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A MORSE INDEX THEOREM FOR PERTURBED GEODESICS ON SEMI-RIEMANNIAN MANIFOLDS

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Dedicated to Kazimierz Gęba

Abstract. Perturbed geodesics are trajectories of particles moving on a semi-Riemannian manifold in the presence of a potential. Our purpose here is to extend to perturbed geodesics on semi-Riemannian manifolds the well known Morse Index Theorem. When the metric is indefinite, the Morse index of the energy functional becomes infinite and hence, in order to obtain a meaningful statement, we substitute the Morse index by its relative form, given by the spectral flow of an associated family of index forms. We also introduce a new counting for conjugate points, which need not to be isolated in this context, and prove that our generalized Morse index equals the total number of conjugate points. Finally we study the relation with the Maslov index of the flow induced on the Lagrangian Grassmannian.

1. Introduction

A semi-Riemannian manifold is a smooth n-dimensional manifold M endowed with a (pseudo) metric given by a nondegenerate symmetric two-form g of constant index ν. We denote by D the associated Levi–Civita connection and by D/dx the covariant derivative of a vector field along a smooth curve γ. Let I be an interval on the real line. Let V be a smooth function defined on $I \times M$. A *perturbed geodesic* abbreviated as *p-geodesic* is a smooth curve $\gamma\colon I \to M$ which

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satisfies the differential equation

$$\frac{D}{dx}\gamma'(x) + \nabla V(x,\gamma(x)) = 0 \tag{1.1}$$

where ∇V denotes gradient of $V(x,-)$ with respect to the metric g.

From the viewpoint of analytical dynamics, the data (g,V) define a mechanical system on the manifold M, with kinetic energy $g(v,v)/2$ and time dependent potential energy V. Solutions of the differential equation (1.1) are trajectories of particles moving on the semi-Riemannian manifold in the presence of the potential V. If the potential vanishes we get trajectories of free particles and hence geodesics on M. This motivates the suggestive name, "perturbed geodesics", already adopted in [36] for the periodic case. If the potential V is time independent then, modulo reparametrization, perturbed geodesics become geodesics of the Jacobi metric associated to (g,V). The total energy

$$e = \frac{1}{2}g(\gamma(x))(\gamma'(x),\gamma'(x)) + V(\gamma(x))$$

is constant along such trajectories and when V is bounded from above the solutions of (1.1) with energy e greater than $\sup_{m\in M} V(m)$ are nothing but reparametrized geodesics for metric $[e-V]g$ on M with total energy one (see [1]).

In what follows we will consider perturbed geodesics connecting two given points of M and we will normalize the domain by taking as I the interval $[0,1]$.

A vector field ξ along γ is called a *Jacobi field* if it verifies the linear differential equation

$$\frac{D^2}{dx^2}\xi(x) + R(\gamma'(x),\xi(x))\gamma'(x) + D_{\xi(x)}\nabla V(x,\gamma(x)) = 0, \tag{1.2}$$

where R is the curvature tensor of D.

Given a p-geodesic γ, an instant $x \in (0,1]$ is said to be a *conjugate instant* if there exists at least one non zero Jacobi field with $\xi(0) = \xi(x) = 0$. The corresponding point $q = \gamma(x)$ on M is said to be a *conjugate point* to the point $p = \gamma(0)$ along γ.

Let $\mathcal{I}$ be the n-dimensional vector space of all Jacobi fields along γ verifying $\xi(0) = 0$. The number $m(x) = \dim\{\xi \in \mathcal{I} : \xi(x) = 0\}$ is called the *geometric multiplicity* of x. Thus x is a conjugate instant if and only if $m(x) > 0$. Let $\mathcal{I}[x] = \{\xi(x) : \xi \in \mathcal{I}\} \subset T_{\gamma(x)}M$. Denoting with $\perp$ the orthogonal with respect to the metric g, the rank theorem applied to the homomorphism $\xi \in \mathcal{I} \to \xi(x)$ gives $m(x) = \operatorname{codim}\mathcal{I}[x] = \dim \mathcal{I}[x]^{\perp}$.

We will say that a conjugate instant x is *regular* if the restriction of the form $g_{\gamma(x)}$ to $\mathcal{I}[x]^{\perp}$ is a non degenerate quadratic form and that γ is a *regular* p-geodesic if all the conjugate instants along γ are regular. It is easy to see that regular conjugate instants are isolated and therefore any regular p-geodesic γ has only a finite number of conjugate instants.

The *conjugate index* of a regular p-geodesic γ is defined by

$$i_{\mathrm{con}}(\gamma) = \sum_{x\in(0,1]} \operatorname{sign}\,(g|_{\mathcal{I}[x]^\perp}) \tag{1.3}$$

where sign denotes the signature of a quadratic form.

It is easy to see that both the regularity and $i_{\mathrm{con}}(\gamma)$ are preserved under small enough p-geodesic perturbations of a regular p-geodesic. For geodesics on a semi-Riemannian manifold this was proved in [19] and [16], where it was also shown that the sum in (1.3) is not a topological invariant of γ if the geodesic is not regular.

In what follows we will shortly describe some known results in the geodesic case. If the metric is Riemannian the conjugate points along a geodesic are always regular and $\operatorname{sign}\,(g|_{\mathcal{I}[x]^\perp}) = m(x)$. The regularity of all conjugate points implies then that $i_{\mathrm{con}}(\gamma)$ is a topological invariant. The celebrated Morse index theorem states that the Morse index $\mu_{\mathrm{Morse}}(\gamma)$ of the geodesic γ, considered as a critical point of the energy functional, equals the total multiplicity $\sum_{x\in]0,1]} m(x)$ of conjugate points along the geodesic.

The regularity property and the Morse index theorem, appropriately restated, continue to be true for time-like and light-like geodesics on Lorentz manifolds. Everything turns to be as before, eventually after passing to a quotient space of the space of Jacobi fields (see [6] for the proof of this result and its applications to relativity). However the above approach breaks down for space-like geodesics or for geodesics of any causal character on general semi-Riemannian manifolds. In this case not only the Morse index of a geodesic fails to be finite but also the conjugate points can accumulate. Moreover, they can disappear under a small perturbation. Thus, in order to formulate the Morse index theorem for geodesics and p-geodesics on semi-Riemannian manifolds, one has to extend (1.3) to a topological invariant of general p-geodesics and find some kind of renormalized Morse index as a substitute for the right hand side.

The first breakthrough for this problem in the geodesic case was obtained by Helfer in [19]. His substitute for the Morse index is a spectral index defined as the sum of Krein signatures of negative real eigenvalues of the Jacobi differential operator, viewed as a self-adjoint operator on a Krein space. Recently, in a series of papers, Piccione, Tausk and their collaborators [27], [33], [16], [8], [32] initiated a sort of critical review of Helfer's work, improving and completing his results. They fixed a technical gap in Helfer's argument in [8]. In [27], following Helfer, they defined conjugate index by equation (1.3) for any geodesic, irrespective of whether it is regular or not, but they gave an example which shows that this naive definition does not work in the non regular case. However the correct expression for the conjugate index of a degenerate conjugate point was not given, and therefore they conclude that the Morse index theorem holds true for regular geodesics only. Their main result in [33], [16] is an expression for the spectral index as the difference between Morse indices of the index form (the Hessian) restricted to some special subspaces of the domain. Similar decomposition was already stated by Helfer in [19] without proof.

What we propose here is a different definition of the two sides in the Morse index theorem which extends to perturbed geodesics as well. This allows us to handle both the problem with the Morse index and the accumulation of conjugate points, providing a new and, we believe, interesting proof of this theorem.

The Hessian h_γ of the energy functional at a p-geodesic γ is a bounded Fredholm quadratic form. If moreover h_γ is nondegenerate, the p-geodesic γ is called nondegenerate. Together with a nondegenerate p-geodesic γ we consider a path $\widetilde{\gamma}$ of perturbed geodesics canonically induced by γ on the manifold $\Omega(M)$ of all H^1-paths on M (see Section 3). As generalized Morse index of γ we take the negative of the spectral flow of the family of Hessians of the energy functional along the canonical path $\widetilde{\gamma}$.

Roughly speaking, the spectral flow of a path of Fredholm quadratic forms or, what is the same, the spectral flow of the path $\{A_t\}_{t\in[a,b]}$ of self-adjoint Fredholm operators arising in the Riesz representation of the forms, is the integer given by the number of negative eigenvalues of A_a that become positive as the parameter t goes from a to b minus the number of positive eigenvalues of A_a that become negative. It is easy to see that if one of (and hence all) the operators in the path have a finite Morse index, then the spectral flow of a path A is nothing but the difference between the Morse indices at the end points. Thus spectral flow appears to be the right substitute of the Morse index in the framework of strongly indefinite functionals. For example, in [12], it substitutes the Morse index in defining a grading for the Floer homology groups. In [11] it plays the same role in the formulation of a bifurcation theorem for critical points for strongly indefinite functionals. In general the spectral flow depends on the homotopy class of the whole path. However in the specific case of the energy functional associated to a mechanical system things are simpler. In this framework, spectral flow depends only on the endpoints of the path and therefore it can be considered as a relative form of Morse index in the sense of [13].

We also change the counting of conjugate points. The papers [19], [33] follow the lines of the proof of the Morse index theorem by Duistermaat [10], which essentially leads to consider as the "total number of conjugate points" along γ the Maslov index of the flow line induced on the Lagrangian Grassmanian by the associated Hamiltonian flow. We define the conjugate index μ_{con} by means of a suspension of the complexified family of boundary value problems defining Jacobi fields. The resulting boundary value problem is parameterized by points of the complex plane. The conjugate instants are in one to one correspondence with points in the complex plane where the determinant of the associated fundamental matrix vanishes. This determinant defines a smooth map from an open subset of the complex plane into the plane. A well known topological invariant that counts algebraically the zeroes of a map is the Brouwer degree. In order to keep signs according to the Riemannian case we define the conjugate index μ_{con} as the minus the degree of this map. With this said, our main theorem takes the standard form.

Theorem 1.1. *Let (M, g) be a semi-Riemannian manifold, $V\colon I \times M \to \mathbb{R}$ a smooth potential and $\gamma\colon I \to M$ a nondegenerate perturbed geodesic. Then $\mu_{\mathrm{spec}}(\gamma) = \mu_{\mathrm{con}}(\gamma)$.*

While $\mu_{\mathrm{spec}}(\gamma)$ has an intrinsic, i.e. coordinate free definition through the spectral flow of Hessians, the conjugate index is constructed using an appropriate choice of coordinates along the perturbed geodesic. The above theorem shows in particular the independence of conjugate index from the various choices involved in the construction.

The classical Morse index theorem is a special case of Theorem 1.1. When the metric is Riemannian $\mu_{\mathrm{spec}}(\gamma) = \mu_{\mathrm{Morse}}(\gamma)$ while $\mu_{\mathrm{con}}(\gamma) = \sum_{x\in]0,1]} m(x)$.

Our interest in this problem was partially motivated by the questions about the stability of the focal index raised in [27]. However the main reason that lead us to the present formulation of the Morse index theorem in terms of the spectral flow is because of the relevance of this invariant to bifurcation of critical points of strongly indefinite functionals found in [11]. Our purpose is to combine the index theorem with the results in [11] in order to study bifurcation of perturbed geodesics on semi-Riemanian manifolds. This is done in [29].

A few words about our definition of conjugate index are in order. It is very close to other topological invariants that arise in bifurcation theory. To some extent it was suggested by the approach to Hopf bifurcation in [20] and that of potential operators in [22] where analogous invariants are treated in this way. The simplest topological invariants which detect gauge anomalies are also of this form [4]. Related ideas in the context of Sturm–Liouville boundary value problems can be found in [14].

Three beautiful papers [5], [18], [34] have strongly influenced the method of proof of Theorem 1.1. This proof has some interest in its own. It was found in trying to understand the relation between the Morse index and regularized determinants for families of boundary value problems discussed in [26]. Although we did not quite succeed in this, yet we believe that the proof of our theorem shed some light on that question. Even in the case of Riemannian manifolds it gives a new proof of the classical Morse Index Theorem. Previous proofs either used the variational characterization of eigenvalues of a self-adjoint operator (which cannot be used in the semi-Riemannian case) or the homotopy properties of

Lagrangian Grassmannian Λ_n. Here we substitute the later with the standard properties of topological degree and operators in the trace class.

Other variants of our proof can be easily conceived. A more topological but less elementary one should be along the following lines: the family of complex self-adjoint operators has an index bundle in $K(S^2) \equiv \mathbb{Z}$ whose first Chern class can be easily related to the spectral index as in [7]. On the other hand one can try to relate this class to the conjugate index by deforming the clutching function of this bundle to the map b_z in Section 4. This will make the Morse index theorem reminiscent of the Atiyah–Singer index theorem for families of *complex self-adjoint Fredholm operators*.

On the other hand, after Floer's work, results of type spectral flow equals Maslov index became increasingly popular. See for example [13], [34] and the references there. The paper of Nicolaescu [30] places them into the realm of index theorems for one parameter families of *real self-adjoint Fredholm operators*. Index theorems for variational problems with general Lagrangians and general boundary conditions, e.g. focal points for geodesics starting from a sub-manifold [37], [32], can be also cast in the above form and indeed can be easily deduced as particular cases of the main theorem in [34].

We consider here only perturbations of the geodesic energy functional by a potential because, as we mentioned above, this is the most general framework on which the generalized Morse index has an intrinsic, geometric meaning. For more general Lagrangians, as for example in [37], it still can be defined but it depends on the choice coordinates. On the contrary our theory is general enough to cover trajectories of a mechanical system while keeping the geometric content of the geodesic case.

The paper is structured as follows: in Section 2 we shortly review the spectral flow of one parameter families of quadratic functionals of Fredholm type. In Section 3 we give the variational formulation of our geometric problem and introduce the spectral index. In Section 4, using the orthonormal parallel trivialization along the p-geodesic, we define the conjugate index and compute it in terms of the associated Green function. Theorem 1.1 is proved in Section 5, while Section 6 is devoted to the relation with the Maslov index.

This is a second, slightly enlarged version, of the paper. The first version circulated in the form of a preprint from January 2003. We were unaware of the paper [37] at that time.

Finally, let us point out that the main idea in [32] (i.e. that on semi Riemannian manifolds the spectral flow detects among conjugate points those that are bifurcation points), the one in [15] (that at degenerate conjugate points this can be computed by means of partial signatures) and the main idea in this paper (that conjugate points can be also counted using topological degree of a Wronskian) belong to the same project and were all conceived in discussions connected with the Ph.D. thesis of A. Portaluri at Polytechnic University of Turin.

2. Spectral flow for paths of Fredholm quadratic forms

The informal description of the spectral flow of a path of self-adjoint operators given in the introduction can be made rigorous in many different ways. Beginning with [5], several different approaches to this invariant appeared in the literature. Here we will use the approach in [11]. We will need their construction in a slightly generalized form since our goal is to give an intrinsic, i.e. coordinate free, construction of the generalized Morse index.

The tangent space to the manifold of paths on a semi-Riemannian manifold has a natural Hilbertable structure but not a natural Riemannian metric on it. Therefore we have to work directly with paths of bounded quadratic (equivalently bilinear) forms arising as Hessians of the energy functional and not with self-adjoint operators representing them with respect to a given scalar product. Fortunately the spectral flow, being a topological invariant, depends only on the path of quadratic forms and not, as the name could misleadingly suggest, on the spectrum of the operators representing the form. Below we will give the very simple proof of this nontrivial fact. Then we will be able to define spectral flow of a family of Fredholm quadratic forms on a Hilbert bundle over the interval $[0,1]$ which is the object that intrinsically arises in our framework.

Let S,T be two invertible self-adjoint operators on a Hilbert space H such that $S-T$ is compact. Then the difference between spectral projections of S and T corresponding to a given spectral set is also compact. Denoting with $E_-(\,\cdot\,)$ and $E_+(\,\cdot\,)$ the negative and positive spectral subspace of an operator, it follows then that $E_-(S)\cap E_+(T)$ and $E_+(S)\cap E_-(T)$ have finite dimension [11], [3]. The *relative Morse index* of the pair (S,T) is defined by

$$\mu_{\mathrm{rel}}(S,T)=\dim\{E_-(S)\cap E_+(T)\}-\dim\{E_+(S)\cap E_-(T)\}\,.$$

It is easy to see that when the negative spectral subspaces of both operators are finite dimensional $\mu_{\mathrm{rel}}(S,T)$ is given by the difference $\mu_{\mathrm{Morse}}(S)-\mu_{\mathrm{Morse}}(T)$ between Morse indices.

A bounded self-adjoint operator A is Fredholm if $\ker A$ is finite dimensional. The topological group $GL(H)$ of all automorphisms of H acts naturally on the space of all self-adjoint Fredholm operators $\Phi_S(H)$ by *cogredience* sending $A\in\Phi_S(H)$ to S^*AS. This induces an action of paths in $GL(H)$ on paths in $\Phi_S(H)$. It was shown in [11, Theorem 2.1] that for any path $A\colon[a,b]\to\Phi_S(H)$ there exist a path $M\colon[a,b]\to GL(H)$, and a symmetry $\mathcal{J}(\mathcal{J}^2=\mathrm{id})$ such that $M^*(t)A(t)M(t)=\mathcal{J}+K(t)$ with $K(t)$ compact for each $t\in[a,b]$.

Let $A: [a, b] \to \Phi_S(H)$ be a path such that $A(a)$ and $A(b)$ are invertible operators.

Definition 2.1. The spectral flow of the path A is the integer

$$\mathrm{sf}(A, [a, b]) \equiv \mu_{\mathrm{rel}}(\mathcal{J} + K(a), \mathcal{J} + K(b)),$$

where $\mathcal{J} + K$ is any compact perturbation of a symmetry cogredient with A.

It follows from the properties of the relative Morse index that the left hand side is independent of $\mathcal{J}, M$ and that the above definition is nothing but a rigorous version of the heuristic description of the spectral flow given in the introduction [11].

The spectral flow $\mathrm{sf}(A, [a, b])$ is additive and invariant under homotopies with invertible end points. It is clearly preserved by cogredience. For paths that are compact perturbations of a fixed operator it coincides with the relative Morse index of its end points.

A Hilbertable structure on a Hilbert space H is the set of all scalar products on H equivalent to the given one. A *Fredholm quadratic form* is a function $q: H \to \mathbb{R}$ such that there exists a bounded symmetric bilinear form $b = b_q: H \times H \to R$ with $q(u) = b(u, u)$ and with $\ker b$ of finite dimension. Here $\ker b = \{u : b(u, v) = 0 \text{ for all } v\}$. The space $Q(H)$ of bounded quadratic forms is a Banach space with the norm defined by $\|q\| = \sup\{|q(u)| : \|u\| = 1\}$. The set $Q_F(H)$ of all Fredholm quadratic forms is an open subset of $Q(H)$ that is stable under perturbations by weakly continuous quadratic forms. A quadratic form is called *nondegenerate* if the map $u \to b_q(u, -)$ is an isomorphism between H and H^*. By Riesz representation theorem, for any choice of scalar product $\langle \cdot , \cdot \rangle$ in the Hilbertable structure, $Q_F(H)$ is isometrically isomorphic to $\Phi_S(H)$. Clearly this isometry sends the set of all non-degenerate quadratic forms onto $GL(H)$. From the Fredholm alternative applied to the representing operator it follows that a Fredholm quadratic form q is non-degenerate if and only if $\ker b_q = 0$.

A path of quadratic forms $q: [a, b] \to Q_F(H)$ with nondegenerate end points $q(a)$ and $q(b)$ will be called *admissible*.

Definition 2.2. The spectral flow of an admissible path $q: [a, b] \to Q_F(H)$ is given by

$$\mathrm{sf}(q, [a, b]) = \mathrm{sf}(A_q, [a, b])$$

where $A_{q(t)}$ is the unique self-adjoint operator such that $\langle A_{q(t)} u, u \rangle = q(t)(u)$ for all $u \in H$.

That this is independent from the choice of the scalar product in a given structure follows from the invariance of the spectral flow under cogredience. Indeed let $\langle \cdot , \cdot \rangle_1$ be a scalar product equivalent to $\langle \cdot , \cdot \rangle$ and let $A_{q(t)}, B_{q(t)}$ be such that $\langle A_{q(t)} u, u \rangle = \langle B_{q(t)} u, u \rangle_1 = q(t)(u)$ for all $u \in H$.

Denoting by H_1 the vector space H endowed with the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_1$, there exists a positive self-adjoint operator $D\colon H \to H_1$ such that $\langle u, v\rangle_1 = \langle Du, v\rangle$ for all $u, v \in H$. Therefore $D = \mathrm{id}^*$ where id is considered as a map from H_1 into H. Moreover, we have that $A_{q(t)} = DB_{q(t)}$. Finally, by invariance under cogredience we get

$$\mathrm{sf}(A_q, [a,b]) = \mathrm{sf}(DB_q, [a,b]) = \mathrm{sf}(\mathrm{id}^* B_q \mathrm{id}, [a,b]) = \mathrm{sf}(B_q, [a,b]),$$

which is what we wanted to show.

We list below some properties of the spectral flow of an admissible path of quadratic forms that we will use later. They need not be proved here since they follow easily from the representation formula in the Definition 2.2 and the analogous properties of the spectral flow for paths of self-adjoint Fredholm operators proven in [11].

- (Normalization) Let $q \in C([a,b]; Q_F(H))$ be such that $q(t)$ is non-degenerate for each $t \in [a,b]$. Then $\mathrm{sf}(q, [a,b]) = 0$.
- (Cogredience) Let $M \in C([a,b]; L(H_1, H))$ be a path of invertible operators between the Hilbert spaces H and H_1 and let p be the path of quadratic forms on H_1 defined by $p(t)(v) = q(t)[M(t)^{-1}v]$. Then

$$\mathrm{sf}(p, [a,b]) = \mathrm{sf}(q, [a,b]).$$

- (Homotopy invariance) Let $h \in C([0,1] \times [a,b]; Q_S(H))$ be such that $h(s,t)$ is non-degenerate for each $s \in [0,1]$ and $t = a, b$. Then

$$\mathrm{sf}(h(0,\,\cdot\,), [a,b]) = \mathrm{sf}(h(1,\,\cdot\,), [a,b]).$$

- (Additivity) Let $c \in (a,b)$ be a parameter value at which $q(c)$ is non-degenerate. Then

$$\mathrm{sf}(q, [a,b]) = \mathrm{sf}(q, [a,c]) + \mathrm{sf}(q, [c,b]).$$

We will also need a formula that leads to the calculation of the spectral flow for paths with only regular crossing points.

If a path $q\colon [a,b] \to Q_F(H)$ is differentiable at t then the derivative $\dot q(t)$ is also a quadratic form. We will say that a point t is a crossing point if $\ker b_{q(t)} \neq \{0\}$, and we will say that the crossing point t is regular if the *crossing form* $\Gamma(q,t)$, defined as the restriction of the derivative $\dot q(t)$ to the subspace $\ker b_{q(t)}$, is nondegenerate. It is easy to see that regular crossing points are isolated and that the property of having only regular crossing forms is generic for paths in $Q_F(H)$. From [11, Theorem 4.1] we obtain:

Proposition 2.3. *If all crossing points of the path are regular then they are finite in number and*

$$\operatorname{sf}(q,[a,b]) = \sum_i \operatorname{sign}\Gamma(q,t_i). \tag{2.1}$$

A *generalized family of Fredholm quadratic forms* parameterized by an interval is a smooth function $q\colon \mathcal{H} \to \mathbb{R}$, where $\mathcal{H}$ is a Hilbert bundle over $[a,b]$ and q is such that its restriction q_t to the fiber $\mathcal{H}_t$ over t is a Fredholm quadratic form. If q_a and q_b are non degenerate, we define the spectral flow $\operatorname{sf}(q) = \operatorname{sf}(q,[a,b])$ of such a family q by choosing a trivialization $M\colon [a,b]\times\mathcal{H}_a \to \mathcal{H}$ and defining

$$\operatorname{sf}(q) = \operatorname{sf}(\widetilde{q},[a,b]) \tag{2.2}$$

where $\widetilde{q}(t)u = q_t(M_t u)$.

It follows from cogredience property that the right hand side of (2.2) is independent of the choice of the trivialization. Moreover, all of the above properties hold true in this more general case, including the calculation of the spectral flow through a non degenerate crossing point given in Proposition 2.3 provided we substitute the usual derivative with the intrinsic derivative of a bundle map.

3. The spectral index

Given a smooth n-dimensional manifold M, let Ω be the manifold of all H^1-paths in M. Elements of $\Omega = H^1(I;M)$ are maps $\gamma\colon I \to M$ such that for any coordinate chart (U,ϕ) on M the composition $\phi\circ\gamma\colon \gamma^{-1}(U)\to\mathbb{R}^n$ belongs to $H^1(\gamma^{-1}(U);\mathbb{R}^n)$. It is well known that Ω is a smooth Hilbert manifold modelled by $H^1(I;\mathbb{R}^n)$. We will denote with $\tau\colon TM \to M$ the projection of the tangent bundle of M to its base, and by $H^1(\gamma)$ the Hilbert space $H^1(\gamma) = \{\xi \in H^1(I;TM) : \tau\circ\xi = \gamma\}$ of all H^1-vector fields along γ. The tangent space $T_\gamma\Omega$ at γ can be identified in a natural way with $H^1(\gamma)$. For all this the basic reference is [23].

By [23, Proposition 2.1], the map

$$\pi\colon \Omega \to M\times M, \quad \pi(\gamma) = (\gamma(0),\gamma(1)) \tag{3.1}$$

is a submersion and therefore for each $(p,q)\in M\times M$ the fiber of π

$$\Omega_{p,q} = \{\gamma\in\Omega : \gamma(0) = p,\ \gamma(1) = q\} \tag{3.2}$$

is a submanifold of codimension $2n$ whose tangent space $T_\gamma\Omega_{p,q} = \ker T_\gamma\pi$ is identified with the subspace $H^1_0(\gamma)$ of $H^1(\gamma)$ defined by

$$H^1_0(\gamma) = \{\xi\in H^1(\gamma) : \xi(0) = \xi(1) = 0\}. \tag{3.3}$$

Since π is a submersion, it follows that the family of Hilbert spaces $H_0^1(\gamma)$ form a Hilbert bundle $TF(\pi) = \ker T\pi$ over Ω, called the *bundle of tangents along the fibers of* π.

To each pair (g, V), where g is a semi-Riemannian metric on M and $V\colon I \times M \to \mathbb{R}$ is a smooth potential where $I = [0, 1]$, there is associated an *energy functional* $E\colon \Omega \to \mathbb{R}$ defined by

$$E(\gamma) = \int_0^1 \frac{1}{2} g(\gamma'(x), \gamma'(x))\, dx - \int_0^1 V(x, \gamma(x))\, dx. \tag{3.4}$$

It is well known that E is a smooth function and hence so are the restrictions $E_{p,q}$ of E to $\Omega_{p,q}$. We will be interested in the critical points of $E_{p,q}$. The differential of $E_{p,q}$ at a point $\gamma \in \Omega_{p,q}$ is given by the restriction of dE to $H_0^1(\gamma)$. It is easy to see that for $\xi \in H_0^1(\gamma)$

$$dE_{p,q}(\gamma)[\xi] = \int_0^1 g\left(\frac{D}{dx}\xi(x), \gamma'(x)\right) dx - \int_0^1 g(\nabla V(x, \gamma(x)), \xi(x))\, ds. \tag{3.5}$$

By standard regularity arguments one shows that if $dE_{p,q}(\gamma)[\xi] = 0$ for all ξ then γ is smooth and then performing integration by parts one obtains that the critical points of $E_{p,q}$ are precisely the smooth paths γ between p and q that verify the equation (1.1) of perturbed geodesics.

Let us recall that if N is a Hilbert manifold and n is a critical point of a smooth function $f\colon N \to \mathbb{R}$ then the *Hessian of* f at n is the quadratic form h_n on T_nN given by $h_n(v) = v(\chi(f))$, where χ is any vector field defined on a neighbourhood of n such that $\chi(n) = v$. Through the identification of $T_\gamma\Omega_{p,q}$ with $H_0^1(\gamma)$ a well know result in Calculus of Variations yields the Hessian of $E_{p,q}$ at γ. This is the quadratic form $h_\gamma\colon H_0^1(\gamma) \to R$ whose associated bilinear form $H_\gamma\colon H_0^1(\gamma) \times H_0^1(\gamma) \to R$ is given by

$$\begin{aligned} H_\gamma(\xi, \eta) = &\int_0^1 g\left(\frac{D}{dx}\xi(x), \frac{D}{dx}\eta(x)\right) dx \\ &- \int_0^1 g(R(\gamma'(x), \xi(x))\gamma'(x) + D_{\xi(x)}\nabla V(x, \gamma(x)), \eta(x))\, dx. \end{aligned}$$

Proposition 3.1. *The form h_γ is a Fredholm quadratic form. Moreover, h_γ is non degenerate if and only if* 1 *is not a conjugate instant.*

Proof. We begin by constructing a Riemannian metric related to g. Since g is a non-degenerate symmetric form, we can split TM as direct sum of T^+M and T^-M such that the restriction of $g_\pm$ to $T^\pm M$ is positive definite and negative definite respectively.

Let j be the endomorphism of $TM = T^+M \oplus T^-M$ given by $j(u^+ + u^-) = u^+ - u^-$. We define a new metric $\overline{g}$ by $\overline{g}(u, v) = g(ju, v)$. Then $\overline{g}$ is a Riemannian metric on M and j represents g with respect to $\overline{g}$.

The metric $\overline{g}$ induces a scalar product in $H_0^1(\gamma)$ given by

$$\langle \xi, \eta \rangle_{H_0^1} = \int_0^1 \overline{g}\left(\frac{D}{dx}\xi(x), \frac{D}{dx}\eta(x)\right) dx.$$

By the very definition of $\overline{g}$ we have

$$\int_0^1 g\left(\frac{D}{dx}\xi(x), \frac{D}{dx}\eta(x)\right) dx = \langle \mathcal{J}_\gamma \xi, \eta \rangle_{H_0^1} \tag{3.6}$$

where $\mathcal{J}_\gamma(\xi)(x) := j(\gamma(x))\xi(x)$ namely $\mathcal{J}_\gamma$ is pointwise j.

Clearly $\mathcal{J}_\gamma$ is bounded with $\mathcal{J}_\gamma^2 = I$, and hence the quadratic form

$$d_\gamma(\xi) = \int_0^1 g\left(\frac{D}{dx}\xi(x), \frac{D}{dx}\xi(x)\right) dx \tag{3.7}$$

is non degenerate being represented by $\mathcal{J}_\gamma \in GL(H_0^1(\gamma))$. On the other hand $h_\gamma = d_\gamma - c_\gamma$ where

$$c_\gamma(\xi) = \int_0^1 g(R(\gamma'(x), \xi(x))\gamma'(x) + D_{\xi(x)}\nabla V(x, \gamma(x)), \xi(x))\, dx. \tag{3.8}$$

The form c_γ is the restriction to $H_0^1(\gamma)$ of a quadratic form defined on the space $C_\gamma^0(TM)$ of all continuous vector fields over γ. Since the inclusion $H_0^1(\gamma) \hookrightarrow C_\gamma^0(TM)$ is a compact operator, it follows that c_γ is weakly continuous and therefore h_γ is Fredholm being a weakly continuous perturbation of a non degenerate form.

For the second assertion we notice that if $H_\gamma(\xi, \eta) = 0$ for all $\eta \in H_0^1(\gamma)$ then, again by regularity, ξ is smooth. Integrating by parts in (3.6), we obtain that ξ must verify the Jacobi equation (1.2) with Dirichlet boundary conditions. The converse is clear. Therefore $\ker h_\gamma = \{0\}$ if and only if the instant 1 is not conjugate to 0. □

Remark 3.2. That the quadratic form h_γ is Fredholm can be proved without introducing the metric $\overline{g}$. However it is of some interest to notice that the use of $\overline{g}$ combined with a parallel trivialization of the tangent bundle along γ produces a concrete construction of the abstract reduction of the path of hessians to a path of compact perturbation of a symmetry $\mathcal{J}$ used in our definition of the spectral flow in Section 2.

From now on let $p, q \in M$ be fixed points and let γ be a p-geodesic from p to q such that 1 is not a conjugate instant. Such a p-geodesic will be called *nondegenerate*. In order to define the spectral index of a nondegenerate p-geodesic γ we will consider the path induced by γ on Ω.

Namely, for each $t \in [0, 1]$ let $\gamma_t \in \Omega$ be the curve defined by $\gamma_t(x) = \gamma(t \cdot x)$. Since γ is a critical point of $E_{p,q}$ it follows from (1.1) that γ_t is a critical point of the functional $E_t\colon \Omega_{p,\gamma(t)} \to \mathbb{R}$ defined by

$$E_t(\gamma) = \int_0^1 \frac{1}{2} g(\gamma'(x), \gamma'(x))\, dx - \int_0^1 t^2 V(tx, \gamma(x))\, dx. \tag{3.9}$$

In other words, each γ_t is a p-geodesic for the potential $V_t(x,m) = t^2 V(tx, m)$.

Let $h_t\colon H_0^1(\gamma_t) \to \mathbb{R}$ be the Hessian of E_t at the critical point γ_t. By Proposition 3.1, h_t is degenerate if and only if 1 is a conjugate instant for γ_t. In particular $h_1 = h_\gamma$ is non degenerate. Moreover, h_0 is nondegenerate as well. Indeed, $\gamma_0 \equiv p$ is a constant path which is a critical point of E_0. An H^1-vector field ξ along p is simply a path $\xi \in H^1(I; T_p(M))$ and hence

$$h_0(\xi) = d_p(\xi) = \int_0^1 g\left(\frac{D}{dx}\xi(x), \frac{D}{dx}\xi(x)\right) dx$$

which is nondegenerate by the previous discussion.

Let us consider the canonical path of p-geodesics $\widetilde{\gamma}\colon [0,1] \to \Omega$ defined by $\widetilde{\gamma}(t) = \gamma_t$. Clearly, the family of Hessians h_t, $0 \le t \le 1$, defines a smooth function h on the total space of the Hilbert bundle $\mathcal{H} = \widetilde{\gamma}^* TF(\pi)$ over $[0,1]$, that is a Fredholm quadratic form at each fiber and non degenerate at 0 and at 1. The spectral flow $\mathrm{sf}(h)$ of such a family is well defined by (2.2) of the previous section.

Definition 3.3. The *generalized Morse index* $\mu_{\text{spec}}(\gamma)$ of a p-geodesic γ is the integer

$$\mu_{\text{spec}}(\gamma) = -\mathrm{sf}(h). \tag{3.10}$$

For pertubed geodesics on Riemannian manifolds the following holds.

Proposition 3.4. *If the metric g is Riemannian then the Morse index $\mu_{\text{Morse}}(\gamma)$ (i.e. the dimension of the maximal negative subspace of the Hessian of h_γ) is finite and*

$$\mu_{\text{spec}}(\gamma) = \mu_{\text{Morse}}(\gamma). \tag{3.11}$$

Proof. The first assertion is well known. It follows from the fact that in the Riemannian case each h_γ is a weakly continuous perturbation of a positive definite form d_γ. The dimension of the maximal negative subspace of this form coincides with the dimension of the negative spectral space of any self-adjont operator representing the form. But this subspace is finite dimensional because the operator is essentially positive, i.e. compact perturbation of a positive one.

In order to prove the second assertion we observe that, with respect to the scalar product defined by formula (3.6) (with $\mathcal{J} = \mathrm{id}$) on $H_0^1(\gamma_t)$, the form h_t is represented by an essentially positive operator of the form $\mathrm{id} - C_t$ with C_t compact self-adjoint. By [11, Proposition 3.9] the spectral flow of a family of essentially positive operators is the difference between the Morse indices at the end points. Applying this to any trivialization of $\mathcal{H}$ we have that

$$\mu_{\mathrm{spec}}(\gamma) = -\mu_{\mathrm{Morse}}(\gamma_0) + \mu_{\mathrm{Morse}}(\gamma_1) = \mu_{\mathrm{Morse}}(\gamma). \qquad \square$$

4. The conjugate index

In this section we introduce a topological invariant that counts the algebraic number of conjugate points along a nondegenerate p-geodesic γ and which coincides with the expression (1.3) in the case of a regular p-geodesic.

Given a perturbed geodesic γ we will use a particular trivialization of $\gamma^*(TM)$ by choosing a g-frame $\mathcal{E}$ along γ made by n parallel vector fields $\{e^1, \dots, e^n\}$. Here a g-frame means that the vector fields e^i are point-wise g-orthogonal and moreover $g(e^i(x), e^i(x)) = \varepsilon_i$, where $\varepsilon_i = 1$ for $i = 1, \dots, n-\nu$, and $\varepsilon_i = -1$ if $i \geq n - \nu + 1$, for all $x \in [0,1]$. Such a frame induces a trivialization

$$M_{\mathcal{E}}\colon I \times \mathbb{R}^n \to \gamma^*(TM) \tag{4.1}$$

of $\gamma^*(TM)$ defined by $M_{\mathcal{E}}(x, u_1 \dots u_n) = \sum_{i=1}^n u_i e^i(x)$.

Writing the vector field ξ along γ as $\xi(x) = \sum_{i=1}^n u_i(x) e^i(x)$, inserting the above expression in the equation (1.2) of Jacobi fields and taking g product with e^j, we reduce the Jacobi equation (1.2) to a linear second order system of ordinary differential equations

$$\varepsilon_i u_i''(x) + \sum_{j=1}^n S_{ij}(x) u_j(x) = 0, \quad 1 \leq i \leq n,$$

where $S_{ij} = g(R(\gamma', e^i)\gamma' + D_{e^i} \nabla V(\,\cdot\,, \gamma), e^j)$.

Putting $u(x) = (u_1(x), \dots, u_n(x))$ and $S(x) = (S_{ij}(x))$, the above system becomes

$$Ju''(x) + S(x)u(x) = 0 \tag{4.2}$$

where J is the symmetry

$$J = \begin{pmatrix} \mathrm{id}_{n-\nu} & 0 \\ 0 & -\mathrm{id}_\nu \end{pmatrix}. \tag{4.3}$$

Under the trivialization $M_{\mathcal{E}}$, the metric g on $\gamma^*(TM)$ goes into indefinite product of index ν on $\mathbb{R}^n$ given by $\langle u, v\rangle_\nu = \langle Ju, v\rangle$ where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the Euclidean scalar product on $\mathbb{R}^n$. Since both $R(\gamma', -)\gamma'$ and the Hessian $D_{-}\nabla V(\cdot, \gamma)$ are g-symmetric endomorphisms of $\gamma^*(TM)$ it follows that the matrix $S(x)$ is symmetric.

Now let us apply the same argument to each p-geodesic γ_t introduced in the previous section, using the induced parallel g-frame $\mathcal{E}_t = \{e_t^1, \dots, e_t^n\}$ with $e_t^i(x) = e^i(t \cdot x)$. The corresponding trivialization $M_{\mathcal{E}_t}$ transforms the equation for Jacobi fields on γ_t into a one parameter family of second order systems of ordinary differential equations

$$Ju''(x) + S_t(x)u(x) = 0 \tag{4.4}$$

where $S(t,x) = S_t(x)$ is smooth in $[0,1] \times [0,1]$ and $S_t^* = S_t$.

By definition of S_{ij} and γ_t it follows that

$$\begin{aligned}(S_t)_{ij}(x) &= g(R(\gamma_t'(x), e_t^i(x))\gamma_t'(x) + D_{e_t^i(x)}\nabla V(x, \gamma_t(x)), e_t^j(x)) \\ &= t^2[g(R(\gamma'(t\cdot x), e^i(t\cdot x))\gamma(t\cdot x) \\ &\quad + D_{e^i(t\cdot x)}\nabla V(t\cdot x, \gamma(t\cdot x)), e^j(t\cdot x))] = t^2 S_{ij}(t\cdot x).\end{aligned}$$

Hence we have

$$S_t(x) = t^2 S(t\cdot x). \tag{4.5}$$

The trivialization $M_{\mathcal{E}_t}$ induces a one to one correspondence between solutions $u = (u_1, \dots, u_n)$ of the Dirichlet problem

$$Ju''(x) + S_t(x)u(x) = 0, \quad u(0) = 0 = u(1) \tag{4.6}$$

and Jacobi fields over γ_t vanishing at $0,1$. On the other hand $\xi \to \xi_t$ is a bijection between Jacobi fields over γ vanishing at 0 and t with the Jacobi fields over γ_t vanishing at $0,1$. It follows then that $t \in (0,1]$ is a conjugate instant for γ if and only if the boundary value problem (4.6) has a nontrivial solution.

We now will take into account the complexified problem (4.6) by considering the operator $Ju'' + S_t(x)u$ acting on complex valued vector functions $u\colon I \to \mathbb{C}^n$.

Let $\mathcal{O}$ be the bounded domain on the complex plane defined by

$$\mathcal{O} = \{z = t + is \in \mathbb{C} : 0 < t < 1,\ -1 < s < 1\}. \tag{4.7}$$

For any $z = t + is \in \overline{\mathcal{O}}$ let us consider the closed unbounded operator

$$\mathcal{A}_z\colon \mathcal{D}(\mathcal{A}_z) \subset L^2(I;\mathbb{C}^n) \to L^2(I;\mathbb{C}^n)$$

with domain $\mathcal{D}(\mathcal{A}_z) = H^2(I;\mathbb{C}^n) \cap H_0^1(I;\mathbb{C}^n)$, defined by

$$\mathcal{A}_z(u)(x) = Ju''(x) + S_z(x)u(x) \tag{4.8}$$

where $S_z(x) = S_t(x) + is\,\mathrm{id}$.

The two parameter family $\mathcal{A}_z$ of unbounded self-adjoint Fredholm operators of $H = L^2(I;\mathbb{C}^n)$ is a perturbation of a fixed unbounded operator Ju'' by smooth family $\mathcal{S}\colon \overline{\mathcal{O}} \to \mathcal{L}(H)$ of bounded operators $\mathcal{S}_z$ defined by the pointwise multiplication by the matrix S_z.

Let us consider now the associated family of first order Hamiltonian systems. Putting $v = Ju'$, the equation $Ju''(x) + S_z u(x) = 0$ becomes equivalent to

$$\begin{cases} u'(x) = Jv(x), \\ v'(x) = -S_z(x)u(x), \end{cases} \quad \text{for all } x \in [0,1].$$

Taking $w = (u,v) \in \mathbb{C}^{2n}$, the above system can be rewritten as the complex Hamiltonian system

$$w'(x) = \sigma H_z(x) w(x) \tag{4.9}$$

where

$$\sigma = \begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \tag{4.10}$$

is the complex symplectic matrix, while $H_z(x)$ is the matrix defined by

$$H_z(x) = \begin{pmatrix} -S_z(x) & 0 \\ 0 & -J \end{pmatrix}. \tag{4.11}$$

Let $\Psi_z(x)$ be the fundamental solution of (4.9). The matrix $\Psi_z(x)$ is the unique solution of the Cauchy problem

$$\begin{cases} \Psi_z'(x) = \sigma H_z(x)\Psi_z(x) & \text{for } x \in [0,1], \\ \Psi_z(0) = \mathrm{id}. \end{cases} \tag{4.12}$$

Consider the block decomposition of $\Psi_z(x)$

$$\Psi_z(x) = \begin{pmatrix} a_z(x) & b_z(x) \\ c_z(x) & d_z(x) \end{pmatrix}$$

and let $b_z = b_z(1)$ be the upper right entry in the block decomposition of $\Psi_z(1)$.

The matrix b_z can be also described directly in terms of the solutions of (4.4) as follows: for each i, $1 \le i \le n$, let u_i be a solution of the initial value problem

$$Ju''(x) + S_z u(x) = 0, \quad u_i(0) = 0, \quad u_i'(0) = e_i,$$

where $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{C}^n$, then $b_z = (u_1(1), \ldots, u_n(1))^T$.

Our definition of the conjugate index is based on the following elementary observation (compare [28, Lemma 1.5]).

Lemma 4.1. *The following three statements are equivalent:*

(a) $\ker \mathcal{A}_z \neq \{0\}$,
(b) $\mathrm{Im}(z) = 0$ *and* $t = Re(z)$ *is a conjugate instant,*
(c) $\det b_z = 0$.

Proof. Since the spectrum of self-adjoint operator is real, $\ker \mathcal{A}_z \neq 0$ can occur only at $z = t + is$ with $s = 0$. But functions belonging to the kernel of $\mathcal{A}_t$ are precisely the solutions of the boundary value problem (4.6). Therefore t must be a conjugate instant. The converse is clear. Hence the equivalence between (a) and (b) is proved.

Now let $u \in \ker \mathcal{A}_z$. From the block decomposition of $w(1) = \Psi_z(1)w(0)$ and the boundary conditions $u(0) = 0 = u(1)$ we get $b_z u'(0) = 0$. Thus, if $\det b_z \neq 0$ we have $u'(0) = 0$ and hence $u \equiv 0$, which yields $\ker \mathcal{A}_z = \{0\}$. On the other hand, if $\det b_z = 0$, taking $0 \neq v_0 \in \ker b_z$, $w_0 = (0, v_0)$ and $u(x)$ equal to the first component of $\Psi_z(x)w_0$, we have $u(x) \not\equiv 0$ and $u \in \ker \mathcal{A}_z$. This proves the equivalence between (b) and (c). □

Consider now $\rho\colon \overline{\mathcal{O}} \to \mathbb{C}$ given by $\rho(z) = \det b_z$. Since $\ker \mathcal{A}_0 = \ker \mathcal{A}_1 = \{0\}$ it follows that $0 \notin \rho(\partial\mathcal{O})$. Under this conditions the Brouwer degree $\deg(\rho, \mathcal{O}, 0)$ of the map ρ in $\mathcal{O}$ with respect to 0 is defined [9]. Brouwer's degree is a topological invariant that counts with multiplicities the number of zeroes of ρ in $\mathcal{O}$ and since the zeroes of ρ correspond to conjugate instants of the p-geodesic we make the following:

Definition 4.2. The conjugate index $\mu_{\text{con}}(\gamma)$ of a nondegenerate p-geodesic γ is the integer

$$\mu_{\text{con}}(\gamma) = -\deg(\rho, \mathcal{O}, 0). \tag{4.13}$$

That the right hand side is independent from the choice of $\mathcal{O}$ follows from the excision property of degree.

The algebraic multiplicity of an isolated, but not necessarily regular, conjugate instant t_0 can now be defined by

$$\mu_{\text{con}}(\gamma, t_0) = -\deg(\rho, \mathcal{O}', 0), \tag{4.14}$$

where $\mathcal{O}'$ is any open neighbourhood of the point $z_0 = (t_0, 0)$ in $\mathcal{O}$ containing no other zeroes of the map ρ.

If all conjugate instants are isolated then $\mu_{\text{con}}(\gamma)$ is the sum of the algebraic multiplicities of the conjugate instants.

In Section 6 we will show that if t_0 is a regular conjugate instant, then $\mu_{\text{con}}(\gamma, t_0)$ coincides with $\operatorname{sign}\big(g|_{\mathcal{I}[t_0]^\perp}\big)$.

If all data are analytic, the multiplicities of the conjugate points can be computed using an algorithm for computation of degree of a polynomial plane vector vector-field whose origins can be traced back to Kroenecker and Ostrogradski. Below we shortly sketch a calculation of this type in a special case.

Assuming M, g and V analytic, by uniqueness of solutions of ordinary differential equations we have that the restriction of ρ to the real axis is a real analytic function which does not vanish identically and hence has only isolated zeroes. Therefore ρ has a finite number of zeroes of the form $z_i = (t_i, 0)$ with $t_1 < \ldots < t_k$. Taking isolating neighbourhoods $\mathcal{O}_i$ as before we get

$$\mu_{\text{con}}(\gamma) = \sum_{i=1}^{k} \mu_{\text{con}}(\gamma, t_i) = -\sum_{i=1}^{k} \deg(\rho, \mathcal{O}_i, 0).$$

Fix a conjugate instant t_j and let $z_j = (t_j, 0)$. With our assumptions ρ is a real analytic map from $\mathbb{C} \cong \mathbb{R}^2$ into itself. Let P and Q be the non vanishing homogeneous polynomials of lowest degree (respectively, m and n) that arise in the (real) Taylor series of the map ρ at the point z_j.

Then we have

$$\rho(z) = \rho(t, s) = (P(t - t_j, s) + f(t, s), Q(t - t_j, s) + g(t, s)) \tag{4.15}$$

with $|f(z)| = O(|z|^{m+1})$ and $|g(z)| = O(|z|^{n+1})$ for z ranging on a bounded set.

We will present the computation of $\mu_{\text{con}}(\gamma, t_j)$ upon the extra assumption:

(H_1) The point z_j is an isolated zero of the homogeneous map $\eta = (P, Q)$.

The general algorithm for the degenerate case can be found in [24, Chapter I, 15] and [25, Appendix].

We first show that the local multiplicity of ρ and η at z_j are the same using a well known argument. Without loss of generality we can assume $z_j = 0$. By homogeneity of P and Q and compactness of the unit circle we can find two positive constants A and B such that for each $z \neq 0$ either $A|z|^m < P(z)$ and $|f(z)| < B(|z|^{m+1})$ or $A|z|^n < Q(z)$ and $|g(z)| < B(|z|^{n+1})$. It follows from this inequalities that the homotopy $h(\lambda, z) = (P(z) + \lambda f(z), Q(z) + \lambda g(z))$, $0 \leq \lambda \leq 1$ does not vanishes on the boundary of the circle with center at 0 and radius $R = A/B$. Therefore $\mu_{\text{con}}(\gamma, t_j) = -\deg(\rho, \mathcal{O}_i, 0) = -\deg(\eta, B(0, R), 0)$. The right hand side can be computed directly from the coefficients of P and Q as follows: we first observe that 0 is an isolated zero for η if and only if $N_0(s) = P(1, s)$ and $N_1(s) = Q(1, s)$ do not have common real root and $|P(0, 1)| + |Q(0, 1)| > 0$.

Assuming $m \geq n$, let us consider polynomials $N_0, \ldots, N_l$, defined inductively by the Euclidean algorithm, namely $N_{i+1}(s)$ is the rest of the division of $N_{i-1}(s)$ by $N_i(s)$. The final polynomial $N_l(r)$ is the greatest common divisor of the polynomials N_0 and N_1.

Choose r in $\mathbb{R}$ such that r is not root of any of the polynomials N_i. We denote by $m(r)$ the number of sign changes of the corresponding values $N_i(r)$. The number $m(r)$ becomes constant for r sufficiently large. We denote with m_+ this value. Analogously, we denote by m_- the common value of $m(r)$ for r negative and of sufficiently large absolute value. From the above discussion and [24, Theorem 10.2] we have the following formula for the conjugate index at a degenerate point.

Lemma 4.3. *If M, V are analytic and if* (H_1) *holds at a conjugate point t_j of γ, then*

$$\mu_{\mathrm{con}}(\gamma, t_j) = -\left(1 + (-1)^{m+n}\right) \frac{m_+ - m_-}{2}.$$

In [27] the authors defined the conjugate index using the formula (1.3) irrespectively whether the geodesic is regular or not. They constructed an example of a geodesic with an isolated conjugate point and such that the associated Maslov index does not coincide with the equation (1.3). The correct multiplicity at such a point is not given by $\operatorname{sign} g|_{\mathcal{I}[t]^\perp}$. It can be computed either using the algorithm for calculation of the degree of an analytic vector field in our approach or a well known formula for the Maslov index in terms of partial signatures in the approach chosen in [15]. Let us remark in this respect that the multiplicity of a conjugate point, as defined in (4.14), coincides with the analogous multiplicity defined in terms of the partial signatures in [15]. This follows immediately from Proposition 6.1. However we were unable to find any direct relation between the invariants $m_\pm$ arising in the above lemma with the partial signatures at the given point.

We close this section with a proposition which provides a way to compute the number defined by the formula (4.13) in terms of the trace of the Green kernel of (4.8). This is essentially a known result [26], [17]. We include the proof here for the sake of completeness.

Let us recall first that a compact operator K is said to be of *trace class* if the series of the square roots of eigenvalues of K^*K is convergent. The trace class $\mathcal{T}$ is a bilateral ideal contained in the ideal of all compact operators $\mathcal{K}$. There is a well defined linear functional Tr on $\mathcal{T}$ which has the usual properties of the trace. In particular if both AB and BA are in $\mathcal{T}$ then $\operatorname{Tr} AB = \operatorname{Tr} BA$.

Our calculations below can be better formalized using operator-valued differential one-forms. By a slight abuse of notation we will denote by $d\mathcal{A}_z$ the differential of the bounded part $\mathcal{S}_z$ of the family $\mathcal{A}_z$, and consequently we will denote by $d\mathcal{A}_z\,\mathcal{A}_z^{-1}$ the operator valued one-form given by

$$d\mathcal{A}_z\,\mathcal{A}_z^{-1} = \partial_t \mathcal{S}_z \mathcal{A}_z^{-1}\,dt + i\partial_s \mathcal{S}_z \mathcal{A}_z^{-1}\,ds = \partial_t \mathcal{S}_t \mathcal{A}_z^{-1}\,dt + i\mathcal{A}_z^{-1}\,ds \tag{4.16}$$

defined on the set $\{z \in \overline{\mathcal{O}} : \mathcal{A}_z \text{ has a bounded inverse}\}$.

This notation incorporates the action on the left (resp. right) of an operator valued function on a one-form in the natural way, by multiplying on the left (or right) the coefficients of the form. In the same vein, the trace $\operatorname{Tr}\theta$ of an operator valued one form $\theta = E\,dt + F\,ds$ is the complex valued one form $\operatorname{Tr} E\,dt + \operatorname{Tr} F\,ds$.

Proposition 4.4. *The form $d\mathcal{A}_z\,\mathcal{A}_z^{-1}$ is a trace class valued form and*

$$\mu_{\mathrm{con}}(\gamma) = -\frac{1}{2\pi i}\int_{\partial\mathcal{O}} \operatorname{Tr} d\mathcal{A}_z\,\mathcal{A}_z^{-1}. \tag{4.17}$$

Proof. By Lemma 4.1, $\mathcal{A}_z$ has a bounded inverse $\mathcal{A}_z^{-1}$ for all $z \in \partial\mathcal{O}$. Moreover, the operator $\mathcal{A}_z^{-1}$ is an integral operator of the form

$$\mathcal{A}_z^{-1}(u)(x) = \int_0^1 K_z(x,y)\, u(y)\, dy \tag{4.18}$$

with the Green kernel $K_z(x,y)$ given by

$$K_z(x,y) = C\widetilde{K}_z(x,y)D^*, \tag{4.19}$$

where $C = (I,0)$, $D = (0,I)$ and $\widetilde{K}_z(x,y)$ is the $2n \times 2n$ matrix defined by

$$\widetilde{K}_z(x,y) = \begin{cases} -\Psi_z(x)P_z\Psi_z(y)^{-1} & 0 \le x < y \le 1, \\ \Psi_z(x)(I-P_z)\Psi_z(y)^{-1} & 0 \le y < x \le 1, \end{cases} \tag{4.20}$$

with

$$P_z = \begin{pmatrix} 0 & 0 \\ b_z^{-1} & 0 \end{pmatrix} \Psi_z(1) \tag{4.21}$$

(see [17, Chapter XIV, Theorem 3.1]).

Since the kernel in (4.19) is of class $C^{0,1}$, it follows from a well known theorem of Fredholm that the operator $\mathcal{A}_z^{-1}$ is of trace class and therefore form $d\mathcal{A}_z\mathcal{A}_z^{-1}$ is $\mathcal{T}$-valued.

In order to prove the formula (4.17) we will first calculate $\operatorname{Tr} d\mathcal{A}_z\,\mathcal{A}_z^{-1}$ using the fact that the trace of an integral operator belonging to the trace class can be computed integrating the trace of its kernel [17]. For $z \in \partial\mathcal{O}$ from (4.10) and (4.11) we get

$$\operatorname{Tr} d\mathcal{A}_z\mathcal{A}_z^{-1} = \int_0^1 \operatorname{Tr}\,[dS_z(x)\, K_z(x,x)]\, dx = -\int_0^1 \operatorname{Tr}\,[\sigma dH_z(x)\widetilde{K}_z(x,x)]\, dx.$$

On the other hand,

$$\begin{aligned} -\operatorname{Tr}\,[\sigma\, dH_z(x)\widetilde{K}_z(x,x)] &= \operatorname{Tr}\,[\sigma dH_z(x)\, \Psi_z(x)P_z\Psi_z^{-1}(x)] \\ &= \operatorname{Tr}\,[d\Psi_z'(x)P_z\Psi_z^{-1}(x) - \sigma H_z(x)\, d\Psi_z(x)P_z\Psi_z^{-1}(x)] \\ &= \operatorname{Tr}\,[d\Psi_z'(x)P_z\Psi_z^{-1}(x) - \Psi_z'(x)\Psi_z^{-1}(x)\, d\Psi_z(x)P_z\Psi_z^{-1}(x)] \\ &= \frac{d}{dx}\operatorname{Tr}\,[d\Psi_z(x)P_z\Psi_z^{-1}(x)]. \end{aligned}$$

From this, integrating in x, a direct computation yields

$$\operatorname{Tr} d\mathcal{A}_z\mathcal{A}_z^{-1} = \operatorname{Tr}\,[d\Psi_z(1)P_z\Psi_z^{-1}(1)] = \operatorname{Tr} db_z b_z^{-1} = d\,\log\rho(z),$$

since $\operatorname{Tr} db_z b_z^{-1}$ coincides with the logarithmic differential of $\det b_z$.

Integrating over $\partial\mathcal{O}$ we finally obtain

$$\frac{1}{2\pi i}\int_{\partial\mathcal{O}} \operatorname{Tr} d\mathcal{A}_z\,\mathcal{A}_z^{-1} = \frac{1}{2\pi i}\int_{\partial\mathcal{O}} d\,\log\,\rho(z)\,,$$

which is precisely the degree $\deg(\rho, \mathcal{O}, 0)$ by the well known formula relating the degree of a map on the open set $\mathcal{O}$ with the winding number of its restriction to the boundary (see [9, Chapter 1, Section 6.6]). $\square$

5. Proof of Theorem 1.1

Let $\widetilde{\gamma}\colon [0,1] \to \Omega$, $\widetilde{\gamma}(t) = \gamma_t$ be the canonical path defined in Section 3 and let $h\colon \mathcal{H} \to \mathbb{R}$ be the generalized family of quadratic forms whose restriction h_t to the fiber $H^1_0(\gamma_t)$ of the Hilbert bundle $\mathcal{H} = \widetilde{\gamma}^* TF(\pi)$ is the Hessian of E_t at γ_t. The parallel trivialization $M_{\mathcal{E}_t}$ of $\gamma_t^* TM$ defined by formula (4.1) induces a trivialization of $\mathcal{H}$, under which h is transformed into the family of Fredholm quadratic forms $\widehat{h}_t$ on $H^1_0([0,1];\mathbb{R}^n)$ given by $\widehat{h}_t(u) = h_t(\sum_{i=1}^n u_i e_t^i)$. Using the computation of the previous section we obtain

$$\widehat{h}_t(u) = \int_0^1 \langle Ju'(x), u'(x)\rangle\, dx - \int_0^1 \langle S_t(x)u(x), u(x)\rangle\, dx \tag{5.1}$$

where J is given by (4.3) and S_t is the smooth family of symmetric matrices introduced in (4.5).

By definition of the generalized Morse index of γ, we have

$$\mu_{\text{spec}}(\gamma) = -\mathrm{sf}(h) = -\mathrm{sf}(\widehat{h}, [0,1]). \tag{5.2}$$

We will reduce the calculation of $\mathrm{sf}(\widehat{h}, [0,1])$ to that of a path having only regular crossing. In order to obtain this we will apply a perturbation result of Robbin and Salamon in [34] to the path of operators $\widetilde{\mathcal{A}} = \{\widetilde{\mathcal{A}}_t\}_{\{t\in[0,1]\}}$ where $\widetilde{\mathcal{A}}_t\colon \mathcal{D}(\widetilde{\mathcal{A}}_t) \subset L^2(I;\mathbb{R}^n) \to L^2(I;\mathbb{R}^n)$ is the closed, real self-adjoint operator defined on $\mathcal{D}(\widetilde{\mathcal{A}}_t) = H^2(I;\mathbb{R}^n) \cap H^1_0(I;\mathbb{R}^n)$ by

$$\widetilde{\mathcal{A}}_t(u)(x) = Ju''(x) + S_t(x)u(x). \tag{5.3}$$

Notice that the restriction $\{\mathcal{A}_t : t \in [0,1]\}$ of the previously defined family $\{\mathcal{A}_z\}$ given by formula (4.8) to the real axis is nothing but the complexification of the path $\widetilde{\mathcal{A}}$.

Since $\widetilde{\mathcal{A}}_t = JD^2 + \mathcal{S}_t$ is a compact differentiable perturbation of JD^2, it verifies all the assumptions in [34, Theorem 4.22] and hence there exist a $\delta > 0$ arbitrarily close to zero such the path $\widetilde{\mathcal{A}}^\delta_t = \widetilde{\mathcal{A}}_t + \delta I$ has only regular crossing points. Regular crossing for paths of unbounded operators has the same meaning as for paths of quadratic forms. Namely, t_0 is a regular crossing point if the crossing form $\Gamma(\widetilde{\mathcal{A}}^\delta, t_0)$, defined as the restriction of the quadratic form

$$\langle \dot{\mathcal{S}}_t u, u\rangle_{L^2} = \int_0^1 \langle \dot{S}_t(x)u(x), u(x)\rangle\, dx \tag{5.4}$$

to $\ker \widetilde{\mathcal{A}}^\delta_{t_0}$, is non degenerate. Regular crossing points are isolated and thus $\operatorname{Ker} \widetilde{\mathcal{A}}^\delta_t \neq \{0\}$ only at a finite number of points $0 < t_1 < \ldots < t_k < 1$.

Let $\widehat{h}_t^\delta(u) = \widehat{h}_t(u) + \delta||u||_{L^2}/2$. From equation (5.1), using once more integration by parts, we obtain that $\operatorname{Ker}\widehat{h}_t^\delta = \operatorname{Ker}\widetilde{\mathcal{A}}_t^\delta$. Moreover, by the very definition of the crossing form, at each crossing point t_j the forms $\Gamma(\widehat{h}^\delta, t_j)$ and $\Gamma(\widetilde{\mathcal{A}}^\delta, t_j)$ coincide. Taking δ small enough, using the homotopy invariance of spectral flow and Proposition 2.3 we obtain

$$\mu_{\mathrm{spec}}(\gamma) = -\mathrm{sf}(\widehat{h}^\delta, [0,1]) = -\sum_{j=1}^k \operatorname{sign}\Gamma(\widehat{h}^\delta, t_j) = -\sum_{j=1}^k \operatorname{sign}\Gamma(\widetilde{\mathcal{A}}^\delta, t_j).$$

On the other hand, the complexified path $\{\mathcal{A}_t^\delta : t \in [0,1]\}$ has the same crossing points as $\widetilde{\mathcal{A}}^\delta$ because $\operatorname{Ker}\mathcal{A}_t^\delta$ is the complexification of $\operatorname{Ker}\widetilde{\mathcal{A}}_t^\delta$ and $\operatorname{sign}\Gamma(\mathcal{A}^\delta, t_j) = \operatorname{sign}\Gamma(\widetilde{\mathcal{A}}^\delta, t_j)$ because a real symmetric matrix and its complexification have the same eigenvalues. This together with the previous calculation gives

$$\mu_{\mathrm{spec}}(\gamma) = -\sum_{j=1}^k \operatorname{sign}\Gamma(\mathcal{A}^\delta, t_j). \tag{5.5}$$

Now let us compute the perturbation on the other side of the equality in Theorem 1.1. Let $b_z^\delta(x)$ be the upper right entry in the block decomposition of the fundamental matrix $\Psi_z^\delta(x)$ associated to the perturbed family $\mathcal{A}_z^\delta$ and let $\rho^\delta(z) = \det b_z^\delta(1)$. By the continuous dependence of the flow with respect to the initial condition, we can take δ small enough so that $|\rho^\delta(z) - \rho(z)| < \inf_{z\in\partial\mathcal{O}} |\rho(z)|$. Then $h(t,z) = t\rho^\delta(z) + (1-t)\rho(z)$ is an admissible homotopy and therefore by the homotopy invariance of degree

$$-\mu_{\mathrm{con}}(\gamma) = \deg(\rho, \mathcal{O}, 0) = \deg(\rho^\delta, \mathcal{O}, 0). \tag{5.6}$$

Taking closed disjoint neighbourhoods $\mathcal{D}_j$ of $(t_j, 0)$ in $\mathcal{O}$ with piecewise smooth boundary from the additivity of degree and from Lemma 4.1 we obtain

$$-\mu_{\mathrm{con}}(\gamma) = \sum_{j=1}^k \deg(\rho^\delta, \mathcal{D}_j, 0) = \sum_{j=1}^k \frac{1}{2\pi i}\int_{\partial\mathcal{D}_j} \operatorname{Tr} d\mathcal{A}_z^\delta(\mathcal{A}_z^\delta)^{-1}. \tag{5.7}$$

Comparing (5.5) with (5.7), we see that the proof of the Theorem 1.1 will be complete if we prove the identity

$$\frac{1}{2\pi i}\int_{\partial\mathcal{D}_j} \operatorname{Tr} d\mathcal{A}_z^\delta\,(\mathcal{A}_z^\delta)^{-1} = \operatorname{sign}\Gamma(\mathcal{A}^\delta, t_j) \quad \text{for any} \quad j = 1, \dots, k. \tag{5.8}$$

The rest of the section will be devoted to the proof of (5.8). On the basis of the previous discussion we can assume that $\mathcal{A}_t$ has only regular crossing points and drop δ from our notations.

The idea of the proof is as follows: in a standard way one can find a path P_t of finite rank projectors defined on a neighbourhood of t_j such that P_t reduces $\mathcal{A}_t$ and hence also $\mathcal{A}_z$ with $\Re z = t$. By a well-known theorem, after a t-dependent unitary change of coordinates, the family can be locally reduced by a single projector. Using this reduction equation (5.8) follows from the corresponding statement in finite dimensions where it can be shown to be true by an elementary calculation. This works well for bounded operators. However in our setting a problem arises. Under change of coordinates the transformed family is not any more a bounded perturbation of a fixed operator and there is a problem with the definition of the one form in (5.8). We will avoid all the technicalities related to the differentiability of general families of unbounded operators by rewriting all the forms in terms of bounded operators. For this we will deal simultaneously with $\mathcal{A}_z$ as closed operators and also as bounded operators with respect to the graph norm on the domain. Notational ambiguities usually arise in such a situation. Hence, even at cost of being clumsy, we will carefully distinguish both operators on our notation and we will do the same with the corresponding families of spectral projectors.

We will denote by W the Hilbert space $H^2(I;\mathbb{C}^n)\cap H^1_0(I;\mathbb{C}^n)$ endowed with the graph norm of Ju'' and, as before, H will denote $L^2(I;\mathbb{C}^n)$. Let $\widetilde{j}$ be the inclusion of W into H. The family $A_z = \mathcal{A}_z \circ \widetilde{j}$ is a family of *bounded* operators. Moreover, since the operator $\widetilde{j}$ is compact we have that $\mathcal{A}_z^{-1} = \widetilde{j} \circ A_z^{-1}$ is a compact operator whenever A_z^{-1} exists and is bounded. On the other hand, being $dA_z = d\mathcal{A}_z \circ \widetilde{j}$, we have also that

$$d\mathcal{A}_z \circ \mathcal{A}_z^{-1} = dA_z \circ A_z^{-1}. \tag{5.9}$$

For a fixed j, choose a positive number $\mu > 0$ such that the only point in the spectrum of $\mathcal{A}_{t_j}$ in the interval $[-\mu,\mu]$ is 0 and then choose η small enough such that neither μ nor $-\mu$ lies in the spectrum of $\mathcal{A}_t$ for $|t-t_j|<\eta$. For such a t, let P_t be the orthogonal projection in H onto the spectral subspace associated to the part of the spectrum of $\mathcal{A}_t$ lying in the interval $[-\mu,\mu]$. Then $\mathcal{A}_t P_t = P_t \mathcal{A}_t$ on the domain of $\mathcal{A}_t$. In other words P_t reduces the operator $\mathcal{A}_t$.

From the integral representation of P_t given by

$$P_t = \frac{1}{2\pi i}\int_{\mathcal{C}} (\mathcal{A}_t + \lambda\,\mathrm{id})^{-1}\, d\lambda, \tag{5.10}$$

where $\mathcal{C}$ is a symmetric curve in the complex plane surrounding the spectrum in $(-\mu,\mu)$, one can show that the projector P_t factors through $\widetilde{j}$. Indeed, defining $R_t\colon H \to W$ by

$$R_t = \frac{1}{2\pi i}\int_{\mathcal{C}} (A_t + \lambda\widetilde{j})^{-1}\, d\lambda,$$

we have $P_t = \widetilde{j} \circ R_t$. Moreover, if $Q_t = R_t \circ \widetilde{j}$, we have that $Q_t^2 = Q_t$ and hence each Q_t is a projector belonging to $\mathcal{L}(W)$.

By [21, Chapter II, Section 6] there exist two smooth paths U and V of unitary operators of H and W respectively, defined in $[t_j-\eta, t_j+\eta]$, such that

$U_{t_j} = \mathrm{id}_H$, $V_{t_j} = \mathrm{id}_W$ and such that

$$P_t U_t = U_t P_{t_j}, \quad Q_t V_t = V_t Q_{t_j}. \tag{5.11}$$

Taking eventually a smaller η, we can consider the smooth operator valued function $N_z = U_t^{-1} A_z V_t$ defined on some open neighbourhood of the closed domain $\mathcal{D}_j = [t_j - \eta, t_j + \eta] \times [-1, 1]$ together with the differential one-form

$$\theta = dN_z\, N_z^{-1} \tag{5.12}$$

We claim that θ take values in $\mathcal{T}(H)$, where $\mathcal{T}(H)$ is the trace class, and that

$$\operatorname{Tr} d\mathcal{A}_z\, \mathcal{A}_z^{-1} = \operatorname{Tr} dN_z\, N_z^{-1}. \tag{5.13}$$

Indeed, denoting by dot the ordinary derivative with respect to t and using $\mathcal{A}_z^{-1} = \widetilde{j} \circ A_z^{-1}$ we obtain

$$\begin{aligned} dN_z N_z^{-1} = &- U_t^{-1} \dot{U}_t\, dt + U_t^{-1} A_z \dot{V}_t V_t^{-1} A_z^{-1} U_t\, dt \\ &+ U_t^{-1}\, \dot{S}_t \mathcal{A}_z^{-1} U_t\, dt + i U_t^{-1} \mathcal{A}_z^{-1} U_t\, ds. \end{aligned} \tag{5.14}$$

The coefficients of all terms in the right hand side of (5.14) belong to the trace class. The last two because $\mathcal{A}_z^{-1} \in \mathcal{T}(H)$. For the first two let us recall that U_t is a solution of the Cauchy problem

$$\begin{cases} \dot{U}_t U_t^{-1} = [\dot{P}_t, P_t], \\ U_{t_j} = I, \end{cases} \tag{5.15}$$

where $[\dot{P}_t, P_t]$ is the commutator. Since the spectral subspace associated with the part of the spectrum of $\mathcal{A}_t$ lying in the interval $[-\mu, \mu]$ is finite dimensional it follows that $[\dot{P}_t, P_t]$ is of trace class and indeed a finite rank operator. Thus by (5.15) we have that $\dot{U}_t = [\dot{P}_t, P_t] U_t$ is of trace class and hence so is $U_t^{-1} \dot{U}_t$. The same argument shows that $\dot{V}_t V_t^{-1} \in \mathcal{T}(H)$ and this completes the proof of the first assertion.

In order to show that (5.13) holds we first notice that by definition of R_t

$$\dot{U}_t U_t^{-1} = [\dot{P}_t, P_t] = \widetilde{j} \dot{R}_t \widetilde{j} R_t - \widetilde{j} R_t \widetilde{j} \dot{R}_t = \widetilde{j} D_t,$$

where $D_t = \dot{R}_t \widetilde{j} R_t - R_t \widetilde{j} \dot{R}_t$. In the same way we get $\dot{V}_t V_t^{-1} = D_t \widetilde{j}$.

By commutativity of the trace

$$\operatorname{Tr} U_t^{-1} \dot{U}_t = \operatorname{Tr} \dot{U}_t U_t^{-1} = \operatorname{Tr} \widetilde{j} D_t = \operatorname{Tr} D_t \widetilde{j} = \operatorname{Tr} \dot{V}_t V_t^{-1}.$$

From this, taking in account that

$$\operatorname{Tr} [U_t^{-1} A_z \dot{V}_t V_t^{-1} A_z^{-1} U_t] = \operatorname{Tr} \dot{V}_t V_t^{-1}$$

it follows that

$$\operatorname{Tr} [-U_t^{-1} \dot{U}_t + U_t^{-1} A_z \dot{V}_t V_t^{-1} A_z^{-1} U_t] \equiv 0.$$

Finally, substituting the above in (5.14) and summing up, we obtain

$$\operatorname{Tr} dN_z N_z^{-1} = \operatorname{Tr}[U_t^{-1}(\dot{S}_t \mathcal{A}_z^{-1} dt + i\mathcal{A}_z^{-1}\, ds)U_t] = \operatorname{Tr} d\mathcal{A}_z \mathcal{A}_z^{-1},$$

since the trace do not change under conjugation. This proves (5.13).

From the very definition of the one form $\theta = dN_z N_z^{-1}$ it follows easily that

$$dN_z N_z^{-1} P_{t_j} = P_{t_j} dN_z N_z^{-1}$$

for every $z \in \mathcal{D}_j$ belonging to its domain and hence P_{t_j} reduces the coefficients of θ.

Let $H_j = \operatorname{Im} P_{t_j} = \operatorname{Ker} \mathcal{A}_{t_j}$. Under the splitting $H = H_j \oplus H_j^{\perp}$ we can write θ as a matrix of one forms

$$\theta = \begin{pmatrix} \theta_0 & 0 \\ 0 & \theta_1 \end{pmatrix}, \tag{5.16}$$

where $\theta_0 = dN_z N_z^{-1}|_{H_j}$ and $\theta_1 = dN_z N_z^{-1}|_{H_j^{\perp}}$.

Taking traces,

$$\frac{1}{2\pi i}\int_{\partial\mathcal{D}_j} \operatorname{Tr}\theta = \frac{1}{2\pi i}\int_{\partial\mathcal{D}_j} \operatorname{Tr}\theta_0 + \frac{1}{2\pi i}\int_{\partial\mathcal{D}_j} \operatorname{Tr}\theta_1. \tag{5.17}$$

We claim that the last term in (5.17) vanishes. In order to prove this we observe first that $N_z^{-1}|_{H_j^{\perp}}$ exists for all $z \in \mathcal{D}_j$ and not only on the boundary. On the other hand since both $U_t\widetilde{j}$ and $\widetilde{j}V_t$ verify the differential equation $\dot{W} = [\dot{P}_t, P_t]W$ with the same initial condition $W(t_j) = \widetilde{j}$ we have that $U_t\widetilde{j} = \widetilde{j}V_t$ everywhere. It follows then that $N_z = U_t^{-1} A_t V_t + is\widetilde{j}$. Substituting this into the expression $dN_z N_z^{-1}|_{H_j^{\perp}}$ and writing it in the form $E_z dt + F_z ds$, a direct computation yields $\partial_s E_z = \partial_t F_z$. Since $\mathcal{D}_j$ is simply connected the one form $\operatorname{Tr}\theta_1$ is exact and its integral over $\partial\mathcal{D}_j$ vanishes.

Combining with (4.8) and (5.13), we obtain

$$\frac{1}{2\pi i}\int_{\partial\mathcal{D}_j} \operatorname{Tr} d\mathcal{A}_z \mathcal{A}_z^{-1} = \frac{1}{2\pi i}\int_{\partial\mathcal{D}_j} \operatorname{Tr}(dN_z N_z^{-1})|_{H_j}, \tag{5.18}$$

where the right hand side is an integral of the trace form on a finite dimensional space H_j.

Now let us turn our attention to the signature. At this point we will identify the finite dimensional subspace $H_j = \operatorname{Ker}\mathcal{A}_{t_j}$ with $\operatorname{Im} Q_{t_j} = \operatorname{Ker} N_{t_j}$. With this identification $\widetilde{j}\colon \operatorname{Im} Q_{t_j} \to H_j$ becomes the identity. Let us consider the path of symmetric endomorphisms $M\colon [t_j - \eta, t_j + \eta] \to \mathcal{L}(H_j)$ given by $M_t = N_t|_{H_j}$. Writing down the definition of N_t we find that for all $u, v \in H_j$

$$\begin{aligned}\langle \dot{M}_t u, v\rangle_{L^2} = &- \langle U_t^{-1}\dot{U}_t U_t^{-1} A_t V_t u, v\rangle_{L^2} \\ &+ \langle U_t^{-1}\dot{A}_t V_t u, v\rangle_{L^2} + \langle U_t^{-1} A_t \dot{V}_t u, v\rangle_{L^2}.\end{aligned}$$

Putting $t = t_j$ in the above formula and using the fact that $\mathcal{A}_t$ is symmetric we notice that the first and the last term in the right hand side of the above equation vanish. Therefore

$$\operatorname{sign}\Gamma(M_t, t_j) = \operatorname{sign}\Gamma(\mathcal{A}, t_j). \tag{5.19}$$

In view of (5.19) and (5.18) Theorem 1.1 follows from the following result

Proposition 5.1. *If M is the path defined above and if $M_z = M_t + is\mathrm{id}$, for $z \in \mathcal{D}_j$, then*

$$\operatorname{sign}\Gamma(M, t_j) = \frac{1}{2\pi i}\int_{\partial\mathcal{D}_j} \operatorname{Tr} dM_z M_z^{-1}.$$

Proof. By Kato's Selection Theorem [21, Chapter II, Theorem 6.8] there exists smooth functions $\lambda_1(t), \ldots, \lambda_{n_j}(t)$ representing for each t the eigenvalues of the symmetric matrix M_t. Equivalently, M_t is similar to a smooth path of matrices Δ_t having the form $\Delta_t = \operatorname{diag}[\lambda_1(t), \ldots, \lambda_{n_j}(t)]$. By [21, Chapter II, Theorem 5.4], $\operatorname{Tr}\dot{M}_t = \sum_{i=1}^{n_j}\dot{\lambda}_i(t)$.

Putting $\Delta_z = \Delta_t + is\,\mathrm{id}$,

$$\operatorname{Tr} dM_z M_z^{-1} = \operatorname{Tr} d\Delta_z\Delta_z^{-1} = \sum_{l=1}^{n_j} d(\lambda_l(t) + is)(\lambda_l(t) + is).^{-1}$$

Now by elementary integration

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial\mathcal{D}_j} & d(\lambda_l(t) + is)(\lambda_l(t) + is)^{-1} \\ &= \begin{cases} 0 & \text{if } \lambda_l(t_j-\eta)\lambda_l(t_j+\eta) > 0, \\ 1 & \text{if } \lambda_l(t_j-\eta)\lambda_l(t_j+\eta) < 0 \text{and } \lambda_l(t_j+\eta) > 0, \\ -1 & \text{if } \lambda_l(t_j-\eta)\lambda_l(t_j+\eta) < 0 \text{and } \lambda_l(t_j+\eta) < 0. \end{cases}\end{aligned} \tag{5.20}$$

Summing over $l = 1, \ldots, n_j$ in (5.20) and using Proposition 5.1 we obtain

$$\frac{1}{2\pi i}\int_{\partial\mathcal{D}_j} \operatorname{Tr} dM_z M_z^{-1} = \mu(M_{t_j-\eta}) - \mu(M_{t_j+\eta}) = \operatorname{sign}\Gamma(M, t_j). \tag{5.21}$$

This completes the proof of the Theorem 1.1. □

6. Relation with the Maslov index

Let us consider again the path $\widetilde{\mathcal{A}}$ of real unbounded self-adjoint operators defined by (5.3). According to the results of Section 4, we have that $t \in [0,1]$ is a conjugate instant along γ if and only if $\ker \widetilde{\mathcal{A}}_t \neq \{0\}$. Let $\Psi_t(x)$ be the solution to the initial value problem

$$\begin{cases} \Psi_t'(x) = \sigma H_t(x)\Psi_t(x) & \text{for } x \in [0,1], \\ \Psi_t(0) = I, \end{cases} \tag{6.1}$$

where σ and $H_t(x)$ are matrices defined as in (4.10) and (4.11) with $z = t$, but with real coefficients this time.

The path $\Psi_t = \Psi_t(1)$, $t \in [0,1]$ is a path of real symplectic matrices i.e. $\Psi_t \sigma \Psi_t^* = \sigma$. Since the symplectic group acts on the manifold Λ_n of all Lagrangian subspaces of $\mathbb{R}^{2n}$, the action of the path Ψ on fixed Lagrangian subspace l produces a path on Λ_n. This path can be used in order to count conjugate points of γ.

Let us recall that in terms of the complex structure σ the manifold Λ_n can be defined as the submanifold of the Grassmanian $G_n(2n)$ whose elements are n-dimensional subspaces l of $\mathbb{R}^{2n}$ such that $\sigma(l)$ is orthogonal to l. Given a path of Lagrangian subspaces $\lambda\colon [a,b] \to \Lambda_n$ and a fixed Lagrangian subspace l such that λ_a and λ_b are transverse to l, there is a well defined integral-valued homotopy invariant $\mu_l\,(\lambda,[a,b])$, called *Maslov index* [35], which counts algebraically the number of crossing points of λ with l i.e. points $t \in [a,b]$ at which λ_t fails to be transverse to l. Here we take $l = \{0\} \times \mathbb{R}^n$ and $\lambda_t = \Psi_t(l)$. With this choice, we have that t is a conjugate instant along γ if and only if $\lambda_t \cap l \neq \{0\}$. Since conjugate instants cannot accumulate at 0 we can find an $\varepsilon > 0$ such that there are no conjugate instants in $[0,\varepsilon]$. The Maslov index of $\lambda\colon [\varepsilon,1] \to \Lambda_n$ is well defined and independent of the choice of ε.

The *Maslov index* of a p-geodesic is defined by $\mu_{\mathrm{Maslov}}(\gamma) = \mu_l\,(\lambda,[\varepsilon,1])$.

Proposition 6.1.

$$\mu_{\mathrm{spec}}(\gamma) = \mu_{\mathrm{con}}(\gamma) = \mu_{\mathrm{Maslov}}(\gamma). \tag{6.2}$$

Proof. We will be rather sketchy here since the arguments are very close to those used in the first part of the previous theorem. A different approach in the geodesic case can be found in [32].

There is a natural identification of the tangent space to the manifold $\Lambda(n)$ at a given point l in $\Lambda(n)$ with the set of quadratic forms on l which allows to express the Maslov index of a generic path as a sum of the signatures of the crossing forms at points of regular crossing analogous to that in Proposition 2.3 for the spectral flow. In terms of this identification the crossing form $\Gamma(\lambda,t)$ of a path λ at a given point t can be constructed as follows: let $M\colon U \to \mathcal{L}(\mathbb{R}^n;\mathbb{R}^{2n})$ be a smooth path of monomorphisms defined on a neighbourhood U of t such that $\operatorname{Im} M_s = \lambda_s$. By [35], the crossing form at t is the quadratic form defined by

$$\Gamma(\lambda,t)v = \langle \sigma \dot{M}_t w, M_t w\rangle \quad \text{for } v \in \lambda_t \cap l,\ M_t w = v. \tag{6.3}$$

The resulting form is independent of the choice of the frame M.

As before, a crossing point t is regular if the crossing form $\Gamma(\lambda,t)$ is non-degenerate. For paths with only regular crossing points the Maslov index can be computed by $\mu_l(\lambda,[a,b])=\sum_t \Gamma(\lambda,t)$.

From the above discussion the paths $\widehat{h}_t, \widetilde{\mathcal{A}}_t$ and the path $\lambda_t=\Psi_t(l)$ defined in $[\varepsilon,1]$ have the same crossing points. The crossing form $\Gamma(\lambda,t)$ can be easily computed in this case taking as M the second column of the block decomposition of

$$\Psi_t=\begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix}.$$

Then $M_t w\in l$ if and only if $b_t w=0$.

It turns out that $\lambda_t\cap l=\{(0,v): v=d_t w\}$ and

$$\Gamma(\lambda,t)(0,v)=-\langle \dot{b}_t w, d_t w\rangle=-\langle J d_t w, d_t w\rangle=-\langle Jv,v\rangle$$

by (6.1). Identifying $\lambda_t\cap l$ with $V_t=\operatorname{Im} d_t$, the crossing form is $-\langle Jv,v\rangle$.

On the other hand, by (5.4), for $u\in\ker\mathcal{A}_t$

$$\Gamma(\mathcal{A},t)u=-\int_0^1\langle \dot{S}_t(x)u(x),u(x)\rangle\,dx. \tag{6.4}$$

where as before $\cdot$ denotes the derivative with respect to t.

For any $s\in(0,1]$, the function $u_s(x)=u(sx/t)$ solves the Cauchy problem

$$\begin{cases} Ju_s''(x)+S_s(x)u_s(x)=0 & \text{for all } x\in[0,1],\\ u_s(0)=0,\quad u_s'(0)=\dfrac{s}{t}u'(0). \end{cases} \tag{6.5}$$

If we differentiate the equation in (6.5) with respect to s and we evaluate at $s=t$, we get

$$\begin{cases} J\dot{u}_t''(x)+\dot{S}_t(x)u_t(x)+S_t(x)\dot{u}_t(x)=0 & \text{for all } s\in[0,1],\\ \dot{u}_t(0)=0,\quad \dot{u}_t'(0)=0. \end{cases}$$

Taking into account that $u\in\ker\widetilde{\mathcal{A}}_t$ and the previous computation, integrating by parts we get

$$\Gamma(\widetilde{\mathcal{A}},t)(u)=\int_0^1\langle J\dot{u}_t''+S_t\dot{u}_t(x),u_t(x)\rangle\,dx=-\langle Ju_t'(1),\dot{u}_t(1)\rangle$$

Since $u_t(x)=u(x)$, $\dot{u}_t(x)=(x/t)u'(x)$ we get

$$\Gamma(\widetilde{\mathcal{A}},t)(u)=-\frac{1}{t}\langle Ju'(1),u'(1)\rangle.$$

The above calculation shows that the isomorphism sending $u\in\ker\widetilde{\mathcal{A}}_t$ into $tu'(1)\in V_t$ transforms $\Gamma(\widetilde{\mathcal{A}},t)$ into $\Gamma(\lambda,t)$. We conclude that the regular crossings of $\widetilde{\mathcal{A}}$ correspond to the regular crossings of λ and moreover the crossing forms have the same signature. Now the assertion (6.2) follows from (6.3) and the perturbation argument used in the proof of the main theorem. $\square$

We close this section by showing, as promised, that in the case of a regular geodesic $\mu_{\rm con}(\gamma) = \mu_{\rm spec}(\gamma)$ coincides with the expression given by formula (1.3).

Let us observe that if $\xi \in \mathcal{I}$ then for any Jacobi field η along γ the function given by $g(D\xi(x)/dx, \eta(x))$ is constant (to prove this it is enough to use (1.2) in the expression for the derivative of this function). In particular, if also $\eta \in \mathcal{I}$ then $g(D\xi(x)/dx, \eta(x)) \equiv 0$. This shows that for any Jacobi field $\xi \in \ker \widetilde{\mathcal{A}}$ the covariant derivative $D\xi(t)/dx$ belongs to $\mathcal{I}[t]^\perp$.

Let $M\colon \ker \widetilde{\mathcal{A}}_t \to \mathcal{I}[t]^\perp$ be the monomorphism defined by

$$Mu = t\frac{D}{dx}\bigg(\sum_{i=1}^n u_i\bigg(\frac{x}{t}\bigg)e^i(x)\bigg).$$

By dimension counting M is an isomorphism. On the other hand, using our previous calculation we get

$$g(Mu, Mu) = -\langle Ju'(1), u'(1)\rangle = t\Gamma(\widetilde{\mathcal{A}}, t).$$

Taking signatures and summing over all conjugate points we obtain the desired conclusion.

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