Topological **M**ethods in **N**onlinear **A**nalysis
Volume 47, No. 2, 2016, 715–737
DOI: 10.12775/TMNA.2016.031

INVARIANCE OF BIFURCATION EQUATIONS FOR HIGH DEGENERACY BIFURCATIONS OF NON-AUTONOMOUS PERIODIC MAPS

Henrique M. Oliveira

Abstract. Bifurcations of the class A_μ, in Arnold's classification, in non-autonomous p-periodic difference equations generated by parameter depending families with p maps, are studied. It is proved that the conditions of degeneracy, non-degeneracy and unfolding are invariant relatively to cyclic order of compositions for any natural number μ. The main tool for the proofs is the local topological conjugacy. Invariance results are essential for proper definition of bifurcations of the class A_μ and associated lower codimension bifurcations, using all possible cyclic compositions of fiber families of maps of the p-periodic difference equation. Finally, we present two examples of the class A_3 or swallowtail bifurcation occurring in period two difference equations for which bifurcation conditions are invariant.

1. Introduction

1.1. Motivation. Our paper is motivated by recent papers [11], [28] which develop bifurcation theory for non-autonomous dynamical systems. As is well-known in this setting there are some difficulties to overcome, both in the case of continuous and discrete time. As a starting point it is necessary to set a proper definition of a dynamical system [6], [12], [22] and of an attractor and repeller [5]. It is also necessary to define clearly the concept of bifurcation. There is a good set of research works on this subject, see e.g. [2], [16], [19]–[22], [26]–[31].

2010 *Mathematics Subject Classification.* Primary: 37G15; Secondary: 39A28.
Key words and phrases. Topological conjugacy; A_μ degenerate bifurcation; non-autonomous map; p-periodic map; alternating system.

In this paper we are concerned with the definition of bifurcation equations for local bifurcations in one-dimensional p-periodic maps or p-periodic difference equations. In particular, we focus our attention on the class A_μ of bifurcations, in Arnold's classification [3], [4], for a positive integer μ. The main result of the paper is the invariance of A_μ bifurcation conditions in respect to the cyclic order of maps in the iteration. Actually, we establish all results for alternating maps, i.e. for $p = 2$ or two fiber maps, and for fixed points of composition maps. This approach has an advantage of being simple in presentation, notation and comfortable to the reader in comparison with the direct study of p compositions and general k-periodic orbits. Next we generalize the results to periodic orbits of p-periodic maps, that is only an exercise of composition and repeated application of methods developed for alternating maps.

Bifurcations of the class A_μ occur in the autonomous case when one has one real dynamic variable x, the parameter space is real μ-dimensional and the related family of mappings satisfies a set of degeneracy conditions. These conditions provide topological equivalence to the unfolding of the germ $x \pm x^{\mu+1}$ at the origin [4]. There are many different approaches in the literature, in this work we follow the definitions of [4] concerning the germ, topological equivalence, unfolding, codimension, and classification of singularities and bifurcations. We suggest as an introduction to the general subject of bifurcations the book [23]. The class A_μ includes the fold, for $\mu = 1$, the cusp, for $\mu = 2$, the swallowtail, for $\mu = 3$, and the butterfly, for $\mu = 4$ (see [4], [13], [35], [36]).

At the end of this paper we consider equations of the swallowtail bifurcation, i.e. the A_3 class, as an example for our results. In this case the bifurcation set ([1]) in the parameter space is made up of three surfaces of fold bifurcations, which meet in two lines of cusp bifurcations and one line of simultaneous double fold, which in turn meet at a single swallowtail bifurcation point as we can see in Figure 1. This bifurcation has codimension three [23], since one needs three independent parameters to completely unfold the bifurcation.

On the subject of codimension see also [4], [8], [14], [17]; we note that the definition of codimension in [14] is different from the one provided by [4] and [23] but the results are basically the same, modulus personal gusto.

The p maps of a family can exhibit a plethora of geometrical behavior not present when we study lower codimension bifurcation. For instance, for $\mu = 3$ and alternating maps the Schwarzian derivative cannot be simultaneously negative at the singularity for two maps, as we will see in the last section. The negative Schwarzian condition restricts severely the geometry of families of mappings [10], [33]. Without the negative Schwarzian, we have in the unfolding of this singularity a variety of dynamic phenomena not usually seen in most of the

([1]) For the definition of bifurcation set see Definition 2.2.

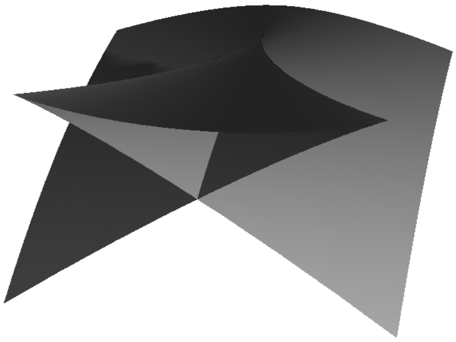

FIGURE 1. The bifurcation set in the parameter space near the origin, the most degenerate point where the A_3 singularity occurs. The control space is real three dimensional. The cut facing the observer is at $a = 1$.

works on one-dimensional discrete dynamics [7], [10]. Obviously, in this scenario one does not benefit from Singer's Theorem [34].

1.2. Overview. We organized this paper in four sections. In Section 2 we introduce basic concepts including a brief recollection of A_μ bifurcation equations for families of autonomous real maps.

In Section 3, the core of this work, we study in detail the equations of bifurcation for alternating systems. We prove that when we perform a change in the order of composition of maps the degeneracy conditions, non-degeneracy conditions and transversality conditions remain invariant. These results imply that this type of bifurcation is well-defined in the general case of alternating systems. Finally we provide a straightforward generalization of these results to periodic orbits of p-periodic maps.

In Section 4 we prove some conditions that override the possibility of A_3 or swallowtail bifurcation in the case of alternating maps, where $p = 2$. Finally, we present two examples concerning alternating maps. These examples show that this class of high degeneracy bifurcations occurs in simple applications without the need of exotic constructions.

2. Preliminaries

2.1. Basic definitions and notation. Let a non-autonomous dynamical system be defined as in [22]. Consider the non-autonomous iteration given by

$$x_{n+1} = f_n(x_n),\ x_n \in I_n, \quad \text{with } n \in \mathbb{N}, \tag{2.1}$$

where I_n are real intervals (not necessarily compact and maybe $\mathbb{R}$) which are fibers of the non-autonomous dynamical system. The usual distance is defined in I_n. The iteration starts at the initial condition $x_0 \in I_0$. Each map f_n is at least continuous and defined so that

$$f_n \colon I_n \to I_{n+1}, \qquad x_n \mapsto f_n(x_n)$$

and $f_n(I_n) \subseteq I_{n+1}$.

The system is periodic of period p if

$$f_{n+p} = f_p \quad \text{and} \quad f_{n+p}(I_n) \subseteq I_n,$$

for every $n \in \mathbb{N}$, where p is the minimal positive integer satisfying the above conditions. When $p = 2$ with fibers I_0 and I_1, such that

$$f_0(I_0) \subseteq I_1 \quad \text{and} \quad f_1(I_1) \subseteq I_0,$$

we say that we have an *alternating system*.

In the sequel, by $\mathcal{C}(I)$ we denote the collection of all continuous maps with domain I, by $\mathcal{C}^1(I)$ the collection of all continuously differentiable elements of $\mathcal{C}(I)$ and, in general, by $\mathcal{C}^s(I)$, $s \geq 1$, the collection of all elements of $\mathcal{C}(I)$ having continuous derivatives up to order s in I.

Let $f \in \mathcal{C}^1(I)$ and let q be a periodic point of period m. Denoting the derivative by D, q is called a *hyperbolic attractor* if $|Df^m(q)| < 1$, a *hyperbolic repeller* if $|Df^m(q)| > 1$, and *non-hyperbolic* if $|Df^m(q)| = 1$.

Definition 2.1. We say that two continuous maps $f \colon I \to I$ and $g \colon J \to J$ are topologically conjugate if there exists a homeomorphism $h \colon I \to J$ such that $h \circ f = g \circ h$. We call h the *topological conjugacy* of f and g.

We use Λ for a vector parameter in $\mathbb{R}^\mu$.

Definition 2.2. If f_Λ is a family of maps, then the *regular values* Λ of parameters are those which have the property that $f_{\widetilde{\Lambda}}$ is topologically conjugate to f_Λ for all $\widetilde{\Lambda}$ in some open neighbourhood of Λ. If Λ is not a regular value, it is called a *bifurcation value*. The collection of all bifurcation values is called the *bifurcation set*, $\Omega \subset \mathbb{R}^\mu$, in the parameter space.

Let f_Λ be a family of maps in $\mathcal{C}^s(I)$. Let Λ_0 be a particular vector parameter and $a \in I$ be a fixed point of f_{Λ_0}, i.e.

$$a = f_{\Lambda_0}(a),$$

the condition of a being non-hyperbolic is necessary for the existence of a local bifurcation. The existence and nature of that bifurcation depends on other symmetry and differentiability conditions that we will discuss below. If there exists a local bifurcation we say that (a, Λ_0) is a *bifurcation point* (when there is no risk of confusion, we say that a is a *bifurcation point*).

Notation 2.3. For notational simplicity we consider the real vector parameter Λ as a standard variable along with the dynamic variable x, i.e., we write

$$f_\Lambda(x) = f(x, \Lambda),$$

keeping in mind that the compositions are always in the dynamic variable x.

When there is no danger of confusion and no operations regarding the parameter, we denote the evaluation of functions depending on the dynamic variable and the parameter omitting the later, for instance $f_\Lambda(x) = f(x, \Lambda)$ will be denoted by $f(x)$ in order to avoid the complicated notation needed for computations of high order chain rules. Nevertheless, all maps in this paper depend on the parameter as well on the dynamic variable, even when that dependence is not visible in some formulas or expressions.

We denote the derivatives related to some variable y by ∂_y. Repeated differentiation relatively to the same variable is denoted by ∂_{y^n}, for instance $\partial_{yyy} = \partial_{y^3}$. When there is no danger of confusion, we denote strict partial derivatives, i.e., not seeing composed functions, by a subscript. For instance, the third partial derivative of f relatively to y is denoted by f_{yyy} or f_{y^3}.

This means, in particular, that when dealing with composition of real scalar functions $g(x,t)$ and $f(x,t)$, such that $g \circ f(x,t) = g(f(x,t),t)$, we have the chain rules

$$\begin{aligned}
\partial_t g(f(x,t),t) &= g_x(f(x,t),t) f_t(x,t) + g_t(f(x,t),t),\\
\partial_x g(f(x,t),t) &= g_x(f(x,t),t) f_x(x,t).
\end{aligned}$$

Throughout this paper we deal with p-periodic sequences of maps $f_0, \dots, f_{p-1}$ on a real dynamic variable x and depending on a real vector parameter Λ, such that

$$f_j \colon I_j \times \Theta \to I_{j+1}, \qquad (x, \Lambda) \mapsto f_0(x, \Lambda)$$

for $j = 0, \dots, p-1$. The fibers I_j for the dynamic variable are intervals of $\mathbb{R}$ and $\Theta \subset \mathbb{R}^\mu$ is the parameter set, $f_j \in \mathcal{C}^{\mu+1}(I_j)$ and $f_j \in \mathcal{C}^1(\Theta)$, with μ a positive integer. Moreover, the property

$$f_j(I_j, \Lambda) \subseteq I_{j+1(\mathrm{mod}\, p)} \quad \text{holds for all } \Lambda \in \Theta.$$

In this paper we use the convention that capital letters are used for compositions of maps in the dynamic variable. Capital F and G will be always used for the direct and reverse composition of alternating maps

$$F = f_1 \circ f_0 \quad \text{and} \quad G = f_0 \circ f_1.$$

Consider the set of indexes $j = 0, \ldots, p-1$ for the p-periodic system. We set the following notation for the p compositions:

$$\begin{aligned} F_0 &= f_{p-1} \circ \ldots \circ f_0, \\ F_1 &= f_0 \circ f_{p-1} \circ \ldots \circ f_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_{p-1} &= f_{p-2} \circ \ldots \circ f_0 \circ f_{p-1}. \end{aligned}$$

The repeated composition (always in the dynamic variable) is denoted by

$$f^k = \underbrace{(f \circ \ldots \circ f)}_{k},$$

where k is a positive integer.

2.2. Conditions for the class A_μ of bifurcations in autonomous systems. In this paragraph, we recall briefly the conditions for the class A_μ of local bifurcations, in Arnold classification, as explained in Theorem on page 20 in Arnold et al. [4]. For the iteration of maps, the normalized germ of the class A_μ is $x \pm x^{\mu+1}$ and has the following principal family [4], also called the prototype polynomial or normal form [23]:

$$x \pm x^{\mu+1} + \lambda_1 + \ldots + \lambda_\mu x^{\mu-1},$$

where λ_j, $j = 1, \ldots, \mu$, are real parameters.

Given an autonomous discrete dynamical system generated by the iteration of f, in order to compute the bifurcation points of the class A_μ one has to solve the following bifurcation equations [23]:

$$\begin{aligned} f(x, \Lambda) &= x \quad \text{(fixed point equation)}, \\ f_x(x, \Lambda) &= 1 \quad \text{(non-hyperbolicity condition)}. \end{aligned} \tag{2.2}$$

The simplest of such local bifurcations is the saddle node bifurcation, i.e. A_1. One assumes, in this case, the generic non-degeneracy condition

$$f_{xx}(x, \lambda) \neq 0 \tag{2.3}$$

and the transversality condition [23]

$$f_\lambda(x, \lambda) \neq 0, \quad \text{with } \lambda \in \mathbb{R}. \tag{2.4}$$

We set generically that $\lambda \in \mathbb{R}$, since one needs only one parameter to locally unfold this singularity [1], [4], [8], [14], [15], [23]. The normalized germ of this bifurcation is $x \pm x^2$, with principal family $x \pm x^2 + \lambda$, which is weak topologically conjugate to any other family [4], [23] satisfying the bifurcation conditions.

Adding degeneracy conditions, one obtains higher degeneracy local bifurcations.

Therefore, the equations for the occurrence of the A_μ class of bifurcations for a general positive integer μ are

$$\begin{aligned} f(x,\Lambda) &= x, \\ f_x(x,\Lambda) &= 1, \\ f_{xx}(x,\Lambda) &= 0, \\ &\cdots\cdots\cdots\cdots \\ f_{x^\mu}(x,\Lambda) &= 0, \end{aligned} \tag{2.5}$$

with the solution (a,Λ_0). It is easy to see that these conditions are satisfied by the normalized germ $x \pm x^{\mu+1}$ at the origin. One has the non-degeneracy condition

$$f_{x^{\mu+1}}(a,\Lambda_0) \neq 0, \tag{2.6}$$

for $\mu = 2$ we have the cusp, for $\mu = 3$ the swallowtail and for $\mu = 4$ the butterfly [4], [8], [9], [14], [15], [23]. The transversality condition (see pages 66, 297, 298 and 303 of [23]) at the solution of the above conditions is given by the condition of non-singularity of the Jacobian matrix of the map $(f, f_x, f_{xx}, \ldots, f_{x^{\mu-1}})$ relatively to parameters at the bifurcation point

$$\det \begin{bmatrix} f_{\lambda_1}(a,\Lambda_0) & \cdots & f_{\lambda_\mu}(a,\Lambda_0) \\ \vdots & \ddots & \vdots \\ f_{x^{\mu-1}\lambda_1}(a,\Lambda_0) & \cdots & f_{x^{\mu-1}\lambda_\mu}(a,\Lambda_0) \end{bmatrix} \neq 0, \tag{2.7}$$

and it assures that the vector parameter is enough to unfold the local bifurcation [23]. This happens since condition (2.7) assures that the μ lower order terms in the Taylor polynomial of f depend uniquely on the μ components of Λ, i.e. $\lambda_1, \ldots, \lambda_\mu$.

3. A_μ class of bifurcation in families of p-periodic maps

3.1. Invariance of bifurcation conditions.

3.1.1. On invariance of degeneracy and non-degeneracy conditions for alternating systems. In this paragraph, we study invariance of degeneracy conditions of alternating families of maps for all singularities of the class A_μ, using topological conjugacy.

Given an initial condition $x_0 \in I_0$, the alternating system is given by the iteration

$$x_{n+1} = f_{n(\mathrm{mod}\,2)}(x_n,\Lambda), \quad x_n \in I_{n(\mathrm{mod}\,2)}. \tag{3.1}$$

If there is a pair (a, I_0) such that after two iterations the iteration returns to (a, I_0) we say that a is a *periodic point* in the fiber I_0 with period 2. We note that the point $b = f_0(a, \Lambda)$ is also a periodic point in the fiber I_1 with period 2. Consider the compositions F and G, we have

$$a = F(a, \Lambda) \quad \text{and} \quad b = G(b, \Lambda).$$

In other words, a (resp. b) is a periodic point with period 2 in the fiber I_0 (resp. I_1) of alternating system (3.1) if and only if a (resp. b) is a fixed point of F (resp. G). Below we give bifurcation equations with $\mu - 1$ degeneracy conditions on derivatives on x for F and G, they are exactly the same as in the non-autonomous case:

$$\begin{cases} F(x,\Lambda) = x, \\ F_x(x,\Lambda) = 1, \\ F_{xx}(x,\Lambda) = 0, \\ \dots\dots\dots\dots \\ F_{x^\mu}(x,\Lambda) = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} G(x,\Lambda) = x, \\ G_x(x,\Lambda) = 1, \\ G_{xx}(x,\Lambda) = 0, \\ \dots\dots\dots\dots \\ G_{x^\mu}(x,\Lambda) = 0. \end{cases} \tag{3.2}$$

These equations have different solutions for the dynamic variable x, depending on the fiber we choose. At the solutions of (3.2), the non-degeneracy conditions are

$$F_{x^{\mu+1}}(a, \Lambda_0) \neq 0 \quad \text{and} \quad G_{x^{\mu+1}}(b, \Lambda_0) \neq 0. \tag{3.3}$$

A natural question arises: *Are the solutions in the parameter space equal for different compositions F and G?*

A similar question was posed in [9], [11] and it was positively solved in particular cases of degeneracy conditions until the cusp, i.e. $\mu = 2$.

In this section we show that the answer to the above question is positive in general case. We prove that if a parameter vector satisfies equations (3.2) for F then it is a solution of the system for G. The next lemma will be used to solve the general problem of the symmetry of bifurcation equations with respect to the order of composition.

LEMMA 3.1. *Let $\mu \geq 1$ and let h and f be real functions satisfying the following conditions*:

(a) *there exists a such that $f(a) = a$ and f is a Lipschitz homeomorphism in some open interval I containing a*;
(b) *h is a Lipschitz homeomorphism with Lipschitz constant L in an open neighbourhood I_h of a, and there exists an open neighbourhood J_b of $h(a) = b$ such that its inverse h^{-1} is also Lipschitz continuous with Lipschitz constant M*;
(c)
$$\lim_{x \to a} \frac{|f(x) - x|}{|x - a|^\mu} = 0 \quad \textit{and} \quad \lim_{x \to a} \frac{|f(x) - x|}{|x - a|^{\mu+1}} > 0.$$

Then g, the conjugate of f by the homeomorphism h,

$$g = h \circ f \circ h^{-1}$$

satisfies

$$\lim_{y\to b} \frac{|g(y)-y|}{|y-b|^{\mu}} = 0 \quad \textit{and} \quad \lim_{y\to b} \frac{|g(y)-y|}{|y-b|^{\mu+1}} > 0. \tag{3.4}$$

PROOF. We first compute the domain J, where y ranges, when we compute limit (3.4). Of course, we take J to be an open interval containing b such that $J \subseteq h(I_h)$. The limit has meaning if J also satisfies

$$h^{-1}(J) \subseteq I, \qquad f(h^{-1}(J)) \subseteq I_h. \tag{3.5}$$

As both f and h^{-1} are homeomorphisms we can choose the open interval J small enough just to satisfy conditions (3.5). We note that $a \in h^{-1}(J)$ and $a \in f(h^{-1}(J))$.

Let us consider limit (3.4). We have

$$\begin{aligned}
0 \le \lim_{y\to b} \frac{|g(y)-y|}{|y-b|^{\mu}} &= \lim_{y\to b} \frac{|(h\circ f\circ h^{-1})(y) - (h\circ h^{-1})(y)|}{|y-b|^{\mu}} \\
&\le L \lim_{y\to b} \frac{|(f\circ h^{-1})(y) - h^{-1}(y)|}{|y-b|^{\mu}} \\
&= L \lim_{y\to b} \frac{|(f\circ h^{-1})(y) - h^{-1}(y)|}{|h^{-1}(y)-a|^{\mu}} \left(\frac{|h^{-1}(y)-a|}{|y-b|}\right)^{\mu} \\
&= L \lim_{y\to b} \frac{|(f\circ h^{-1})(y) - h^{-1}(y)|}{|h^{-1}(y)-a|^{\mu}} \left(\frac{|h^{-1}(y)-h^{-1}(b)|}{|y-b|}\right)^{\mu} \\
&\le LM^{\mu} \lim_{y\to b} \frac{|(f\circ h^{-1})(y) - h^{-1}(y)|}{|h^{-1}(y)-a|^{\mu}}.
\end{aligned}$$

As $\lim_{y\to b} h^{-1}(y) = a$, if we set $h^{-1}(y) = x$, it follows that

$$0 \le \lim_{y\to b} \frac{|g(y)-y|}{|y-b|^{\mu}} \le LM^{\mu} \lim_{x\to a} \frac{|f(x)-x|}{|x-a|^{\mu}} = 0.$$

As $\lim_{x\to a} |f(x)-x|/|x-a|^{\mu+1} > 0$, we apply a similar reasoning to f to get

$$0 < \lim_{x\to a} \frac{|f(x)-x|}{|x-a|^{\mu+1}} \le ML^{\mu+1} \lim_{y\to b} \frac{|g(y)-y|}{|y-b|^{\mu+1}},$$

therefore

$$\lim_{y\to b} \frac{|g(y)-y|}{|y-b|^{\mu+1}} > 0,$$

as desired. □

Using the previous lemma we can easily prove the next result.

THEOREM 3.2. *Let $\mu \ge 2$ and let there be an alternating family of maps with individual mappings $f_0 \in \mathcal{C}^{\mu+1}(I_0)$, $f_1 \in \mathcal{C}^{\mu+1}(I_1)$ in the dynamic variable and the compositions $F = f_1 \circ f_0$ and $G = f_0 \circ f_1$ such that:*

(a) *There exist a, b, fixed points of F and G respectively,*

$$a = F(a, \Lambda), \qquad b = G(b, \Lambda).$$

(b) *The non-hyperbolicity condition for F*

$$\partial_x F(x, \Lambda)|_{x=a} = \partial_x f_1(x, \Lambda)|_{x=b}\, \partial_x f_0(x, \Lambda)|_{x=a} = 1$$

holds.

(c) *Higher degeneracy conditions for F*

$$\partial_{x^i} F(x, \Lambda)|_{x=a} = 0, \quad \textit{for every } 2 \leq i \leq \mu$$

hold.

(d) *The non-degeneracy condition for F*

$$\partial_{x^{\mu+1}} F(x, \Lambda)|_{x=a} \neq 0$$

holds.

Then $\partial_{x^i} G(x, \Lambda)|_{x=b} = 0$, for every $2 \leq i \leq \mu$ and $\partial_{x^{\mu+1}} G(x, \Lambda)|_{x=b} \neq 0$.

PROOF. Properties (3.2) and (3.2) imply that f_0, f_1 are diffeomorphisms in suitable neighbourhoods of a and b, respectively. Therefore, we can define local inverses. Being local diffeomorphisms, f_0 and f_1 are also local Lipschitz continuous and so are their inverses. In particular, $f_0(a) = b$ and $f_0^{-1}(b) = a$. We apply Lemma 3.1 to F and G making the identification $f = F$, $g = G$ and $h = f_0$. By (3.2) and (3.2),

$$\lim_{x \to a} \frac{|F(x) - x|}{|x - a|^\mu} = \lim_{x \to a} \frac{|(F(x) - x) - (F(a) + a)|}{|x - a|^\mu} = 0.$$

Therefore, $h = f_0$ and F satisfy the hypotheses of Lemma 3.1, and hence the thesis with $G(x) = (f_0 \circ F \circ f_0^{-1})(x)$. Thus, we obtain

$$\lim_{x \to b} \frac{|(f_0 \circ F \circ f_0^{-1})(x) - x|}{|x - b|^\mu} = \lim_{x \to b} \frac{|G(x) - x|}{|x - b|^\mu} = 0$$

and

$$\lim_{x \to b} \frac{|(f_0 \circ F \circ f_0^{-1})(x) - x|}{|x - b|^{\mu+1}} = \lim_{x \to b} \frac{|G(x) - x|}{|x - b|^{\mu+1}} > 0,$$

that is, the first μ derivatives of $G(x) - x$ are zero at b and the non-degeneracy condition holds as well. □

3.1.2. Example using the Faà di Bruno formula. Although seeming needless after the previous results, the next example will be important to deduce properties on the geometrical behavior of the composition of maps related to the swallowtail bifurcation at the beginning of Section 4. On the other hand, it is interesting to recover the Faà di Bruno formula [18], [24], since we will use it to prove the invariance of the transversality conditions. We think that it is possible to establish combinatorial results, using this theorem on both ends of the general formula for the derivatives of the compositions. This is an interesting open line of research for readers interested in Bell polynomials and other relevant combinatorial quantities associated with the Faà di Bruno formula, see [18], [25] and [32].

EXAMPLE 3.3 (Alternating maps). Given two real maps f and g defined in real intervals I_0 and I_1, we prove directly that if the second derivative relatively to the dynamic variable of any of the two maps $g \circ f$ and $f \circ g$ is zero, then the other one must be zero as well, disregarding the order of composition. The same holds for the third derivatives. We do this directly, using the chain rule for computing the derivatives of composed maps and its generalization, the *Faà di Bruno formula* [18].

Let f and g be $\mathcal{C}^3$ functions satisfying the following conditions:

(1) $(g \circ f)(a) = a$ and $(f \circ g)(b) = b$, which is $f(a) = b$ and $g(b) = a$.

(2) $\dfrac{d(g \circ f)}{dx}(x)\Big|_{x=a} = g'(b)f'(a) = 1.$

(3) $\dfrac{d^m(g \circ f)}{dx^m}(x)\Big|_{x=a} = 0$ for $m = 2, 3.$

Let us recall the formula of Faà di Bruno for the derivatives of composition

$$\frac{d^m(g \circ f)}{dx^m}(x) = m! \sum g^{(n)}(f(x)) \prod_{j=1}^{m} \frac{1}{\beta_j!} \left(\frac{f^{(j)}(x)}{j!} \right)^{\beta_j}, \tag{3.6}$$

where the sum is over all different solutions β_j in nonnegative integers $\beta_1, \dots, \beta_m$ of the linear Diophantine equations

$$\sum_{j=1}^{m} j\beta_j = m \quad \text{and} \quad n := \sum_{j=1}^{m} \beta_j.$$

To avoid overload with indexes we use the notation used in [32]:

$$\begin{gathered} f_0 = f(a), \qquad f_1 = f'(x)\big|_{x=a}, \dots, \qquad f_m = f^{(m)}(x)\big|_{x=a}, \\ g_0 = g(b), \qquad g_1 = g'(x)\big|_{x=b}, \dots, \qquad g_m = g^{(m)}(x)\big|_{x=b}, \\ \frac{d^m(g \circ f)}{dx^m}(x)\Big|_{x=a} = (gf)_m, \qquad \frac{d^m(f \circ g)}{dx^m}(x)\Big|_{x=b} = (fg)_m. \end{gathered}$$

With this notation, and taking into account hypotheses (1)–(3), Faà di Bruno's formula gives

$$(gf)_m = m! \sum g_n \prod_{j=1}^{m} \frac{1}{\beta_j!} \left(\frac{f_j}{j!} \right)^{\beta_j}, \tag{3.7}$$

$$(fg)_m = m! \sum f_n \prod_{j=1}^{m} \frac{1}{\beta_j!} \left(\frac{g_j}{j!} \right)^{\beta_j}. \tag{3.8}$$

Condition (2) in this notation is

$$f_1 g_1 = 1. \tag{3.9}$$

Let us consider the first two cases: $m = 2$ and $m = 3$, the cusp and swallowtail. Let $m = 2$. We shall use formula (3.6), therefore we have to solve the equation $\beta_1 + 2\beta_2 = 2$ for all possible values of the vector (β_1, β_2) in $\mathbb{N} \times \mathbb{N}$. The only solutions are $(\beta_1, \beta_2) = (0, 1)$, which gives $n = 1$, and $(\beta_1, \beta_2) = (2, 0)$, which gives $n = 2$. So we have

$$\begin{aligned}(gf)_2 &= 2!\left(g_1 \frac{1}{0!}\left(\frac{f_1}{1!}\right)^0 \frac{1}{1!}\left(\frac{f_2}{2!}\right)^1 + g_2 \frac{1}{2!}\left(\frac{f_1}{1!}\right)^2 \frac{1}{0!}\left(\frac{f_2}{2!}\right)^0\right) = 0 \\ &= g_1 f_2 + g_2 f_1^2 = g_1 f_2 + \frac{g_2}{g_1^2} = 0\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}(fg)_2 &= 2!\left(f_1 \frac{1}{0!}\left(\frac{g_1}{1!}\right)^0 \frac{1}{1!}\left(\frac{g_2}{2!}\right)^1 + f_2 \frac{1}{2!}\left(\frac{g_1}{1!}\right)^2 \frac{1}{0!}\left(\frac{g_2}{2!}\right)^0\right) \\ &= f_1 g_2 + f_2 g_1^2 = \frac{g_2}{g_1} + f_2 g_1^2.\end{aligned} \tag{3.11}$$

We solve the system with equations (3.9) and (3.10) for g_2, to obtain

$$g_2 = g_2(f_1, f_2) = -\frac{f_2}{f_1^3}. \tag{3.12}$$

By substituting $g_1 = 1/f_1$ and $g_2(f_1, f_2)$ in (3.11), we get

$$(fg)_2 = -f_1 \frac{f_2}{f_1^3} + f_2 \frac{1}{f_1^2} = 0.$$

Let $m = 3$. By Faà di Bruno's formula and taking into account the hypotheses, we obtain

$$(gf)_3 = g_1 f_3 + 3 g_2 f_1 f_2 + g_3 f_1^3 = g_1 f_3 + \frac{3 g_2 f_2}{g_1} + \frac{g_3}{g_1^3} = 0. \tag{3.13}$$

We solve the system with equations (3.9), (3.10) and (3.13) for g_3, keeping in mind that g_1 and g_2 have been computed earlier. Hence, we get

$$g_3 = -\frac{f_3}{f_1^4} - \frac{3(-f_2/f_1^3) f_2}{f_1^2} = \frac{3 f_2^2}{f_1^5} - \frac{f_3}{f_1^4}. \tag{3.14}$$

By replacing g_1, g_2 and g_3 by the previously obtained solutions, we find

$$\begin{aligned}(fg)_3 &= f_1 g_3 + 3 f_2 g_1 g_2 + f_3 g_1^3 \\ &= f_1\left(\frac{3 f_2^2}{f_1^5} - \frac{f_3}{f_1^4}\right) + 3 f_2 \frac{1}{f_1}\left(-\frac{f_2}{f_1^3}\right) + \frac{f_3}{f_1^3} = 0.\end{aligned} \tag{3.15}$$

3.1.3. On invariance of degeneracy and non-degeneracy conditions for periodic orbits of p-periodic systems. What we have shown in the previous paragraph is the invariance of bifurcation equations with respect to interchange in the composition of alternating maps. In this paragraph we generalize these to general p-periodic non-autonomous systems.

THEOREM 3.4. *Let $\mu \geq 2$ and let there be a p-periodic family of maps with individual mappings $f_0, \ldots, f_{p-1}$ with $f_j \in \mathcal{C}^{\mu+1}(I_j)$ for a fixed $j \in \{0, \ldots, p-1\}$, and a periodic point $a_j \in I_j$ with period p, i.e. a fixed point of F_j, such that*:

(a) *There exist $a_0, \ldots, a_{p-1}$, fixed points of $F_0, \ldots, F_{p-1}$, respectively, that is*

$$F_0(a_0) = a_0, \ldots, F_j(a_j) = a_j, \ldots, F_{p-1}(a_{p-1}) = a_{p-1}.$$

(b) *The non-hyperbolicity condition*

$$\partial_x F_j(x)\big|_{x=a_j} = \prod_{i=0}^{p-1} \partial_x f_i(a_j) = 1$$

holds.

(c) *Higher degeneracy conditions*

$$\partial_{x^i} F_j(x)\big|_{x=a_j} = 0, \quad \text{for every } 2 \leq i \leq \mu$$

hold.

(d) *The non-degeneracy condition*

$$\partial_{x^{\mu+1}} F_j(x)\big|_{x=a_j} \neq 0$$

holds.

Then, all compositions F_m, $0 \leq m \leq p-1$, satisfy

$$\partial_{x^i} F_m(x)\big|_{x=a_m} = 0, \quad \text{for every } 2 \leq i \leq \mu, \quad \text{and} \quad \partial_{x^{\mu+1}} F_m(x)\big|_{x=a_m} \neq 0.$$

PROOF. Without loss of generality we assume that $j = 0$, what can be done re-indexing maps of the p-periodic system. We now apply Theorem 3.2 to the alternating system f_0, $f_{p-1} \circ \ldots \circ f_1$ with compositions $F = F_0$ and $G = F_1 = f_0 \circ F \circ f_0^{-1}$, making $a = a_0$, $b = a_1$, and getting the result for $j = 1$. Applying the same argument repeatedly, the result follows immediately for all cyclic compositions F_m, $0 \leq m \leq p-1$. □

REMARK 3.5. The same result holds for k-periodic points of compositions F_j, i.e. $k \times p$-periodic points of the alternating system, since in that case we apply Theorem 3.2 to the alternating system with compositions $F = F_0^k$ and $G = F_1^k = f_0 \circ F^k \circ f_0^{-1}$.

Now we can choose the composition order that makes the bifurcation equations easier to solve.

3.1.4. Invariance of transversality conditions. In this paragraph we prove the symmetry for the transversality conditions concerning cyclic compositions of maps. We follow the same technique of proving the result as for alternating maps (2) f and g and generalizing it to periodic points of p-periodic systems. Suppose that there exists a solution Λ_0 in the parameter space of bifurcation equations (3.2), that the non-degeneracy condition (3.3) holds at Λ_0, and that a solution coexists with fixed points a and b for compositions F and G.

Consider the map $\mathcal{F} = (F, F_x, \ldots, F_{x^{\mu-1}})$ with the derivatives of the composition F and the Jacobian determinant of $\mathcal{F}$, now as a function of Λ. It is the determinant

$$J_\Lambda \mathcal{F}(x,\Lambda) = \det \begin{bmatrix} F_{\lambda_1}(x,\Lambda) & \cdots & F_{\lambda_\mu}(x,\Lambda) \\ \vdots & \ddots & \vdots \\ F_{x^{\mu-1}\lambda_1}(x,\Lambda) & \cdots & F_{x^{\mu-1}\lambda_\mu}(x,\Lambda) \end{bmatrix}. \tag{3.16}$$

Consider similar definitions for $\mathcal{G}(x,\Lambda)$ and $J_\Lambda\mathcal{G}(x)$ relatively to the composition G.

LEMMA 3.6. *The Jacobians $J_\Lambda\mathcal{F}(x,\Lambda)$ and $J_\Lambda\mathcal{G}(x,\Lambda)$ computed at the solutions of the bifurcation conditions* (3.2) *and* (3.3) *satisfy the equality*

$$J_\Lambda \mathcal{F}(a,\Lambda_0) = (f_x(a,\Lambda_0))^{(3\mu-\mu^2)/2} J_\Lambda \mathcal{G}(b,\Lambda_0). \tag{3.17}$$

PROOF. The proof rests on the fact that we can obtain the lines of the Jacobian matrix $[J_\Lambda\mathcal{F}(a,\Lambda_0)]$ of $\mathcal{F}$ using the Gaussian manipulation of the Jacobian matrix $[J_\Lambda\mathcal{G}(b,\Lambda_0)]$ of $\mathcal{G}$. Consider

$$[J_\Lambda \mathcal{F}(a,\Lambda_0)] = \begin{bmatrix} L_1 \\ \vdots \\ L_\mu \end{bmatrix},$$

where L_i denotes the i-th line of the matrix. We have to prove that

$$[J_\Lambda \mathcal{G}(b,\Lambda_0)] = \begin{bmatrix} \alpha_{11}L_1 \\ \alpha_{21}L_1 + \alpha_{22}L_2 \\ \vdots \\ \displaystyle\sum_{j=1}^{\mu} \alpha_{\mu j} L_j \end{bmatrix},$$

(2) Notation that we adopt in this paragraph to simplify the presentation of the next results and proofs. We replace f_0 by f and f_1 by g. The compositions are $F = g \circ f$ and $G = f \circ g$.

i.e.

$$[J_\Lambda \mathcal{G}(b,\Lambda_0)] = \begin{bmatrix} \alpha_{11} & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{\mu 1} & \alpha_{\mu 2} & \cdots & \alpha_{\mu\mu} \end{bmatrix} \begin{bmatrix} L_1 \\ \vdots \\ L_\mu \end{bmatrix},$$

which is

$$[J_\Lambda \mathcal{G}(b,\Lambda_0)] = A\,[J_\Lambda \mathcal{F}(a,\Lambda_0)]\,,$$

where

$$A = \begin{bmatrix} \alpha_{11} & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{\mu 1} & \alpha_{\mu 2} & \cdots & \alpha_{\mu\mu} \end{bmatrix}.$$

We also prove that $\det A$ is different from 0.

The fact that for general $\mu \geq 1$ the matrix A is a lower triangular matrix is trivial. Below, we prove that each entry of the main diagonal is

$$\alpha_{jj} = (f_x(a,\Lambda_0))^{2-j}, \quad 1 \leq j \leq \mu, \tag{3.18}$$

this equality implies that all such entries are different from zero due to the non-hyperbolicity condition at the bifurcation

$$\partial_x F(x,\Lambda)|_{x=a,\Lambda=\Lambda_0} = g_x(b,\Lambda_0) f_x(a,\Lambda_0) = 1.$$

Moreover, (3.18) implies that the determinant of A is

$$\det A = \prod_{j=1}^{\mu} (f_x(a,\Lambda_0))^{2-j} = (f_x(a,\Lambda_0))^{(3\mu-\mu^2)/2},$$

as desired.

We prove now equality (3.18). Note that $G \circ f = f \circ g \circ f = f \circ F$. We differentiate this local conjugacy in order to λ_i, with $i = 1,\ldots,\mu$. We have

$$\partial_{\lambda_i} G(f(x,\Lambda),\Lambda) = \partial_{\lambda_i} f(F(x,\Lambda),\Lambda), \tag{3.19}$$

with

$$\begin{aligned} \partial_{\lambda_i} G(f(x,\Lambda),\Lambda) &= G_{\lambda_i}(f(x,\Lambda),\Lambda) + G_x(x,\Lambda) f_{\lambda_i}(x,\Lambda), \\ \partial_{\lambda_i} f(F(x,\Lambda),\Lambda) &= f_{\lambda_i}(x,\Lambda) + f_x(F(x,\Lambda),\Lambda) F_{\lambda_i}(x,\Lambda). \end{aligned} \tag{3.20}$$

At the points (a,Λ_0) and (b,Λ_0) equating the second members of (3.20), one has

$$G_{\lambda_i}(b,\Lambda_0) + G_x(b,\Lambda_0) f_{\lambda_i}(a,\Lambda_0) = f_{\lambda_i}(a,\Lambda_0) + f_x(a,\Lambda_0) F_{\lambda_i}(a,\Lambda_0),$$

which due to conditions (3.2) at the bifurcation point is

$$G_{\lambda_i}(b,\Lambda_0) = f_x(a,\Lambda_0) F_{\lambda_i}(a,\Lambda_0), \tag{3.21}$$

this equality gives the relation between the first rows of the Jacobians.

To get the relations between the second rows we consider the derivative relatively to x of (3.20), we present only the terms that matter for the computation of the main diagonal of A

$$\begin{aligned} \partial_{\lambda_i x} G(f(x,\Lambda),\Lambda) &= G_{\lambda_i x}(f(x,\Lambda),\Lambda) f_x(x,\Lambda) + \dots, \\ \partial_{\lambda_i x} f(F(x,\Lambda),\Lambda) &= \dots + f_x(F(x,\Lambda),\Lambda) F_{\lambda_i x}(x,\Lambda). \end{aligned} \tag{3.22}$$

which due to conditions (3.2) gives, equalizing the right-hand sides of (3.22) at the bifurcation value,

$$G_{x\lambda_i}(b,\Lambda_0) = \text{l.o.t.} + F_{x\lambda_i}(a,\Lambda_0), \tag{3.23}$$

where l.o.t. stands for "lower order terms" in terms of derivatives on the dynamical variable of G_{λ_i} and F_{λ_i}, terms that do not appear in the main diagonal of A. This expression gives the relation between the second rows of the Jacobians.

To get the relation between the third rows of the Jacobians, we consider the derivative of (3.22) regarding x

$$\begin{aligned} \partial_{\lambda_i x^2} G(f(x,\Lambda),\Lambda) &= G_{\lambda_i x^2}(f(x,\Lambda),\Lambda) f_x^2(x,\Lambda) + \dots, \\ \partial_{\lambda_i x^2} f(F(x,\Lambda),\Lambda) &= \dots + f_x(F(x,\Lambda),\Lambda) F_{\lambda_i x^2}(x,\Lambda), \end{aligned}$$

which due to conditions (3.2) gives at the bifurcation value

$$G_{x^2\lambda_i}(b,\Lambda_0) f_x^2(a,\Lambda_0) = \text{l.o.t.} + f_x(a,\Lambda_0) F_{x^2\lambda_i}(a,\Lambda_0).$$

Repeating this process, using the Faà di Bruno formula (3.6) and the bifurcation equations (3.2), knowing that the lower order terms in derivatives relatively to x (order less than $j-1$) do not contribute to the diagonal of A, we have for $1 \leq j \leq \mu$

$$G_{x^{j-1}\lambda_i}(b,\Lambda_0)(f_x(a,\Lambda_0))^{j-1} = \text{l.o.t.} + f_x(a,\Lambda_0) F_{x^{j-1}\lambda_i}(a,\Lambda_0),$$

dividing it by $(f_x(a,\Lambda_0))^{j-1}$, which cannot be zero due to the second line of equations (3.2), we obtain

$$G_{x^{j-1}\lambda_i}(b,\Lambda_0) = \text{l.o.t.} + (f_x(a,\Lambda_0))^{2-j} F_{x^{j-1}\lambda_i}(a,\Lambda_0),$$

the desired result. □

In the general setting of p-periodic maps and considering the last lemma, the transversality conditions for A_μ bifurcations of $k \times p$-periodic points of the first two possible compositions of maps are such that

$$J_\Lambda \mathcal{F}_0^k(a_0,\Lambda_0) \neq 0 \Rightarrow J_\Lambda \mathcal{F}_1^k(a_1,\Lambda_0) \neq 0, \tag{3.24}$$

where J_Λ was defined in (3.16), because F_0^k and F_1^k are two compositions of alternating maps as we have seen in Remark 3.5. The generalization to periodic points of all cyclic compositions of p-periodic maps makes no difficulties and the proof is obtained by repeated application of Lemma 3.6.

THEOREM 3.7. *If one of transversality conditions of the class A_μ bifurcations for $k \times p$-periodic orbits of p-periodic maps at $\Lambda = \Lambda_0$ is satisfied, say*

$$J_\Lambda \mathcal{F}_j^k(a_j, \Lambda_0) \neq 0,$$

then it is satisfied for all cyclic compositions of individual maps, i.e.

$$J_\Lambda \mathcal{F}_0^k(a_0, \Lambda_0) \neq 0, \ldots, J_\Lambda \mathcal{F}_{p-1}^k(a_{p-1}, \Lambda_0) \neq 0.$$

3.2. Conclusion. The invariance of degeneracy and transversality conditions implies that the bifurcation problem with μ degeneracy conditions on the iteration variable is independent on the choice of cyclic order in the composition of maps when maps are sufficiently differentiable. This invariance is fundamental, since it means that we can define local bifurcations in a unique way for families of p-periodic maps using any compositions of particular maps. Hence, the bifurcation set in the parameter space is the same for all F_j^k.

The main conclusion of this study, is that it suffices to solve bifurcation conditions applied to one of F_j^k possible compositions to obtain the bifurcation set. The bifurcation is well-defined using bifurcation conditions on composition families. Each fiber replicates the behavior of the others. Hence, the local bifurcations studied in this work of p-periodic difference equations are defined by the same rules of local bifurcations of autonomous systems.

4. Examples

We conclude this work with study of a particular case of alternating maps. We establish some useful criteria on existence of swallowtail singularity for alternating maps and give two examples exhibiting this type of bifurcation.

Let $f \in \mathcal{C}^3(I)$. The Schwarzian derivative of f is

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2,$$

defined for every x in I that is not a critical point of f.

PROPOSITION 4.1. *Consider an alternating system with families $f = f_0$ and $g = f_1$ satisfying all conditions of the class A_3 bifurcation, i.e., swallowtail bifurcation, together with transversality conditions. If one of the maps say, without loss of generality, g, has Schwarzian derivative different from zero at b, $Sg(b) \neq 0$, then the product of Schwarzian derivatives must be negative at the swallowtail bifurcation point, i.e.*

$$Sg(b) \cdot Sf(a) < 0.$$

PROOF. Recall Example 3.3 with the same notation for derivatives. Consider f and g under conditions of that example. Recall the equalities obtained and the notation. If g is assumed to have negative Schwarzian derivative at b, one has

$$\frac{g_3}{g_1} - \frac{3}{2}\left(\frac{g_2}{g_1}\right)^2 < 0,$$

then, from (3.9) and (3.14) one has

$$\frac{g_3}{g_1} = \frac{3(f_1)^2}{(f_1)^4} - \frac{f_3}{(f_1)^3} = \frac{1}{(f_1)^2}\left(3\left(\frac{f_2}{f_1}\right)^2 - \frac{f_3}{f_1}\right) < \frac{3}{2}\left(\frac{g_2}{g_1}\right)^2,$$

and, by equality (3.12)

$$\frac{g_2}{g_1} = -\frac{f_2}{(f_1)^2}.$$

The inequality above becomes

$$3\left(\frac{f_2}{f_1}\right)^2 - \frac{f_3}{f_1} < \frac{3}{2}\left(\frac{f_2}{f_1}\right)^2.$$

Therefore,

$$\frac{f_3}{f_1} - \frac{3}{2}\left(\frac{f_2}{f_1}\right)^2 > 0.$$

Therefore, if g has negative Schwarzian derivative at b, then f must have positive Schwarzian derivative at a.

On the other hand, if f is assumed to have negative Schwarzian derivative, then by similar reasonings,

$$\frac{f_3}{f_1} = -\frac{g_3}{g_1}(f_1)^2 + 3\left(\frac{f_2}{f_1}\right)^2 < \frac{3}{2}\left(\frac{f_2}{f_1}\right)^2.$$

This implies

$$\frac{g_3}{g_1}(f_1)^2 > \frac{3}{2}\left(\frac{f_2}{f_1}\right)^2$$

and, as$g_2/g_1 = -f_2/(f_1)^2$, it follows that

$$\frac{g_3}{g_1} > \frac{3}{2}\left(\frac{f_2}{(f_1)^2}\right)^2 = \frac{3}{2}\left(\frac{g_2}{g_1}\right)^2.$$

Hence, if f has negative Schwarzian derivative then g must have positive Schwarzian derivative. $\square$

As in [9], while working on pitchfork bifurcation, we can state the following two results on the A_3 degenerate bifurcation. The proofs are similar.

PROPOSITION 4.2. *Let f and g be $\mathcal{C}^3$ alternating maps. If f is strictly increasing and g is strictly decreasing in x (analogously, if f is strictly decreasing and g is strictly increasing), then the alternating system associated with f and g cannot have an A_3 degenerate bifurcation.*

PROPOSITION 4.3. *Let f and g be $\mathcal{C}^3$ alternating maps. If f and g are both strictly increasing in the dynamic variable and one of the following two situations takes place:*

$$\min_{x\in I_0,I_1}(\partial_{x^2}f(x),\partial_{x^2}g(x))>0 \quad (\textit{both convex})$$

or

$$\max_{x\in I_0,I_1}(\partial_{x^2}f(x),\partial_{x^2}g(x))<0 \quad (\textit{both concave}),$$

then the alternating system generated by them cannot have a swallowtail bifurcation.

PROPOSITION 4.4. *If f and g are both strictly decreasing in the dynamic variable and the following takes place:*

$$\max_{x\in I_0,I_1}(\partial_{x^2}f(x),\partial_{x^2}g(x))>0 \quad \textit{and} \quad \min_{x\in I_0,I_1}(\partial_{x^2}f(x),\partial_{x^2}g(x))<0,$$

i.e. one is concave and the other is convex or vice-versa, then the alternating system generated by them cannot have an A_3 degenerate bifurcation.

Now, we consider two particular examples. The first one is an alternating system with polynomial families, and the second one with the tangent family $\lambda\tan x$ and a polynomial family. The first example is relatively easy to compute, but the second one has elusive roots due to its high degeneracy. Thus, some numeric work is necessary. We present only the solutions and discard the tedious computations of the second example.

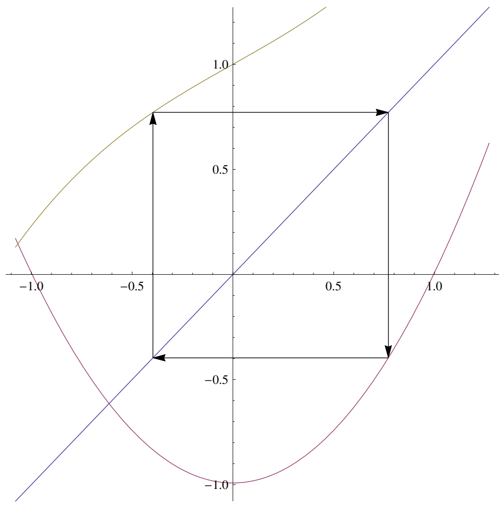

FIGURE 2. The geometry of individual maps at the swallowtail bifurcation point A_3 for Example 4.5. Note that one map is convex and the other is concave.

EXAMPLE 4.5. The alternating system f and g with a quadratic polynomial $f_0 = x^2 + \lambda_1$ and a cubic polynomial $f_1 = \lambda_3 x^3 + \lambda_2 x + 1$, defined in the real line. The compositions are

$$F(x) = f_1 \circ f_0(x) = \lambda_3 x^6 + 3\lambda_3\lambda_1 x^4 + (3\lambda_3\lambda_1^2 + \lambda_2)x^2 + 1 + \lambda_2\lambda_1 + \lambda_3\lambda_1^3,$$
$$G(x) = f_0 \circ f_1(x) = \lambda_3^2 x^6 + 2\lambda_2\lambda_3 x^4 + 2\lambda_3 x^3 + \lambda_2^2 x^2 + 2\lambda_2 x + 1 + \lambda_1.$$

The bifurcation equations (3.2) for F or G have solutions

$$\lambda_1 = -\frac{3^5}{5 \cdot 7^2}, \quad \lambda_2 = \frac{5^2 \cdot 7}{2^2 \cdot 3^4} \quad \text{and} \quad \lambda_3 = \frac{3^5 \cdot 7^5}{2^4 \cdot 3^{15}},$$

with

$$a = \frac{3^3}{5 \cdot 7}, \qquad b = -\frac{2 \cdot 3^5}{5^2 \cdot 7^2},$$

such that $f_0(a) = b$ and $f_1(b) = a$. The Schwarzian derivatives are

$$Sf(a) = \frac{1}{b} = -\frac{5^2 \cdot 7^2}{2 \cdot 3^5}, \qquad Sg(b) = \frac{1}{6}\frac{1}{b^2} = \frac{5^4 \cdot 7^4}{2^3 \cdot 3^{11}},$$

naturally, with opposite signs at the bifurcation points, according to Proposition 4.1. Obviously, $SF(a) = SG(b) = 0$ at bifurcation points.

The transversality condition (3.24) is, by Lemma 3.6, the same for F and G (since $3.3 - 3^2/2 = 0$) and gives

$$J_\Lambda \mathcal{F}(a, \Lambda_0) = J_\Lambda \mathcal{G}(b, \Lambda_0) = -\frac{2^4 \cdot 3^{10}}{5^4 \cdot 7^3} \neq 0.$$

We can see in Figure 2 the geometry of individual maps at the A_3 singularity. Both functions are increasing at suitable neighbourhoods of a and b, one function is concave and the other is convex. The bifurcation set is similar to the one depicted in Figure 1. We have the same behavior of two possible compositions F and G.

EXAMPLE 4.6. Consider now the family of real alternating maps f_0 and f_1 with $f_0(x) = -x^4 + \lambda_1 x^2 + x + \lambda_2$ and $f_2(x) = \lambda_3 \tan x$, defined on suitable open sets near solutions of swallowtail bifurcation equations. We have solutions of bifurcation conditions $a \simeq 0.0797053$, $b \simeq 0.0793675$, $\lambda_1 \simeq -0.0400839$, $\lambda_2 \simeq -0.0000428492$, $\lambda_3 \simeq 1.00215$. The non-degeneracy condition gives $F^{(iv)}(b) \simeq -26.7$. We note that a and b are very near each other and the maps are almost parallel at bifurcation points, Figure 3. The Schwarzian derivatives are $Sf(a) = -1.96648$, $Sg(b) = 2$.

The transversality condition (3.24) is again the same for F and G and has the form

$$J_\Lambda \mathcal{F}(a, \Lambda_0) = J_\Lambda \mathcal{G}(b, \Lambda_0) = -2.08013.$$

These two examples, exhibiting the swallowtail bifurcation, produce evidence that high degeneracy bifurcations can occur in particular cases and the theory has applications.

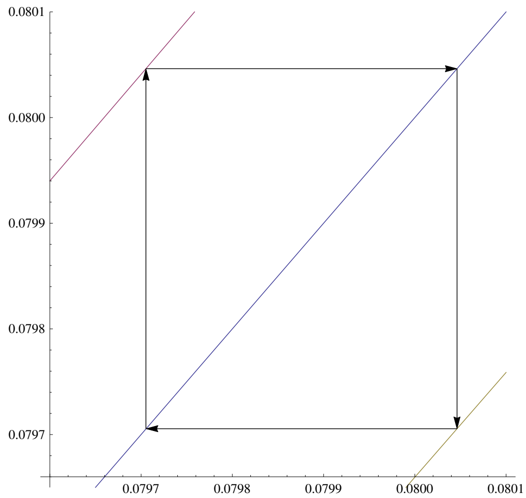

FIGURE 3. The orbit of bifurcation points a and b of f_0, f_1 in Example 4.6 viewed as a cobweb diagram. The maps, one concave and the other convex, are almost parallel.

Acknowledgements. The author thanks the anonymous referee for his remarks and suggestions which helped to improve this paper. The author also thanks MichałMisiurewicz for a fruitful discussion about the proof of Lemma 3.1. The author was partially funded through the project PEst-OE/EEI/LA0009/2013 for CAMGSD.

REFERENCES

[1] D.J. ALLWRIGHT, *Hypergraphic functions and bifurcations in recurrence relations*, SIAM J. Appl. Math. **34** (4) (1978), 687–691.

[2] J.F. ALVES AND L. SILVA, *Nonautonomous graphs and topological entropy of nonautonomous Lorenz systems*, Internat. J. Bifur. Chaos **25**, No. 6 (2015), 1550079 (9 pages).

[3] V.I. ARNOLD, *Critical points of smooth functions*, Proceedings of ICM **74** (1) (1979), 19–40.

[4] ______, *Dynamical Systems.* V. *Bifurcation Theory and Catastrophe Theory*, Encyclopedia of Mathematical Sciences, vol. 5, Springer, Berlin, 1994.

[5] B. AULBACH, M. RASMUSSEN AND S. SIEGMUND, *Approximation of attractors of nonautonomous dynamical systems*, Discrete Contin. Dyn. Syst. **5** (2) (2005), 215–238.

[6] W. BEYN, T. HLS AND M. SAMTENSCHNIEDER, *On r-periodic orbits of k-periodic maps*, J. Difference Equ. Appl. **8** (14) (2008), 865–887.

[7] K. BRUCKS AND H. BRUIN, *Topics from One-Dimensional Dynamics*, vol. 62, Cambridge University Press, 2004.

[8] S. CHOW AND J. HALE, *Methods of Bifurcation Theory*, vol. 251, Springer, 1982.

[9] E. D'ANIELLO AND H.M. OLIVEIRA, *Pitchfork bifurcation for non-autonomous interval maps*, Differ. Equ. Appl. **15** (3) (2009), 291–302.

[10] W. DE MELO AND S. STRIEN, *One-Dimensional Dynamics*, Springer, Berlin, Heildelberg, 1993.

[11] S. ELAYDI, R. LUIS AND H. OLIVEIRA, *Local bifurcation in one dimensional non-autonomous periodic difference equations*, Internat. J. Bifur. Chaos **23** (3) (2013), 1–18.

[12] S. ELAYDI AND R. SACKER, *Skew-product dynamical systems: Applications to difference equations*, Proceedings of the Second Annual Celebration of Mathematics, 2005.

[13] R. GILMORE, *Catastrophe Theory for Scientists and Engineers*, Dover Publications, 1993.

[14] M. GOLUBITSKY AND D. SCHAEFFER, *Singularities and Groups in Bifurcation Theory*, vol. 51, Appl. Math. Sci., 1985.

[15] J. GUCKENHEIMER, *On the bifurcation of maps of the interval*, Invent. Math. **39** (2) (1977), 165–178.

[16] T. HÜLS, *A model function for non-autonomous bifurcations of maps*, Discrete Contin. Dyn. Syst. Ser. B **7** (2) (2007), 351.

[17] G. IOOSS, *Bifurcations of Maps and Applications*, vol. 36, Mathematics Studies, (North-Holland, Amsterdam, New York, Oxford), France, 1979.

[18] W. JOHNSON, *The curious history of Faà di Bruno's formula*, Amer. Math. Monthly **109** (3) (2002), 217–234.

[19] P. KLOEDEN, C. PÖTZSCHE AND M. RASMUSSEN, *Discrete time nonautonomous dynamical systems*, Manuscript, 2011.

[20] P. Kloeden, M. Rasmussen, Nonautonomous dynamical systems, Mathematical Surveys and Monographs, vol. 176, Mathematical Surveys and Monographs, 2011.

[21] P. KLOEDEN AND S. SIEGMUND, *Bifurcations and continuous transitions of attractors in autonomous and nonautonomous systems*, Internat. J. Bifur. Chaos **15** (3) (2005), 743–762.

[22] S. KOLYADA, M. MISIUREWICZ, AND L. SNOHA, *Topological entropy of nonautonomous piecewise monotone dynamical systems on the interval*, Fund. Math. **160** (1997), 161–181.

[23] I.A. KUZNETSOV, *Elements of Applied Bifurcation Theory*, vol. 112, 3rd ed., Springer, New York, Berlin, Heidelberg, 1998.

[24] S.-F. LACROIX, *Traité du Calcul différentiel et du Calcul intégral*, 2e Édition Revue et Augmentée, vol. 1, Courcier, Paris, 1810.

[25] S. NOSCHESE AND P. RICCI, *Differentiation of multivariable composite functions and bell polynomials*, Comput. Anal. Appl. **5** (3) (2003), 333–340.

[26] C. PÖTZSCHE, *Geometric Theory of Discrete Nonautonomous Dynamical Systems*, Springer, Berlin, 2010.

[27] ———, *Nonautonomous bifurcation of bounded solutions* I: *A Lyapunov–Schmidt approach*, Discrete Contin. Dyn. Syst. Ser. B **14** (2) (2010), 739–776.

[28] ———, *Bifurcations in a periodic discrete-time environment*, Real World Appl. **14** (1) (2013), 53–82.

[29] ———, *Nonautonomous bifurcation of bounded solutions* II: *A shovel bifurcation pattern*, Discrete Contin. Dyn. Syst. Ser. A **31** (3) (2013), 941–973.

[30] M. RASMUSSEN, *Towards a bifurcation theory for nonautonomous difference equations*, Difference Equations and Applications **12** (3–4) (2006), 297–312.

[31] ———, *Attractivity and Bifurcation for Nonautonomous Dynamical Systems*, vol. 1907, Springer, Berlin, Heidelberg, 2007.

[32] S. ROMAN, *The formula of Faa di Bruno*, Amer. Math. Monthly **87** (10) (1980), 805–809.

[33] A. SHARKOVSKY, I. MAISTRENKOAND E. ROMANENKO, *Difference Equations and their Applications*, vol. 250, Springer, Berlin, Heidelberg, 1993.

[34] D. SINGER, *Stable orbits and bifurcation of maps of the interva*l, Appl. Math. **35** (2) (1978), 260–267.

[35] R. THOM, *Stabilité structurelle et Morphogenèse*, Interéditions, 1977.

[36] E. ZEEMAN, *Catastrophe Theory*, Addison–Wesley, 1977, selected papers.

Manuscript received November 6, 2014

accepted May 19, 2015

HENRIQUE M. OLIVEIRA
Center for Mathematical Analysis,
Geometry and Dynamical Systems
Mathematics Department
Instituto Superior Técnico
Universidade de Lisboa
Av. Rovisco Pais
1049-001 Lisboa, PORTUGAL

E-mail address: holiv@math.ist.ulisboa.pt