in \widetilde{L}^+(p)$. Then there is a sequence $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ with $s_n \xrightarrow{n\to+\infty} +\infty$ such that $\widetilde{\pi}(p, s_n) \xrightarrow{n\to+\infty} q$. Since

$$d(\widetilde{\pi}(x, h_p^{-1}(s_n)), q) \leq d(\widetilde{\pi}(x, h_p^{-1}(s_n)), \widetilde{\pi}(p, s_n)) + d(\widetilde{\pi}(p, s_n), q)$$

and x is asymptotically almost $\widetilde{\pi}$-periodic, we have

$$d(\widetilde{\pi}(x, h_p^{-1}(s_n)), q) \xrightarrow{n\to+\infty} 0.$$

But $h_p^{-1}(s_n) \to +\infty$ as $n \to +\infty$, which implies that $q \in \widetilde{L}^+(x)$. Thus $\overline{\widetilde{\pi}^+(p)} \subset \widetilde{L}^+(x)$.

On the other hand, take $q \in \widetilde{L}^+(x)$. Then there is a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $\lambda_n \xrightarrow{n\to+\infty} +\infty$ and $\widetilde{\pi}(x, \lambda_n) \xrightarrow{n\to+\infty} q$. By compactness of $\overline{\widetilde{\pi}^+(p)}$, the sequence $\{\widetilde{\pi}(p, h_p(\lambda_n))\}_{n\in\mathbb{N}}$ admits a convergent subsequence. We may assume without loss of generality that $\lim_{n\to+\infty} \widetilde{\pi}(p, h_p(\lambda_n)) = y \in \overline{\widetilde{\pi}^+(p)}$. As

$$d(\widetilde{\pi}(x, \lambda_n), y) \leq d(\widetilde{\pi}(x, \lambda_n), \widetilde{\pi}(p, h_p(\lambda_n))) + d(\widetilde{\pi}(p, h_p(\lambda_n)), y)$$

and x is asymptotically almost $\widetilde{\pi}$-periodic, we have

$$d(\widetilde{\pi}(x, \lambda_n), y) \xrightarrow{n\to+\infty} 0.$$

Thus, by uniqueness, we have $q = y \in \overline{\widetilde{\pi}^+(p)}$ which implies that $\widetilde{L}^+(x) \subset \overline{\widetilde{\pi}^+(p)}$. Consequently, $\widetilde{L}^+(x) = \overline{\widetilde{\pi}^+(p)}$ and assertion (b) is proved. □

Now, we intend to give sufficient conditions for a point to be asymptotically almost $\widetilde{\pi}$-periodic and asymptotically $\widetilde{\pi}$-periodic. Before that, we recall the definition of Zhukovskiĭ quasi $\widetilde{\pi}$-stability which will be useful in the sequel. The reader may consult [17] where this concept was introduced for impulsive systems.

DEFINITION 3.15. A point $x \in X \setminus M$ is called *Zhukovskiĭ quasi $\widetilde{\pi}$-stable* with respect to the set $A \subset X$, if for every $\varepsilon > 0$ there is $\delta = \delta(x, \varepsilon) > 0$ such that if $d(x, y) < \delta$ for $y \in A$, then one can find a time reparametrization h_y such that

$$d(\widetilde{\pi}(x, t), \widetilde{\pi}(y, h_y(t))) < \varepsilon, \quad \text{for all } t \geq 0.$$

A subset $B \subset X \setminus M$ is *Zhukovskiĭ quasi $\widetilde{\pi}$-stable* with respect to the set $A \subset X$, if each point $z \in B$ is Zhukovskiĭ quasi $\widetilde{\pi}$-stable with respect to A.

THEOREM 3.16. *Let $x \in X \setminus M$ satisfy the following conditions*:

(a) $\overline{\widetilde{\pi}^+(x)}$ *is compact*;
(b) $\widetilde{L}^+(x) \setminus M$ *is Zhukovskiĭ quasi $\widetilde{\pi}$-stable with respect to the set* $\widetilde{\pi}^+(x)$;
(c) $\widetilde{L}^+(x)$ *coincides with the closure of an almost $\widetilde{\pi}$-periodic orbit.*

Then x is asymptotically almost $\widetilde{\pi}$-periodic.

PROOF. Let $\varepsilon > 0$ be given. By condition (c), there exists an almost $\widetilde{\pi}$-periodic point $p \in X$ such that $\widetilde{L}^+(x) = \overline{\widetilde{\pi}^+(p)}$. Thus, there are $T = T(\varepsilon) > 0$ and $\tau_n \in [n, n + T]$ such that

$$d(\widetilde{\pi}(p, t + \tau_n), \widetilde{\pi}(p, t)) < \frac{\varepsilon}{2}, \quad \text{for all } t \geq 0 \text{ and for all } n \in \mathbb{N}. \tag{3.12}$$

According to condition (a), we may assume that $\widetilde{\pi}(x, \tau_n) \xrightarrow{n\to+\infty} q$. Since $\tau_n \xrightarrow{n\to+\infty} +\infty$, one can conclude that $q \in \widetilde{L}^+(x) = \overline{\widetilde{\pi}^+(p)}$.

Case 1. $q \in \overline{\widetilde{\pi}^+(p)} \setminus M$.

By Theorem 3.2, the point q is almost $\widetilde{\pi}$-periodic with the same family of almost period of p. Then

$$d(\widetilde{\pi}(q, t + \tau_n), \widetilde{\pi}(q, t)) < \frac{\varepsilon}{2}, \quad \text{for all } t \geq 0 \text{ and for all } n \in \mathbb{N}. \tag{3.13}$$

Since $q \in \widetilde{L}^+(x) \setminus M$, it follows by (b) that there is $\delta = \delta(q, \varepsilon/2) > 0$ such that if $y \in \widetilde{\pi}^+(x)$ and $d(q, y) < \delta$, one can find a time reparametrization h_y such that

$$d(\widetilde{\pi}(y, h_y(t)), \widetilde{\pi}(q, t)) < \frac{\varepsilon}{2}, \quad \text{for all } t \geq 0.$$

By the convergence $\widetilde{\pi}(x, \tau_n) \xrightarrow{n\to+\infty} q$, we may choose $n_0 \in \mathbb{N}$ for which $d(\widetilde{\pi}(x, \tau_{n_0}), q) < \delta$. Consequently, we may find a time reparametrization h_0 such that

$$d(\widetilde{\pi}(x, \tau_{n_0} + h_0(t)), \widetilde{\pi}(q, t)) < \frac{\varepsilon}{2}, \quad \text{for all } t \geq 0. \tag{3.14}$$

Let us define $h_q \colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$h_q(t) = \begin{cases} h_0(t - \tau_{n_0}) + \tau_{n_0} & \text{if } t > \tau_{n_0}, \\ t & \text{if } t \in [0, \tau_{n_0}]. \end{cases} \tag{3.15}$$

Note that h_q is a time reparametrization. Using (3.13) and (3.14), we have

$$\begin{aligned} d(\widetilde{\pi}(x, h_q(t)), \widetilde{\pi}(q, t)) &= d(\widetilde{\pi}(x, h_0(t - \tau_{n_0}) + \tau_{n_0}), \widetilde{\pi}(q, t)) \\ &\leq d(\widetilde{\pi}(x, \tau_{n_0} + h_0(t - \tau_{n_0})), \widetilde{\pi}(q, t - \tau_{n_0})) + d(\widetilde{\pi}(q, t - \tau_{n_0}), \widetilde{\pi}(q, t)) < \varepsilon, \end{aligned}$$

for all $t \geq \tau_{n_0}$. Hence, x is asymptotically almost $\widetilde{\pi}$-periodic.

Case 2. $q \in \overline{\widetilde{\pi}^+(p)} \cap M$.

Since M satisfies STC, in notation of Remark 2.8 we write

$$H_1^{(q)} = F(L_q, (\lambda_q, 2\lambda_q]) \cap B(q, \eta_q) \quad \text{and} \quad H_2^{(q)} = F(L_q, [0, \lambda_q]) \cap B(q, \eta_q).$$

Now, we need to consider the cases when $\{\widetilde{\pi}(x, \tau_n)\}_{n\in\mathbb{N}}$ admits a subsequence in $H_1^{(q)}$ and when $\{\widetilde{\pi}(x, \tau_n)\}_{n\in\mathbb{N}}$ admits a subsequence in $H_2^{(q)}$. For that, we shall consider the cases $\{\widetilde{\pi}(x, \tau_n)\}_{n\in\mathbb{N}} \subset H_2^{(q)}$ and $\{\widetilde{\pi}(x, \tau_n)\}_{n\in\mathbb{N}} \subset H_1^{(q)}$.

Subcase 2.1. $\{\widetilde{\pi}(x, \tau_n)\}_{n\in\mathbb{N}} \subset H_2^{(q)}$.

By Theorem 3.3, the point q is almost $\widetilde{\pi}$-periodic with the same family of almost period of p. Then, for the $\varepsilon > 0$ given before, we have

$$d(\widetilde{\pi}(q, t + \tau_n), \widetilde{\pi}(q, t)) < \frac{\varepsilon}{2}, \quad \text{for all } t \geq 0 \text{ and for all } n \in \mathbb{N}. \tag{3.16}$$

Set $\overline{q} = \widetilde{\pi}(q, s)$ for some $s \in (0, \phi(q))$. By the tube condition we have $\widetilde{\pi}(x, \tau_n + s) \xrightarrow{n\to+\infty} \widetilde{\pi}(q, s) = \overline{q}$, that is, $\overline{q} \in \widetilde{L}^+(x) \setminus M$. Moreover, defining $t_n = \tau_n + s$, it follows by condition (b), there is $n_1 \in \mathbb{N}$ such that one can find a time reparametrization h_1 such that

$$d(\widetilde{\pi}(x, t_{n_1} + h_1(t)), \widetilde{\pi}(\overline{q}, t)) < \frac{\varepsilon}{2}, \quad \text{for all } t \geq 0. \tag{3.17}$$

Suppose that $t \geq t_{n_1} = \tau_{n_1} + s$. From (3.16), we obtain

$$d(\widetilde{\pi}(\overline{q}, t - t_{n_1}), \widetilde{\pi}(q, t)) = d(\widetilde{\pi}(q, t - \tau_{n_1}), \widetilde{\pi}(q, t)) < \frac{\varepsilon}{2}. \tag{3.18}$$

Let $h_q\colon \mathbb{R}_+ \to \mathbb{R}_+$ be given by

$$h_q(t) = \begin{cases} h_1(t - t_{n_1}) + t_{n_1} & \text{if } t > t_{n_1}, \\ t & \text{if } t \in [0, t_{n_1}]. \end{cases} \tag{3.19}$$

Then, h_q is a time reparametrization. Using (3.17) and (3.18), we get

$$\begin{aligned} d(\widetilde{\pi}(x, h_q(t)), \widetilde{\pi}(q,t)) &= d(\widetilde{\pi}(x, h_1(t - t_{n_1}) + t_{n_1}), \widetilde{\pi}(q,t)) \\ &\le d(\widetilde{\pi}(x, t_{n_1} + h_1(t - t_{n_1})), \widetilde{\pi}(\overline{q}, t - t_{n_1})) + d(\widetilde{\pi}(\overline{q}, t - t_{n_1}), \widetilde{\pi}(q,t)) < \varepsilon, \end{aligned}$$

for all $t \ge t_{n_1}$. Hence, x is asymptotically almost $\widetilde{\pi}$-periodic.

Subcase 2.2. $\{\widetilde{\pi}(x, \tau_n)\}_{n\in\mathbb{N}} \subset H_1^{(q)}$.

In virtue of Theorem 3.3, the point $I(q)$ is almost $\widetilde{\pi}$-periodic with the same family of almost period of p. Then, for the $\varepsilon > 0$ given before, we have

$$d(\widetilde{\pi}(I(q), t + \tau_n), \widetilde{\pi}(I(q), t)) < \frac{\varepsilon}{2}, \quad \text{for all } t \ge 0 \text{ and for all } n \in \mathbb{N}. \tag{3.20}$$

Let $y_n = \widetilde{\pi}(x, \tau_n)$ and $z_n = \widetilde{\pi}(y_n, \phi(y_n))$, for $n \in \mathbb{N}$. Note that $y_n \xrightarrow{n\to+\infty} q$, $\phi(y_n) \xrightarrow{n\to+\infty} 0$ and

$$z_n = \widetilde{\pi}(y_n, \phi(y_n)) = I(\pi(y_n, \phi(y_n))) \xrightarrow{n\to+\infty} I(\pi(q,0)) = I(q).$$

Thus $I(q) \in \widetilde{L}^+(x) \setminus M$. Moreover, since $\tau_n \xrightarrow{n\to+\infty} +\infty$, we can choose $n_2 \in \mathbb{N}$ such that $\phi(y_n) < \tau_n$ for every $n \ge n_2$. By condition (b), there are $n_3 \in \mathbb{N}$ with $n_3 > n_2$ and a time reparametrization h_3 such that

$$d(\widetilde{\pi}(z_{n_3}, h_3(t)), \widetilde{\pi}(I(q), t)) < \frac{\varepsilon}{2}, \quad \text{for all } t \ge 0. \tag{3.21}$$

Define $h_{I(q)}\colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$h_{I(q)}(t) = \begin{cases} h_3(t - \tau_{n_3}) + \tau_{n_3} + \phi(y_{n_3}) & \text{if } t > \tau_{n_3}, \\ t + \phi(y_{n_3}) & \text{if } t \in (\phi(y_{n_3}), \tau_{n_3}], \\ 2t & \text{if } t \in [0, \phi(y_{n_3})]. \end{cases} \tag{3.22}$$

Then $h_{I(q)}$ is a time reparametrization and using (3.20), (3.21) and (3.22) we have

$$\begin{aligned} d(\widetilde{\pi}(x, h_{I(q)}(t)), \widetilde{\pi}(I(q), t)) &= d(\widetilde{\pi}(x, h_3(t - \tau_{n_3}) + \tau_{n_3} + \phi(y_{n_3})), \widetilde{\pi}(I(q), t)) \\ &= d(\widetilde{\pi}(z_{n_3}, h_3(t - \tau_{n_3})), \widetilde{\pi}(I(q), t)) \\ &\le d(\widetilde{\pi}(z_{n_3}, h_3(t - \tau_{n_3})), \widetilde{\pi}(I(q), t - \tau_{n_3})) \\ &\quad + d(\widetilde{\pi}(I(q), t - \tau_{n_3}), \widetilde{\pi}(I(q), t)) < \varepsilon, \end{aligned}$$

for all $t > \tau_{n_3}$. Therefore, x is asymptotically almost $\widetilde{\pi}$-periodic and the result is proved. □

Corollary 3.17. *Let $x \in X \setminus M$ satisfy the following conditions*:

(a) $\overline{\widetilde{\pi}^+(x)}$ *is compact*;
(b) $\widetilde{L}^+(x) \setminus M$ *is Zhukovskiĭ quasi* $\widetilde{\pi}$*-stable with respect to the set* $\widetilde{\pi}^+(x)$;
(c) $\widetilde{L}^+(x)$ *coincides with the closure of a* $\widetilde{\pi}$*-periodic orbit.*

Then x *is asymptotically* $\widetilde{\pi}$*-periodic.*

Proof. Using the same arguments as in the proof of Theorem 3.16, we get the result. □

Next we present a result which gives us conditions for a set to be Zhukovskiĭ quasi $\widetilde{\pi}$-stable. In [17], this result was presented for a restricted system where the phase space is $X \setminus M$. Thus, our result generalizes [17, Theorem 3.4].

Theorem 3.18. *Let* $(X, \pi; M, I)$ *be an impulsive system and* $x \in X \setminus M$. *Suppose that* $T = \sup_{k\geq 1} \phi(x_k^+) < +\infty$ *and the following assumptions hold*:

(a) $d(I(p), I(q)) \leq \lambda_1 d(p,q)$ *for all* $p, q \in M$ *and* $d(\pi(p,\phi(p)), \pi(q,\phi(q))) \leq \lambda_2 d(p,q)$ *for all* $p, q \in \overline{\widetilde{\pi}^+(x)} \setminus M$, *where* $\lambda_1, \lambda_2 > 0$ *and* $\lambda_1\lambda_2 \leq 1$;
(b) $|\phi(p_1^+) - \phi(q_1^+)| \leq |\phi(p) - \phi(q)|$ *for all* $p, q \in \overline{\widetilde{\pi}^+(x)} \setminus M$;
(c) $\overline{\widetilde{\pi}^+(x)}$ *is compact.*

Then every subset A *in* $\overline{\widetilde{\pi}^+(x)} \setminus M$ *is Zhukovskiĭ quasi* $\widetilde{\pi}$*-stable with respect to* $\widetilde{\pi}^+(x)$.

Proof. Let $\varepsilon > 0$ be given and $z \in \overline{\widetilde{\pi}^+(x)} \setminus M$. Since π is uniformly continuous on $K = \overline{\widetilde{\pi}^+(x)} \times [0,T]$, there is $\delta_1 = \delta_1(K,\varepsilon) > 0$, $\delta_1 < \varepsilon$, such that if $y \in \widetilde{\pi}^+(x)$, $t_1, t_2 \in [0,T]$ and $\max\{d(y,z), |t_1 - t_2|\} < \delta_1$, then

$$d(\pi(z,t_1), \pi(y,t_2)) < \varepsilon. \tag{3.23}$$

Now, since ϕ is continuous on $X \setminus M$, there is $\delta = \delta(z,\delta_1) > 0$, $\delta < \delta_1$, such that if $y \in X$ and $d(y,z) < \delta$ then

$$|\phi(y) - \phi(z)| < \delta_1. \tag{3.24}$$

Thus, if $d(y,z) < \delta$, it follows by conditions (a), (b) and (3.24) that

$$d(z_1^+, y_1^+) \leq \lambda_1\lambda_2 d(z,y) < \delta \quad \Rightarrow \quad |\phi(z_1^+) - \phi(y_1^+)| \leq |\phi(z) - \phi(y)| < \delta_1,$$

where $z_1 = \pi(z,\phi(z))$, $y_1 = \pi(y,\phi(y))$, $z_1^+ = I(z_1)$ and $y_1^+ = I(y_1)$. Using the principle of induction, if $d(z,y) < \delta$ then $d(z_n^+, y_n^+) < \delta$, and therefore,

$$|\phi(z_n^+) - \phi(y_n^+)| < \delta_1, \quad \text{for all } n = 0, 1, \ldots$$

Define the time reparametrization $h_y \colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$h_y(t) = t_n(y) + \frac{\phi(y_n^+)}{\phi(z_n^+)}(t - t_n(z)) \quad \text{if } t \in [t_n(z), t_{n+1}(z)),\ n = 0, 1, 2, \ldots$$

Thus, if $t \in [t_n(z), t_{n+1}(z))$, $n = 0, 1, \ldots$, we have

$$d(\widetilde{\pi}(z,t), \widetilde{\pi}(y, h_y(t))) = d\bigg(\pi(z_n^+, t - t_n(z)), \pi\bigg(y_n^+, \frac{\phi(y_n^+)}{\phi(z_n^+)}(t - t_n(z))\bigg)\bigg).$$

If $d(z,y) < \delta$ then $d(z_n^+, y_n^+) < \delta$ for all $n = 0, 1, \dots$, consequently,

$$\left| t - t_n(z) - \frac{\phi(y_n^+)}{\phi(z_n^+)} (t - t_n(z)) \right| < |\phi(z_n^+) - \phi(y_n^+)| < \delta_1,$$

and by (3.23), we obtain $d(\widetilde{\pi}(z,t), \widetilde{\pi}(y, h_y(t))) < \varepsilon$. Since $t \geq 0$ was taken arbitrary, we may conclude that every point $z \in A \subset \overline{\widetilde{\pi}^+(x)} \setminus M$ is Zhukovskiĭ quasi $\widetilde{\pi}$-stable with respect to the set $\widetilde{\pi}^+(x)$ and the proof is complete. □

Corollary 3.19. *Let $(X, \pi; M, I)$ be an impulsive system satisfying conditions* (a) *and* (b) *of Theorem* 3.18*, X be complete and $x \in X \setminus M$ be asymptotically almost $\widetilde{\pi}$-periodic. If $\phi(x_k^+) < +\infty$ for all $k \in \mathbb{N}$ then $\widetilde{L}^+(x) \setminus M$ is Zhukovskiĭ quasi $\widetilde{\pi}$-stable with respect to $\widetilde{\pi}^+(x)$.*

Proof. By Theorem 3.14, the set $\overline{\widetilde{\pi}^+(x)}$ is compact. Thus $\overline{\{x_k^+ : k \in \mathbb{N}\}}$ is compact. By the compactness of $\overline{\{x_k^+ : k \in \mathbb{N}\}}$, hypothesis (H2) and, since $\{x_k^+\}_{k \in \mathbb{N}} \subset I(M)$, we have $\overline{\{x_k^+ : k \in \mathbb{N}\}} \cap M = \emptyset$ and therefore, $\sup_{k \geq 1} \phi(x_k^+) < +\infty$ because ϕ is uniformly continuous on compact sets $K \subset X \setminus M$. The result follows from Theorem 3.18. □

Example 3.20. Consider the impulsive semidynamical system $(\mathbb{R}^2, \pi; M, I)$, where $(\mathbb{R}^2, \pi)$ is a continuous semidynamical system given by

$$\pi((x,y),t) = (x+t, y), \quad (x,y) \in \mathbb{R}^2 \text{ and } t \geq 0,$$

$M = \{(x,y) \in \mathbb{R}^2 : x = 2\}$ and $I \colon M \to X$ is given by $I(x,y) = (1, y/2)$, $(x,y) \in M$. Note that the impulse operator I satisfies

$$d(I(p), I(q)) \leq \frac{1}{2} d(p,q), \quad \text{for all } p, q \in M.$$

It is easy to see that if $(x,y) \in \mathbb{R}^2$ and $x < 2$ then

$$\phi(x,y) = 2 - x \quad \text{and} \quad \pi((x,y), \phi(x,y)) = (2, y).$$

We claim that the point $(1,1)$ is asymptotically $\widetilde{\pi}$-periodic. In fact, it is enough to show that the conditions of Corollary 3.17 are satisfied.

At first, we note that $\overline{\widetilde{\pi}^+(1,1)}$ is a compact set. If $p, q \in \overline{\widetilde{\pi}^+(1,1)} \setminus M$ then $|\phi(p_1^+) - \phi(q_1^+)| = 0 \leq |\phi(p) - \phi(q)|$, $d(\pi(p, \phi(p)), \pi(q, \phi(q))) \leq d(p,q)$ and $\phi((1,1)_k^+) = 1$ for all $k = 1, 2, \dots$ By Theorem 3.18, $\widetilde{L}^+(1,1) \setminus M$ is Zhukovskiĭ quasi $\widetilde{\pi}$-stable with respect to the set $\widetilde{\pi}^+(1,1)$.

Now, it is not difficult to see that $\widetilde{L}^+(1,1) = [1,2] \times \{0\}$. Clearly, $(1,0)$ is $\widetilde{\pi}$-periodic and $\overline{\widetilde{\pi}^+(1,0)} = [1,2] \times \{0\} = \widetilde{L}^+(1,1)$. Hence, the conditions of Corollary 3.17 are satisfied and hence $(1,1)$ is asymptotically $\widetilde{\pi}$-periodic.

Theorem 3.21. *Let $x \in X \setminus M$ be asymptotically $\widetilde{\pi}$-periodic. Then there is $\tau > 0$ such that $\{\widetilde{\pi}(x, n\tau) : n \in \mathbb{N}\}$ is relatively compact in X.*

Proof. Since $x \in X \setminus M$ is asymptotically $\widetilde{\pi}$-periodic, there exist $p \in X$, $\widetilde{\pi}$-periodic with period $\tau > 0$, and a time reparametrization $h_p\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\lim_{t\to+\infty} d(\widetilde{\pi}(x,t), \widetilde{\pi}(p, h_p(t))) = 0.$$

In particular,

$$\lim_{n\to+\infty} d(\widetilde{\pi}(x,n\tau), \widetilde{\pi}(p, h_p(n\tau))) = 0. \tag{3.25}$$

For each $n \in \mathbb{N}$, there are $k_n \in \mathbb{N}$ and $r_n \in [0,\tau)$ such that $h_p(n\tau) = k_n\tau + r_n$. We may assume without loss of the generality that $r_n \xrightarrow{n\to+\infty} r_0 \in [0,\tau]$. Since $\widetilde{\pi}(p, h_p(n\tau)) = \widetilde{\pi}(p, r_n)$, by Lemmas 2.4 and 2.6, $\{\widetilde{\pi}(p, h_p(n\tau))\}_{n\in\mathbb{N}}$ admits a convergent subsequence in X. Using (3.25), we obtain that the sequence $\{\widetilde{\pi}(x, n\tau)\}_{n\in\mathbb{N}}$ possesses a convergent subsequence in X. □

The next result provides sufficient conditions to obtain the converse of Theorem 3.21.

Theorem 3.22. *Suppose that* $(X,\pi;M,I)$ *satisfies conditions* (a) *and* (b) *of Theorem* 3.18, $x \in X \setminus M$ *and* $\sup_{k\geq 1}\phi(x_k^+) < +\infty$. *If the sequence* $\{\widetilde{\pi}(x,n\tau)\}_{n\in\mathbb{N}}$ *converges in* $X \setminus M$, *for some* $\tau > 0$, *then* x *is asymptotically* $\widetilde{\pi}$*-periodic.*

Proof. In order to show this result, we will show that the conditions of Corollary 3.17 are satisfied. Let $p \in X \setminus M$ be such that $\widetilde{\pi}(x, n\tau) \xrightarrow{n\to+\infty} p$.

Firstly, let us prove that $\overline{\widetilde{\pi}^+(x)}$ is compact. Indeed, let $\{y_n\}_{n\in\mathbb{N}} \subset \widetilde{\pi}^+(x)$ be an arbitrary sequence. Then there is a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $y_n = \widetilde{\pi}(x, t_n)$, for every $n \in \mathbb{N}$.

If $\{t_n\}_{n\in\mathbb{N}}$ admits a convergent subsequence then $\{y_n\}_{n\in\mathbb{N}}$ also admits one. Now, if $t_n \xrightarrow{n\to+\infty} +\infty$, then for each $n \in \mathbb{N}$ there are $a_n \in \mathbb{N}$ and $b_n \in [0,\tau)$ such that $t_n = a_n\tau + b_n$. We may assume that $b_n \xrightarrow{n\to+\infty} b_0 \in [0,\tau]$. Since $a_n\tau \xrightarrow{n\to+\infty} +\infty$, $b_n \xrightarrow{n\to+\infty} b_0$ and $\widetilde{\pi}(x, a_n\tau) \xrightarrow{n\to+\infty} p$, it follows that the sequence $\{y_n\}_{n\in\mathbb{N}}$ ($y_n = \widetilde{\pi}(\widetilde{\pi}(x, a_n\tau), b_n)$, $n = 1,2,\ldots$) admits a convergent subsequence in $\overline{\widetilde{\pi}^+(x)}$, see Lemmas 2.4 and 2.6. Thus, $\overline{\widetilde{\pi}^+(x)}$ is compact.

According to Theorem 3.18, $\widetilde{L}^+(x) \setminus M$ is Zhukovskiĭ quasi $\widetilde{\pi}$-stable with respect to the set $\widetilde{\pi}^+(x)$.

Now, we need to prove that $\widetilde{L}^+(x)$ is the closure of a $\widetilde{\pi}$-periodic orbit. Recall that $p = \lim_{n\to+\infty} \widetilde{\pi}(x, n\tau)$.

Since $p \notin M$, it follows by Lemma 2.2, that there is a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ with $\varepsilon_n \xrightarrow{n\to+\infty} 0$ such that

$$\widetilde{\pi}(\widetilde{\pi}(x,n\tau), \tau + \varepsilon_n) \xrightarrow{n\to+\infty} \widetilde{\pi}(p,\tau).$$

On the other hand, using Lemma 2.3, we get

$$\widetilde{\pi}(\widetilde{\pi}(x,n\tau), \tau + \varepsilon_n) = \widetilde{\pi}(\widetilde{\pi}(x,(n+1)\tau), \varepsilon_n) \xrightarrow{n\to+\infty} \widetilde{\pi}(p,0) = p.$$

By the uniqueness of limit, we get $p = \widetilde{\pi}(p, \tau)$, that is, p is $\widetilde{\pi}$-periodic.

We claim that $\widetilde{L}^+(x) = \overline{\widetilde{\pi}^+(p)} = \overline{\widetilde{\pi}(p, [0, \tau])}$. Indeed, if $q \in \widetilde{L}^+(x)$, then there is a sequence $\{\tau_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\tau_n \xrightarrow{n \to +\infty} +\infty$ and $\widetilde{\pi}(x, \tau_n) \xrightarrow{n \to +\infty} q$. There are $k_n \in \mathbb{N}$ and $r_n \in [0, \tau)$ such that $\tau_n = k_n \tau + r_n$, $n \in \mathbb{N}$. Using Lemmas 2.4 and 2.6, we may assume (we take a subsequence if necessary) that

$$\widetilde{\pi}(x, \tau_n) = \widetilde{\pi}(\widetilde{\pi}(x, k_n \tau), r_n) \xrightarrow{n \to +\infty} q \in \overline{\widetilde{\pi}(p, [0, \tau])}.$$

On the other hand, let $q \in \overline{\widetilde{\pi}(p, [0, \tau])}$, then there is a sequence $\{s_n\}_{n \in \mathbb{N}} \subset [0, \tau]$ such that $q = \lim_{n \to +\infty} \widetilde{\pi}(p, s_n)$. We may assume without loss of generality that $s_n \xrightarrow{n \to +\infty} s_0 \in [0, \tau]$. If $s_0 \neq t_k(p)$, for every $k = 1, 2, \ldots$, then by Lemma 2.4, we get $q = \widetilde{\pi}(p, s_0)$. Let $t_n = n\tau + s_0$, thus $t_n \xrightarrow{n \to +\infty} +\infty$ and, using Lemma 2.4, again we have

$$\widetilde{\pi}(x, t_n) = \widetilde{\pi}(\widetilde{\pi}(x, n\tau), s_0) \xrightarrow{n \to +\infty} \widetilde{\pi}(p, s_0) = q.$$

Then, $q \in \widetilde{L}^+(x)$.

Now, if $s_0 = t_k(p)$ for some $k \in \mathbb{N}$, then looking at the proof of Lemma 2.6 and taking in account that $\widetilde{\pi}(x, n\tau) \xrightarrow{n \to +\infty} p$, we get that either

$$\widetilde{\pi}(x, t_n) = \widetilde{\pi}(\widetilde{\pi}(x, n\tau), s_0) \xrightarrow{n \to +\infty} \widetilde{\pi}(p, s_0) = p_k^+ = q$$

or

$$\widetilde{\pi}(x, t_n) = \widetilde{\pi}(\widetilde{\pi}(x, n\tau), s_0) \xrightarrow{n \to +\infty} p_k = q.$$

In both of cases, we have $q \in \widetilde{L}^+(x)$. Hence, $\widetilde{L}^+(x) = \overline{\widetilde{\pi}(p, [0, \tau])}$. By Corollary 3.17 we have x is asymptotically $\widetilde{\pi}$-periodic. □

Corollary 3.23. *Suppose that* $(X, \pi; M, I)$ *satisfies conditions* (a) *and* (b) *of Theorem* 3.18, $x \in X \setminus M$ *and* $\phi(x_k^+) < +\infty$ *for all* $k \in \mathbb{N}$. *If the sequence* $\{\widetilde{\pi}(x, n\tau)\}_{n \in \mathbb{N}}$ *converges in* $X \setminus M$, *for some* $\tau > 0$, *then* x *is asymptotically* $\widetilde{\pi}$*-periodic.*

Proof. By the proof of Theorem 3.22 we have that $\overline{\widetilde{\pi}^+(x)}$ is compact. Thus $\overline{\{x_k^+ : k \in \mathbb{N}\}}$ is compact. By the compactness of $\overline{\{x_k^+ : k \in \mathbb{N}\}}$, hypothesis (H2) and, since $\{x_k^+\}_{k \in \mathbb{N}} \subset I(M)$, we have $\overline{\{x_k^+ : k \in \mathbb{N}\}} \cap M = \emptyset$ and therefore, $\sup_{k \geq 1} \phi(x_k^+) < +\infty$ because ϕ is uniformly continuous on compact sets $K \subset X \setminus M$. The result follows from Theorem 3.22. □

We finish this section presenting necessary and sufficient conditions for a point to be asymptotically $\widetilde{\pi}$-stationary.

Theorem 3.24. *Let* $(X, \pi; M, I)$ *be an impulsive system. Then* $x \in X$ *is asymptotically* $\widetilde{\pi}$*-stationary if and only if the sequence* $\{\widetilde{\pi}(x, t_n)\}_{n \in \mathbb{N}}$ *converges in* X, *where* $t_n = \sum_{k=1}^{n} 1/k$, $n \in \mathbb{N}$.

Proof. First, we suppose that x is asymptotically $\widetilde{\pi}$-stationary. Then, there is a stationary point $p \in X$ such that

$$\lim_{t\to+\infty} d(\widetilde{\pi}(x,t),p) = 0. \tag{3.26}$$

Since $t_n = \sum_{k=1}^{n} 1/k \xrightarrow{n\to+\infty} +\infty$, it follows from (3.26) that $\{\widetilde{\pi}(x,t_n)\}_{n\in\mathbb{N}}$ converges to $p \in X$.

Conversely, suppose that $\widetilde{\pi}(x,t_n) \xrightarrow{n\to+\infty} p \in X$, where $t_n = \sum_{k=1}^{n} 1/k$, $n \in \mathbb{N}$. Let $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be an arbitrary sequence such that $s_n \xrightarrow{n\to+\infty} +\infty$. There are $n_0 \in \mathbb{N}$ and a sequence $\{m_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ such that $t_{m_n} < s_n \leq t_{m_n+1}$, for all $n \geq n_0$. Note that $0 < s_n - t_{m_n} \leq t_{m_n+1} - t_{m_n} = 1/(m_n+1)$ for $n \geq n_0$.

Case 1. $p \notin M$. In this case we have

$$\widetilde{\pi}(x,s_n) = \widetilde{\pi}(\widetilde{\pi}(x,t_{m_n}), s_n - t_{m_n}) \xrightarrow{n\to+\infty} p,$$

since $\widetilde{\pi}(x,t_{m_n}) \xrightarrow{n\to+\infty} p$ and $(s_n - t_{m_n}) \xrightarrow{n\to+\infty} 0$. Since $\{s_n\}_{n\in\mathbb{N}}$ is arbitrary, we have

$$\lim_{t\to+\infty} d(\widetilde{\pi}(x,t),p) = 0.$$

If $0 \leq s < \phi(p)$ then, by Lemma 2.4, we get

$$p = \lim_{t\to+\infty} \widetilde{\pi}(x,t+s) = \pi(p,s) = \widetilde{\pi}(p,s).$$

Hence, p is stationary and x is asymptotically $\widetilde{\pi}$-stationary.

Case 2. $p \in M$. By hypothesis (H1), the set M satisfies STC. Using Remark 2.8, we may write $H_1^{(p)} = F(L_p,(\lambda_p,2\lambda_p]) \cap B(p,\eta_p)$ and $H_2^{(p)} = F(L_p,[0,\lambda_p]) \cap B(p,\eta_p)$. We may assume that $\{\widetilde{\pi}(x,t_{m_n})\}_{n\in\mathbb{N}} \subset H_1^{(p)}$. In fact, suppose that there is a subsequence $\{m_{n_k}\}_{k\in\mathbb{N}}$ such that $\widetilde{\pi}(x,t_{m_{n_k}}) \in H_2^{(p)}$ for all $k \in \mathbb{N}$. Let $\{r_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ be an arbitrary sequence such that $r_k \xrightarrow{k\to+\infty} +\infty$. As we did before, we may assume that $t_{m_{n_k}} < r_k \leq t_{m_{n_k}+1}$ for all $k \in \mathbb{N}$. Then

$$\widetilde{\pi}(x,r_k) = \widetilde{\pi}(\widetilde{\pi}(x,t_{m_{n_k}}), r_k - t_{m_{n_k}}) \xrightarrow{k\to+\infty} p \in M.$$

Since $\{r_k\}_{k\in\mathbb{N}}$ is arbitrary, we have $\lim_{t\to+\infty} d(\widetilde{\pi}(x,t),p) = 0$. This shows that p is stationary since

$$p = \lim_{k\to+\infty} \widetilde{\pi}(x, s + t_{m_{n_k}}) = \pi(p,s)$$

for all $s \in [0,\phi(p))$. But $p \in M$ and it contradicts the definition of an impulsive system.

Then, let us assume $\{\widetilde{\pi}(x,t_{m_n})\}_{n\in\mathbb{N}} \subset H_1^{(p)}$. In this case, we may also suppose that $t_{m_n} < s_n - \phi(\widetilde{\pi}(x,t_{m_n})) \leq t_{m_n+1}$, for all $n \geq n_0$, since $\phi(\widetilde{\pi}(x,t_{m_n})) \xrightarrow{n\to+\infty} 0$. Consequently,

$$0 < s_n - t_{m_n} - \phi(\widetilde{\pi}(x,t_{m_n})) \leq t_{m_n+1} - t_{m_n} = \frac{1}{m_n+1}.$$

Since

$$\widetilde{\pi}(x, t_{m_n} + \phi(\widetilde{\pi}(x, t_{m_n}))) \xrightarrow{n\to+\infty} I(p) \quad \text{and} \quad [s_n - t_{m_n} - \phi(\widetilde{\pi}(x, t_{m_n}))] \xrightarrow{n\to+\infty} 0,$$

we have

$$\widetilde{\pi}(x, s_n) = \widetilde{\pi}(\widetilde{\pi}(x, t_{m_n} + \phi(\widetilde{\pi}(x, t_{m_n}))), s_n - t_{m_n} - \phi(\widetilde{\pi}(x, t_{m_n}))) \xrightarrow{n\to+\infty} I(p).$$

Since $\{s_n\}_{n\in\mathbb{N}}$ is arbitrary, we have $\lim_{t\to+\infty} d(\widetilde{\pi}(x, t), I(p)) = 0$. Hence, $I(p)$ is stationary and x is asymptotically $\widetilde{\pi}$-stationary. Therefore, the result is proved.□

3.3. Discrete dynamical systems in the sense of Kaul: almost periodic and asymptotically almost periodic motions. In [21], the author considers an impulsive semidynamical system $(\Omega, \widetilde{\pi})$, where $\Omega \subset X$ is an open set in a metric space X and the continuous impulse function I is defined on the boundary $\partial\Omega$ of Ω in X and takes values in Ω. Kaul defines a discrete dynamical system associated to the impulsive semidynamical system $(\Omega, \widetilde{\pi})$ and presents a study of Lyapunov stability and recursive properties in $(\Omega, \widetilde{\pi})$ by relating them to the corresponding discrete system, see [21] and [22].

Following Kaul's ideas we also may define a discrete system associated to a given impulsive system $(X, \pi; M, I)$ as follows. Let $H = \{x \in I(M) : \phi(x_n^+) < +\infty$ for all $n = 0, 1, \ldots\}$. Now, define the mapping $g\colon H \to H$ by

$$g(x) = \widetilde{\pi}(x, \phi(x)) = I(\pi(x, \phi(x))) = I(x_1) = x_1^+, \tag{3.27}$$

for every $x \in H$. Note that g is a continuous function on H,

$$g^0(x) = x \quad \text{and} \quad g^{k+1}(x) = g(g^k(x)) = x_{k+1}^+,$$

for each $k = 0, 1, \ldots$, and $x \in H$. The pair (H, g) is called the *discrete dynamical system associated* to the system $(X, \pi; M, I)$ *in the sense of Kaul.*

Definition 3.25. A point $x \in X$ is called *almost $\widetilde{\pi}$-periodic by time reparametrization*, if given $\varepsilon > 0$, there is a number $T = T(\varepsilon) > 0$ such that for every $\alpha \geq 0$, the interval $[\alpha, \alpha + T]$ contains a number $\tau_\alpha > 0$ and one can obtain a time reparametrization h_α such that

$$d(\widetilde{\pi}(x, h_\alpha(t) + \tau_\alpha), \widetilde{\pi}(x, t)) < \varepsilon, \quad \text{for all } t \geq 0. \tag{3.28}$$

Definition 3.26. Let $\sigma > 0$. A *time σ-reparametrization* is a time reparametrization $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $|h(t) - t| < \sigma$ for all $t \in \mathbb{R}_+$. If $0 \leq h(t) - t < \sigma$ for all $t \in \mathbb{R}_+$, we say that h is a *positive time σ-reparametrization.*

Definition 3.27. A point $x \in X$ is called *almost $\widetilde{\pi}$-periodic by time σ-reparametrization*, if given $\varepsilon > 0$, there is a number $T = T(\varepsilon) > 0$ such that for every $\alpha \geq 0$, the interval $[\alpha, \alpha + T]$ contains a number $\tau_\alpha > 0$ and one can obtain a time σ-reparametrization h_α such that (3.28) holds. A point $x \in X$ is called *almost $\widetilde{\pi}$-periodic by a positive time σ-reparametrization*, if the reparametrization h_α is a positive time σ-reparametrization.

The next lemma shows that every point from a positive orbit $\widetilde{\pi}^+(x)$ is almost $\widetilde{\pi}$-periodic by time σ-reparametrization provided x has this positive property.

Lemma 3.28. *Let $\sigma > 0$ and $x \in X$ be almost $\widetilde{\pi}$-periodic by a positive time σ-reparametrization. Then every point $y \in \widetilde{\pi}^+(x)$ is almost $\widetilde{\pi}$-periodic by time σ-reparametrization.*

Proof. Let $\varepsilon > 0$ be given. Since $x \in X$ is almost $\widetilde{\pi}$-periodic by a positive time σ-reparametrization, there is $T = T(\varepsilon) > 0$ such that for every $\alpha \geq 0$, the interval $[\alpha, \alpha + T]$ contains a number $\tau_\alpha > 0$ and one can find a time σ-reparametrization h_α such that

$$d(\widetilde{\pi}(x, h_\alpha(t) + \tau_\alpha), \widetilde{\pi}(x, t)) < \varepsilon, \quad \text{for all } t \geq 0.$$

Take $y \in \widetilde{\pi}^+(x)$, then $y = \widetilde{\pi}(x, s)$ for some $s \geq 0$. Let $T_s = T + \sigma$. For each $\alpha \geq 0$ consider the number $\tau_\alpha > 0$ and the function $h_\alpha \colon \mathbb{R}_+ \to \mathbb{R}_+$ chosen above and define

$$\tau_\alpha^s = \tau_\alpha + h_\alpha(s) - s \quad \text{and} \quad H_\alpha(t) = h_\alpha(t + s) - h_\alpha(s), \quad \text{for all } t \geq 0.$$

Then $\tau_\alpha^s \in [\alpha, \alpha + T_s]$, $H_\alpha \colon \mathbb{R}_+ \to \mathbb{R}_+$ is a time reparametrization and for all $t \geq 0$ we have

$$-\sigma < H_\alpha(t) - t = h_\alpha(t+s) - h_\alpha(s) - t = (h_\alpha(t+s) - (t+s)) - (h_\alpha(s) - s) < \sigma.$$

Thus H_α is a time σ-reparametrization and

$$\begin{aligned} d(\widetilde{\pi}(y, t), &\widetilde{\pi}(y, H_\alpha(t) + \tau_\alpha^s)) \\ &= d(\widetilde{\pi}(x, s + t), \widetilde{\pi}(x, s + h_\alpha(t + s) - h_\alpha(s) + \tau_\alpha + h_\alpha(s) - s)) \\ &= d(\widetilde{\pi}(x, s + t), \widetilde{\pi}(x, h_\alpha(t + s) + \tau_\alpha)) < \varepsilon \end{aligned}$$

for all $t \geq 0$. The proof is complete. □

The concept of almost periodicity for discrete systems was introduced in [21]. Let us consider $g \colon H \to H$ as defined in (3.27).

Definition 3.29. A point $x \in H$ is said to be *almost g-periodic* if, given $\varepsilon > 0$, there is $N_1 > 0$ such that for each $n_1 \in \mathbb{Z}_+$, the interval $[n_1, n_1 + N_1]$ contains a number $m_{n_1} \in \mathbb{Z}_+$ such that

$$d(g^n(x), g^{n+m_{n_1}}(x)) = d(x_n^+, x_{n+m_{n_1}}^+) < \varepsilon, \quad \text{for all } n \in \mathbb{Z}_+.$$

In the next result, we present sufficient conditions for a point to be almost $\widetilde{\pi}$-periodic by time reparametrization provided this point is almost g-periodic in its associated discrete system (H, g).

Theorem 3.30. *Let $(X, \pi; M, I)$ be an impulsive system and (H, g) be its associated discrete system in the sense of Kaul. If $x \in H$ is a point almost g-periodic, $\overline{g^+(x)} \cap M = \emptyset$ and $\overline{g^+(x)}$ is compact, then x is almost $\widetilde{\pi}$-periodic by time reparametrization.*

Proof. Let $\varepsilon > 0$ be given. Since $\overline{g^+(x)}$ is compact and $\overline{g^+(x)} \cap M = \emptyset$, we have $T = \sup_{k\geq 0} \phi(x_k^+) < +\infty$ because ϕ is uniformly continuous on the compact set $\overline{g^+(x)}$. The mapping π is uniformly continuous on $\overline{g^+(x)} \times [0,T]$, then there is $\delta \in (0,\varepsilon)$ such that if $y,z \in \overline{g^+(x)}$ and $t_1, t_2 \in [0,T]$ satisfying $\max\{d(y,z), |t_1 - t_2|\} < \delta$, then

$$d(\pi(y,t_1), \pi(z,t_2)) < \varepsilon. \tag{3.29}$$

By the uniform continuity of ϕ on $\overline{g^+(x)}$, one can obtain $\delta_1 \in (0,\delta)$ such that if $y,z \in \overline{g^+(x)}$ with $d(y,z) < \delta_1$, then $|\phi(y) - \phi(z)| < \delta$.

By hypothesis, $x \in H$ is almost g-periodic. For $\delta_1 > 0$ chosen above, there is $N_0 \in \mathbb{Z}_+$ such that for each $m \in \mathbb{Z}_+$, the interval $[m, m+N_0]$ contains a number $n_m \in \mathbb{Z}_+$ such that

$$d(g^n(x), g^{n+n_m}(x)) = d(x_n^+, x_{n+n_m}^+) < \delta_1, \quad \text{for all } n \in \mathbb{Z}_+.$$

Let $T_1 = (N_0+1)T$. We claim that T_1 satisfies Definition 3.25. Indeed, given $\alpha \geq 0$, there is $k \in \mathbb{Z}_+$ such that $t_k(x) \leq \alpha < t_{k+1}(x)$. Thus there is $n_k \in [k+1, k+1+N_0] \cap \mathbb{Z}_+$ such that

$$d(x_n^+, x_{n+n_k}^+) < \delta_1, \quad \text{for all } n \in \mathbb{Z}_+.$$

Let $\tau_\alpha = t_{n_k}(x)$. Then $\tau_\alpha \in [\alpha, \alpha + T_1]$ because

$$\alpha < t_{k+1}(x) \leq t_{n_k}(x) = \tau_\alpha < t_{k+1+N_0}(x) = t_k(x) + \sum_{i=k}^{k+N_0} \phi(x_i^+) \leq \alpha + (N_0+1)T.$$

Now, we define the time reparametrization $h_\alpha \colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$h_\alpha(t) = t_n(x_{n_k}^+) + \frac{\phi(x_{n+n_k}^+)}{\phi(x_n^+)}\,(t - t_n(x)), \quad t \in [t_n(x), t_{n+1}(x)), \quad n = 0,1,\dots$$

If $t = t_n(x)$, $n \in \mathbb{Z}_+$, we have

$$\begin{aligned} d(\widetilde{\pi}(x,t), \widetilde{\pi}(x, h_\alpha(t) + \tau_\alpha)) &= d(x_n^+, \widetilde{\pi}(x, t_n(x_{n_k}^+) + t_{n_k}(x))) \\ &= d(x_n^+, x_{n+n_k}^+) < \delta_1 < \varepsilon. \end{aligned}$$

If $t \in (t_n(x), t_{n+1}(x))$, $n \in \mathbb{Z}_+$, we have

$$\begin{aligned} &d(\widetilde{\pi}(x,t), \widetilde{\pi}(x, h_\alpha(t) + \tau_\alpha)) \\ &\qquad = d\bigg(\pi(x_n^+, t - t_n(x)), \pi\bigg(x_{n+n_k}^+, \frac{\phi(x_{n+n_k}^+)}{\phi(x_n^+)}\,(t - t_n(x))\bigg)\bigg). \end{aligned}$$

Since $d(x_n^+, x_{n+n_k}^+) < \delta_1$ and

$$\left| t - t_n(x) - \frac{\phi(x_{n+n_k}^+)}{\phi(x_n^+)}\,(t - t_n(x)) \right| < |\phi(x_n^+) - \phi(x_{n+n_k}^+)| < \delta,$$

we conclude by (3.29) that $d(\widetilde{\pi}(x,t),\widetilde{\pi}(x,h_\alpha(t)+\tau_\alpha)) < \varepsilon$. Hence,

$$d(\widetilde{\pi}(x,t),\widetilde{\pi}(x,h_\alpha(t)+\tau_\alpha)) < \varepsilon, \quad \text{for all } t \geq 0. \qquad \square$$

Definition 3.31. A point $x \in H$ is called *strongly almost g-periodic* if, given $\varepsilon > 0$, there is $N_1 > 0$ such that for each $n_1 \in \mathbb{Z}_+$, the interval $[n_1, n_1 + N_1]$ contains a number $m_{n_1} \in \mathbb{Z}_+$ such that

$$d(x_n^+, x_{n+m_{n_1}}^+) < \varepsilon \quad \text{and} \quad |t_k(x_n^+) - t_k(x_{n+m_{n_1}}^+)| < \varepsilon, \quad \text{for all } n, k \in \mathbb{Z}_+.$$

Theorem 3.32. *Let $(X,\pi;M,I)$ be an impulsive system and (H,g) be its associated discrete system in the sense of Kaul. If $x \in H$ is a point strongly almost g-periodic, $\overline{g^+(x)} \cap M = \emptyset$ and $\overline{g^+(x)}$ is compact, then for every $\varepsilon > 0$, the point x is almost $\widetilde{\pi}$-periodic by time ε-reparametrization.*

Proof. Let $\varepsilon > 0$ be given. Using the proof of Theorem 3.30, we need just to note that if $t \in [t_n(x), t_{n+1}(x))$, $n = 0,1,\ldots$, then

$$\begin{aligned} |h_\alpha(t) - t| &= \left| t_n(x_{n_k}^+) + \frac{\phi(x_{n+n_k}^+)}{\phi(x_n^+)}(t - t_n(x)) - t \right| \\ &\leq \max\{|t_n(x_{n_k}^+) - t_n(x)|, |t_{n+1}(x_{n_k}^+) - t_{n+1}(x)|\} < \varepsilon, \end{aligned}$$

where we have used the second condition of Definition 3.31. Therefore, x is almost $\widetilde{\pi}$-periodic by time ε-reparametrization. $\square$

Definition 3.33. A point $x \in H$ is said to be *asymptotically almost g-periodic* if there is an almost g-periodic point $p \in H$ such that

$$\lim_{n\to+\infty} d(g^n(x), g^n(p)) = 0. \tag{3.30}$$

Definition 3.34. A point $x \in X$ is called *asymptotically almost $\widetilde{\pi}$-periodic by time reparametrization*, if there are an almost $\widetilde{\pi}$-periodic by time reparametrization point p and a reparametrization h_p such that

$$\lim_{t\to+\infty} d(\widetilde{\pi}(x,t),\widetilde{\pi}(p,h_p(t))) = 0.$$

Next, we present sufficient conditions for a point in H to be asymptotically almost $\widetilde{\pi}$-periodic by time reparametrization.

Theorem 3.35. *Let $(X,\pi;M,I)$ be an impulsive system and (H,g) be its associated discrete system in the sense of Kaul. Suppose that X is a complete metric space. If $x \in H$ is a point asymptotically almost g-periodic and $\{x_n^+\}_{n\in\mathbb{N}}$ is convergent in H, then x is asymptotically almost $\widetilde{\pi}$-periodic by time reparametrization.*

Proof. By hypothesis, there exists an almost g-periodic point $p \in H$ such that

$$\lim_{n\to+\infty} d(g^n(x), g^n(p)) = 0. \tag{3.31}$$

Since $\{x_n^+\}_{n\in\mathbb{N}}$ converges in H, we have that $\overline{g^+(x)}$ is compact and $\overline{g^+(x)}\cap M=\emptyset$ ($\overline{g^+(x)}\subset H\subset I(M)$ and we have hypothesis (H2)). Now, using (3.31), we obtain that $\overline{g^+(p)}$ is compact and $\overline{g^+(p)}\cap M=\emptyset$. Hence, by Theorem 3.30, the point p is almost $\widetilde{\pi}$-periodic by time reparametrization.

Let $\varepsilon>0$ be given. By the uniform continuity of π on $(\overline{g^+(x)}\cup\overline{g^+(p)})\times[0,T]$, there is $\delta\in(0,\varepsilon)$ such that if $y,z\in\overline{g^+(x)}\cup\overline{g^+(p)}$ and $t_1,t_2\in[0,T]$ satisfying $\max\{d(y,z),|t_1-t_2|\}<\delta$, then

$$d(\pi(y,t_1),\pi(z,t_2))<\varepsilon. \tag{3.32}$$

Let $\lim\limits_{n\to+\infty}x_n^+=\lim\limits_{n\to+\infty}p_n^+=z\in H$. By the continuity of ϕ at z, there is $\delta_1\in(0,\delta)$ such that if $d(y,z)<\delta_1$ then $|\phi(y)-\phi(z)|<\delta/2$. Let $n_1\in\mathbb{N}$ be such that $d(x_n^+,z)<\delta_1$ and $d(p_n^+,z)<\delta_1$ for all $n\geq n_1$.

By (3.31), there is $n_2\in\mathbb{N}$, $n_2\geq n_1$, such that

$$d(x_n^+,p_n^+)=d(g^n(x),g^n(p))<\delta_1,\quad \text{for all } n\geq n_2.$$

Now, we define a time reparametrization $h_p\colon\mathbb{R}_+\to\mathbb{R}_+$ by

$$h_p(t)=t_n(p)+\frac{\phi(p_n^+)}{\phi(x_n^+)}(t-t_n(x)),\quad t\in[t_n(x),t_{n+1}(x)),\ n=0,1,\dots$$

If $t=t_n(x)$ for $n\geq n_2$, we have

$$d(\widetilde{\pi}(x,t),\widetilde{\pi}(p,h_p(t)))=d(x_n^+,p_n^+)<\delta_1<\varepsilon.$$

If $t\in(t_n(x),t_{n+1}(x))$, for $n\geq n_2$, we have

$$d(\widetilde{\pi}(x,t),\widetilde{\pi}(p,h_p(t)))=d\bigg(\pi(x_n^+,t-t_n(x)),\pi\bigg(p_n^+,\frac{\phi(p_n^+)}{\phi(x_n^+)}(t-t_n(x))\bigg)\bigg).$$

Since $d(x_n^+,p_n^+)<\delta_1$ and

$$\begin{aligned}\left|t-t_n(x)-\frac{\phi(p_n^+)}{\phi(x_n^+)}(t-t_n(x))\right|&<|\phi(x_n^+)-\phi(p_n^+)|\\&\leq|\phi(x_n^+)-\phi(z)|+|\phi(z)-\phi(p_n^+)|<\delta,\end{aligned}$$

for $n\geq n_2$, we may conclude by (3.32) that

$$d(\widetilde{\pi}(x,t),\widetilde{\pi}(p,h_p(t)))<\varepsilon,\quad \text{for all } t\geq t_{n_2}(x).$$

Therefore, x is asymptotically almost $\widetilde{\pi}$-periodic by time reparametrization. □

3.4. Lyapunov stability and Zhukovskiĭ quasi stability. In this last subsection, we present sufficient conditions to obtain Zhukovskiĭ quasi stability via Lyapunov stability. The concept of Lyapunov stability for discrete systems in the sense of Kaul was introduced in [22] as presented below.

Definition 3.36. A point $x \in H$ is called *Lyapunov g-stable* with respect to a set $P \subset H$ if $x \in \overline{P}$ and, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $d(x,p) < \delta$ with $p \in P$, then

$$d(g^n(x), g^n(p)) = d(x_n^+, p_n^+) < \varepsilon, \quad \text{for all } n \in \mathbb{Z}_+.$$

A subset $A \subset H$ is called *Lyapunov g-stable* with respect to $P \subset H$ if $A \subset \overline{P}$ and each point $x \in A$ is Lyapunov g-stable with respect to $P \subset H$.

If $A \subset X$ is a set such that $\phi(a) < +\infty$ for every $a \in A$, then we may define

$$\widetilde{A} = \bigcup_{x \in A} \widetilde{\pi}(x, \phi(x)).$$

Note that $\widetilde{A} \subset I(M)$. If $\phi(a_j^+) < +\infty$ for all $j = 0, 1, \ldots,$ and for all $a \in A$, then $\widetilde{A} \subset H$.

Lemma 3.37. *If $B \subset X$ is positively $\widetilde{\pi}$-invariant then $\overline{B} \setminus M$ is positively $\widetilde{\pi}$-invariant.*

Proof. Let $b \in \overline{B} \setminus M$ and $t \geq 0$. Then there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subset B$ such that $x_n \xrightarrow{n \to +\infty} b$. By Lemma 2.2, there is a sequence $\{\varepsilon_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\varepsilon_n \xrightarrow{n \to +\infty} 0$ and $\widetilde{\pi}(x_n, t + \varepsilon_n) \xrightarrow{n \to +\infty} \widetilde{\pi}(b, t)$. Since $\{\widetilde{\pi}(x_n, t+\varepsilon_n)\}_{n \in \mathbb{N}} \subset B$, we have $\widetilde{\pi}(b, t) \in \overline{B} \setminus M$ as we have hypothesis (H2). □

Theorem 3.38. *Let $(X, \pi; M, I)$ be an impulsive system and (H, g) be its associated discrete system in the sense of Kaul, where H is a closed set. Let $A, B \subset X \setminus M$, $A \subset \overline{B}$ and B be a relatively compact positively $\widetilde{\pi}$-invariant set. Suppose that $\phi(b) < +\infty$ for all $b \in \overline{B} \setminus M$. If $\widetilde{\overline{A}}$ is Lyapunov g-stable with respect to $\widetilde{\overline{B}}$, then any set $\mathcal{O} \subset \overline{A} \setminus M$ is Zhukovskiĭ quasi $\widetilde{\pi}$-stable with respect to $\overline{B} \setminus M$.*

Proof. Let $\mathcal{O} \subset \overline{A} \setminus M$, $x \in \mathcal{O}$ be fixed and note that $\phi(x) < +\infty$ by hypothesis. Since $\overline{B}$ is compact and H is closed, we have $\widetilde{\overline{B}}$ compact and $\widetilde{\overline{B}} \cap M = \emptyset$. Note that $\widetilde{B} \subset H$ because B is positively $\widetilde{\pi}$-invariant. The continuity of ϕ on the compactness of the set $\widetilde{\overline{B}} \cup \{x\}$ implies that

$$T = \sup_{a \in \widetilde{\overline{B}} \cup \{x\}} \phi(a) < +\infty.$$

Let $\varepsilon > 0$ be given. Using the uniform continuity of π in $\overline{B} \times [0, T]$, one can obtain $\delta_1 \in (0, \varepsilon)$ such that for each $y, z \in \overline{B}$ and $t_1, t_2 \in [0, T]$ satisfying $\max\{d(y, z), |t_1 - t_2|\} < \delta_1$ we have

$$d(\pi(y, t_1), \pi(z, t_2)) < \varepsilon. \tag{3.33}$$

Note that ϕ is uniformly continuous on the compact $\overline{\widetilde{B}}$. Thus there is $\delta_2 \in (0, \delta_1)$ such that if $y, z \in \overline{\widetilde{B}}$ and $d(y, z) < \delta_2$ then

$$|\phi(y) - \phi(z)| < \delta_1. \tag{3.34}$$

By the Lyapunov g-stability of $x_1^+ \in \overline{\widetilde{A}}$ with respect to $\overline{\widetilde{B}}$, there is $\delta_3 \in (0, \delta_2)$ such that if $p \in \overline{\widetilde{B}}$ with $d(x_1^+, p) < \delta_3$ then

$$d(g^n(x_1^+), g^n(p)) < \delta_2, \quad \text{for all } n \in \mathbb{Z}_+. \tag{3.35}$$

Since π is continuous in $X \times \mathbb{R}_+$, I is continuous in M and ϕ is continuous in $X \setminus M$ one can find $\delta_4 \in (0, \delta_3)$ such that if $y \in \overline{B} \setminus M$ with $d(x, y) < \delta_4$ then

$$d(x_1^+, y_1^+) = d(I(\pi(x, \phi(x))), I(\pi(y, \phi(y)))) < \delta_3.$$

Consequently, by (3.35), we have

$$d(x_n^+, y_n^+) < \delta_2, \quad \text{for all } n \in \mathbb{Z}_+. \tag{3.36}$$

Since B is positively $\widetilde{\pi}$-invariant, we have that $\overline{B} \setminus M$ is positively $\widetilde{\pi}$-invariant, see Lemma 3.37. Thus, $x_n^+, y_n^+ \in \overline{\widetilde{B}}$ for all $n \in \mathbb{N}$, where $y \in \overline{B} \setminus M$. Then if $y \in \overline{B} \setminus M$ with $d(x, y) < \delta_4$ it follows by (3.36) and (3.34) that

$$|\phi(x_n^+) - \phi(y_n^+)| < \delta_1, \quad \text{for every } n \in \mathbb{Z}_+. \tag{3.37}$$

For $y \in \overline{B} \setminus M$ such that $d(x, y) < \delta_4$ we define the time reparametrization $h_y \colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$h_y(t) = t_n(y) + \frac{\phi(y_n^+)}{\phi(x_n^+)}\,(t - t_n(x)), \quad t \in [t_n(x), t_{n+1}(x)), \quad n = 0, 1, \ldots$$

If $t = t_n(x)$, $n \in \mathbb{N}$, we have

$$d(\widetilde{\pi}(x, t), \widetilde{\pi}(y, h_y(t))) = d(x_n^+, y_n^+) < \delta_2 < \varepsilon.$$

If $t \in (t_n(x), t_{n+1}(x))$, $n \in \mathbb{N}$, we have

$$d(\widetilde{\pi}(x, t), \widetilde{\pi}(y, h_y(t))) = d\left(\pi(x_n^+, t - t_n(x)), \pi\left(y_n^+, \frac{\phi(y_n^+)}{\phi(x_n^+)}\,(t - t_n(x))\right)\right).$$

Since $x_n^+, y_n^+ \in \overline{\widetilde{B}}$, $d(x_n^+, y_n^+) < \delta_2 < \delta_1$ (see (3.36)) and

$$\left|t - t_n(x) - \frac{\phi(y_n^+)}{\phi(x_n^+)}\,(t - t_n(x))\right| < |\phi(x_n^+) - \phi(y_n^+)| < \delta_1$$

(see (3.37)), we conclude by (3.33) that $d(\widetilde{\pi}(x, t), \widetilde{\pi}(y, h_y(t))) < \varepsilon$. Thus, every point $x \in \mathcal{O} \subset \overline{A} \setminus M$ is Zhukovskiĭ quasi $\widetilde{\pi}$-stable with respect to the set $\overline{B} \setminus M$ and the proof is complete. □

Corollary 3.39. *Let $(X,\pi;M,I)$ be an impulsive system and (H,g) be its associated discrete system in the sense of Kaul, where H is a closed set. Let $x \in X \setminus M$, $\overline{\widetilde{\pi}^+(x)}$ be compact, $\phi(x_n^+) < +\infty$ for all $n \in \mathbb{Z}_+$ and $\{x_n^+\}_{n\in\mathbb{N}}$ be convergent in H. If $\overline{g^+(x_1^+)}$ is Lyapunov g-stable with respect to itself, then any set $\mathcal{O} \subset \overline{\widetilde{\pi}^+(x)} \setminus M$ is Zhukovskiĭ quasi $\widetilde{\pi}$-stable with respect $\overline{\widetilde{\pi}^+(x)} \setminus M$.*

Proof. It is enough to note that $\phi(y) < +\infty$ for all $y \in \overline{\widetilde{\pi}^+(x)} \setminus M$. In fact, since

$$\overline{\widetilde{\pi}^+(x)} \setminus M = \pi^+(x) \cup (\widetilde{L}^+(x) \setminus M)$$

and $\phi(y) < +\infty$ for all $y \in \widetilde{\pi}^+(x)$, we need to show that $\phi(y) < +\infty$ for all $y \in \widetilde{L}^+(x) \setminus M$. Let $y \in \widetilde{L}^+(x) \setminus M$ then there is a sequence $\{s_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that $s_n \xrightarrow{n\to+\infty} +\infty$ and $\widetilde{\pi}(x,s_n) \xrightarrow{n\to+\infty} y$. For each $n \in \mathbb{N}$ there is $k_n \in \mathbb{Z}_+$ such that $t_{k_n}(x) \le s_n < t_{k_n+1}(x)$. Then

$$\widetilde{\pi}(x,s_n) = \pi(x_{k_n}^+, s_n - t_{k_n}) \quad \text{and} \quad \phi(\widetilde{\pi}(x,s_n)) = \phi(x_{k_n}^+) - (s_n - t_{k_n}(x)).$$

Thus

$$\phi(\widetilde{\pi}(x,s_n)) \le \phi(x_{k_n}^+), \quad \text{for all } n \in \mathbb{N}. \tag{3.38}$$

Since $\{x_n^+\}_{n\in\mathbb{N}}$ is convergent in H, we may write $\lim_{n\to+\infty} x_n^+ = z \in H$. Then, using the continuity of ϕ in $X \setminus M$ as $n \to +\infty$ in (3.38), we get $\phi(y) \le \phi(z)$. Now, since $z \in H$, we have $\phi(z) < +\infty$. Hence, $\phi(y) < +\infty$ for all $y \in \widetilde{L}^+(x) \setminus M$. The result follows by Theorem 3.38. □

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