Topological Methods in Nonlinear Analysis
Volume 48, No. 2, 2016, 493–506
DOI: 10.12775/TMNA.2016.055

LINEARIZATION OF PLANAR HOMEOMORPHISMS WITH A COMPACT ATTRACTOR

Armengol Gasull — Jorge Groisman — Francesc Mañosas

Abstract. Kerékjártó proved in 1934 that a planar homeomorphism with an asymptotically stable fixed point is conjugated, on its basin of attraction, to one of the maps $z \mapsto z/2$ or $z \mapsto \overline{z}/2$, depending on whether f preserves or reverses the orientation. We extend this result to planar homeomorphisms with a compact attractor.

1. Introduction

Consider the discrete dynamical system generated by a planar homeomorphism f. It is well-known that if f has an asymptotically stable fixed point, then its basin of attraction $\mathcal{U}$ is an open and simply connected subset of the plane. Moreover, Kerékjártó ([7], [8]) proved that f restricted to $\mathcal{U}$ is either conjugated to $L_1(z) = z/2$ or to $L_2(z) = \overline{z}/2$ in $\mathbb{C}$, depending on whether f preserves or reverses the orientation. A different proof of this result is also given in [4]. This result has been extended, with clear modifications, to $\mathbb{R}^3$ in [5] and to $\mathbb{R}^m$ for $m \neq 4, 5$ in [6], when f preserves orientation.

In this paper we will focus on the planar case and we extend Kerékjártó's result to the case where f has a compact attractor. To state our result we need

2010 *Mathematics Subject Classification.* Primary: 37C15; Secondary: 37C70, 54H20.
Key words and phrases. Kerékjártó's theorem; attractor; linearization; Lyapunov function.
The first author is supported by the MINECO grant No. MTM2013-40998-P and by the CIRIT grant No. 2014SGR568. The third author is supported by the MINECO grant No. MTM2014-52209-C2-1-P. The first and second authors thank for the support of the Grupo de Sistemas Dinámicos de la Udelar.

to introduce a new concept, the stabilizer of a compact attractor. This notion is analogous to the one proposed in [3] for ordinary differential equations.

Let K be a compact attractor, not necessarily stable, and with basin of attraction $\mathcal{A}(K)$. Define the new compact set

$$\widetilde{K} := \{x \in \mathcal{A}(K) : \alpha(x) \cap K \neq \emptyset\},$$

where $\alpha(x)$ denotes the *alpha-limit* of the orbit passing through x, which we call the *stabilizer of* K. We will see that $\widetilde{K}$ is a compact stable attractor with the same basin of attraction as K. Our main result is the following theorem:

Theorem 1.1. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism, let K be a compact attractor and let $\mathcal{U}$ be its basin of attraction. Assume that $\mathcal{U}$ is connected and simply connected. Then $\mathcal{U} \setminus \widetilde{K}$ is homeomorphic to $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ and $f|_{\mathcal{U}\setminus\widetilde{K}}$ is conjugated to $L_1(z) = z/2$ or $L_2(z) = \overline{z}/2$ on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$.*

As corollaries of the above theorem we get Kerékjártó's result and the following extension:

Corollary 1.2. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism and let K be a global compact attractor. Then $\mathbb{R}^2 \setminus \widetilde{K}$ is homeomorphic to $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ and $f|_{\mathbb{R}^2\setminus\widetilde{K}}$ is conjugated either to L_1 or to L_2 on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$.*

Let us recall the main steps of Kerékjártó's proof. If γ is a Jordan curve surrounding the fixed point, p, then clearly there exists n such that $f^n(\gamma)$ is also a Jordan curve which surrounds p and lies in the bounded component of $\mathcal{U} \setminus \gamma$. Then, using all the curves $f^j(\gamma)$, $j = 0, 1, \ldots, n-1$, and some topological reasonings he constructs a new curve, say Γ, for which the same holds but with $n = 1$. Then, the closed annulus $\mathcal{A}$ with boundaries Γ and $f(\Gamma)$ constitutes a fundamental domain on which he constructs the conjugacy ψ between f and L_j, $j = 1$ or 2. In fact, ψ must send $\mathcal{A}$ to the set $A := \{z \in \mathbb{C} : 1/2 \leq |z| \leq 1\}$, with some natural restrictions on the boundary. Then, this ψ can be extended to $\mathcal{U}$ in a natural way by iteration.

Our proof of Theorem 1.1 follows a similar approach, but with two main differences. The first one is that the curve Γ with the property described above is constructed by using a different idea. First, we prove the existence of a continuous Lyapunov function L associated to the asymptotically stable compact set $\widetilde{K}$, by adapting a similar construction developed in [1] for ordinary differential equations. Afterwards, we show how to smoothen some of the level sets of L by using Sard's theorem and the classification of one dimensional manifolds. One of these smooth levels will be Γ. A second difference is that we use an extension of Jordan's curve theorem known as Schoenflies' theorem ([2], [9]) to prove the existence of a continuous conjugacy ψ between the respective domains $\mathcal{A}$ and A, satisfying a suitable boundary condition.

In [10], J. Lewowicz proposed to use the *Lyapunov metrics* (see the next section for a precise definition) to study structural stability of homeomorphisms on compact manifolds, and similar concepts, such as topological stability and persistence. Since then, the method has been successfully applied to a wide spectrum of dynamical systems, such as hyperbolic and almost hyperbolic diffeomorphisms on manifolds, geodesic flows, pseudo-Anosov maps, billiards, expansive homeomorphisms on compact manifolds, in particular on surfaces, and on expansive homeomorphisms of the plane.

Our second result relates usual Lyapunov functions, Lyapunov metrics and global asymptotically stable fixed points in the plane.

THEOREM 1.3. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism and let p be a fixed point. Then the following statements are equivalent*:

(a) *p is globally asymptotically stable.*
(b) *There exists a Lyapunov function for f and p.*
(c) *f is conjugate either to $L_1(z) = z/2$ or to $L_2(z) = \overline{z}/2$.*
(d) *There exists a Lyapunov metric D for which p is D-stable.*

In fact, by using the extension of Kerékjártó's theorem to dimension 3, or to dimensions $m > 5$ in the orientation preserving case, Theorem 1.3 can be generalized to these situations in a natural way.

The paper is organized as follows. In Section 2 we introduce the stabilizer of an attractor, we give some examples and prove its basic properties in $\mathbb{R}^m$. We also prove some basic facts about globally asymptotically stable compact sets and its relation with the existence of Lyapunov functions in $\mathbb{R}^m$. In this setting we also introduce some properties of the Lyapunov metrics. We conclude with some results in the planar case. In Section 3 we prove Theorems 1.1 and 1.3 and some corollaries, including Kerékjártó's theorem.

2. General definitions and preliminary results

We begin by reviewing some definitions and basic facts about the dynamics of homeomorphisms and we also introduce the stabilizer of a compact attractor.

Let X be a locally compact topological space and let $f\colon X \to X$ be a homeomorphism.

We say that $Y \subset X$ is *invariant* if $f(Y) = Y$. For $x \in X$ the *omega limit set of* x, denoted by $\omega(x)$, is the set of accumulation points of the sequence $\{f^n(x)\}_{n\in\mathbb{N}}$. Analogously the *alpha limit set of* x, denoted by $\alpha(x)$, is the set of accumulation points of the sequence $\{f^{-n}(x)\}_{n\in\mathbb{N}}$. The alpha and omega limit sets are closed and invariant subsets of X.

We will say that a compact set $K \subset X$ is an *attractor* if this set is invariant and there exists a neighbourhood $\mathcal{U}$ of K such that for all $x \in \mathcal{U}$, $\omega(x) \neq \emptyset$ and

$\omega(x) \subset K$. If K is an attractor, the set

$$\mathcal{A}(K) = \{x \in X : \omega(x) \neq \emptyset \text{ and } \omega(x) \subset K\}$$

is invariant and open and we call it the *basin of attraction of* K. In the case that $\mathcal{A}(K) = X$ we will say that K is a *global attractor*.

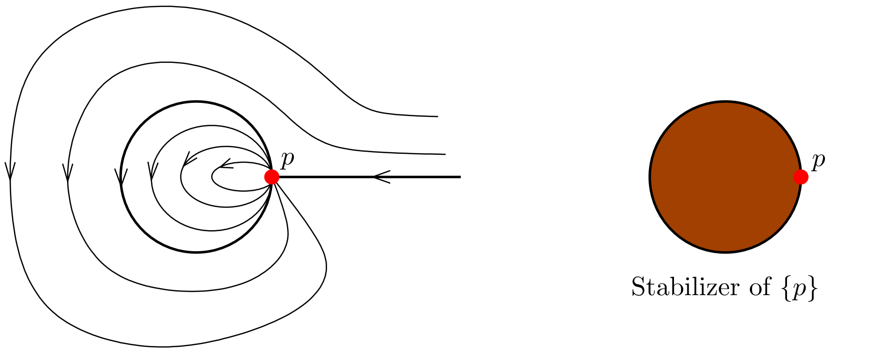

Figure 1. Flow that defines f_1 and stabilizer of $\{p\}$.

We will say that a compact set $K \subset X$ is *asymptotically stable* if it is an attractor and for all neighbourhoods $\mathcal{U}$ of K there exists $\mathcal{V} \subset \mathcal{U}$, a neighbourhood of K, such that $f^n(x) \in \mathcal{U}$ for all $x \in \mathcal{V}$ and $n \in \mathbb{N}$. In the case that $\mathcal{A}(K) = X$ we will say that K is *globally asymptotically stable*.

Let K be an attractor. Recall that we have introduced its stabilizer as $\widetilde{K} := \{x \in \mathcal{A}(K) : \alpha(x) \cap K \neq \emptyset\}$. Let us show some examples. Let f_1 and f_2 be the *time one maps* given by the flows with phase portraits given in Figures 1 and 2, respectively. In these figures the stabilizers of different compact attractors are displayed. It is interesting to notice that when we consider $K = \{p\}$ for f_2, then this set is a compact attractor but Theorem 1.1 cannot be applied because its basin of attraction $\mathcal{U} = \mathbb{R}^2 \setminus \{q\}$ is not simply connected. On the other hand, taking $K = \{p, q\}$ we get a global attractor and, by Corollary 1.2, we obtain that f_2 is conjugated with L_1 on $\mathbb{R}^2 \setminus \widetilde{K}$.

Our first objective is to show that in our situation $\widetilde{K}$ is asymptotically stable with the same basin as K. We need some preliminary results.

From now on we restrict our attention to the case when X is an open subset of $\mathbb{R}^m$. If K is an attractor, we can always assume that K is a global attractor by considering $f|_{\mathcal{A}(K)}$ which is also a homeomorphism. If $\mathcal{U} \subset X$ we will denote by $\overline{\mathcal{U}}, \mathring{\mathcal{U}}$ and $\mathcal{U}^c$ the closure of $\mathcal{U}$, its interior and $X \setminus \mathcal{U}$, respectively.

Lemma 2.1. *Let $f\colon X \to X$ be a homeomorphism where X is an open subset of $\mathbb{R}^m$. Assume that $K \subset X$ is a global attractor. Let Q_1 be a compact neighbourhood of K. Then there exists a compact neighbourhood Q_2 such that $Q_1 \subset Q_2$ and $f(Q_2) \subset Q_2$.*

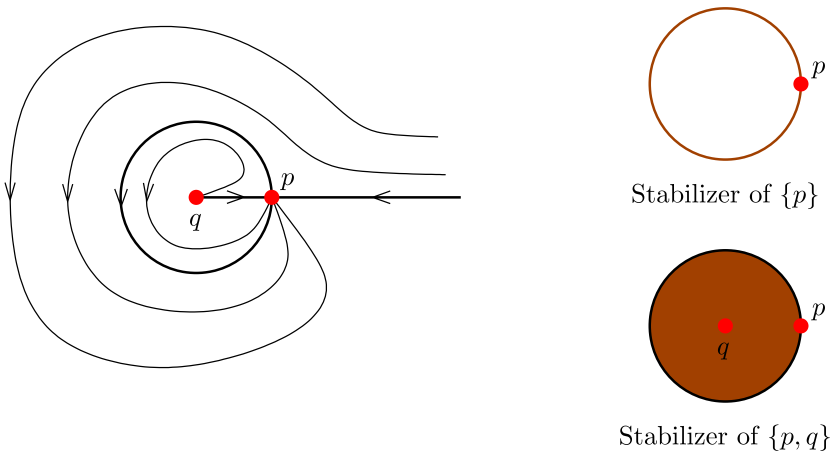

FIGURE 2. Flow that defines f_2 and different stabilizers.

PROOF. Let $x \in Q_1$ and consider the first natural number $n_x \geq 1$ such that $f^{n_x}(x) \in \mathring{Q}_1$. Let B_x be an open neighbourhood of x such that $\overline{B}_x$ is compact and $f^{n_x}(B_x) \subset \mathring{Q}_1$. Then $\{B_x\}_{x \in Q_1}$ is an open cover of Q_1. Let $x_1, \dots, x_l$ be such that $Q_1 \subset \bigcup_{i=1}^{l} B_{x_i}$. Define

$$Q_2 = \bigcup_{i=1}^{l} \bigcup_{j=0}^{n_{x_i}} f^j(\overline{B}_{x_i}).$$

Clearly, Q_2 is a compact set and $f(Q_2) \subset Q_2$. □

LEMMA 2.2. *Let $X \subset \mathbb{R}^m$ be open, $f\colon X \to X$ be a homeomorphism and let $K \subset X$ be a global attractor. Given any $x \in X$, we have that $x \in \widetilde{K}$ or $\alpha(x) = \emptyset$ (i.e. the negative orbit of x leaves any compact set).*

PROOF. Let $x \in X$ and let $M \subset X$ be a compact set such that the negative orbit of x does not leave M. Then $\alpha(x) \neq \emptyset$. Take $y \in \alpha(x)$. Then $w(y) \subset K$ and $w(y) \subset \alpha(x)$, since $\alpha(x)$ is an invariant set. So $\alpha(x) \cap K \neq \emptyset$. Thus $x \in \widetilde{K}$.□

PROPOSITION 2.3. *Let $X \subset \mathbb{R}^m$ be open, $f\colon X \to X$ be a homeomorphism and let $K \subset X$ be an attractor. Then $\widetilde{K}$ is asymptotically stable with the same basin of attraction as K. Moreover, K is asymptotically stable if and only if $\widetilde{K} = K$.*

PROOF. We can restrict our attention to the case where K is a global attractor. Notice that by definition $\widetilde{K}$ is invariant. Now, we prove that it is compact. By Lemma 2.1, there exists a compact neighbourhood Q of K such that $f(Q) \subset Q$. Thus, $f^{-1}(Q^c) \subset Q^c$ and then $\widetilde{K} \subset Q$. Thus $\widetilde{K}$ is bounded. We will prove that $\widetilde{K}^c$ is an open set which will imply the compactness of $\widetilde{K}$. Let

$p \in \widetilde{K}^{\rm c}$. By Lemma 2.2, there exists $n \geq 0$ such that $f^{-n}(p) \in Q^{\rm c}$. Let $\mathcal{V}$ be a neighbourhood of p such that $f^{-n}(\mathcal{V}) \subset Q^{\rm c}$. Since $Q^{\rm c}$ is invariant for f^{-1}, then $\alpha(x) \subset \overline{Q^{\rm c}}$ for all $x \in \mathcal{V}$, so $\mathcal{V} \subset \widetilde{K}^{\rm c}$. Then $\widetilde{K}$ is a compact set. By definition $K \subset \widetilde{K}$, so $\widetilde{K}$ is a global attractor. Let us see that $\widetilde{K}$ is asymptotically stable. Suppose that this is not true. Then there exists an open neighbourhood $\mathcal{U}$ of $\widetilde{K}$ such that for all open neighbourhoods $\mathcal{V}$ of $\widetilde{K}$ there exist $x \in \mathcal{V}$ and $n_x \geq 0$ such that $f^{n_x}(x)$ does not belong to $\mathcal{U}$. Let Q be the compact and positive invariant neighbourhood of $\widetilde{K}$. Then there exists a sequence $\{x_k\}_k$ of Q such that:

- $f^{n_k}(x_k)$ does not belong to $\mathcal{U}$ for some $n_k \geq 0$,
- $\lim x_k = x \in \widetilde{K}$.

Since $x_k \in Q$ and Q is a positive invariant set, we have that (taking a subsequence if necessary) $f^{n_k}(x_k)$ tends to a point $y \in Q \setminus \mathcal{U}$. By Lemma 2.2, there exist $j > 0$ and a neighbourhood $\mathcal{V}$ of y such that $f^{-j}(y) \in Q^{\rm c}$ and $f^{-j}(\mathcal{V}) \subset Q^{\rm c}$. We claim that the sequence $\{n_k\}_k$ is unbounded. If not, we can take a subsequence $\{x_{k_i}\}_i$ that tends to x and $f^n(x_k) \to y$ for some constant n. Thus, $y = f^n(x)$ which contradicts the invariance of $\widetilde{K}$. So, let $k_0 \in \mathbb{N}$ be such that $f^{n_k}(x_k) \in \mathcal{V}$, for all $k > k_0$. Then $f^{n_k - j}(x_k) \in Q^{\rm c}$, for all $k > k_0$, and since the sequence $\{n_k\}_k$ is unbounded there exists some k such that $n_k - j > 0$ which contradicts the positive invariance of Q. Therefore $\widetilde{K}$ is asymptotically stable. Since by definition $K \subset \widetilde{K}$, it follows that it is globally asymptotically stable. Thus the first statement holds. Clearly if $\widetilde{K} = K$, it follows that K is asymptotically stable. Lastly assume that K is asymptotically stable and we will see that $\widetilde{K} = K$. Suppose to the contrary that $x \in \widetilde{K} \setminus K$. Then since $\alpha(x) \cap K \neq \emptyset$, it follows that for any ε the set $B_\varepsilon = \{y \in \mathbb{R}^m : d(y, K) < \varepsilon\}$ contains some pre-image of x which contradicts the stability of K. This ends the proof of the proposition. □

2.1. Lyapunov functions in $\mathbb{R}^m$. A classical tool to investigate the stability of compact invariant sets is the existence of the so-called Lyapunov functions. Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a homeomorphism and $K \subset \mathbb{R}^m$ be a compact invariant set. We will say that a proper and continuous map $L\colon \mathbb{R}^m \to \mathbb{R}$ is a *Lyapunov function* for K and f, if it satisfies the following two properties:

(i) For all $x \in \mathbb{R}^m$, $L(x) \geq 0$ and $L(x) = 0$ if and only if $x \in K$.
(ii) For all $x \in \mathbb{R}^m \setminus K$, $L(f(x)) < L(x)$.

The following result relates the global asymptotical stability with the existence of Lyapunov functions.

Proposition 2.4. *Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a homeomorphism and K be a compact invariant subset. Then the following statements are equivalent*:

(a) *There exists a Lyapunov function for f and K.*
(b) *K is globally asymptotically stable.*

PROOF. Assume that (a) holds and let $L\colon \mathbb{R}^m \to \mathbb{R}$ be a Lyapunov function for K and f. First we show that K is a global attractor. Let $x \in \mathbb{R}^m$ and define $k = L(x)$. Then the positive orbit of x is contained in $L^{-1}[0,k]$ which is a compact set. Therefore $\omega(x) \neq \emptyset$. We claim that $\omega(x) \subset K$. If not, there exists $y \in \omega(x)$ such that $L(y) = \ell \neq 0$. Therefore $L(f(y)) = \ell' < \ell$. Since $f(y) \in \omega(x)$, this implies that there exists a sequence $0 < n_1 < \ldots < n_i \ldots$ such that $\lim f^{n_i}(x) = f(y)$. In particular, for j large enough we will have $L(f^{n_j}(x)) = s < \ell$. Thus we will obtain that $L(f^n(x)) < s < \ell = L(y)$ for $n > n_j$, which contradicts the fact that $y \in \omega(x)$. This proves the claim and shows that K is a global attractor.

Now we show that K is asymptotically stable. Let U be an open neighbourhood of K and set $M = \min\{L(x) : x \notin U\}$ which exists because L is proper. Moreover, since $K \subset U$ it follows that $M \neq 0$. Assume, to the contrary, that for all $j > 0$ there exists x_j and $k_j \geq 0$ such that $d(x_j, K) < 1/j$ and $f^{k_j}(x_j) \notin U$. Thus we will obtain a sequence x_j tending to K with $L(x_j) > M$, which contradicts the continuity of L. Therefore there exists a value n such that $B_n = \{x \in \mathbb{R}^m : d(x,K) < 1/n\}$ satisfies that for all $y \in B_n$ and for all $r \geq 0$, $f^r(y) \in U$. This ends the proof of the first implication.

Now assume that K is globally asymptotically stable. Define $s\colon \mathbb{R}^m \to \mathbb{R}$ by

$$s(x) = d(x,K) = \min\{\|x-y\| : y \in K\}$$

which clearly is a positive continuous function that only vanishes at K. Since K is globally asymptotically stable for any $x \in \mathbb{R}^m \setminus K$ there exists $k_x \in \mathbb{N}$ such that $s(f^n(x)) < s(x)/2$ for all $n > k_x$. Thus

$$\Phi(x) = \sup\{s(f^n(x)) : n \in \mathbb{N}\} = \max\{s(f^n(x)) : n \in \{0,\ldots,k_x\}\}$$

is well-defined. Clearly it is continuous and since $\Phi(x) \geq s(x)$ it is also proper. By definition, Φ satisfies property (i) and also satisfies that $\Phi(f(x)) \leq \Phi(x)$ for all $x \in \mathbb{R}^m$. Now consider

$$L(x) := \sum_{n=0}^{\infty} \frac{\Phi(f^n(x))}{2^n}.$$

Clearly this series converges uniformly on compact subsets of $\mathbb{R}^m$ and hence defines a continuous function on $\mathbb{R}^m$. Moreover, since $L(x) \geq \Phi(x) \geq s(x)$ for all $x \in \mathbb{R}^m$, it follows that L is proper and satisfies property (i). On the other hand,

$$L(f(x)) - L(x) = \sum_{i=0}^{\infty} 2^{-n}(\Phi(f^{n+1}(x)) - \Phi(f^n(x))) \leq 0 \quad \text{and} \quad L(f(x)) - L(x) = 0$$

if and only if $\Phi(x) = \Phi(f^n(x))$ for all $n \in \mathbb{N}$ which implies that $x \in K$. $\square$

2.2. Lyapunov metrics in $\mathbb{R}^m$. To formulate the notion of *Lyapunov metric* we need to introduce some notation. Given a homeomorphism $f\colon \mathbb{R}^m \to \mathbb{R}^m$ and a continuous map $G\colon \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, we denote by $\Delta(G)$ the map $\Delta(G)\colon \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ defined by

$$\Delta(G)(x,y) = G(f(x), f(y)) - G(x,y).$$

We denote by $\Delta^2(G)$ the map $\Delta(\Delta(G))$.

Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a homeomorphism. We say that $D\colon \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is a *Lyapunov metric* for f if it is continuous, vanishes only on the diagonal, and is such that both $D(x,y)$ and $\Delta^2(D)(x,y)$ are positive for (x,y) whenever $x \neq y$.

Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a homeomorphism of the plane that admits a Lyapunov metric D. A fixed point $x \in \mathbb{R}^m$ is said to be *D-stable* if the map $D_x\colon \mathbb{R}^m \to \mathbb{R}$ defined by $D_x(y) = D(x,y)$ is proper and given any $k' > 0$ there exists $k > 0$ such that $D(x, f^m(y)) < k'$ for all $m \geq 0$, whenever $D(x,y) < k$.

Proposition 2.5. *Let $f\colon \mathbb{R}^m \to \mathbb{R}^m$ be a homeomorphism and let D be a Lyapunov metric for f. If p is a fixed point of f that is D-stable then p is globally asymptotically stable.*

Proof. We will show that the map D_p is a Lyapunov function for f and p. By hypothesis it is positive and proper and only vanishes at p. So it only remains to show that $D_p(f(x)) < D_p(x)$ for all $x \in \mathbb{R}^m$, $x \neq p$. That is $\Delta(D)(x,p) < 0$ for all $x \neq p$. Set

$$\mathcal{W} := \{x \in \mathbb{R}^m : \Delta(D)(f^n(x), p) < 0 \text{ for all } n \geq 0\} \cup \{p\}.$$

We will show that $\mathcal{W} = \mathbb{R}^m$.

First of all we claim that there exists a neighbourhood $\mathcal{U}$ of p, such that $\mathcal{U} \subset \mathcal{W}$. To prove this note that since

$$\Delta(D)(f(x), p) = \Delta(D)(x,p) + \Delta^2(D)(x,p),$$

we have that $\Delta(D)(f(x), p) > \Delta(D)(x,p)$, and consequently, for all $n > 0$, $\Delta(D)(f^n(x), p) > \Delta(D)(x,p)$. Therefore we get

$$D(f^n(x), p) = D(x,p) + \sum_{i=0}^{n-1} \Delta(D)(f^i(x), p) > D(x,p) + n\Delta(D)(x,p).$$

Thus if $\Delta(D)(x,p) > 0$ we obtain that $\lim_{n\to\infty} D(f^n(x), p) = \infty$. We obtain the same conclusion when $\Delta(D)(x,p) = 0$, simply observing that for $y = f(x)$ we have $\Delta(D)(y,p) > 0$. Lastly note that if $\Delta(D)(f^j(x), p) \geq 0$ then $\lim_{n\to\infty} D(f^n(f^j(x)), p) = \infty$ and so $\lim_{n\to\infty} D(f^n(x), p) = \infty$.

Now we prove the claim. If such a neighbourhood $\mathcal{U}$ does not exist this implies that there is a sequence $\{x_i\}_i$ such that $\lim x_i = p$ and $\lim_{n\to\infty} D(f^n(x_i), p) = \infty$

for all i. This clearly contradicts the fact that p is D-stable. In particular, we have showed that $\mathcal{W}$ is invariant.

Next we will show that $\mathcal{W}$ is closed. Let $\{x_k\}_k$ be a convergent sequence of points from $\mathcal{W}$ and let x be its limit point. If x does not belong to $\mathcal{W}$ then there exists $n \in \mathbb{N}$ such that $D(f^n(x), p) > 0$. Therefore, there exists a neighbourhood $\mathcal{B}$ of $f^n(x)$ such that $D(y, p) > 0$ for all $y \in \mathcal{B}$. Thus for a sufficiently large k, we have that $f^n(x_k) \in \mathcal{B}$ which contradicts the fact that $x_k \in \mathcal{W}$.

The next step is to show that $D(f^n(x), p)$ tends to zero when n tends to infinity and $x \in \mathcal{W}$. Let us suppose that there exists a point $x \in \mathcal{W}$ such that $D(f^n(x), p)$ does not converge to zero. Note that since $x \in \mathcal{W}$, the function D_p is decreasing over the positive orbit of x. Then the sequence $\{D(f^n(x), p)\}_{n\geq 0}$ is bounded by $D(x, p)$. Since D_p is a proper function we obtain that the set $\{f^n(x) : n \geq 0\}$ is bounded and therefore there exists an accumulation point $q \neq p$. Since $\mathcal{W}$ is closed, we have that $q \in \mathcal{W}$, so $D(f(q), p) < D(q, p)$. This implies the existence of $n_1 < n_2$ such that $D(f^{n_1}(x), p) < D(f^{n_2}(x), p)$ which contradicts that $x \in \mathcal{W}$.

Lastly we prove that $\mathcal{W} = \mathbb{R}^m$. It only remains to show that $\mathcal{W}$ is open. Let x be an arbitrary point of $\mathcal{W}$. By the above observation, we know that $\lim\limits_{n\to\infty} D(f^n(x), p) = 0$. We claim that $\lim\limits_{n\to\infty} f^n(x) = p$. If this is not the case, then $\{f^n(x)\}_{n\geq 0}$ must accumulate in a point $q \neq p$ in view of the fact that D_p is a proper function. So, there exists some subsequence $\{f^{n_k}(x)\}_{n_k}$ converging to q. Using the continuity of D, we have that $D(f^{n_k}(x), p)$ tends to $D(q, p)$ which is a positive number. This fact is in contradiction with the fact that $\lim\limits_{n\to\infty} D(f^n(x), p) = 0$. Therefore $\lim_{n\to\infty} f^n(x) = p$ and there exists k such that $f^k(x) \in \mathcal{U}$. Let $\mathcal{V}$ be an open neighbourhood of $f^k(x)$ contained in $\mathcal{U}$. Then $f^{-k}(\mathcal{V})$ is an open neighbourhood of x contained in $\mathcal{W}$. Therefore $\mathcal{W}$ is open and then $\mathcal{W} = \mathbb{R}^m$. Thus D_p is a Lyapunov function for f and p and the result follows from Proposition 2.4. □

2.3. The planar case. In this section we focus on $\mathbb{R}^2$. Given a Jordan curve $J \subset \mathbb{R}^2$, we denote by $\operatorname{int}(J)$ the bounded connected component of $\mathbb{R}^2 \setminus J$. Given two Jordan curves J_1 and J_2, we write $J_1 \prec J_2$ if $J_1 \subset \operatorname{int}(J_2)$. Note that if $J_1 \prec J_2$, in particular, $J_1 \cap J_2 = \emptyset$. Note also that if $J_1 \prec J_2$ and $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism then $h(J_1) \prec h(J_2)$. Thus $h^n(J_1) \prec h^n(J_2)$ for all $n \in \mathbb{Z}$. The next lemma plays a crucial role in the proof of Theorem 1.1.

LEMMA 2.6. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism and K globally asymptotically stable. Then there exists an analytic Jordan curve J with $K \subset \operatorname{int}(J)$ satisfying that $f(J) \prec J$ and $K \subset \operatorname{int}(f(J))$.*

PROOF. In view of Proposition 2.4 there exists a Lyapunov continuous function $L\colon \mathbb{R}^2 \to \mathbb{R}$, proper and strictly decreasing over the orbits (not contained in K) of f. The first objective is to modify L to obtain another Lyapunov function $\overline{L}$ which is analytic (in fact polynomial) in some open range of levels. First we note that given $b > 0$ there exists $0 < a < b$ such that $f(L^{-1}[a,b]) \cap L^{-1}[a,b] = \emptyset$. To do this consider $c < b$ and set

$$\widehat{a} = \max\{L(f(x)) : x \in L^{-1}[c,b]\}.$$

This value is well-defined because L is proper and L and f are continuous, and clearly $\widehat{a} < b$ because L is a Lyapunov function. Then if $\widehat{a} < c$ it suffices to choose $a = c$. Otherwise we can choose any $a \in (\widehat{a}, b)$.

Now choose $0 < a < b$ such that $M := L^{-1}[a,b]$ satisfies that $M \cap f(M) = \emptyset$. Choose also $0 < \varepsilon < (b-a)/4$ and denote $M_\varepsilon = L^{-1}([a+\varepsilon, b-\varepsilon])$ and $M_{2\varepsilon} = L^{-1}([a+2\varepsilon, b-2\varepsilon])$ which are compact sets because L is proper. By the Stone–Weierstrass theorem, there exists a polynomial map P such that

$$\max\{|P(x) - L(x)| : x \in M\} < \min(a, \varepsilon/4).$$

In particular, $P(x)$ is positive in M. Set also $\mathcal{U} = L^{-1}((a+\varepsilon/2, b-\varepsilon/2))$ and $g \colon \mathbb{R}^2 \to [0,1]$ a $\mathcal{C}^\infty$ function such that $g(x) = 1$ for all $x \in M_\varepsilon$ and $g(x) = 0$ for all $x \notin \mathcal{U}$. Lastly define $\overline{L} = gP + (1-g)L$. Simple computations show that $\overline{L}$ satisfies the following properties:

(1) $\overline{L}(x) = P(x)$ when $x \in M_\varepsilon$ and $\overline{L}(x) = L(x)$ when $x \notin \mathcal{U}$.
(2) $\overline{L}^{-1}([a+2\varepsilon, b-2\varepsilon]) \subset M_\varepsilon$.
(3) $\overline{L}(x) \in [a,b]$ for all $x \in M$.

Now we claim that $\overline{L}$ is still a proper Lyapunov function. It is proper because it coincides with L outside of M. By construction, it follows that $\overline{L}$ satisfies property (i) of a Lyapunov function. Now we show that it is decreasing over the orbits. Set $x \notin K$, we need to show that $\overline{L}(f(x)) < \overline{L}(x)$. This is clear when x and $f(x)$ do not belong to M because in this case L and $\overline{L}$ coincide. Suppose that $x \in M$. By construction, $f(x) \notin M$ and $\overline{L}(f(x)) = L(f(x)) < a$. On the other hand, by (3), $\overline{L}(x) \geq a$. Lastly, suppose that $f(x) \in M$. Also by construction, $x \notin M$ and $\overline{L}(x) = L(x) > b$. Also by (3) we have that $\overline{L}(f(x)) \in [a,b]$. This concludes the proof of the claim.

Note also that from (1) and (2) it follows that $\overline{L}$ is polynomial on the set $\overline{L}^{-1}([a+2\varepsilon, b-2\varepsilon])$. By Sard's theorem, we get that for almost all values $z \in (a+2\varepsilon, b-2\varepsilon)$, $\overline{L}^{-1}(z)$ is an analytic compact manifold. Now we choose $z \in (a+2\varepsilon, b-2\varepsilon)$ with this property and we get that $L^{-1}(z)$ is a finite union of Jordan curves. Since $\overline{L}$ is proper it follows that $\overline{L}^{-1}(z)$ separates K and a circle of radius R large enough. Thus at least one of the Jordan curves of $L^{-1}(z)$ must surround K. Denote one of them by J.

To end the proof of the lemma we will show that $f(J) \prec J$. First of all note that since J surrounds K, $f(K) = K$ and f is a homeomorphism, we have that $f(J)$ also surrounds K. Furthermore, since $L(f(x)) < z$ for all $x \in J$, it follows that either $J \prec f(J)$ or $f(J) \prec J$. However, if $J \prec f(J)$ we also get $f(J) \prec f^2(J)$ because f is a homeomorphism and hence $J \prec f^2(J)$. Iterating this argument, we have $J \prec f^n(J)$ for all $n \in \mathbb{N}$ which contradicts the fact that K is globally asymptotically stable. This ends the proof of the lemma. □

We end this section stating the classical Schoenflies' theorem, [2], [9]. As we will see, we will use it together with some ideas of [9, Corollary 2.1] for proving Theorem 1.1.

THEOREM 2.7. *Let J_1 and J_2 be simple closed curves. Then any homeomorphism $h\colon J_1 \to J_2$ can be extended to a homeomorphism from* $\operatorname{int}(J_1) \cup J_1$ *onto* $\operatorname{int}(J_2) \cup J_2$.

3. Proof of the main results

PROOF OF THEOREM 1.1. Since $\mathcal{U}$ is connected and simply connected it is homeomorphic to $\mathbb{R}^2$ by the Riemann theorem. Therefore, we can restrict our attention to the case where $\mathcal{U} = \mathbb{R}^2$ and K is a global attractor. From Proposition 2.3 the set $\widetilde{K}$ is globally asymptotically stable and from Lemma 2.6 we know that there exists a Jordan curve J surrounding $\widetilde{K}$ such that $f(J) \prec J$. First of all we claim that the ring determined by J and $f(J)$ that is $J \cup (\operatorname{int}(J) \setminus \operatorname{int}(f(J)))$ is contained in $\mathbb{R}^2 \setminus \widetilde{K}$. This is because if $z \in J \cup (\operatorname{int}(J) \setminus \operatorname{int}(f(J)))$ then $f^{-1}(z)$ belongs to the ring determined by $f^{-1}(J)$ and J. Therefore $f^{-1}(z) \notin \widetilde{K}$ and, since $\widetilde{K}$ is invariant, $z \notin \widetilde{K}$. So the claim is proved. For $i = 1, 2$ denote

$$S_i = \{z \in \mathbb{R}^2 : \|z\| = 1/i\},$$
$$S_i^+ = \{(x,y) \in S_i : y \geq 0\}, \quad S_i^- = \{(x,y) \in S_i : y \leq 0\}$$

and consider a homeomorphism $g_1\colon J \to S_1$. Denote

$$J^+ = g_1^{-1}(S_1^+), \quad J^- = g_1^{-1}(S_1^-), \quad a = g_1^{-1}(-1,0), \quad b = g_1^{-1}(1,0)$$

and consider two simple and disjoint paths γ_1, γ_2 contained in the annulus determined by J and $f(J)$ joining a with $f(a)$ and b with $f(b)$. Denote also by $\widehat{J}$ the Jordan curve obtained by gluing $J^+, f(J^+), \gamma_1$ and γ_2. There are two possibilities. Either $\operatorname{int}(\widehat{J}) \cap \operatorname{int}(f(J)) = \emptyset$ or $\operatorname{int}(f(J)) \subset \operatorname{int}(\widehat{J})$. In the first case consider homeomorphisms g_3 and g_4 from γ_1 to $I_1 = \{(t,0) : t \in [-1,-1/2]\}$ and from γ_2 to $I_2 = \{(t,0) : t \in [1/2,1]\}$ and $g_2\colon f(J) \to S_2$ given by $g_2(x) = g_1(f^{-1}(x))/2$. Gluing g_1, g_2, g_3 and g_4 we obtain and homeomorphism from $\widehat{J}$ to the Jordan curve formed by S_1^+, S_2^+, I_1 and I_2. Thus, by Schoenflies' theorem, this homeomorphism can be extended to a homeomorphism g^+ between the interiors of both Jordan curves, see Figure 3.

A similar argument shows that gluing g_1, g_2, g_3 and g_4 we obtain a homeomorphism between the Jordan curve formed by $J^-, f(J^-), \gamma_1, \gamma_2$ to the Jordan

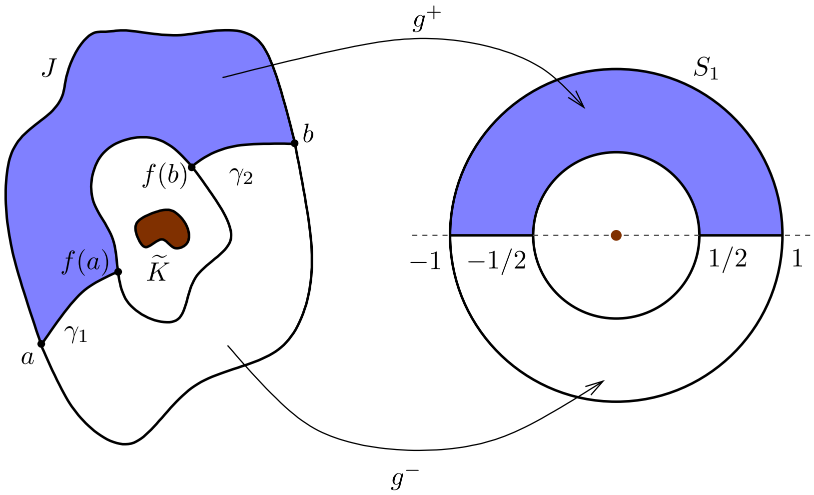

Figure 3. Construction of g in the preserving orientation case.

curve formed by S_1^-, S_2^-, I_1 and I_2. We also extend these homeomorphisms to a homeomorphism g^- between its interiors. Since g^+ and g^- coincide in γ_1 and γ_2, we can glue both homeomorphisms to obtain a homeomorphism g between the annulus determined by J and $f(J)$, which we denote by $\mathcal{A}$, and the annulus determined by S_1 and S_2.

Now we claim that $\mathbb{R}^2 \setminus \widetilde{K} = \bigcup_{i\in\mathbb{Z}} f^i(\mathcal{A})$. Let $z \in \mathbb{R}^2 \setminus \widetilde{K}$ and assume that $z \notin \mathcal{A}$. If $z \in \bigcup_{i\in\mathbb{Z}} f^n(J)$ there is nothing to prove. From Lemma 2.2 it follows that there exists $i \leq 0$ such that $f^i(z) \notin \operatorname{int}(J)$. Now, since K is an attractor there exists a first $j > 0$ such that $f^{i+j}(z) \in \operatorname{int}(J)$. This implies that $f^{(i+j)}(z) \in \mathcal{A}$ and hence $z \in f^{-(i+j)}(\mathcal{A})$.

Lastly let $h\colon \mathbb{R}^2 \setminus \widetilde{K} \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ be defined by $h(z) = 2^n g(f^n(z))$ when $f^n(z) \in \mathcal{A}$. Note that if $z \notin \bigcup_{n\in\mathbb{Z}} f^n(J)$ then the map h is uniquely defined. If $z \in f^n(J)$ for some $n \in Z$ then $f^{-n}(z) \in J \subset \mathcal{A}$ and $f^{-n+1}(z) \in f(J) \subset \mathcal{A}$. Thus $h(z) = 2^{-n} g(f^{-n}(z)) = 2^{-n} g_1(f^{-n}(z))$ or

$$\begin{aligned} h(z) &= 2^{-n+1} g(f^{-n+1}(z)) = 2^{-n+1} g_2(f^{-n+1}(z)) \\ &= 2^{-n+1} \frac{g_1(f^{-n}(z))}{2} = 2^{-n} g_1(f^{-n}(z)). \end{aligned}$$

Therefore, h is a well-defined homeomorphism. Lastly, we have

$$h(z) = 2^n g(f^n(z)) = 2(2^{n-1} g(f^{n-1}(f(z)))) = 2h(f(z))$$

which implies that $h(f(h^{-1}(z))) = z/2$. In particular, f preserves orientation. This ends the proof of the theorem in this case.

When $\text{int}(f(J)) \subset \text{int}(\tilde{J})$ the proof follows in a similar way considering the Jordan curve J^* obtained joining $J^+, f(J^-), \gamma_1, \gamma_2$ and the map g_2^* given by $g_2^*(x) = \overline{g_1(f^{-1}(x))}/2$ instead of $\tilde{J}$ and g_2. In this case the corresponding map h^* satisfies $h^*(f((h^*)^{-1}(z))) = \overline{z}/2$ and hence f reverses orientation. This ends the proof of the theorem. □

The proof of Corollary 1.2 is straightforward from Theorem 1.1 because in this case the basin of attraction is the whole plane, that is $\mathcal{U} = \mathbb{R}^2$ and hence it is clearly connected and simply connected. Next, we prove Kerékjártó's theorem.

PROOF OF KERÉKJÁRTÓ'S THEOREM. We want to apply Theorem 1.1 when $K = \{p\}$, where p is an asymptotically stable fixed point and $\mathcal{U}$ is its basin of attraction. Hence the result follows once we prove that the set $\mathcal{U}$ is connected and also simply connected. This is so, because we obtain a homeomorphism $h\colon \mathcal{U} \setminus \{p\} \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ that linearizes f. Clearly, this homeomorphism extends to a homeomorphism between $\mathcal{U}$ and $\mathbb{R}^2$ by putting $h(p) = \mathbf{0}$.

The above properties of $\mathcal{U}$ are well-known, but we prove them for the sake of completeness. First, let us see that $\mathcal{U}$ is arc-connected. To do this let $D \subset \mathcal{U}$ be an open disc containing p and $z_1, z_2 \in \mathcal{U}$. Since p is asymptotically stable, there exists n such that $f^n(z_1), f^n(z_2) \in D$. Now set $\alpha \subset D$ be an arc joining $f^n(z_1)$ and $f^n(z_2)$. Then $\overline{\alpha} = f^{-n}(\gamma)$ is an arc joining z_1 and z_2. Since $f^n(\overline{\alpha}) \subset D \subset \mathcal{U}$ and $\mathcal{U}$ is f-invariant, it follows that $\overline{\alpha} \subset \mathcal{U}$.

Now we show that $\mathcal{U}$ is simply connected. Let $J \subset \mathcal{U}$ be a Jordan curve and we will see that $\text{int}(J) \subset \mathcal{U}$. As before let $D \subset \mathcal{U}$ be an open disc containing p and $n \in \mathbb{N}$ be such that $f^n(J) \subset D$. Since f is a homeomorphism, $f^n(J)$ is also a Jordan curve and $f^n(\text{int}(J)) = \text{int}(f^n(J)) \subset D \subset \mathcal{U}$. Since $\mathcal{U}$ is f-invariant, $\text{int}(J) \subset \mathcal{U}$ and the curve J can be deformed in $\mathcal{U}$ to a point. □

Collecting all our results in the plane we can prove Theorem 1.3.

PROOF OF THEOREM 1.3. The equivalence (a) ⇔ (b) follows from Proposition 2.4, while the double implication (a) ⇔ (c) follows from Kerékjártó's theorem. Also it follows from Proposition 2.5 that statement (d) implies statement (a). To finish the proof it suffices to show that statement (c) implies (d). This is a consequence of the fact that the usual Euclidian metric D is a Lyapunov metric for L_1 and L_2 for which $(0,0)$ is D-stable. Then the metric transported via the conjugation gives the desired Lyapunov metric for f. □

Acknowledgments. We thank Rafael Ortega for sending us his interesting paper [11], from which we learned about Kerékjártó's result and its extensions.

REFERENCES

[1] N.P. BHATIA AND G.P. SZEGÖ, *Stability theory of dynamical systems*, Reprint of the 1970 original, Classics in Mathematics, Springer–Verlag, Berlin, 2002.

[2] S.S. Cairns, *An elementary proof of the Jordan–Schoenflies theorem*, Proc. Amer. Math. Soc. **2** (1951), 860–867.

[3] A. Cima, F. Mañosas and J. Villadelprat, *A Poincaré–Hopf theorem for noncompact manifolds*, Topology **37** (1998), 261–277.

[4] T. Homma and S. Kinoshita, *On the regularity of homeomorphisms of E^n*, J. Math. Soc. Japan **5** (1953), 365–371.

[5] ______, *On a topological characterization of the dilatation in E^3*, Osaka Math. J. **6** (1954), 135–144.

[6] L.S. Husch, *A topological characterization of the dilation in E^n*, Proc. Amer. Math. Soc. **28** (1971), 234–236.

[7] B. de Kerékjártó, *Sur le caractère topologique des représentations conformes*, C.R. Acad. Sci. Paris **198** (1934), 317–320 (in French).

[8] B. von Kerékjártó, *Topologische Charakterisierung der linearen Abbildungen*, Acta Litt. Acad. Sei. Szeged. **6** (1934), 235–262 (in German).

[9] B. Kolev, *Plane Topology and Dynamical Systems*, École thématique, Summer School "Systèmes Dynamiques et Topologie en Petites Dimensions", Grenoble, France, 1994, pp. 35. ⟨cel-00719540⟩

[10] J. Lewowicz, *Lyapunov functions and topological stability*, J. Differential Equations **38** (1980), 192–209.

[11] P. Ortega and F.R. Ruiz del Portal, *Attractors with vanishing rotation number*, J. Eur. Math. Soc. (JEMS) **13** (2011), 1569–1590.

Manuscript received May 22, 2015

accepted September 21, 2015

Armengol Gasull and Francesc Mañosas
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra
Barcelona, SPAIN

E-mail address: gasull@mat.uab.cat, manyosas@mat.uab.cat

Jorge Groisman
IMERL, Facultad de Ingeniería
Universidad de la República
Montevideo, URUGUAY

E-mail address: manyosas@mat.uab.cat