MULTIPLICITY OF SOLUTIONS
OF SOME QUASILINEAR EQUATIONS IN $\mathbb{R}^N$
WITH VARIABLE EXPONENTS AND CONCAVE-CONVEX
NONLINEARITIES

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ABSTRACT. We prove multiplicity of solutions for a class of quasilinear problems in $\mathbb{R}^N$ involving variable exponents and nonlinearities of concave-convex type. The main tools used are variational methods, more precisely, Ekeland's variational principle and Nehari manifolds.

1. Introduction

In this paper, we consider the existence and multiplicity of solutions for the following class of quasilinear problems involving variable exponents:

$$(\mathrm{P}_{\lambda,k}) \quad \begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = \lambda g(k^{-1}x)|u|^{q(x)-2}u + f(k^{-1}x)|u|^{r(x)-2}u \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{cases}$$

where λ and k are positive parameters with $k \in \mathbb{N}$, the operator $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, named the $p(x)$-Laplacian, is a natural extension of the p-Laplace operator with p being a positive constant.

We assume that $p, q, r\colon \mathbb{R}^N \to \mathbb{R}$ are positive Lipschitz continuous functions, $\mathbb{Z}^N$-periodic, that is,

(p$_1$) $p(x + z) = p(x), \quad q(x + z) = q(x) \quad \text{and} \quad r(x + z) = r(x),$

for $x \in \mathbb{R}^N$, $z \in \mathbb{Z}^N$, verifying

(p$_2$) $1 < q_- \leq q(x) \leq q_+ < p_- \leq p(x) \leq p_+ < r_- \leq r \ll p^*,$

almost everywhere in $\mathbb{R}^N$, where $p_+ = \operatorname*{ess\,sup}_{x \in \mathbb{R}^N} p(x)$, $p_- = \operatorname*{ess\,inf}_{x \in \mathbb{R}^N} p(x)$ and

(P) $p^*(x) = \begin{cases} Np(x)/(N - p(x)) & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$

Hereafter, the notation $u \ll v$ means that $\inf_{x \in \mathbb{R}^N}(v(x) - u(x)) > 0$.

Furthermore, we assume the condition

(H) $\dfrac{q_+}{p_-} < \dfrac{(r_+ - q_+)}{(r_+ - p_-)} \dfrac{(r_- - p_+)}{(r_- - q_-)}.$

Here, we would like to point out that this condition is equivalent to $0 < q < p$ for the case where the exponent is constant. This technical condition will be needed, especially in the proof of Lemma 3.7.

Regarding the functions f and g, we assume the following conditions:

(g$_1$) $g\colon \mathbb{R}^N \to \mathbb{R}$ is a nonnegative measurable function with $g \in L^{\Theta(x)}(\mathbb{R}^N)$ where $\Theta(x) = r(x)/(r(x) - q(x))$,

(f$_1$) $f\colon \mathbb{R}^N \to \mathbb{R}$ is a positive continuous function such that

$$\lim_{|x| \to \infty} f(x) = f_\infty$$

and $0 < f_\infty < f(x)$ for all $x \in \mathbb{R}^N$,

(f$_2$) there exist ℓ points $a_1, \ldots, a_\ell$ in $\mathbb{Z}^N$ with $a_1 = 0$, such that

$$1 = f(a_i) = \max_{\mathbb{R}^N} f(x), \quad \text{for } 1 \leq i \leq \ell.$$

Problems with variable exponents appear in various applications. The reader is referred to Růžička [39] and Kristály, Radulescu and Varga [30] for several questions in mathematical physics where such class of problems appears. In recent years, these problems have attracted an increasing attention. We would like to mention [3], [5]–[7], [14], [18], [23], [34], [35], [36], [38], as well as the survey papers [8], [16], [41] for the advances and references in this field.

Problem (P$_{\lambda,k}$) has been considered in the literature for the case where the exponents are constants, see, for example, Adachi and Tanaka [1], Autuori and Pucci [9], Cao and Noussair [12], Cao and Zhou [13], Hirano [24], Hirano and Shioji [25], Hsu, Lin and Hu [26], Hu and Tang [27], Jeanjean [28], Lin [31], Pucci and Radulescu [37], Tarantello [42], Wu [45], [46] and their references.

In [12], Cao and Noussair have studied the existence and multiplicity of positive and nodal solutions for the following problem:

$$(\mathrm{P}_1) \qquad \begin{cases} -\Delta u + u = f(\varepsilon x)|u|^{r-2}u & \text{in } \mathbb{R}^N, \\ u \in H^{1,2}(\mathbb{R}^N), \end{cases}$$

where ε is a positive real parameter, $r \in (2, 2^*)$ and f verifies conditions (f_1)–(f_2). By using variational methods, the authors showed the existence of at least ℓ positive solutions and ℓ nodal solutions if ε is small enough. Later on, Wu in [45] considered the perturbed problem

$$(\mathrm{P}_2) \qquad \begin{cases} -\Delta u + u = f(\varepsilon x)|u|^{r-2}u + \lambda g(\varepsilon x)|u|^{q-2}u & \text{in } \mathbb{R}^N, \\ u \in H^{1,2}(\mathbb{R}^N), \end{cases}$$

where λ is a positive parameter and $q \in (0, 1)$. In [45], the author showed the existence of at least ℓ positive solutions for (P_2) when ε and λ are small enough.

In [26], Hsu, Lin and Hu have considered the following class of quasilinear problems:

$$(\mathrm{P}_3) \qquad \begin{cases} -\Delta_p u + |u|^{p-2}u = f(\varepsilon x)|u|^{r-2}u + \lambda g(\varepsilon x) & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases}$$

with $N \geq 3$ and $2 \leq p < N$. In that paper, the authors have proved the same type of results found in [12] and [45].

In [37], Pucci and Radulescu studied the existence and multiplicity of solution for the following class of quasilinear problems:

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + u^{p-1} = \lambda u^{q-1} - h(x)u^{r-1} & \text{in } \mathbb{R}^N, \\ u \geq 0 & \text{in } \mathbb{R}^N, \end{cases}$$

where λ is a positive parameter, $2 \leq p < q < \min\{r, p^*\}$ and $h(x) > 0$ satisfies

$$\int_{\mathbb{R}^N} h(x)^{q/(r-q)}\, dx < +\infty.$$

Motivated by the above paper, Autuori and Pucci in [9] also considered the existence of multiple solution for more general quasilinear problems like

$$-\mathrm{div}(A(x, \nabla u)) + a(x)|u|^{p-2}u = \lambda w(x)|u|^{q-2}u - h(x)|u|^{r-2}u \quad \text{in } \mathbb{R}^N,$$

and they improved the results obtained in [37]. In [38], Pucci and Zhang considered a class of quasilinear variable exponent of the type

$$-\mathrm{div}(A(x, \nabla u)) + a(x)|u|^{p(x)-2}u = \lambda w(x)|u|^{q(x)-2}u - h(x)|u|^{r(x)-2}u \quad \text{in } \mathbb{R}^N,$$

improving some results found in [9] and [37].

Motivated by results proved in [12], [26] and [45], by a similar approach we intend to prove the existence of multiple solutions for problem $(\mathrm{P}_{\lambda,k})$. However,

once we are working with variable exponents, some estimates that hold for the constant case are not immediate for the variable case, and so, a careful analysis is necessary. More precisely, when the exponents are constant each term in the nonlinearity is homogeneous, what allows to get some estimates involving the energy functional, however if the exponents are not constant we loose this property. This difficulty is overcome by using Lemmas 3.8 and 3.9. We added further explanations immediately before the statement of each of these lemmas. Finally, we also would like to point out that our main result complements the study made in [38], because in that paper the first result establishes the existence of only two solutions for λ large, while the second result shows a multiplicity result assuming some conditions which imply that the energy functional associated with the problem is even. Here, we have showed that the multiplicity of solution is affected by the number of maximum points of function f, and in our approach there is no symmetry condition on the energy functional.

Our main result is the following.

THEOREM 1.1. *Assume that* (p_1)–(p_2), (g_1), (f_1)–(f_2) *and* (H) *are satisfied. Then, there are positive numbers* k_* *and* $\Lambda_* = \Lambda(k_*)$, *such that problem* $(P_{\lambda,k})$ *admits at least* $\ell + 1$ *solutions for* $0 < \lambda < \Lambda_*$ *and* $k > k_*$.

Notation. The following notations will be used:

- C and c_i denote generic positive constants, which may vary from line to line.
- We denote by $\int u$ the integral $\int_{\mathbb{R}^N} u \, dx$, for any measurable function u.
- $B_R(z)$ denotes the open ball in $\mathbb{R}^N$ with center at z and radius R.

2. Preliminaries on Lebesgue and Sobolev spaces with variable exponents in $\mathbb{R}^N$

In this section, we recall definitions and some results involving the spaces $L^{h(x)}(\mathbb{R}^N)$ and $W^{1,h(x)}(\mathbb{R}^N)$. We refer to [15], [20], [19], [29] for the fundamental properties of these spaces.

Hereafter, let us denote by $L_+^\infty(\mathbb{R}^N)$ the set

$$L_+^\infty(\mathbb{R}^N) = \left\{ u \in L^\infty(\mathbb{R}^N) : \operatorname*{ess\,inf}_{x \in \mathbb{R}^N} u \geq 1 \right\},$$

and we will assume that $h \in L_+^\infty(\mathbb{R}^N)$. The variable exponent Lebesgue space $L^{h(x)}(\mathbb{R}^N)$ is defined by

$$L^{h(x)}(\mathbb{R}^N) = \left\{ u \colon \mathbb{R}^N \to \mathbb{R} \text{ is measurable} : \int |u(x)|^{h(x)} < +\infty \right\},$$

and its usual norm is

$$\|u\|_{h(x)} = \inf \left\{ t > 0 : \int \left| \frac{u(x)}{t} \right|^{h(x)} \leq 1 \right\}.$$

On the space $L^{h(x)}(\mathbb{R}^N)$, we consider the *modular function* $\rho\colon L^{h(x)}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\rho(u) = \int |u(x)|^{h(x)}.$$

In what follows, let us denote by h_- and h_+ the following real numbers:

$$h_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^N} h(x) \quad\text{and}\quad h_+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} h(x).$$

PROPOSITION 2.1. *Let* $u \in L^{h(x)}(\mathbb{R}^N)$ *and* $\{u_n\}_{n\in\mathbb{N}} \subset L^{h(x)}(\mathbb{R}^N)$. *Then:*

(a) *If* $u \neq 0$, *then* $\|u\|_{h(x)} = a$ *if and only if* $\rho(u/a) = 1$.
(b) $\|u\|_{h(x)} < 1$ $(= 1;\ > 1)$ *if and only if* $\rho(u) < 1 (= 1; > 1)$.
(c) *If* $\|u\|_{h(x)} > 1$ *then* $\|u\|_{h(x)}^{h_-} \leq \rho(u) \leq \|u\|_{h(x)}^{h_+}$.
(d) *If* $\|u\|_{h(x)} < 1$ *then* $\|u\|_{h(x)}^{h_+} \leq \rho(u) \leq \|u\|_{h(x)}^{h_-}$.
(e) $\lim\limits_{n\to+\infty} \|u_n\|_{h(x)} = 0$ *if and only if* $\lim\limits_{n\to\infty} \rho(u_n) = 0$.
(f) $\lim\limits_{n\to+\infty} \|u_n\|_{h(x)} = +\infty$ *if and only if* $\lim\limits_{n\to\infty} \rho(u_n) = +\infty$.

As usual, we denote by $h'(x) = h(x)/(h(x)-1)$ the conjugate exponent function of $h(x)$, and define

$$h^*(x) = \begin{cases} \dfrac{Nh(x)}{N - h(x)} & \text{if } h(x) < N, \\[2mm] +\infty & \text{if } h(x) \geq N. \end{cases}$$

We have the following Hölder inequality for Lebesgue spaces with variable exponents.

PROPOSITION 2.2 (Hölder-type inequality). *Let* $u \in L^{h(x)}(\mathbb{R}^N)$ *and* $v \in L^{h'(x)}(\mathbb{R}^N)$. *Then,* $uv \in L^1(\mathbb{R}^N)$ *and*

$$\int |u(x)v(x)| \leq \left(\frac{1}{h_-} + \frac{1}{h'_-}\right) \|u\|_{h(x)} \|v\|_{h'(x)}.$$

LEMMA 2.3. *Let* $h, b \in L_+^\infty(\mathbb{R}^N)$ *with* $h(x) \leq b(x)$ *almost everywhere in* $\mathbb{R}^N$ *and* $u \in L^{b(x)}(\mathbb{R}^N)$. *Then,* $|u|^{h(x)} \in L^{b(x)/h(x)}(\mathbb{R}^N)$,

$$\||u|^{h(x)}\|_{b(x)/h(x)} \leq \max\left\{\|u\|_{b(x)}^{h_+}, \|u\|_{b(x)}^{h_-}\right\},$$

and further

$$\||u|^{h(x)}\|_{b(x)/h(x)} \leq \|u\|_{b(x)}^{h_+} + \|u\|_{b(x)}^{h_-}.$$

The next three results are important tools to study the properties of some energy functionals, and their proofs can be found in [5].

PROPOSITION 2.4 (Brezis–Lieb's lemma, first version). *Let*

$$\{\eta_n\} \subset L^{h(x)}(\mathbb{R}^N, \mathbb{R}^m) \quad \text{with } m \in \mathbb{N}$$

verifying:

(a) $\eta_n(x) \to \eta(x)$, *almost everywhere in* $\mathbb{R}^N$;
(b) $\sup_{n \in \mathbb{N}} |\eta_n|_{L^{h(x)}(\mathbb{R}^N, \mathbb{R}^m)} < \infty$.

Then, $\eta \in L^{h(x)}(\mathbb{R}^N, \mathbb{R}^m)$ *and*

$$\int (|\eta_n|^{h(x)} - |\eta_n - \eta|^{h(x)} - |\eta|^{h(x)}) = o_n(1).$$

PROPOSITION 2.5 (Brezis–Lieb's lemma, second version). *Let*

$$\{\eta_n\} \subset L^{h(x)}(\mathbb{R}^N, \mathbb{R}^m)$$

verifying:

(a) $\eta_n(x) \to \eta(x)$, *almost everywhere in* $\mathbb{R}^N$;
(b) $\sup_{n \in \mathbb{N}} |\eta_n|_{L^{h(x)}(\mathbb{R}^N, \mathbb{R}^m)} < \infty$.

Then $\eta_n \rightharpoonup \eta$ *in* $L^{h(x)}(\mathbb{R}^N, \mathbb{R}^m)$.

The next proposition is a Brezis–Lieb type result.

PROPOSITION 2.6 (Brezis–Lieb's lemma, third version). *Let*

$$\{\eta_n\} \subset L^{h(x)}(\mathbb{R}^N, \mathbb{R}^m)$$

such that:

(a) $\eta_n(x) \to \eta(x)$, *almost everywhere in* $\mathbb{R}^N$;
(b) $\sup_{n \in \mathbb{N}} |\eta_n|_{L^{h(x)}(\mathbb{R}^N, \mathbb{R}^m)} < \infty$.

Then

$$(2.1) \qquad \int \left| |\eta_n|^{h(x)-2}\eta_n - |\eta_n - \eta|^{h(x)-2}(\eta_n - \eta) - |\eta|^{h(x)-2}\eta \right|^{h'(x)} = o_n(1).$$

The variable exponent Sobolev space $W^{1,h(x)}(\mathbb{R}^N)$ is defined by

$$W^{1,h(x)}(\mathbb{R}^N) = \left\{ u \in W^{1,1}_{\text{loc}}(\mathbb{R}^N) : u \in L^{h(x)}(\mathbb{R}^N) \text{ and } |\nabla u| \in L^{h(x)}(\mathbb{R}^N) \right\}.$$

The corresponding norm for this space is

$$\|u\|_{W^{1,h(x)}(\mathbb{R}^N)} = \|u\|_{h(x)} + \|\nabla u\|_{h(x)}.$$

The spaces $L^{h(x)}(\mathbb{R}^N)$ and $W^{1,h(x)}(\mathbb{R}^N)$ are separable and reflexive Banach spaces when $h_- > 1$.

On the space $W^{1,h(x)}(\mathbb{R}^N)$, we consider the *modular function*

$$\rho_1 \colon W^{1,h(x)}(\mathbb{R}^N) \to \mathbb{R}$$

given by

$$\rho_1(u) = \int \left(|\nabla u(x)|^{h(x)} + |u(x)|^{h(x)} \right).$$

If, we define

$$(2.2) \qquad \|u\| = \inf \left\{ t > 0 : \int \frac{(|\nabla u|^{h(x)} + |u|^{h(x)})}{t^{h(x)}} \leq 1 \right\},$$

then $\| \cdot \|_{W^{1,h(x)}(\mathbb{R}^N)}$ and $\| \cdot \|$ are equivalent norms on $W^{1,h(x)}(\mathbb{R}^N)$.

PROPOSITION 2.7. *Let $u \in W^{1,h(x)}(\mathbb{R}^N)$ and $\{u_n\} \subset W^{1,h(x)}(\mathbb{R}^N)$. Then, the conclusion of Proposition 2.1 takes place for $\| \cdot \|$ and ρ_1.*

The next result is a Sobolev's embedding theorem for variable exponent, whose proof can be found in [15] and [20].

THEOREM 2.8 (Sobolev's embedding theorem). *Let $h \colon \mathbb{R}^N \to \mathbb{R}$ be Lipschitz continuous with $1 < h_- \leq h_+ < N$. Consider $s \in L_+^\infty(\mathbb{R}^N)$ satisfying $h(x) \leq s(x) \leq h^*(x)$ almost everywhere in $\mathbb{R}^N$. Then there is a continuous embedding*

$$W^{1,h(x)}(\mathbb{R}^N) \hookrightarrow L^{s(x)}(\mathbb{R}^N).$$

3. Technical lemmas

For convenience, throughout the paper we shall denote

$$g_k(x) = g(k^{-1}x) \quad \text{and} \quad f_k(x) = f(k^{-1}x) \quad \text{for all } x \in \mathbb{R}^N.$$

Associated with problem $(\mathrm{P}_{\lambda,k})$, we have the energy functional

$$J_{\lambda,k} \colon W^{1,p(x)}(\mathbb{R}^N) \to \mathbb{R}$$

defined by

$$J_{\lambda,k}(u) = \int \frac{1}{p(x)}\big(|\nabla u|^{p(x)} + |u|^{p(x)}\big) - \lambda \int \frac{g_k(x)}{q(x)}|u|^{q(x)} - \int \frac{f_k(x)}{r(x)}|u|^{r(x)}.$$

A direct computation gives $J_{\lambda,k} \in C^1(W^{1,p(x)}(\mathbb{R}^N), \mathbb{R})$ with

$$J_{\lambda,k}'(u)v = \int (|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv)$$

$$- \lambda \int g_k(x)|u|^{q(x)-2}uv - \int f_k(x)|u|^{r(x)-2}uv,$$

for each $u, v \in W^{1,p(x)}(\mathbb{R}^N)$. Therefore, the critical points of $J_{\lambda,k}$ are precisely the (weak) solutions of $(\mathrm{P}_{\lambda,k})$.

Since $J_{\lambda,k}$ is not bounded from below on $W^{1,p(x)}(\mathbb{R}^N)$, we will work on the *Nehari manifold* $\mathcal{M}_{\lambda,k}$ associated with $J_{\lambda,k}$, given by

$$\mathcal{M}_{\lambda,k} = \{u \in W^{1,p(x)}(\mathbb{R}^N) \setminus \{0\} \colon J_{\lambda,k}'(u)u = 0\}.$$

In what follows, we denote by $c_{\lambda,k}$ the real number $c_{\lambda,k} = \inf\limits_{u \in \mathcal{M}_{\lambda,k}} J_{\lambda,k}(u)$. Using well-known arguments, it is easy to prove that $c_{\lambda,k}$ is the mountain pass level of $J_{\lambda,k}$.

For $f \equiv 1$ and $\lambda = 0$, we consider the problem

$$(\mathrm{P}_\infty) \qquad \begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = |u|^{r(x)-2}u & \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N). \end{cases}$$

Associated with problem (P_∞), we have the energy functional

$$J_\infty : W^{1,p(x)}(\mathbb{R}^N) \to \mathbb{R}$$

given by

$$J_\infty(u) = \int \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)}) - \int \frac{1}{r(x)}|u|^{r(x)},$$

the mountain pass level $c_\infty = \inf\limits_{u \in \mathcal{M}_\infty} J_\infty(u)$, and the Nehari manifold

$$\mathcal{M}_\infty = \left\{ u \in W^{1,p(x)}(\mathbb{R}^N) \setminus \{0\} : J'_\infty(u)u = 0 \right\}.$$

For $f \equiv f_\infty$ and $\lambda = 0$, we fix the problem

(P_{f_∞})
$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = f_\infty|u|^{r(x)-2}u, & \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{cases}$$

and as above, we denote by $J_{f_\infty}, c_{f_\infty}$ and $\mathcal{M}_{f_\infty}$ the energy functional, the mountain pass level, and Nehari manifold associated with (P_{f_∞}), respectively.

Hereafter, let us fix $K > 1$ such that

(3.1) $$\|v\|_{r(x)} \leq K\|v\| \quad \text{for any } v \in W^{1,p(x)}(\mathbb{R}^N),$$

which exists by Theorem 2.8.

The next lemma is a technical result, which will be used in Section 5.

LEMMA 3.1 (Local property). *For each $k \in \mathbb{N}$, there are positive constants $\lambda^* = \lambda^*(k)$, β and σ independent of k, such that $J_{\lambda,k}(u) \geq \beta > 0$ for all $\lambda \in (0, \lambda^*)$ with $\|u\| = \sigma$.*

PROOF. Combining the definition of $J_{\lambda,k}$ with Hölder's inequality, Sobolev's embedding theorem and Proposition 2.3, we derive

$$J_{\lambda,k}(u) \geq \frac{1}{p_+}\int (|\nabla u|^{p(x)} + |u|^{p(x)})$$
$$- 2\frac{\lambda}{q_-}\|g_k\|_{\Theta(x)}K^{q_+}\max\{\|u\|^{q_-}, \|u\|^{q_+}\} - \int \frac{1}{r_-}|u|^{r(x)}.$$

If $\|u\| < 1$, by Proposition 2.7 and Theorem 2.8,

$$J_{\lambda,k}(u) \geq \frac{1}{p_+}\|u\|^{p_+} - 2\frac{\lambda}{q_-}\|g_k\|_{\Theta(x)}K^{q_+}\|u\|^{q_-} - \frac{K^{r_+}}{r_-}\|u\|^{r_-}.$$

Since $p_+ < r_-$, by fixing σ small enough such that

$$\frac{1}{p_+}\sigma^{p_+} - \frac{K^{r_+}}{r_-}\sigma^{r_-} \geq \frac{1}{2p_+}\sigma^{p_+},$$

we obtain

$$J_{\lambda,k}(u) \geq \frac{1}{2p_+}\sigma^{p_+} - 2\frac{\lambda}{q_-}\|g_k\|_{\Theta(x)}K^{q_+}\sigma^{q_-},$$

for $\|u\| = \sigma$. Now, fix $\lambda^* = \lambda^*(k) > 0$ satisfying

$$(3.2) \qquad \lambda^*\|g_k\|_{\Theta(x)} < \frac{q_-}{8p_+K^{q_+}}\sigma^{p_+ - q_-}.$$

Then, if $0 < \lambda < \lambda^*$,

$$J_{\lambda,k}(u) > \frac{1}{2p_+}\sigma^{p_+} - 2\frac{\lambda^*}{q_-}\|g_k\|_{\Theta(x)}K^{q_+}\sigma^{q_-} > \frac{1}{4p_+}\sigma^{p_+} = \beta > 0$$

on $\partial B_\sigma(0)$, proving the result. $\qquad\square$

The next result concerns the behavior of $J_{\lambda,k}$ on $\mathcal{M}_{\lambda,k}$.

LEMMA 3.2. *The energy functional $J_{\lambda,k}$ is coercive and bounded from below on $\mathcal{M}_{\lambda,k}$.*

PROOF. For $u \in \mathcal{M}_{\lambda,k}$, we have $J'_{\lambda,k}(u)u = 0$. Therefore,

$$\int f_k(x)|u|^{r(x)} = \int \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) - \lambda \int g_k(x)|u|^{q(x)},$$

loading to

$$J_{\lambda,k}(u) \geq \frac{1}{p_+}\int \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) - \frac{\lambda}{q_-}\int g_k(x)|u|^{q(x)} - \frac{1}{r_-}\int f_k(x)|u|^{r(x)}$$

$$= \left(\frac{1}{p_+} - \frac{1}{r_-}\right)\int \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) - \lambda\left(\frac{1}{q_-} - \frac{1}{r_-}\right)\int g_k(x)|u|^{r(x)}.$$

If $\|u\| > 1$, Propositions 2.3 and 2.7 together with Hölder's inequality and Theorem 2.8 give

$$J_{\lambda,k}(u) \geq \left(\frac{1}{p_+} - \frac{1}{r_-}\right)\|u\|^{p_-} - \left(\frac{1}{q_-} - \frac{1}{r_-}\right)2\lambda K^{q^+}\|g_k\|_{\Theta(x)}\|u\|^{q_+}$$

$$= \|u\|^{q_+}\left\{\left(\frac{1}{p_+} - \frac{1}{r_-}\right)\|u\|^{p_- - q_+} - 2\lambda\left(\frac{1}{q_-} - \frac{1}{r_-}\right)K^{q^+}\|g_k\|_{\Theta(x)}\right\}.$$

Since $q_+ < p_-$, the last inequality implies that $J_{\lambda,k}$ is coercive and bounded from below on $\mathcal{M}_{\lambda,k}$. $\qquad\square$

From now on, let $E_{\lambda,k}(v) = J'_{\lambda,k}(v)v$ for any $v \in W^{1,p(x)}(\mathbb{R}^N)$. Employing the functional $E_{\lambda,k}$, we split $\mathcal{M}_{\lambda,k}$ into three parts:

$$\mathcal{M}_{\lambda,k}^+ = \{v \in \mathcal{M}_{\lambda,k} : E'_{\lambda,k}(v)v > 0\},$$
$$\mathcal{M}_{\lambda,k}^0 = \{v \in \mathcal{M}_{\lambda,k} : E'_{\lambda,k}(v)v = 0\},$$
$$\mathcal{M}_{\lambda,k}^- = \{v \in \mathcal{M}_{\lambda,k} : E'_{\lambda,k}(v)v < 0\}.$$

In the next lemma, we prove that the critical points of $J_{\lambda,k}$ restricted to $\mathcal{M}_{\lambda,k}$ which do not belong $\mathcal{M}_{\lambda,k}^0$ are in fact critical points of $J_{\lambda,k}$ on $W^{1,p(x)}(\mathbb{R}^N)$.

LEMMA 3.3. *If $u_0 \in \mathcal{M}_{\lambda,k}$ is a critical point of $J_{\lambda,k}$ restricted to $\mathcal{M}_{\lambda,k}$ and $u_0 \notin \mathcal{M}_{\lambda,k}^0$, then u_0 is a critical point of $J_{\lambda,k}$.*

PROOF. By Lagrange's multiplier theorem, there is $\tau \in \mathbb{R}$ such that

$$J'_{\lambda,k}(u_0) = \tau E'_{\lambda,k}(u_0) \quad \text{in } (W^{1,p(x)}(\mathbb{R}^N))^*,$$

and so, $0 = J'_{\lambda,k}(u_0)u_0 = \tau E'_{\lambda,k}(u_0)u_0$.

If $u_0 \notin \mathcal{M}^0_{\lambda,k}$, we must have $E'_{\lambda,k}(u_0)u_0 \neq 0$. Hence, $\tau = 0$ and $J'_{\lambda,k}(u_0) = 0$ in $(W^{1,p(x)}(\mathbb{R}^N))^*$, showing the lemma. $\qquad\square$

LEMMA 3.4. *Under assumptions* (p_2), (g_1) *and* (f_2), *we have that* $\mathcal{M}^0_{\lambda,k} = \emptyset$ *for all* $k \in \mathbb{N}$ *and* $0 < \lambda < \Lambda_1 = \Lambda_1(k)$, *where*

$$(3.3) \qquad \Lambda_1 = \frac{K^{-q_+}}{2\|g_k\|_{\Theta(x)}} \left(\frac{r_- - p_+}{r_+ - q_-} \right) \left[\left(\frac{p_- - q_+}{r_+ - p_+} \right) K^{-r_+} \right]^{(p_+ - q_-)/(r_- - p_+)}.$$

PROOF. Arguing by contradiction, if the lemma does not hold, we have $\mathcal{M}^0_{\lambda,k} \neq \emptyset$ for some $\lambda_0 \in (0, \Lambda_1)$ and $k \in \mathbb{N}$. Thereby, for $u \in \mathcal{M}^0_{\lambda_0,k}$,

$$
\begin{aligned}
0 &= E'_{\lambda_0,k}(u)u \\
&= \int p(x)(|\nabla u|^{p(x)} + |u|^{p(x)}) - \lambda_0 \int q(x)g_k(x)|u|^{q(x)} - \int r(x)f_k(x)|u|^{r(x)} \\
&\leq p_+ \int (|\nabla u|^{p(x)} + |u|^{p(x)}) - \lambda_0 q_- \int g_k(x)|u|^{q(x)} - r_- \int f_k(x)|u|^{r(x)} \\
&= \lambda_0(r_- - q_-) \int g_k(x)|u|^{q(x)} - (r_1 - p_+) \int (|\nabla u|^{p(x)} + |u|^{p(x)}).
\end{aligned}
$$

By Propositions 2.7 and 2.3, Hölder's inequality and Sobolev's embedding theorem,

$$(3.4) \quad \min\{\|u\|^{p_-}, \|u\|^{p_+}\} \leq 2\lambda_0 \left(\frac{r_- - q_-}{r_- - p_+} \right) \|g_k\|_{\Theta(x)} K^{q_+} \max\{\|u\|^{q_-}, \|u\|^{q_+}\}.$$

Similarly,

$$
\begin{aligned}
0 &= E'_{\lambda_0,k}(u)u \\
&\geq p_- \int (|\nabla u|^{p(x)} + |u|^{p(x)}) - \lambda_0 q_+ \int g_k(x)|u|^{q(x)} - r_+ \int f_k(x)|u|^{r(x)} \\
&= (p_- - q_+) \int (|\nabla u|^{p(x)} + |u|^{p(x)}) - (r_+ - q_+) \int f_k(x)|u|^{r(x)}.
\end{aligned}
$$

Hence,

$$(3.5) \qquad \left(\frac{p_- - q_+}{r_+ - q_+} \right) \min\{\|u\|^{p_-}, \|u\|^{p_+}\} \leq K^{r_+} \max\{\|u\|^{r_-}, \|u\|^{r_+}\}.$$

If $\|u\| \geq 1$, it follows from (3.4) that

$$(3.6) \qquad \|u\| \leq \left[2\lambda_0 \left(\frac{r_- - q_-}{r_- - p_+} \right) \|g_k\|_{\Theta(x)} K^{q_+} \right]^{1/(p_- - q_+)}.$$

On the other hand, by (3.5),

$$(3.7) \qquad \|u\| \geq \left[\left(\frac{p_- - q_+}{r_+ - q_+} \right) K^{-r_+} \right]^{1/(r_+ - p_-)}.$$

Combining (3.6) and (3.7), we derive that

$$(3.8) \qquad \lambda_0 \geq \frac{K^{-q_+}}{2\|g_k\|_{\Theta(x)}} \left(\frac{r_- - p_+}{r_+ - q_-} \right) \left[\left(\frac{p_- - q_+}{r_+ - p_+} \right) K^{-r_+} \right]^{(p_- - q_+)/(r_+ - p_-)}.$$

Since

$$0 < \left(\frac{p_- - q_+}{r_+ - p_+} \right) K^{-r_+} < 1 \quad \text{and} \quad \frac{p_+ - q_-}{r_- - p_+} > \frac{p_- - q_+}{r_+ - p_-},$$

we deduce

$$\lambda_0 \geq \frac{K^{-q_+}}{2\|g_k\|_{\Theta(x)}} \left(\frac{r_- - p_+}{r_+ - q_-} \right) \left[\left(\frac{p_- - q_+}{r_+ - p_+} \right) K^{-r_+} \right]^{(p_+ - q_-)/(r_- - p_+)},$$

which is a contradiction.

Now, if $\|u\| < 1$, we get from (3.4),

$$(3.9) \qquad \|u\| \leq \left[2\lambda_0 \left(\frac{r_- - q_-}{r_- - p_+} \right) \|g_k\|_{\Theta(x)} K^{q_+} \right]^{1/(p_+ - q_-)}.$$

But, by (3.5),

$$(3.10) \qquad \|u\| \geq \left[\left(\frac{p_- - q_+}{r_+ - q_+} \right) K^{-r_+} \right]^{1/(r_- - p_+)}.$$

Combining (3.9) and (3.10),

$$(3.11) \qquad \lambda_0 \geq \frac{K^{-q_+}}{2\|g_k\|_{\Theta(x)}} \left(\frac{r_- - p_+}{r_+ - q_-} \right) \left[\left(\frac{p_- - q_+}{r_+ - p_+} \right) K^{-r_+} \right]^{(p_+ - q_-)/(r_- - p_+)}$$

so a new contradiction, finishing the proof. $\qquad\qquad\qquad\qquad\qquad$ $\square$

By Lemma 3.4, for $0 < \lambda < \Lambda_1$, we can write $\mathcal{M}_{\lambda,k} = \mathcal{M}^+_{\lambda,k} \cup \mathcal{M}^-_{\lambda,k}$. Therefore, hereafter we will consider the following numbers:

$$\alpha_{\lambda,k} = \inf_{u \in \mathcal{M}_{\lambda,k}} J_{\lambda,k}(u), \quad \alpha^+_{\lambda,k} = \inf_{u \in \mathcal{M}^+_{\lambda,k}} J_{\lambda,k}(u) \quad \text{and} \quad \alpha^-_{\lambda,k} = \inf_{u \in \mathcal{M}^-_{\lambda,k}} J_{\lambda,k}(u).$$

The next five lemmas establish important properties about the sets $\mathcal{M}^+_{\lambda,k}$ and $\mathcal{M}^-_{\lambda,k}$.

LEMMA 3.5. *Assume* (p$_2$), (g$_1$), (f$_1$) *and* (H). *If* $0 < \lambda < \Lambda_1$, *then* $J_{\lambda,k}(u) < 0$ *for all* $u \in \mathcal{M}^+_{\lambda,k}$. *Consequently,* $\alpha_{\lambda,k} \leq \alpha^+_{\lambda,k} < 0$.

PROOF. Let $u \in \mathcal{M}^+_{\lambda,k}$. Then, by the definition of $E'_{\lambda,k}(u)u$,

$$0 < E'_{\lambda,k}(u)u \leq (r_- - q_-)\lambda \int g_k(x)|u|^{q(x)} - (r_- - p_+) \int (|\nabla u|^{p(x)} + |u|^{p(x)}),$$

from where it follows

$$(3.12) \qquad \lambda \int g_k(x)|u|^{q(x)} > \left(\frac{r_- - p_+}{r_- - q_-}\right) \int (|\nabla u|^{p(x)} + |u|^{p(x)}).$$

By the definition of $J_{\lambda,k}(u)$,

$$J_{\lambda,k}(u) \leq \frac{1}{p_-} \int (|\nabla u|^{p(x)} + |u|^{p(x)}) - \frac{\lambda}{q_+} \int g_k(x)|u|^{q(x)} - \frac{1}{r_+} \int f_k(x)|u|^{r(x)}$$

$$= \left(\frac{1}{p_-} - \frac{1}{r_+}\right) \int (|\nabla u|^{p(x)} + |u|^{p(x)}) - \lambda \left(\frac{1}{q_+} - \frac{1}{r_+}\right) \int g_k(x)|u|^{q(x)}.$$

By (3.12) and (H),

$$J_{\lambda,k}(u) \leq \left[\frac{1}{p_-} - \frac{1}{r_+} - \left(\frac{1}{q_+} - \frac{1}{r_+}\right)\left(\frac{r_- - p_+}{r_- - q_-}\right)\right] \int (|\nabla u|^{p(x)} + |u|^{p(x)})$$

$$= \left[\frac{r_+ - p_-}{p_- r_+} - \left(\frac{r_+ - q_+}{q_+ r_+}\right) \cdot \left(\frac{r_- - p_+}{r_- - q_-}\right)\right] \int (|\nabla u|^{p(x)} + |u|^{p(x)})$$

$$= \frac{(r_+ - p_-)}{r_+}\left[\frac{1}{p_-} - \frac{1}{q_+} \cdot \left(\frac{r_+ - q_+}{r_+ - p_-}\right) \cdot \left(\frac{r_- - p_+}{r_- - q_-}\right)\right]$$

$$\cdot \int (|\nabla u|^{p(x)} + |u|^{p(x)}) < 0. \qquad \square$$

LEMMA 3.6. *We have the following inequalities*:

(a) $\int g_k(x)|u|^{q(x)} > 0$ *for each* $u \in \mathcal{M}_{\lambda,k}^+$;

(b) *for each* $u \in \mathcal{M}_{\lambda,k}^+$

$$\|u\| < \left[2\left(\frac{r_- - q_-}{r_- - p_+}\right)K^{q_+}\right]^{1/(p_- - q_+)}$$

$$\cdot \max\left\{(\lambda\|g_k\|_{\Theta(x)})^{1/(p_+ - q_-)}, (\lambda\|g_k\|_{\Theta(x)})^{1/(p_- - q_+)}\right\};$$

(c) $\|u\| > \left[\left(\frac{p_- - q_+}{r_+ - q_+}\right)K^{-r_+}\right]^{1/(r_+ - p_-)} \quad$ *for each* $u \in \mathcal{M}_{\lambda,k}^-$.

PROOF. (a) An immediate consequence of (3.12).

(b) Similarly to the proof of Lemma 3.4,

$$\min\{\|u\|^{p_-}, \|u\|^{p_+}\} < 2\lambda\left(\frac{r_- - q_-}{r_- - p_+}\right)\|g_k\|_{\Theta(x)}K^{q_+}\max\{\|u\|^{q_-}, \|u\|^{q_+}\}.$$

If $\|u\| < 1$, the above inequality gives

$$\|u\| \leq \left[2\lambda\left(\frac{r_- - q_-}{r_- - p_+}\right)\|g_k\|_{\Theta(x)}K^{q_+}\right]^{1/(p_+ - q_-)}.$$

Now, if $\|u\| \geq 1$, we will get

$$\|u\| \leq \left[2\lambda\left(\frac{r_- - q_-}{r_- - p_+}\right)\|g_k\|_{\Theta(x)}K^{q_+}\right]^{1/(p_- - q_+)},$$

showing (b).

(c) Let $u \in \mathcal{M}_{\lambda,k}^-$. Similarly to the proof of Lemma 3.4,

$$\left(\frac{p_- - q_+}{r_+ - q_+}\right) \min\{\|u\|^{p_-}, \|u\|^{p_+}\} < K^{r_+} \max\{\|u\|^{r_-}, \|u\|^{r_+}\}.$$

If $\|u\| < 1$,

$$(3.13) \qquad \|u\| > \left[\left(\frac{p_- - q_+}{r_+ - q_+}\right) K^{-r_+}\right]^{1/(r_- - p_+)},$$

and for $\|u\| \geq 1$,

$$(3.14) \qquad \|u\| > \left[\left(\frac{p_- - q_+}{r_+ - q_+}\right) K^{-r_+}\right]^{1/(r_+ - p_-)}.$$

Thus, the last two inequalities imply that (c) holds. $\qquad \square$

LEMMA 3.7. *Assume that $0 < \lambda < (q_-/p_+)\Lambda_1$ and (g_1). Then there exists a positive constant $d_1 = d_1(p_\pm, q_\pm, r_\pm, K, \|g_k\|_{\Theta(x)})$ such that $J_{\lambda,k}(u) > 0$ for each $u \in \mathcal{M}_{\lambda,k}^-$.*

PROOF. Let $u \in \mathcal{M}_{\lambda,k}^-$. Then, using the definitions of $J_{\lambda,k}$ and $\mathcal{M}_{\lambda,k}$, we can write

$$J_{\lambda,k}(u) \geq \left(\frac{1}{p_+} - \frac{1}{r_-}\right) \int (|\nabla u|^{p(x)} + |u|^{p(x)}) - \lambda\left(\frac{1}{q_-} - \frac{1}{r_-}\right) \int g_k(x)|u|^{q(x)}$$

$$\geq \left(\frac{1}{p_+} - \frac{1}{r_-}\right) \min\{\|u\|^{p_-}, \|u\|^{p_+}\}$$

$$- 2\lambda\left(\frac{1}{q_-} - \frac{1}{r_-}\right) \|g_k\|_{\Theta(x)} K^{q_+} \max\{\|u\|^{q_-}, \|u\|^{q_+}\}.$$

If $\|u\| < 1$, it follows that

$$J_{\lambda,k}(u) \geq \left(\frac{1}{p_+} - \frac{1}{r_-}\right) \|u\|^{p_+} - 2\lambda\left(\frac{1}{q_-} - \frac{1}{r_-}\right) \|g_k\|_{\Theta(x)} K^{q_+} \|u\|^{q_-}$$

$$= \|u\|^{q_-} \left[\left(\frac{1}{p_+} - \frac{1}{r_-}\right) \|u\|^{p_+ - q_-} - 2\lambda\left(\frac{1}{q_-} - \frac{1}{r_-}\right) \|g_k\|_{\Theta(x)} K^{q_+}\right].$$

Thereby, by Lemma 3.6 (c),

$$J_{\lambda,k}(u) > \left[\left(\frac{p_- - q_+}{r_+ - q_+}\right) K^{-r_+}\right]^{q_-/(r_+ - p_-)}$$

$$\cdot \left\{\left(\frac{1}{p_+} - \frac{1}{r_-}\right) \left[\left(\frac{p_- - q_+}{r_+ - q_+}\right) K^{-r_+}\right]^{(p_- - q_+)/(r_+ - p_-)}\right.$$

$$\left. - 2\lambda\left(\frac{1}{q_-} - \frac{1}{r_-}\right) \|g_k\|_{\Theta(x)} K^{q_+}\right\} = d_1.$$

Similarly, if $\|u\| \geq 1$,

$$
J_{\lambda,k}(u) > \left[\left(\frac{p_- - q_+}{r_+ - q_+}\right) K^{-r_+}\right]^{q_+/(r_+ - p_-)}
$$

$$
\cdot \left\{\left(\frac{1}{p_+} - \frac{1}{r_-}\right) \left[\left(\frac{p_- - q_+}{r_+ - q_+}\right) K^{-r_+}\right]^{(p_- - q_+)/(r_+ - p_-)} \right.
$$

$$
\left. - 2\lambda\left(\frac{1}{q_-} - \frac{1}{r_-}\right) \|g_k\|_{\Theta(x)} K^{q_+}\right\} = d_1.
$$

From the above estimates, the lemma follows if $0 < \lambda < (q_-/p_+)\Lambda_1$. $\qquad\square$

The lemma below is crucial in our arguments because it shows a condition for the existence of exactly two nontrivial zeros for a special class of functions.

LEMMA 3.8. *Let* $g_i\colon [0, +\infty) \to [0, +\infty)$, $i \in \{1, 2, 3\}$, *be increasing continuous functions, with* $g_i(0) = 0$ *verifying the following conditions*:

    (a) $\lim_{t\to 0^+} g_3(t)/g_1(t) = 0$;
    (b) $\lim_{t\to +\infty} g_2(t) = +\infty$;
    (c) $\lim_{t\to 0^+} (g_1(t) - g_3(t))/g_2(t) = 0$;
    (d) *the function* $\phi = g_1 - g_3$ *has only one maximum point and* $\phi(t) \to -\infty$
         as $t \to +\infty$.

Suppose there exists $\widetilde{t} \in (0, t_{\max})$ *with* $\phi(t_{\max}) = \max_{t \geq 0} \phi(t)$ *such that* $(g_1 - g_3)/g_2$ *is increasing on* $(0, \widetilde{t})$. *Then, there is* $\lambda_* > 0$ *such that* $\psi = g_1 - \lambda g_2 - g_3$ *has only two nontrivial zeros for all* $0 < \lambda < \lambda_*$.

PROOF. From (a), it is clear that $\phi(t) > 0$ for all $t > 0$ sufficiently small. Since $(g_1 - g_3)/g_2$ is positive and increasing in the interval $(0, \widetilde{t})$, for each $0 < \lambda < \phi(\widetilde{t})$ there is a unique $t_\lambda \in (0, \widetilde{t})$ such that $\lambda = (g_1(t_\lambda) - g_3(t_\lambda))/g_2(t_\lambda)$. Then, by hypothesis that $(g_1 - g_3)/g_2$ is increasing on $(0, \widetilde{t})$, we derive

$$
\lambda g_2(t) < g_1(t) - g_3(t) \quad \text{for all } t \in (t_\lambda, \widetilde{t}).
$$

Now, fix $\lambda^* > 0$ such that

$$
\lambda g_2(t) < g_1(t) - g_3(t) \quad \text{for all } t \in (t_\lambda, t_{\max}) \quad \text{and} \quad \lambda \in (0, \lambda^*).
$$

Since ϕ is decreasing in the interval $(t_{\max}, \infty)$, g_2 is increasing and $g_2(t) \to \infty$ as $t \to \infty$, there is a unique number $t_1 > t_{\max}$ such that $\lambda g_2(t_1) = \phi(t_1)$. Therefore, t_λ and t_1 are the unique nontrivial zeros of ψ for $\lambda \in (0, \lambda^*)$. $\qquad\square$

With the help of Lemma 3.8, we get the following result, which is similar to the constant case, see [10] and [17].

LEMMA 3.9. *For each* $u \in W^{1,p(x)}(\mathbb{R}^N) \setminus \{0\}$, *we have the following*:

(a) if $\int g_k(x)|u|^{q(x)} = 0$, then there exists a unique positive number $t^- = t^-(u)$ such that $t^- u \in \mathcal{M}_{\lambda,k}^-$ and $J_{\lambda,k}(t^- u) = \sup\limits_{t \geq 0} J_{\lambda,k}(tu)$;

(b) if $0 < \lambda < \Lambda_1$ and $\int g_k(x)|u|^{q(x)} > 0$, then there exist $t^* > 0$ and unique positive numbers $t^+ = t^+(u) < t^- = t^-(u)$ such that $t^+ u \in \mathcal{M}_{\lambda,k}^+$, $t^- u \in \mathcal{M}_{\lambda,k}^-$ and

$$J_{\lambda,k}(t^+ u) = \inf_{0 \leq t \leq t^*} J_{\lambda,k}(tu), \qquad J_{\lambda,k}(t^- u) = \sup_{t \geq t^*} J_{\lambda,k}(tu).$$

PROOF. By direct calculations, we see that

$$E'_{\lambda,k}(tu)tu = t\frac{d}{dt}(J_{\lambda,k}(tu)) + t^2 \frac{d^2}{dt^2}(J_{\lambda,k}(tu)).$$

Thus, if $t = \bar{t}$ is a critical point of $J_{\lambda,k}(tu)$,

$$(3.15) \qquad E'_{\lambda,k}(\bar{t}u)\bar{t}u = \bar{t}^2 \frac{d^2}{dt^2}(J_{\lambda,k}(tu))\Big|_{t=\bar{t}}.$$

Using (3.15) and the same ideas of the proof of [17, Lemma 3.6], we get item (a).

To prove item (b), we will apply Lemma 3.8 with the functions

$$g_1(t) = \int t^{p(x)-1}(|\nabla u|^{p(x)} + |u|^{p(x)}),$$

$$g_2(t) = \int t^{q(x)-1} g_k(x)|u|^{q(x)},$$

$$g_3(t) = \int t^{r(x)-1} f_k(x)|u|^{r(x)}.$$

The reader is invited to check that g_1, g_2 and g_3 satisfy the conditions of Lemma 3.8, and so, the function $\psi(t) = g_1(t) - \lambda g_2(t) - g_3(t) = J'_{\lambda,k}(tu)u$ has only two nontrivial zeros, $t^+ < t^-$. Let $\varphi(t) = J_{\lambda,k}(tu)$ on $[0, \infty)$. Then, it is clear that $\varphi(0) = 0$ and $\varphi(t)$ is negative if $t > 0$ is small, implying that φ has a local minimum in $t = t^+$. Consequently,

$$E'_{\lambda,k}(t^+ u)t^+ u > 0,$$

from where it follows that $t^+ u \in \mathcal{M}_{\lambda,k}^+$. Since t^+ and t^- are the unique critical points of φ, we deduce that φ has a global maximum in $t = t^-$, thus

$$E'_{\lambda,k}(t^+ u)t^+ u < 0$$

and $t^- u \in \mathcal{M}_{\lambda,k}^-$. Using Lemmas 3.5 and 3.7, it follows that $J_{\lambda,k}(t^+ u) < 0$ and $J_{\lambda,k}(t^- u) > 0$. Let $t_* > 0$ be the unique zero of φ in (t^+, t^-). Then it is clear that

$$J_{\lambda,k}(t^+ u) = \inf_{0 \leq t \leq t_*} J_{\lambda,k}(tu) \quad \text{and} \quad J_{\lambda,k}(t^- u) = \max_{t \geq t_*} J_{\lambda,k}(tu). \qquad \square$$

LEMMA 3.10. *Assume that g satisfies* (g$_1$) *and let* $\{u_n\}$ *be a* (PS)$_d$ *sequence in* $W^{1,p(x)}(\mathbb{R}^N)$ *for* $J_{\lambda,k}$. *Then* $\{u_n\}$ *is bounded in* $W^{1,p(x)}(\mathbb{R}^N)$.

PROOF. It is clear that

$$J_{\lambda,k}(u_n) - \frac{1}{r_-}J'_{\lambda,k}(u_n)u_n \geq \left(\frac{1}{p_+} - \frac{1}{r_-}\right)\int\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)$$
$$+ \lambda\left(\frac{1}{r_-} - \frac{1}{q_-}\right)\int g_k(x)|u_n|^{q(x)}.$$

Assume that $\|u_n\| \geq 1$ for some $n \in \mathbb{N}$. Then, by Hölder's inequality and Sobolev's embedding theorem, we derive the inequality

$$d + 1 + \|u_n\| \geq \left(\frac{1}{p_+} - \frac{1}{r_-}\right)\|u_n\|^{p_-} - \lambda\left(\frac{1}{q_-} - \frac{1}{r_-}\right)\|g_k\|_{\Theta(x)}K^{q_+}\|u_n\|^{q_+}.$$

Since $1 < q_+ < p_-$, the last inequality yields $\{u_n\}$ is bounded in $W^{1,p(x)}(\mathbb{R}^N)$.□

Now, combining standard arguments with the boundedness of $\{u_n\}$ and Sobolev's imbedding theorem (see [4]), we have the below result.

THEOREM 3.11. *Assume that g satisfies* (g_1). *If* $\{u_n\}$ *is a sequence in* $W^{1,p(x)}(\mathbb{R}^N)$ *such that* $u_n \rightharpoonup u$ *in* $W^{1,p(x)}(\mathbb{R}^N)$ *and* $J'_{\lambda,k}(u_n) \to 0$ *as* $n \to \infty$, *then for some subsequence, still denoted by* $\{u_n\}$, $\nabla u_n(x) \to \nabla u(x)$ *almost everywhere in* $\mathbb{R}^N$ *and* $J'_{\lambda,k}(u) = 0$.

The next theorem is a compactness result on Nehari manifolds. The case for constant exponent is due to Alves [2].

THEOREM 3.12. *Suppose that* (p_2) *holds and let* $\{u_n\} \subset \mathcal{M}_\infty$ *be a sequence with* $J_\infty(u_n) \to c_\infty$. *Then*:
  (a) $u_n \to u$ *in* $W^{1,p(x)}(\mathbb{R}^N)$, *or*
  (b) *there is* $\{y_n\} \subset \mathbb{Z}^N$ *with* $|y_n| \to +\infty$ *and* $w \in W^{1,p(x)}(\mathbb{R}^N)$ *such that* $w_n(x) = u_n(x + y_n) \to w$ *in* $W^{1,p(x)}(\mathbb{R}^N)$ *and* $J_\infty(w) = c_\infty$.

PROOF. Similarly to Corollary 3.10, there is $u \in W^{1,p(x)}(\mathbb{R}^N)$ and a subsequence of $\{u_n\}$, still denoted as $\{u_n\}$, such that $u_n \rightharpoonup u$ in $W^{1,p(x)}(\mathbb{R}^N)$. Applying Ekeland's variational principle, we can assume that

(3.16) $$J'_\infty(u_n) - \tau_n E'_\infty(u_n) = o_n(1),$$

where $(\tau_n) \subset \mathbb{R}$ and $E_\infty(w) = J'_\infty(w)w$, for any $w \in W^{1,p(x)}(\mathbb{R}^N)$.

Since $\{u_n\} \subset \mathcal{M}_\infty$, (3.16) leads to $\tau_n E'_\infty(u_n)u_n = o_n(1)$. Next, we will show that there exists $\eta > 0$ such that

(3.17) $$|E'_\infty(u_n)u_n| > \eta \quad \text{for all } n \in \mathbb{N}.$$

Indeed, first we claim that there exists $\eta_0 > 0$ satisfying $\|u\| > \eta_0$ for any $u \in \mathcal{M}_\infty$. Suppose, by contradiction, that the claim is false. Then, there is $\{v_n\} \subset \mathcal{M}_\infty$ such that $\|v_n\| \to 0$ as $n \to \infty$. Since $\{v_n\} \subset \mathcal{M}_\infty$, we derive

$$\int(|\nabla v_n|^{p(x)} + |v_n|^{p(x)}) = \int|v_n|^{r(x)}.$$

On the other hand, using the fact that $\|v_n\| < 1$ for n large enough, it follows from Propositions 2.1 and 2.7,

$$\|v_n\|^{p+} \leq C \max\{\|v_n\|^{r_-}, \|v_n\|^{r_+}\} = C\|v_n\|^{r_-},$$

leading to $(1/C)^{1/(r_- - p_+)} \leq \|v_n\|$, which is absurd. Therefore, by Proposition 2.7, there is $\varsigma > 0$ such that $\rho_1(u) \geq \varsigma$, $u \in \mathcal{M}_\infty$. By the definition of $E_\infty(u)$,

$$E'_\infty(u_n)u_n \leq p_+ \int (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) - r_- \int |u_n|^{r(x)}$$

$$= (p_+ - r_-) \int (|\nabla u_n|^{p(x)} + |u_n|^{p(x)})$$

$$= (p_+ - r_-)\rho_1(u_n) < (p_+ - r_-)\varsigma,$$

proving (3.17). Now, combining (3.16) and (3.17), we see that $\tau_n \to 0$, and so,

$$J_\infty(u_n) \to c_\infty \quad \text{and} \quad J'_\infty(u_n) \to 0.$$

Next, we will study the following possibilities: $u \neq 0$ or $u = 0$.

CASE 1. $u \neq 0$. Similarly to Theorem 3.11, it follows that the below limits are valid for some subsequence:

- $u_n(x) \to u(x)$ and $\nabla u_n(x) \to \nabla u(x)$ almost everywhere in $\mathbb{R}^N$,
- $\int |\nabla u_n(x)|^{p(x)-2} \nabla u_n(x) \nabla v \to \int |\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla v$,
- $\int |u_n|^{p(x)-2} u_n v \to \int |u|^{p(x)-2} u_n v$,
- $\int |u_n|^{r(x)-2} u_n v \to \int |u|^{r(x)-2} uv$,

for any $v \in W^{1,p(x)}(\mathbb{R}^N)$. Consequently, u is a critical point of J_∞. By Fatou's lemma, it is easy to check that

$$c_\infty \leq J_\infty(u) = J_\infty(u) - \frac{1}{r_-} J'_\infty(u)u$$

$$= \int \left(\frac{1}{p(x)} - \frac{1}{r_-}\right)(|\nabla u|^{p(x)} + |u|^{p(x)}) + \int \left(\frac{1}{r_-} - \frac{1}{r(x)}\right)|u|^{r(x)}$$

$$\leq \liminf_{n \to \infty} \left\{\int \left(\frac{1}{p(x)} - \frac{1}{r_-}\right)(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) + \int \left(\frac{1}{r_-} - \frac{1}{r(x)}\right)|u_n|^{r(x)}\right\}$$

$$= \liminf_{n \to \infty} \left\{J_\infty(u_n) - \frac{1}{r_-} J'_\infty(u_n)u_n\right\} = c_\infty.$$

Hence,

$$\lim_{n \to \infty} \int (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) = \int (|\nabla u|^{p(x)} + |u|^{p(x)}),$$

implying that $u_n \to u$ in $W^{1,p(x)}(\mathbb{R}^N)$.

CASE 2. $u = 0$. In this case, we claim that there are $R, \xi > 0$ and $\{y_n\} \subset \mathbb{R}^N$ satisfying

(3.18) $$\limsup_{n \to \infty} \int_{B_R(y_n)} |u_n|^{p(x)} \geq \xi.$$

If the claim is false, we must have

$$\limsup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^{p(x)} = 0.$$

Thus, by a Lions type result for variable exponent proved in [21, Lemma 3.1], $u_n \to 0$ in $L^{s(x)}(\mathbb{R}^N)$, for any $s \in C(\mathbb{R}^N)$ with $p \ll s \ll p^*$.

Recalling $J'_\infty(u_n)u_n = o_n(1)$, the last limit yields

$$\int (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) = o_n(1),$$

or, equivalently, $u_n \to 0$ in $W^{1,p(x)}(\mathbb{R}^N)$, leading to $c_\infty = 0$, which is absurd. This way, (3.18) is true. By a routine argument, we can assume that $y_n \in \mathbb{Z}^N$ and $|y_n| \to \infty$ as $n \to \infty$. Setting $w_n(x) = u_n(x + y_n)$, and using the fact that p and r are $\mathbb{Z}^N$-periodic, a change of variable gives

$$J_\infty(w_n) = J_\infty(u_n) \quad \text{and} \quad \|J'_\infty(w_n)\| = \|J'_\infty(u_n)\|,$$

showing that $\{w_n\}$ is a sequence $(PS)_{c_\infty}$ for J_∞. If $w \in W^{1,p(x)}(\mathbb{R}^N)$ denotes the weak limit of $\{w_n\}$, from (3.18),

$$\int_{B_R(0)} |w|^{p(x)} \geq \xi,$$

showing that $w \neq 0$.

Repeating the same argument of the first case for the sequence $\{w_n\}$, we deduce that $w_n \to w$ in $W^{1,p(x)}(\mathbb{R}^N)$, $w \in \mathcal{M}_\infty$ and $J_\infty(w) = c_\infty$. $\qquad\square$

Our next result will be very useful in the study of the compactness of some functionals.

LEMMA 3.13. *Let $u \in W^{1,(x)}(\mathbb{R}^N)$ be a nontrivial critical point of $J_{\lambda,k}$. Then, there exists a constant $M = M(k) > 0$, independent of λ, such that*

$$J_{\lambda,k}(u) \geq -M\left(\lambda^{p_+/(p_+ - q_-)} + \lambda^{p_-/(p_- - q_+)}\right).$$

PROOF. By hypothesis, $J'_{\lambda,k}(u)u = 0$. Arguing as in the proof of Lemma 3.7, if $\|u\| \geq 1$, then

$$J_{\lambda,k}(u) \geq \left(\frac{1}{p_+} - \frac{1}{r_-}\right)\|u\|^{p_-} - \left(\frac{1}{q_-} - \frac{1}{r_-}\right)2\lambda\|g_k\|_{\Theta(x)}K^{q_+}\|u\|^{q_+}.$$

Applying Young's inequality with $p_1 = p_-/q_+$ and $p_2 = p_-/(p_- - q_+)$, we obtain

$$J_{\lambda,k}(u) \geq \left(\frac{1}{p_+} - \frac{1}{r_-}\right)\|u\|^{p_-} - \varepsilon\left(\frac{1}{q_-} - \frac{1}{r_-}\right)\|u\|^{p_-}$$
$$- \left(\frac{1}{q_-} - \frac{1}{r_-}\right)C_1(\varepsilon)(2\lambda\|g_k\|_{\Theta(x)}K^{q_+})^{p_-/(p_- - q_+)}$$

where

$$C_1(\varepsilon) = \frac{p_- - q_+}{p_-}\left(\frac{q_+}{\varepsilon p_-}\right)^{q_+/(p_- - q_+)}.$$

Choosing

$$\varepsilon = \left(\frac{1}{q_-} - \frac{1}{r_-}\right)^{-1}\left(\frac{1}{p_+} - \frac{1}{r_-}\right),$$

we get

$$J_{\lambda,k}(u) \geq -\left(\frac{1}{q_-} - \frac{1}{r_-}\right)C_1(\varepsilon)(2\lambda\|g_k\|_{\Theta(x)}K^{q_+})^{p_-/(p_- - q_+)}.$$

Analogously, if $\|u\| < 1$, we will get

$$J_{\lambda,k}(u) \geq -\left(\frac{1}{q_-} - \frac{1}{r_-}\right)C_2(\varepsilon)(2\lambda\|g_k\|_{\Theta(x)}K^{q_+})^{p_+/(p_+ - q_-)},$$

where

$$C_2(\varepsilon) = \frac{p_+ - q_-}{p_+}\left(\frac{q_-}{\varepsilon p_+}\right)^{q_-/(p_+ - q_-)}.$$

Therefore, $J_{\lambda,k}(u) \geq -M\left(\lambda^{p_+/(p_+ - q_-)} + \lambda^{p_-/(p_- - q_+)}\right)$ with

$$M = \left(\frac{1}{q_-} - \frac{1}{r_-}\right)(2K^{q_+})^{p_-/(p_- - p_+)}$$
$$\cdot \max\{C_1(\varepsilon)\|g_k\|_{\Theta(x)}^{p_-/(p_- - q_+)}, C_2(\varepsilon)\|g_k\|_{\Theta(x)}^{p_+/(p_+ - q_-)}\}. \qquad \square$$

The next result is an important step to prove the existence of solutions, because it establishes the behavior of the (PS) sequences of functional $J_{\lambda,k}$.

LEMMA 3.14. *Let $\{v_n\}$ be a* $(PS)_d$ *sequence for the functional $J_{\lambda,k}$ with $v_n \rightharpoonup v$ in $W^{1,p(x)}(\mathbb{R}^N)$. Then*

(3.19) $$J_{\lambda,k}(v_n) - J_{0,k}(w_n) - J_{\lambda,k}(v) = o_n(1),$$

(3.20) $$\|J'_{\lambda,k}(v_n) - J'_{0,k}(w_n) - J'_{\lambda,k}(v)\| = o_n(1),$$

where $w_n = v_n - v$.

PROOF. Similarly to Theorem 3.11, the below limits occur

$$\nabla v_n(x) \to \nabla v(x) \quad \text{and} \quad v_n(x) \to v(x) \quad \text{a.e. in } \mathbb{R}^N.$$

Then, by Proposition 2.4,

$$J_{\lambda,k}(v_n) = J_{0,k}(w_n) + J_{\lambda,k}(v) + o_n(1),$$

showing (3.19). Equality (3.20) follows from Propositions 2.5 and 2.6. $\qquad \square$

The proof of the next result follows the same steps found in [31], and so, it will be omitted.

Lemma 3.15.

(a) *There exists a* $(\mathrm{PS})_{\alpha_{\lambda,k}}$ *sequence in* $\mathcal{M}_{\lambda,k}$ *for* $J_{\lambda,k}$;
(b) *there exists a* $(\mathrm{PS})_{\alpha_{\lambda,k}^+}$ *sequence in* $\mathcal{M}_{\lambda,k}^+$ *for* $J_{\lambda,k}$;
(c) *there exists a* $(\mathrm{PS})_{\alpha_{\lambda,k}^-}$ *sequence in* $\mathcal{M}_{\lambda,k}^-$ *for* $J_{\lambda,k}$.

4. Existence of a ground state solution

The first lemma in this section establishes the interval where the functional $J_{\lambda,k}$ satisfies the Palais–Smale condition and its statement is the following.

Lemma 4.1. *Under assumptions* (g_1) *and* (f_1), *if* $0 < \lambda < \Lambda_1$, *then the functional* $J_{\lambda,k}$ *satisfies the* $(\mathrm{PS})_d$ *condition for*

$$d < c_{f_\infty} - M\big(\lambda^{p_+/(p_+ - q_-)} + \lambda^{p_-/(p_- - q_+)}\big).$$

Proof. Let $\{v_n\} \subset W^{1,p(x)}(\mathbb{R}^N)$ be a $(\mathrm{PS})_d$ sequence for the functional $J_{\lambda,k}$ with $d < c_{f_\infty} - M\big(\lambda^{p_+/(p_+ - q_-)} + \lambda^{p_-/(p_- - q_+)}\big)$. By Lemma 3.10, $\{v_n\}$ is a bounded sequence in $W^{1,p(x)}(\mathbb{R}^N)$, and so, for some subsequence, still denoted by $\{v_n\}$, $v_n \rightharpoonup v$ in $W^{1,p(x)}(\mathbb{R}^N)$, for some $v \in W^{1,p(x)}(\mathbb{R}^N)$. Since $J'_{\lambda,k}(v) = 0$ and $J_{\lambda,k}(v) \geq 0$, from (3.19)–(3.20), $w_n = v_n - v$ is a $(\mathrm{PS})_{d^*}$ sequence for $J_{0,k}$ with $d^* = d - J_{\lambda,k}(v) < c_{f_\infty}$.

Claim 4.2. *There is* $R > 0$ *such that*

$$\limsup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |w_n|^{p(x)} = 0.$$

Assuming by a moment the claim, we have

$$\int |w_n|^{r(x)} \to 0.$$

On the other hand, by (3.20), we know that $J'_{0,k}(w_n) = o_n(1)$, then

$$\int (|\nabla w_n|^{p(x)} + |w_n|^{p(x)}) = o_n(1),$$

showing that $w_n \to 0$ in $W^{1,p(x)}(\mathbb{R}^N)$.

Proof of Claim 4.2. If the claim is not true, for each $R > 0$ given, we find $\eta > 0$ and $\{y_n\} \subset \mathbb{Z}^N$ verifying

$$\limsup_{n \to \infty} \int_{B_R(y_n)} |w_n|^{p(x)} \geq \eta > 0.$$

Once $w_n \rightharpoonup 0$ in $W^{1,p(x)}(\mathbb{R}^N)$, it follows that $\{y_n\}$ is an unbounded sequence. Setting $\widetilde{w}_n = w_n(\cdot + y_n)$, we have that $\{\widetilde{w}_n\}$ is also a $(\mathrm{PS})_{d^*}$ sequence for $J_{0,k}$, and so, it must be bounded. Then, there are $\widetilde{w} \in W^{1,p(x)}(\mathbb{R}^N)$ and a subsequence

of $\{\widetilde{w}_n\}$, still denoted as $\{\widetilde{w}_n\}$, such that $\widetilde{w}_n \rightharpoonup \widetilde{w} \in W^{1,p(x)}(\mathbb{R}^N)\backslash\{0\}$. Moreover, since $J'_{0,k}(w_n)\phi(\,\cdot\,-y_n) = o_n(1)$ for each $\phi \in W^{1,p(x)}(\mathbb{R}^N)$, it is possible to prove that $\nabla\widetilde{w}_n(x) \to \nabla\widetilde{w}(x)$ almost everywhere in $\mathbb{R}^N$. Therefore,

$$\int(|\nabla\widetilde{w}|^{p(x)-2}\nabla\widetilde{w}\nabla\phi + |\widetilde{w}|^{p(x)-2}\widetilde{w}\phi) = \int f_\infty|\widetilde{w}|^{r(x)-2}\widetilde{w}\phi,$$

from where it follows that $\widetilde{w}$ is a weak solution of problem (P_{f_∞}). Consequently, after some routine calculations,

$$c_{f_\infty} \le J_{f_\infty}(\widetilde{w}) - \frac{1}{r_-}J'_{f_\infty}(\widetilde{w})\widetilde{w} \le \liminf_{n\to\infty}\left\{J_{0,k}(w_n) - \frac{1}{r_-}J'_{0,k}(w_n)w_n\right\} = d^*$$

which is a contradiction. Then, Claim 4.2 is true. $\qquad\square$

The next theorem shows both the existence of a ground state and that it lies in $\mathcal{M}^+_{\lambda,k}$.

THEOREM 4.3. *Assume that* (g_1) *and* (f_1) *hold. Then, there exists* $0 < \Lambda_* < \Lambda_1$, *such that for* $\lambda \in (0, \Lambda_*)$ *problem* $(\mathrm{P}_{\lambda,k})$ *has at least one ground state solution* u_0. *Moreover, we have that* $u_0 \in \mathcal{M}^+_{\lambda,k}$ *and*

$$J_{\lambda,k}(u_0) = \alpha_{\lambda,k} = \alpha^+_{\lambda,k} \ge -M\big(\lambda^{p_+/(p_+-q_-)} + \lambda^{p_-/(p_--q_+)}\big).$$

PROOF. By Lemma 3.15 (a), there is a minimizing sequence $\{u_n\} \subset \mathcal{M}_{\lambda,k}$ for $J_{\lambda,k}$ such that $J_{\lambda,k}(u_n) = \alpha_{\lambda,k} + o_n(1)$ and $J'_{\lambda,k}(u_n) = o_n(1)$. Since $c_{f_\infty} > 0$, there is $0 < \Lambda_* < \Lambda_1$ such that

$$\alpha_{\lambda,k} < 0 < c_{f_\infty} - M\big(\lambda^{p_+/(p_+-q_-)} + \lambda^{p_-/(p_--q_+)}\big) \quad \text{for any } 0 < \lambda < \Lambda_*.$$

By Lemma 4.1, there is a subsequence of $\{u_n\}$, still denoted as $\{u_n\}$, and $u_0 \in W^{1,p(x)}(\mathbb{R}^N)$ such that $u_n \to u_0$ in $W^{1,p(x)}(\mathbb{R}^N)$. Thereby, u_0 is a solution of $(\mathrm{P}_{\lambda,k})$ and $J_{\lambda,k}(u_0) = \alpha_{\lambda,k}$. We assert that $u_0 \in \mathcal{M}^+_{\lambda,k}$. Otherwise, since $\mathcal{M}^0_{\lambda,k} = \emptyset$ for $0 < \lambda < \Lambda_*$, we have $u_0 \in \mathcal{M}^-_{\lambda,k}$. Hence

$$(4.1) \qquad\qquad \int \lambda g_k(x)|u_0|^{q(x)} > 0.$$

Indeed, if $0 = \int \lambda g_k(x)|u_0|^{q(x)}$, then

$$0 = \int \lambda g_k(x)|u_n|^{q(x)} + o_n(1) = \int(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) - \int f_k(x)|u_n|^{r(x)} + o_n(1).$$

Therefore,

$$\begin{aligned}
\alpha_{\lambda,k} + o_n(1) &= J_{\lambda,k}(u_n)\\
&= \int\frac{1}{p(x)}\big(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\big) - \lambda\int\frac{g_k(x)}{q(x)}|u_n|^{q(x)} - \int\frac{f_k(x)}{r(x)}|u_n|^{r(x)}\\
&\ge \frac{1}{p_+}\int\big(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\big) - \frac{\lambda}{q_-}\int g_k(x)|u_n|^{r(x)} - \frac{1}{r_-}\int f_k(x)|u_n|^{r(x)}\\
&= \left(\frac{1}{p_+} - \frac{1}{r_-}\right)\int\big(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\big) + o_n(1)
\end{aligned}$$

leading to

$$\alpha_{\lambda,k} \geq \left(\frac{1}{p_+} - \frac{1}{r_-} \right) \limsup_{n \in \mathbb{N}} \int \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right)$$

which is absurd, because $\alpha_{\lambda,k} < 0$, showing that (4.1) holds.

By Lemma 3.9 (b), there are numbers $t^+ < t^- = 1$ such that $t^+ u_0 \in \mathcal{M}_{\lambda,k}^+$, $t^- u_0 \in \mathcal{M}_{\lambda,k}^-$ and $J_{\lambda,k}(t^+ u_0) < J_{\lambda,k}(t^- u_0) = J_{\lambda,k}(u_0) = \alpha_{\lambda,k}$, which is a contradiction. Thereby, $u_0 \in \mathcal{M}_{\lambda,k}^+$ and

$$-M\left(\lambda^{p_+/(p_+ - q_-)} + \lambda^{p_-/(p_- - q_+)} \right) \leq J_{\lambda,k}(u_0) = \alpha_{\lambda,k} = \alpha_{\lambda,k}^+. \qquad \square$$

5. Existence of ℓ solutions

In this section, we will show that $(\mathrm{P}_{\lambda,k})$ has at least ℓ nontrivial solutions belonging to $\mathcal{M}_{\lambda,k}^-$.

5.1. Estimates involving the minimax levels. The main goal of this subsection is to prove some estimates involving the minimax levels $c_{\lambda,k}, c_{0,k}, c_\infty$ and c_{f_∞}. First of all, we recall the inequalities

$$J_{\lambda,k}(u) \leq J_{0,k}(u) \quad \text{and} \quad J_\infty(u) \leq J_{0,k}(u) \quad \text{for all } u \in W^{1,p(x)}(\mathbb{R}^N),$$

which imply $c_{\lambda,k} \leq c_{0,k}$ and $c_\infty \leq c_{0,k}$.

LEMMA 5.1. *The minimax levels $c_{0,k}$ and c_{f_∞} satisfy the inequality $c_{0,k} < c_{f_\infty}$. Hence, $c_\infty < c_{f_\infty}$.*

PROOF. In a manner analogous to Theorem 3.12, there is $U \in W^{1,p(x)}(\mathbb{R}^N)$ verifying $J_{f_\infty}(U) = c_{f_\infty}$ and $J_{f_\infty}'(U) = 0$. Similar to Lemma 3.9, there exists $t > 0$ such that $tU \in \mathcal{M}_{0,k}$. Thus,

$$c_{0,k} \leq J_\infty(tU) = \int \frac{t^{p(x)}}{p(x)} \left(|\nabla U|^{p(x)} + |U|^{p(x)} \right) - \int \frac{t^{r(x)}}{r(x)} |U|^{r(x)}.$$

By (f_1), $f_\infty < f(x)$ for all $x \in \mathbb{R}^N$, and so, $f_\infty < 1$. Then,

$$c_{0,k} < J_{f_\infty}(tU) \leq \max_{s \geq 0} J_{f_\infty}(sU) = J_{f_\infty}(U) = c_{f_\infty}. \qquad \square$$

In what follows, let us fix $\rho_0, r_0 > 0$ satisfying:

- $\overline{B_{\rho_0}(a_i)} \cap \overline{B_{\rho_0}(a_j)} = \emptyset$ for $i \neq j$ and $i, j \in \{1, \dots, \ell\}$,
- $\bigcup_{i=1}^{\ell} B_{\rho_0}(a_i) \subset B_{r_0}(0)$,
- $K_{\rho_0/2} = \bigcup_{i=1}^{\ell} \overline{B_{\rho_0/2}(a_i)}$.

Furthermore, we define the function $Q_k \colon W^{1,p(x)}(\mathbb{R}^N) \to \mathbb{R}^N$ by

$$Q_k(u) = \frac{\displaystyle\int \chi(k^{-1}x)|u|^{p_+}}{\displaystyle\int |u|^{p_+}},$$

where $\chi\colon \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$\chi(x) = x \quad \text{if } |x| \le r_0 \quad \text{and} \quad \chi(x) = r_0\frac{x}{|x|} \quad \text{if } |x| > r_0.$$

The next two lemmas give important information on the function Q_k and the level c_∞.

LEMMA 5.2. *There are $\delta_0 > 0$ and $k_1 \in \mathbb{N}$ such that if $u \in \mathcal{M}_{0,k}$ and $J_{0,k}(u) \le c_\infty + \delta_0$, then $Q_k(u) \in K_{\rho_0/2}$ for $k \ge k_1$.*

PROOF. If the lemma does not occur, there must be $\delta_n \to 0$, $k_n \to +\infty$ and $u_n \in \mathcal{M}_{0,k_n}$ satisfying $J_{0,k_n}(u_n) \le c_\infty + \delta_n$ and $Q_{k_n}(u_n) \notin K_{\rho_0/2}$. Fixing $s_n > 0$ such that $s_n u_n \in \mathcal{M}_\infty$, we have that

$$c_\infty \le J_\infty(s_n u_n) \le J_{0,k_n}(s_n u_n) \le \max_{t \ge 0} J_{0,k_n}(t u_n) = J_{0,k_n}(u_n) \le c_\infty + \delta_n.$$

Hence, $\{s_n u_n\} \subset \mathcal{M}_\infty$ and $J_\infty(s_n u_n) \to c_\infty$.

Applying Ekeland's variational principle, we can assume without loss of generality that $\{s_n u_n\} \subset \mathcal{M}_\infty$ is a $(PS)_{c_\infty}$ sequence for J_∞, that is,

$$J_\infty(s_n u_n) \to c_\infty \quad \text{and} \quad J'_\infty(s_n u_n) \to 0.$$

From Theorem 3.12, we must consider the ensuing cases:

1. $s_n u_n \to U \ne 0$ in $W^{1,p(x)}(\mathbb{R}^N)$;
2. there exists $\{y_n\} \subset \mathbb{Z}^N$ with $|y_n| \to +\infty$ such that $v_n(x) = s_n u(x + y_n)$ is convergent in $W^{1,p(x)}(\mathbb{R}^N)$ for some $V \in W^{1,p(x)}(\mathbb{R}^N) \setminus \{0\}$.

By a direct computation, we can suppose that $s_n \to s_0$ for some $s_0 > 0$. Therefore, without loss of generality, we can assume that

$$u_n \to U \quad \text{or} \quad v_n = u(\,\cdot\, + y_n) \to V \quad \text{in } W^{1,p(x)}(\mathbb{R}^N).$$

Case 1. By Lebesgue's dominated convergence theorem

$$Q_{k_n}(u_n) = \frac{\displaystyle\int \chi(k_n^{-1}x)|u_n|^{p+}}{\displaystyle\int |u_n|^{p+}} \to \frac{\displaystyle\int \chi(0)|U|^{p+}}{\displaystyle\int |U|^{p+}} = 0,$$

implying $Q_{k_n}(u_n) \in K_{\rho_0/2}$ for n large, because $0 \in K_{\rho_0/2}$. However, this a contradiction, because we are supposing $Q_{k_n}(u_n) \notin K_{\rho_0/2}$ for all n.

Case 2. Using again Ekeland's variational principle, we can suppose that $J'_{0,k_n}(u_n) = o_n(1)$. Hence, $J'_{0,k_n}(u_n)\phi(\,\cdot\, - y_n) = o_n(1)$ for any $\phi \in W^{1,p(x)}(\mathbb{R}^N)$, and so,

$$(5.1) \quad o_n(1) = \int \left(|\nabla v_n|^{p(x)-2}\nabla v_n \nabla \phi + |v_n|^{p(x)-2}v_n\phi\right)$$

$$- \int f(k_n^{-1}(x + y_n))|v_n|^{r(x)-2}v_n\phi.$$

The last limit implies that, for some subsequence, $\nabla v_n(x) \to \nabla V(x)$ and $v_n(x) \to V(x)$ almost everywhere in $\mathbb{R}^N$.

Now, we will study two cases:

(i) $|k_n^{-1}y_n| \to +\infty$,

(ii) $k_n^{-1}y_n \to y$, for some $y \in \mathbb{R}^N$.

If (i) holds, we see that

$$\int \left(|\nabla V|^{p(x)-2}\nabla V \nabla \phi + |V|^{p(x)-2}V\phi \right) = \int f_\infty |V|^{r(x)-2}V\phi,$$

showing that V is a nontrivial weak solution of problem (P_{f_∞}). Now, combining the condition $f_\infty < 1$ with Fatou's lemma, we get

$$c_{f_\infty} \leq J_{f_\infty}(V) = J_{f_\infty}(V) - \frac{1}{r_-}J'_{f_\infty}(V)V$$

$$\leq \liminf_{n\to\infty}\left\{ J_\infty(u_n) - \frac{1}{r_-}J'_\infty(u_n)u_n \right\} = c_\infty,$$

or equivalently, $c_{f_\infty} \leq c_\infty$, contradicting Lemma 5.1.

Now, if $k_n^{-1}y_n \to y$ for some $y \in \mathbb{R}^N$, then V is a weak solution of the following problem:

$$(\mathrm{P}_{f(y)}) \qquad \begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = f(y)|u|^{r(x)-2}u & \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N). \end{cases}$$

Repeating the previous arguments, we deduce that

(5.2) $$c_{f(y)} \leq c_\infty,$$

where $c_{f(y)}$ is the mountain pass level of the functional $J_{f(y)}\colon W^{1,p(x)}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$J_{f(y)}(u) = \int \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)} \right) - \int \frac{f(y)}{r(x)}|u|^{r(x)}.$$

If $f(y) < 1$, a similar argument explored in the proof of Lemma 5.1 shows that $c_{f(y)} > c_\infty$, contradicting the inequality (5.2). Thereby, $f(y) = 1$ and $y = a_i$ for some $i = 1, \dots \ell$. Hence,

$$Q_{k_n}(u_n) = \frac{\displaystyle\int \chi(k_n^{-1}x)|u_n|^{p_+}}{\displaystyle\int |u_n|^{p_+}}$$

$$= \frac{\displaystyle\int \chi(k_n^{-1}x + k_n^{-1}y_n)|v_n|^{p_+}}{\displaystyle\int |v_n|^{p_+}} \to \frac{\displaystyle\int \chi(y)|V|^{p_+}}{\displaystyle\int |V|^{p_+}} = a_i,$$

implying that $Q_{k_n}(u_n) \in K_{\rho_0/2}$ for n large, which is a contradiction, since by assumption $Q_{k_n}(u_n) \notin K_{\rho_0/2}$. $\square$

LEMMA 5.3. *Let* $\delta_0 > 0$ *given in Lemma* 5.2 *and* $k_3 = \max\{k_1, k_2\}$. *Then, there is* $\Lambda^* = \Lambda^*(k) > 0$ *such that*

$$Q_k(u) \in K_{\rho_0/2}, \quad \text{for all } (u, \lambda, k) \in \mathcal{A}_{\lambda,k} \times [0, \Lambda^*) \times ([k_3, +\infty) \cap \mathbb{N}),$$

where $\mathcal{A}_{\lambda,k} := \{u \in \mathcal{M}_{\lambda,k}^- : J_{\lambda,k}(u) < c_\infty + \delta_0/2\}$.

PROOF. Observe that

$$J_{\lambda,k}(u) = J_{0,k}(u) - \lambda \int \frac{g_k(x)}{q(x)} |u|^{q(x)} \quad \text{for all } u \in W^{1,p(x)}(\mathbb{R}^N).$$

In what follows, let $t_u > 0$ be such that $t_u u \in \mathcal{M}_{0,k}$. Then,

$$(5.3) \qquad J_{0,k}(t_u u) = J_{\lambda,k}(t_u u) + \lambda \int \frac{g_k(x)}{q(x)} (t_u)^{q(x)} |u|^{q(x)}$$

$$\leq \max_{t \geq 0} J_{\lambda,k}(tu) + \lambda \int \frac{g_k(x)}{q(x)} (t_u)^{q(x)} |u|^{q(x)}.$$

CLAIM 5.4. (a) *Given* $\Lambda > 0$, *there is a constant* $R > 0$ *such that* $\mathcal{A}_{\lambda,k} \subset B_R(0)$, *for all* $k \geq k_1$ *and* $\lambda \in [0, \Lambda]$, *that is,* $\mathcal{A}_{\lambda,k}$ *is a bounded set, where* k_1 *was given in Lemma* 5.2. *Moreover,* R *is independent of* k.

(b) *Let* $u \in \mathcal{A}_{\lambda,k}$ *and* $t_u > 0$ *be such that* $t_u u \in \mathcal{M}_{0,k}$. *Then, given* $\Lambda > 0$, *there are* $C > 0$ *and* $k_2 \in \mathbb{N}$ *such that*

$$0 \leq t_u \leq C, \quad \text{for all } (u, \lambda, k) \in \mathcal{A}_{\lambda,k} \times [0, \Lambda] \times ([k_2, +\infty) \cap \mathbb{N}).$$

PROOF OF (a). Let $u \in \mathcal{M}_{\lambda,k}^- \subset \mathcal{M}_{\lambda,k}$ such that $J_{\lambda,k}(u) < c_\infty + \delta_0/2$ for $k \geq k_1$. Then,

$$\int \left(|\nabla u|^{p(x)} + |u|^{p(x)} \right) - \lambda \int g_k(x) |u|^{q(x)} - \int f_k(x) |u|^{r(x)} = 0,$$

$$\int \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + |u|^{p(x)} \right) - \lambda \int \frac{g_k(x)}{q(x)} |u|^{q(x)} - \int \frac{f_k(x)}{r(x)} |u|^{r(x)} < c_\infty + \frac{\delta_0}{2}.$$

Combining the last two expressions, we obtain

$$\left(\frac{1}{p_+} - \frac{1}{r_-} \right) \int \left(|\nabla u|^{p(x)} + |u|^{p(x)} \right) + \left(\frac{1}{q_-} - \frac{1}{r_-} \right) \lambda \int g_k(x) |u|^{q(x)} < c_\infty + \frac{\delta_0}{2}.$$

By previous calculations, we have

$$\left(\frac{1}{p_+} - \frac{1}{r_-} \right) \min\{\|u\|^{p_-}, \|u\|^{p_+}\}$$

$$- \Lambda \left(\frac{1}{q_-} - \frac{1}{r_-} \right) 2 \|g_k\|_{\Theta(x)} K^{q_+} \max\{\|u\|^{q_-}, \|u\|^{q_+}\} < c_\infty + \frac{\delta_0}{2}.$$

Since $q_+ < p_-$, it follows that there is $R > 0$ such that

$$\|u\| \leq R \quad \text{for all } (u, \lambda, k) \in \mathcal{A}_{\lambda,k} \times [0, \Lambda] \times ([k_1, +\infty) \cap \mathbb{N})$$

proving (a).

Proof of (b). Supposing by contradiction that the lemma does not hold. Then, there is $\{u_n\} \subset \mathcal{A}_{\lambda_n,k_n}$ with $\lambda_n \to 0$ and $k_n \to +\infty$ such that $t_{u_n} u_n \in \mathcal{M}_{0,k_n}$ and $t_{u_n} \to \infty$ as $n \to \infty$. Without loss of generality, we can assume that $t_{u_n} \geq 1$. Since $t_{u_n} u_n \in \mathcal{M}_{0,k_n}$ and $f_\infty < f(x)$ for all $x \in \mathbb{R}^N$, we derive

$$(t_{u_n})^{p_+} \int \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right) \geq f_\infty (t_{u_n})^{r_-} \int |u_n|^{r(x)},$$

or equivalently,

$$(5.4) \qquad \int \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right) \geq f_\infty t_{u_n}^{r_- - p_+} \int |u_n|^{r(x)}$$

for n large enough.

Now, we claim that there is $\eta_1 > 0$ such that

$$(5.5) \qquad \int |u_n|^{r(x)} > \eta_1 \quad \text{for all } n \in \mathbb{N}.$$

Indeed, arguing by contradiction, there is a subsequence, still denoted by $\{u_n\}$ such that

$$\int |u_n|^{r(x)} = o_n(1) \quad \text{as } n \to \infty.$$

As $u_n \in \mathcal{M}_{\lambda_n,k_n}^- \subset \mathcal{M}_{\lambda_n,k_n}$, we get

$$(p_- - q_+) \int \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right) - (r_+ - q_+) \int f_k(x)|u_n|^{r(x)} < 0.$$

By item (a), there are positive constants c_1 and c_2 such that $c_1 < \rho_1(u_n) < c_2$. Thus,

$$\frac{p_- - q_+}{r_+ - q_+} < \frac{\displaystyle\int f_k(x)|u_n|^{r(x)}}{\displaystyle\int \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right)} < \frac{\displaystyle\int |u_n|^{r(x)}}{c_1} = o_n(1)$$

which is a contradiction, proving the claim. Thereby, from inequality (5.4),

$$\rho_1(u_n) = \int \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right) \to +\infty,$$

implying that $\{u_n\}$ is a unbounded sequence. However, this is impossible, because by item (a), $\{u_n\}$ is bounded, showing that (b) holds.

By Claim 5.4 (b) and Hölder's inequality, it follows from (5.3) that

$$J_{0,k}(t_u u) \leq J_{\lambda,k}(u) + \frac{\lambda}{q_-} C^{q_+} \|g_k\|_{\Theta(x)} \||u|^{q(x)}\|_{r(x)/q(x)}.$$

Once that $u \in \mathcal{A}_{\lambda,k}$, we get

$$J_{0,k}(t_u u) < c_\infty + \frac{\delta_0}{2} + \lambda c_2 \|g_k\|_{\Theta(x)} \||u|^{q(x)}\|_{r(x)/q(x)}.$$

Using Sobolev's embedding theorem combined with Claim 5.4 (a), we obtain

$$J_{0,k}(t_u u) < c_\infty + \frac{\delta_0}{2} + \lambda c_3 \|g_k\|_{\Theta(x)} \quad \text{for all } u \in \mathcal{A}_{\lambda,k},$$

where c_3 is a positive constant. Setting $\Lambda^* := \delta_0/(2c_3\|g_k\|_{\Theta(x)})$ and $\lambda \in [0, \Lambda^*)$, we get $t_u u \in \mathcal{M}_{0,k}$ and $J_{0,k}(t_u u) < c_\infty + \delta_0$. Then, by Lemma 5.2, $Q_k(t_u u) \in K_{\frac{\rho_0}{2}}$. Now, it remains to note that $Q_k(u) = Q_k(t_u u)$, to conclude the proof of lemma. $\qquad\square$

From now on, we will use the ensuing notations

- $\theta^i_{\lambda,k} = \left\{ u \in \mathcal{M}^-_{\lambda,k} : |Q_k(u) - a_i| < \rho_0 \right\}$,
- $\partial\theta^i_{\lambda,k} = \left\{ u \in \mathcal{M}^-_{\lambda,k} : |Q_k(u) - a_i| = \rho_0 \right\}$,
- $\beta^i_{\lambda,k} = \displaystyle\inf_{u \in \theta^i_{\lambda,k}} J_{\lambda,k}(u)$,
- $\widetilde{\beta}^i_{\lambda,k} = \displaystyle\inf_{u \in \partial\theta^i_{\lambda,k}} J_{\lambda,k}(u)$.

The above numbers are very important in our approach, because we will prove that there is a (PS) sequence of $J_{\lambda,k}$ associated with each $\theta^i_{\lambda,k}$ for $i = 1, \ldots, \ell$. To this end, we need of the following technical result:

LEMMA 5.5. *There are $0 < \Lambda_\sharp < \Lambda^*$ and $k \geq k_\sharp$ such that*

$$\beta_{\lambda,k} < c_{f_\infty} - M\left(\lambda^{p_+/(p_+-q_-)} + \lambda^{p_-/p_--q_+}\right) \quad and \quad \beta^i_{\lambda,k} < \widetilde{\beta}^i_{\lambda,k}$$

for all $\lambda \in [0, \Lambda_\sharp)$ and $k \geq k_\sharp$.

PROOF. From now on, $U \in W^{1,p(x)}(\mathbb{R}^N)$ is a ground state solution associated with (P_∞), that is, $J_\infty(U) = c_\infty$ and $J'_\infty(U) = 0$ (see Theorem 3.12).

For $1 \leq i \leq \ell$ and $k \in \mathbb{N}$, we define the function $\widehat{U}^i_k \colon \mathbb{R}^N \to \mathbb{R}$ by

$$\widehat{U}^i_k(x) = U(x - ka_i).$$

CLAIM 5.6. *For all $i \in \{1, \ldots, \ell\}$, we have that*

$$\limsup_{k \to +\infty} \left(\sup_{t \geq 0} J_{\lambda,k}\left(t\widehat{U}^i_k\right) \right) \leq c_\infty.$$

Indeed, since p, q and r are $\mathbb{Z}^N$-periodic, and $a_i \in \mathbb{Z}^N$, making a change of variable gives

$$J_{\lambda,k}\left(t\widehat{U}^i_k\right) = \int \frac{t^{p(x)}}{p(x)}\left(|\nabla U|^{p(x)} + |U|^{p(x)}\right)$$
$$- \lambda \int g(k^{-1}x + a_i)\frac{t^{q(x)}}{q(x)}|U|^{q(x)} - \int f(k^{-1}x + a_i)\frac{t^{r(x)}}{r(x)}|U|^{r(x)}.$$

Moreover, we know that there exists $s = s(k) > 0$ such that

$$\max_{t \geq 0} J_{\lambda,k}\left(t\widehat{U}^i_k\right) = J_{\lambda,k}\left(s\widehat{U}^i_k\right) \geq \beta,$$

where β was given in Lemma 3.1. By a direct computation, it is possible to prove that $s(k) \not\to 0$ and $s(k) \not\to \infty$ as $k \to \infty$. Thus, without loss of generality,

we can assume $s(k) \to s_0 > 0$ as $k \to \infty$. Thereby,

$$\lim_{k\to\infty}\Big(\max_{t\geq 0} J_{\lambda,k}(t\widehat{U}_k^i)\Big) \leq \int \frac{s_0^{p(x)}}{p(x)}(|\nabla U|^{p(x)} + |U|^{p(x)}) - \int f(a_i)\frac{s_0^{r(x)}}{r(x)}|U|^{r(x)}$$

$$\leq J_\infty(s_0 U) \leq \max_{s\geq 0} J_\infty(sU) = J_\infty(U) = c_\infty.$$

Consequently, $\limsup_{k\to+\infty}\Big(\sup_{t\geq 0} J_{\lambda,k}(t\widehat{U})\Big) \leq c_\infty$ for $i \in \{1,\dots,\ell\}$, showing the claim.

By Lemma 5.1, there is $0 < \Lambda_\sharp < \Lambda^*$ such that

$$c_\infty < c_{f_\infty} - M\big(\lambda^{p_+/(p_+-q_-)} + \lambda^{p_-/(p_--q_+)}\big) \quad \text{for any } \lambda \in [0,\Lambda_\sharp).$$

Choosing $0 < \overline{\delta} < \delta_0$, we have that

$$c_\infty + \overline{\delta} < c_{f_\infty} - M\big(\lambda^{p_+/((p_+-q_-)} + \lambda^{p_-/(p_--q_+)}\big) \quad \text{for any } \lambda \in [0,\Lambda_\sharp).$$

Since $Q_k(U_k^i) \to a_i$ as $k \to \infty$, then $U_k^i \in \theta_{\lambda,k}^i$ for all k large enough. On the other hand, by Claim 5.6, $J_{\lambda,k}(U_k^i) < c_\infty + \overline{\delta}/2$ holds also for k large enough and $\lambda \in [0,\Lambda_\sharp)$. This way, there exists $k_4 \in \mathbb{N}$ such that

$$\beta_{\lambda,k}^i < c_\infty + \frac{\overline{\delta}}{2} < c_{f_\infty} - M\big(\lambda^{p_+/(p_+-q_-)} + \lambda^{p_-/(p_--q_+)}\big),$$

for all $\lambda \in [0,\Lambda_\sharp)$ and $k \geq k_4$.

In order to prove the other inequality, we observe that Lemma 5.3 yields $J_{\lambda,k}(U_k^i) \geq c_\infty + \delta_0/2$ for all $u \in \partial\theta_{\lambda,k}^i$, if $\lambda \in [0,\Lambda_\sharp)$ and $k \geq k_3$. Therefore,

$$\widetilde{\beta}_{\lambda,k}^i \geq c_\infty + \frac{\delta_0}{2}, \quad \text{for } \lambda \in [0,\Lambda_*) \text{ and } k \geq k_3.$$

Fixing $k_\sharp = \max\{k_3,k_4\}$, we derive that $\beta_{\lambda,k}^i < \widetilde{\beta}_{\lambda,k}^i$, for $\lambda \in [0,\Lambda_\sharp)$ and $k \geq k_\sharp$. $\square$

LEMMA 5.7. *For each $1 \leq i \leq \ell$, there exists a* $(\text{PS})_{\beta_{\lambda,k}^i}$ *sequence, $\{u_n^i\} \subset \theta_{\lambda,k}^i$ for the functional $J_{\lambda,k}$.*

PROOF. By Lemma 5.5, we know that $\beta_{\lambda,k}^i < \widetilde{\beta}_{\lambda,k}^i$. Then, the result follows adapting the same ideas explored in [31]. $\square$

6. Proof of Theorem 1.1

Let $\{u_n^i\} \subset \theta_{\lambda,k}^i$ be a $(\text{PS})_{\beta_{\lambda,k}^i}$ sequence in $\mathcal{M}_{\lambda,k}^-$ for the functional $J_{\lambda,k}$ given by Lemma 5.7. Since $\beta_{\lambda,k}^i < c_{f_\infty} - M\big(\lambda^{p_+/(p_+-q_-)} + \lambda^{p_-/(p_--q_+)}\big)$, by Lemma 4.1, there is u^i such that $u_n^i \to u^i$ in $W^{1,p(x)}(\mathbb{R}^N)$. Thus,

$$u^i \in \theta_{\lambda,k}^i, \quad J_{\lambda,k}(u^i) = \beta_\lambda^i \quad \text{and} \quad J_{\lambda,k}'(u^i) = 0.$$

Now, we infer that $u^i \neq u^j$ for $i \neq j$ as $1 \leq i,j \leq \ell$. To see why, it remains to observe that

$$Q_k(u^i) \in \overline{B_{\rho_0}(a_i)} \quad \text{and} \quad Q_k(u^j) \in \overline{B_{\rho_0}(a_j)}.$$

Since $\overline{B_{\rho_0}(a_i)} \cap \overline{B_{\rho_0}(a_j)} = \emptyset$ for $i \neq j$, it follows that $u^i \neq u^j$ for $i \neq j$. From this, $J_{\lambda,k}$ has at least ℓ critical points in $\mathcal{M}_{\lambda,k}^-$ for $\lambda \in [0, \Lambda_\sharp)$ and $k \geq k_\sharp$. By Theorem 4.3, it follows that problem $(\mathrm{P}_{\lambda,k})$ admits at least $\ell + 1$ solutions for $\lambda \in [0, \Lambda_\sharp)$ and $k \geq k_\sharp$. $\qquad\qquad\square$

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