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SOME GENERAL EXISTENCE PRINCIPLES FOR ORDINARY DIFFERENTIAL EQUATIONS

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(*Submitted by A. Granas*)

Dedicated to the memory of Juliusz Schauder

1. Introduction

In this paper some general existence principles for the first order initial value problem

$$\begin{cases} y' = f(t,y), & 0 < t < T, \\ y(0) = a \end{cases} \tag{1.1}$$

and the second order boundary value problem

$$\begin{cases} (py')' = f(t,y,py'), & 0 < t < 1, \\ -\alpha y(0) + \beta \lim\limits_{t\to 0^+} p(t)y'(t) = c & \alpha \geq 0,\ \beta \geq 0, \\ ay(1) + b \lim\limits_{t\to 1^-} p(t)y'(t) = d & a \geq 0,\ b \geq 0 \text{ and } \alpha + a > 0 \end{cases} \tag{1.2}$$

are established. The literature on problems of the above form is extensive, see [1, 3–6, 8–9, 11–13] and their references. In all of these papers f is assumed to be continuous or a Carathéodory function. However, in many applications this class

of functions may be too restrictive. For example the initial value problem

$$\begin{cases} y' + \frac{1}{t}y = 1, & 0 < t < T, \\ y(0) = 0 \end{cases}$$

has a solution $y(t) = \frac{t}{2}$. In this case $f(t,u) = 1 - \frac{u}{t}$ is not an L^1–Carathéodory function. Our goal is to establish existence principles (and theory) which would include problems of this type. The results of this paper extend and complement the existing theory in the literature. Even in the case of the boundary value problem (1.2) with $p \equiv 1$ the results are new.

We use a fixed point approach to establish our existence principles. In particular we use a nonlinear alternative of Leray–Schauder type [9] which is an immediate consequence of the topological transversality theorem [2, 7, 8] of Granas. For completeness we state the result. By a map being *compact* we mean it is continuous with relatively compact range. A map is *completely continuous* if it is continuous and the image of every bounded set in the domain is contained in a compact set of the range.

THEOREM 1.1. *Assume U is a relatively open subset of a convex set K in a Banach space E. Let $G : \overline{U} \to K$ be a compact map, $p^\star \in U$ and*

$$N_\lambda(u) = N(u,\lambda) : \overline{U} \times [0,1] \to K$$

a family of compact maps (i.e. $N(\overline{U} \times [0,1])$ is contained in a compact subset of K and $N : \overline{U} \times [0,1] \to K$ is continuous) with $N_1 = G$ and $N_0 = p^\star$, the constant map to $p^\star$. Then either

(i) *G has a fixed point in $\overline{U}$; or*
(ii) *there is a point $u \in \partial U$ and $\lambda \in (0,1)$ such that $u = N_\lambda u$.*

□

Also the following definition from the literature will be used throughout this paper. Let k be a positive integer and $-\infty < a < b < \infty$. A function $g : [a,b] \times \mathbf{R}^k \to \mathbf{R}$ is an L^1–Carathéodory function provided that, if $g = g(t,z)$, then

(a) the map $z \to g(t,z)$ is continuous for almost all $t \in [a,b]$,
(b) the map $t \to g(t,z)$ is measurable for all $z \in \mathbf{R}^k$,
(c) for a given $r > 0$ there exists $h_r \in L^1[a,b]$ such that $|z| \leq r$ implies $|g(t,z)| \leq h_r(t)$ for almost all $t \in [a,b]$.

We remark here as well that conditions (a) and (b) imply for $t \in [a,b]$ that $g(t,u(t))$ is measurable for any measurable and almost everywhere finite function $u(t)$. This is a result of Carathéodory, see [10, 14]. Also, (c) implies that $g(t,u(t))$ is integrable.

2. Initial Value Problems

We begin by obtaining two existence principles for the initial value problem

$$\text{(2.1)} \qquad \begin{cases} y' + \phi'(t)y = f(t,y), & 0 < t < T, \\ y(0) = a \end{cases}$$

and then these principles will be used to obtain two general existence theorems.

By a solution to (2.1) we mean a function $y \in C[0,T]$ with y differentiable almost everywhere on $[0,T]$, $y(0) = a$ with y satisfying the differentiable equation almost everywhere on $[0,T]$.

THEOREM 2.1. *Suppose*

$$\text{(2.2)} \qquad \begin{cases} \phi : (0,T] \to \mathbf{R} \cup \{+\infty\} \text{ with } \phi \text{ differentiable almost everywhere} \\ \text{and } e^{-\phi} \text{ continuous on } (0,T], \end{cases}$$

$$\text{(2.3)} \qquad \begin{cases} e^{\phi} f \text{ is an } L^1_{\phi}\text{-Carathéodory function. By this we mean that, if} \\ g(t,z) = e^{\phi(t)} f(t,z) \text{ then } g : [0,T] \times \mathbf{R} \to \mathbf{R} \cup \{-\infty, \infty\} \text{ with} \\ \quad \text{(a) the map } z \to g(t,z) \text{ continuous for almost all } t \in [0,T], \\ \quad \text{(b) the map } t \to g(t,z) \text{ measurable for all } z \in \mathbf{R}, \\ \quad \text{(c) for a given } r > 0 \text{ there exists } h_r \in L^1[0,T] \text{ such that} \\ \qquad |z| \le r \text{ implies } |g(t,z)| \le h_r(t) \text{ for almost all } t \in [0,T]. \\ \text{Also, } h_r \text{ must satisfy } \lim_{t \to 0^+} e^{-\phi(t)} \int_0^t h_r(s)\, ds = 0 \end{cases}$$

and

$$\text{(2.4)} \qquad \begin{cases} \text{If } a \neq 0 \text{ assume } \phi(0) = \lim_{t \to 0^+} \phi(t) \in \mathbf{R}. \text{ If } a = 0 \text{ assume} \\ \text{there exist constants } k \text{ and } \delta > 0 \text{ with } \phi(t) \le k \text{ for } t \le \delta \end{cases}$$

are satisfied. In addition assume there is a constant M, independent of λ, such that

$$|y|_0 = \sup_{[0,T]} |y(t)| \le M$$

for any solution y to

$$\text{(2.5)}_\lambda \qquad \begin{cases} y' + \phi'(t)y = \lambda f(t,y), & 0 < t < T, \\ y(0) = a \end{cases}$$

for each $\lambda \in (0,1)$. Then (2.1) has at least one solution.

PROOF. We begin by showing that solving $(2.5)_\lambda$ is equivalent to finding a $y \in C[0,T]$ which satisfies

$$y(t) = a\exp(\phi(0) - \phi(t)) + \lambda e^{-\phi(t)} \int_0^t e^{\phi(s)} f(s, y(s))\, ds \tag{2.6}$$

with the convention that $a\exp(\phi(0) - \phi(t)) = 0$ if $a = 0$. To see this, notice that, if y is a solution of $(2.5)_\lambda$ then $(e^{\phi} y)' = \lambda e^{\phi} f$ almost everywhere on $[0,T]$. Integration from t_1 ($t_1 > 0$) to t yields

$$e^{\phi(t)} y(t) = e^{\phi(t_1)} y(t_1) + \lambda \int_{t_1}^t e^{\phi(s)} f(s, y(s))\, ds.$$

This, together with assumptions (2.3) and (2.4), yields

$$e^{\phi(t)} y(t) = a e^{\phi(0)} + \lambda \int_0^t e^{\phi(s)} f(s, y(s))\, ds$$

and so

$$y(t) = a\exp(\phi(0) - \phi(t)) + \lambda e^{-\phi(t)} \int_0^t e^{\phi(s)} f(s, y(s))\, ds.$$

On the other hand, if $y \in C[0,T]$ satisfies (2.6) then since $\int_0^t e^{\phi(s)} f(s, y(s))\, ds \in AC[0,T]$ we have

$$y'(t) = -a\phi'(t)\exp(\phi(0) - \phi(t)) + \lambda f(t, y(t)) - \lambda \phi'(t) e^{-\phi(t)} \int_0^t e^{\phi(s)} f(s, y(s))\, ds$$

almost everywhere on $[0,T]$. Thus

$$y' + \phi' y = \lambda f(t, y) \qquad \text{almost everywhere on } [0,T].$$

Also, $y(0) = a$ from (2.3) and (2.6).

Define the operator

$$N_\lambda : C_a[0,T] \to C_a[0,T] = \{u \in C[0,T] : \ u(0) = a\}$$

by

$$N_\lambda u(t) = a\exp(\phi(0) - \phi(t)) + \lambda e^{-\phi(t)} \int_0^t e^{\phi(s)} f(s, u(s))\, ds.$$

Now (2.6) is equivalent to the fixed point problem $y = N_\lambda y$. We *claim* that $N_\lambda : C_a[0,T] \to C_a[0,T]$ is continuous. Let $u_n \to u$ in $C_a[0,T]$, i.e. $u_n \to u$ uniformly on $[0,T]$. Now there exists $r > 0$ with $|u_n(s)| \le r$, $|u(s)| \le r$ for $s \in [0,T]$. Let $g(s,z) = e^{\phi(s)} f(s,z)$. By the above uniform convergence we have $g(s, u_n(s)) \to g(s, u(s))$ pointwise almost everywhere on $[0,T]$. Also, there exists an integrable

function h_r with $|g(s,u_n(s))| \le h_r(s)$ for almost all $s \in [0,T]$. The Lebesgue dominated convergence theorem implies that

$$N_\lambda u_n(t) = a\exp(\phi(0)-\phi(t)) + \lambda e^{-\phi(t)}\int_0^t e^{\phi(s)} f(s,u_n(s))\,ds$$
$$\to a\exp(\phi(0)-\phi(t)) + \lambda e^{-\phi(t)}\int_0^t e^{\phi(s)} f(s,u(s))\,ds = N_\lambda u(t)$$

pointwise for each $t \in [0,T]$. Next we show that the convergence is uniform. If this is true then N_λ is continuous.

To show that the convergence above is uniform notice that for $t, t_1 \in (0,T]$ we have

$$\begin{aligned}|N_\lambda u_n(t_1) - N_\lambda u_n(t)| \le{}& |a|e^{\phi(0)}\left|e^{-\phi(t_1)} - e^{-\phi(t)}\right| + e^{-\phi(t_1)}\left|\int_t^{t_1} e^{\phi(s)} f(s,u_n(s))\,ds\right| \\ &+ \left|e^{-\phi(t_1)} - e^{-\phi(t)}\right|\left|\int_0^t e^{\phi(s)} f(s,u_n(s))\,ds\right| \\ \le{}& \left(|a|e^{\phi(0)} + \int_0^T h_r(s)\,ds\right)\left|e^{-\phi(t_1)} - e^{-\phi(t)}\right| \\ &+ e^{-\phi(t_1)}\left|\int_t^{t_1} h_r(s)\,ds\right|.\end{aligned}$$

Also if $t_1 = 0$ and $t > 0$ then

$$|N_\lambda u_n(t_1) - N_\lambda u_n(t)| \le |a|\,|\exp(\phi(0)-\phi(t)) - 1| + \left|e^{-\phi(t)}\int_0^t h_r(s)\,ds\right|.$$

We have similar bounds for $|N_\lambda u(t_1) - N_\lambda u(t)|$. Let $\varepsilon > 0$ be given. There exists a $\tau > 0$ such that $t, t_1 \in [0,T]$ and $|t - t_1| < \tau$ imply

$$|N_\lambda u_n(t_1) - N_\lambda u_n(t)| < \frac{\varepsilon}{3} \qquad \text{for all } n \tag{2.7}$$

and

$$|N_\lambda u(t_1) - N_\lambda u(t)| < \frac{\varepsilon}{3}. \tag{2.8}$$

Then (2.7), (2.8) together with the fact that $N_\lambda u_n(t) \to N_\lambda u(t)$ pointwise imply that the convergence is uniform.

Consequently $N_\lambda : C_a[0,T] \to C_a[0,T]$ is continuous. In addition the Arzela–Ascoli theorem guarantees that N_λ is completely continuous. To see this let $\Omega \subseteq C_a[0,T]$ be bounded, i.e. there exists $r > 0$ with $\sup_{[0,T]} |v(t)| \le r$ for each $v \in \Omega$. Then $N_\lambda\Omega$ is uniformly bounded since

$$|N_\lambda v(t)| \le |a|e^{\phi(0)} \sup_{[0,T]} e^{-\phi(t)} + \sup_{[0,T]}\left|e^{-\phi(t)}\int_0^t h_r(s)\,ds\right|$$

for each $v \in \Omega$. The equicontinuity of $N_\lambda\Omega$ on $[0,T]$ follows from the inequalities obtained above for $|N_\lambda u(t_1) - N_\lambda u(t)|$.

Thus $N_\lambda : C_a[0,T] \to C_a[0,T]$ is continuous and completely continuous. Set

$$U = \{u \in C_a[0,T] : |u|_0 < M+1\}, \quad K = C_a[0,T] \quad \text{and} \quad E = C[0,T].$$

□

REMARK. Let $N(x,\lambda) = N_\lambda(x)$ and notice that $N(\overline{U} \times [0,1])$ is contained in a compact subset of $C_a[0,T]$. To see this let $N(u_n,\lambda_n)$ be any sequence in $N(\overline{U} \times [0,1])$. Then the above arguement implies that $N(u_n,\lambda_n)$ is uniformly bounded and equicontinuous, so the Arzela–Ascoli theorem yields the result.

Now Theorem 1.1 applies with $p^\star = a\exp(\phi(0) - \phi(t))$, but with this choice of U the possibility (ii) of theorem 1.1 is ruled out and we deduce that N_1 has a fixed point, i.e. (2.1) has a solution y. □

In the last existence principle we assumed that $\phi : (0,T] \to \mathbf{R} \cup \{+\infty\}$ and $\limsup_{t\to 0+} \phi(t) < \infty$. We now establish a result where $\limsup_{t\to 0+} \phi(t) = +\infty$ is allowed. Here we discuss the initial value problem

$$\text{(2.9)} \qquad \begin{cases} y' + \phi'(t)y = f(t,y), & 0 < t < T, \\ y(0) = 0. \end{cases}$$

THEOREM 2.2. *Assume* (2.2) *holds and that* f *has the decomposition* $f(t,u) = f^\star(t,u) + \omega(t)$ *with*

$$\text{(2.10)} \qquad \begin{cases} e^{\phi} f^\star \text{ is an } L^1_\phi\text{-Carathéodory function with } \omega : [0,T] \to \mathbf{R}, \\ e^{-\phi(t)} \int_t^T e^{\phi(s)}\omega(s)\,ds \in C[0,T] \cap C^1(0,T] \text{ and} \\ \lim_{t\to 0+} e^{-\phi(t)} \int_t^T e^{\phi(s)}\omega(s)\,ds = 0. \end{cases}$$

In addition assume there is a constant M, *independent of* λ, *such that*

$$|y|_0 = \sup_{[0,T]} |y(t)| \le M$$

for any solution $y \in C[0,T]$ *to*

$$(2.11)_\lambda \qquad y(t) = \lambda e^{-\phi(t)} \int_0^t e^{\phi(s)} f^\star(s,y(s))\,ds - \lambda e^{-\phi(t)} \int_t^T e^{\phi(s)}\omega(s)\,ds$$

for each $\lambda \in (0,1)$. *Then* (2.9) *has at least one solution.*

PROOF. We first show that if $y \in C[0,T]$ satisfies $(2.11)_\lambda$ then y is a solution of

$$(2.12)_\lambda \qquad \begin{cases} y' + \phi'(t)y = \lambda f(t,y), & 0 < t < T, \\ y(0) = 0. \end{cases}$$

To see this, notice that, if $y \in C[0,T]$ satisfies $(2.11)_\lambda$ then since

$$\int_0^t e^{\phi(s)} f^*(s, y(s))\, ds \in AC[0,T]$$

we have

$$y'(t) = \lambda f(t, y(t)) - \lambda\phi'(t)e^{-\phi(t)} \int_0^t e^{\phi(s)} f^*(s, y(s))\, ds + \lambda\phi'(t)e^{-\phi(t)} \int_t^T e^{\phi(s)}\omega(s)\, ds$$

almost everywhere on $[0,T]$. Thus

$$y' + \phi' y = \lambda f(t,y) \qquad \text{almost everywhere on } [0,T].$$

Also, $y(0) = 0$ from (2.10) and $(2.11)_\lambda$. Thus y is a solution of $(2.12)_\lambda$. □

REMARK. We note that if y is a solution of $(2.12)_\lambda$ then y need not satisfy $(2.11)_\lambda$. In fact, if $\lim_{t\to 0+} \phi(t) = \infty$ then any solution of $(2.12)_\lambda$ may be written as

$$y(t) = \lambda e^{-\phi(t)} \int_0^t e^{\phi(s)} f^*(s, y(s))\, ds - \lambda e^{-\phi(t)} \int_t^T e^{\phi(s)}\omega(s)\, ds + Ae^{-\phi(t)}$$

where A is an arbitrary constant.

Define the operator $N_\lambda : C_0[0,T] \to C_0[0,T]$ by

$$N_\lambda u(t) = \lambda e^{-\phi(t)} \int_0^t e^{\phi(s)} f^*(s, u(s))\, ds - \lambda e^{-\phi(t)} \int_t^T e^{\phi(s)}\omega(s)\, ds.$$

Essentially the same analysis as in theorem 2.1 implies that N_1 has a fixed point, i.e. $(2.11)_1$ has a solution, i.e. (2.9) has a solution. □

REMARK. Under assumptions (2.2), (2.10) and $\phi(0) = \lim_{t\to 0+} \phi(t) \in \mathbf{R}$ we may deduce the existence of a solution to the boundary value problem (2.1) with $a \neq 0$. □

Theorems 2.1 and 2.2 may now be used to establish some existence results for initial value problems.

THEOREM 2.3. *Suppose* (2.2), (2.3) *and* (2.4) *are satisfied. In addition assume*

$$\begin{cases} |e^{\phi} f(t,y)| \le \psi(e^{\phi}|y|) \text{ for almost all } t \in [0,T] \text{ where} \\ \psi : [0,\infty) \to (0,\infty) \text{ is a Borel function} \end{cases} \tag{2.13}$$

and

$$e^{-\phi(t)} I^{-1}(t) \text{ is bounded for } t \in [0,T] \tag{2.14}$$

hold. Here

$$I(z) = \int_{|a|}^{z} \frac{du}{\psi(u)}, \qquad z > |a|.$$

Then (2.1) *has a solution on* $[0,T]$ *for each* $T < T_\infty$. *Here*

$$T_\infty = \int_{|a|}^{\infty} \frac{du}{\psi(u)}.$$

PROOF. By theorem 2.1 we need only establish *a priori* bounds for solutions y to $(2.5)_\lambda$ which are defined on $[0,T]$ for $T < T_\infty$. Notice that since $|e^\phi y|^2 = (e^\phi y)^2$ we have

$$|e^\phi y|' = \frac{e^\phi y (e^\phi y)'}{|e^\phi y|} \le |(e^\phi y)'|$$

almost everywhere on the set where $e^\phi y \neq 0$. Fix $t \in [0,T]$. Suppose $|y(t)| > |a|$. Then there exists an $\mu \in [0,t)$ such that $|y(s)| > |a|$ for $s \in (\mu, t]$ and $|y(\mu)| = |a|$. Since $(e^\phi y)' = \lambda e^\phi f(t,y)$ almost everywhere on $[0,T]$ we have

$$|e^\phi y|' \le |(e^\phi y)'| \le \psi(e^\phi |y|)$$

almost everywhere on $(\mu, t]$. Now the change of variables formula [4, 9] yields, since (2.6) with $\lambda = 1$ implies $e^\phi y \in AC[0,1]$,

$$I(|e^{\phi(t)} y(t)|) = \int_{|a|}^{|e^{\phi(t)} y(t)|} \frac{du}{\psi(u)} = \int_\mu^t \frac{|e^{\phi(s)} y(s)|' ds}{\psi(e^{\phi(s)} |y(s)|)} \le t - \mu \le t.$$

Consequently

$$|e^{\phi(t)} y(t)| \le I^{-1}(t)$$

since $T < T_\infty$. This together with (2.14) yields

$$|y(t)| \le \sup_{[0,T]} |e^{-\phi(t)} I^{-1}(t)|.$$

Consequently

$$|y|_0 \le \max\{\sup_{[0,T]} |e^{-\phi(t)} I^{-1}(t)|, |a|\}$$

and the result follows from theorem 2.1. □

EXAMPLE. The initial value problem

$$\text{(2.15)} \qquad \begin{cases} y' + \frac{1}{t} y = y^\alpha + 1, & 0 < t < T, \\ y(0) = 0, & 0 \le \alpha \le 1, \end{cases}$$

has a solution for each $T > 0$.

To see this we apply theorem 2.3. In this case $\phi(t) = \ln t$, $f(t,y) = y^\alpha + 1$ and $\psi(x) = x + 2T$. Clearly (2.2), (2.3) and (2.4) are satisfied. In addition (2.13) holds since

$$|e^\phi f(t,y)| = |t(y^\alpha + 1)| \le t|y|^\alpha + t \le t(|y| + 1) + t \le |ty| + 2T = \psi(e^\phi |y|).$$

Also, $I(z) = \int_0^z \frac{du}{\psi(u)} = \ln\left(\frac{z+2T}{2T}\right)$, $T_\infty = \infty$ and so $I^{-1}(z) = 2T(e^z - 1)$. Thus

$$|e^{-\phi(t)} I^{-1}(t)| = \frac{2T}{t}(e^t - 1)$$

and so (2.14) holds. Consequently theorem 2.3 implies that (2.15) has a solution.

THEOREM 2.4. *Suppose f has the decomposition $f(t,u) = f^*(t,u) + \omega(t)$ and assume* (2.2) *and* (2.10) *hold. In addition suppose that*

$$\text{(2.16)} \qquad \begin{cases} \text{there exists a constant } \gamma,\ 0 \le \gamma < 1 \text{ with} \\ |f^*(t,u)| \le q_1(t)|u|^\gamma + q_2(t)|u| + q_3(t) \\ \text{for almost all } t \in [0,T]. \text{ Here } e^\phi q_j \in L^1[0,T], j = 1,2,3 \\ \text{and there exists a constant } K_0 \text{ with} \\ \sup_{[0,T]}(e^{-\phi(t)} \left| \int_t^T e^{\phi(s)}\omega(s)\,ds \right|) \le K_0 \end{cases}$$

$$\text{(2.17)} \qquad \begin{cases} \text{there exist constants } H_j,\ j = 1,2,3 \text{ with} \\ \sup_{[0,T]}(e^{-\phi(t)} \int_0^t e^{\phi(s)} q_j(s)\,ds) \le H_j,\ j = 1,2,3 \end{cases}$$

and

$$\text{(2.18)} \qquad H_2 < 1$$

are satisfied. Then (2.9) *has a solution on* $[0,T]$.

PROOF. By theorem 2.2 we need only establish *a priori* bounds for solutions $y \in C[0,T]$ to $(2.11)_\lambda$. Now for $t \in [0,T]$ we have

$$\begin{aligned} |y(t)| &= \left| \lambda e^{-\phi(t)} \int_0^t e^{\phi(s)} f^*(s, y(s))\,ds - \lambda e^{-\phi(t)} \int_t^T e^{-\phi(s)}\omega(s)\,ds \right| \\ &\le e^{-\phi(t)} \left(|y|_0^\gamma \int_0^t e^{\phi(s)} q_1\,ds + |y|_0 \int_0^t e^{\phi(s)} q_2\,ds + \int_0^t e^{\phi(s)} q_3\,ds \right) \\ &\quad + e^{-\phi(t)} \left| \int_t^T e^{\phi(s)}\omega\,ds \right| \end{aligned}$$

where $|y|_0 = \sup_{[0,T]} |y(t)|$. Consequently for each $t \in [0,T]$ we have

$$|y(t)| \le H_1 |y|_0^\gamma + H_2 |y|_0 + H_3 + K_0.$$

Thus

$$(1 - H_2)|y|_0 \le H_1 |y|_0^\alpha + H_3 + K_0.$$

Now since $0 \le \gamma < 1$ and $H_2 < 1$ then there exists a constant M with $|y|_0 \le M$ and the result follows from theorem 2.2. □

EXAMPLE. The initial value problem

$$\text{(2.19)} \qquad \begin{cases} y' - \frac{1}{t}y = A_0 t^\beta y^\alpha + 1, & 0 < t < T, \\ y(0) = 0, & 0 \le \alpha < 1,\ \beta > 0,\ A_0 \text{ a constant}, \end{cases}$$

has a solution for each $T > 0$.

To see this we apply theorem 2.4. In this case $\phi(t) = -\ln t$, $f^\star(t,y) = A_0 t^\beta y^\alpha$, $\omega(t) = 1$, $q_1(t) = |A_0| t^\beta$ and $q_2 = q_3 = 0$ so (2.2) and (2.16) hold. Also, $H_2 = H_3 = 0$ and $H_1 = |A_0| \sup_{[0,T]}(t \int_0^t s^{(\beta-1)}\, ds)$ in (2.17). Finally, (2.10) is true since

$$e^{-\phi(t)} \int_t^T e^{\phi(s)} \omega(s)\, ds = t(\ln T - \ln t) \in C[0,T] \cap C^1(0,T]$$

and $\lim_{t\to 0^+} e^{-\phi(t)} \int_t^T e^{\phi(s)}\omega(s)\, ds = 0$ by l'Hospital's rule.

Consequently theorem 2.4 implies that (2.19) has a solution.

3. Boundary Value Problems

We begin this section by establishing two existence principles for the singular two point boundary value problem

$$\text{(3.1)} \qquad \begin{cases} (py')' + \phi' p y' = f(t, y, py'), & 0 < t < 1, \\ \lim\limits_{t\to 0^+} p(t)y'(t) = c, \\ ay(1) + b \lim\limits_{t\to 1^-} p(t)y'(t) = d; & a > 0,\ b \ge 0. \end{cases}$$

By a solution to (3.1) we mean a function $y \in C[0,1] \cap C^1(0,1)$ with $py' \in C[0,1]$ and y' differentiable almost everywhere on $[0,1]$. Also, y satisfies the stated boundary data and the differential equation almost everywhere on $[0,1]$.

THEOREM 3.1. *Suppose*

$$\text{(3.2)} \qquad \begin{cases} \phi : (0,1] \to \mathbf{R} \cup \{+\infty\} \text{ with } \phi \text{ differentiable almost everywhere,} \\ \text{and } e^{-\phi} \text{ continuous on } (0,1], \end{cases}$$

$$\text{(3.3)} \qquad p \in C[0,1] \cap C^1(0,1) \text{ with } p > 0 \text{ on } (0,1),$$

$$\text{(3.4)} \qquad \begin{cases} \text{if } c \neq 0 \text{ assume } \phi(0) = \lim_{t\to 0^+}\phi(t) \in \mathbf{R} \text{ and } e^{-\phi}/p \in L^1[0,1]; \\ \text{if } c = 0 \text{ there exist constants } k \text{ and } \delta \text{ with } \phi(t) \le k \text{ for } t \le \delta \end{cases}$$

and

$$(3.5)\quad \begin{cases} e^{\phi} f \text{ is an } L^1_{\phi}\text{-Carathéodory function. By this we mean that if} \\ g(t,z_1,z_2) = e^{\phi(t)} f(t,z_1,z_2) \text{ then } g : [0,1]\times \mathbf{R}^2 \to \mathbf{R}\cup\{-\infty,\infty\} \text{ with} \\ \quad \text{(a) the map } (z_1,z_2) \to g(t,z_1,z_2) \text{ continuous for almost all } t\in[0,1], \\ \quad \text{(b) the map } t \to g(t,z_1,z_2) \text{ measurable for all } (z_1,z_2)\in\mathbf{R}^2, \\ \quad \text{(c) for a given } r>0 \text{ there exists } h_r \in L^1[0,1] \text{ such that} \\ \qquad |z_1|\le r,\ |z_2|\le r, \text{ implies } |g(t,z_1,z_2)|\le h_r(t) \text{ for almost all} \\ \qquad t\in[0,1]. \text{ Also } h_r \text{ must satisfy } \lim_{t\to 0+} e^{-\phi(t)}\int_0^t h_r(s)\,ds = 0 \\ \qquad \text{and } (e^{-\phi(t)}/p(t))\int_0^t h_r(s)\,ds \in L^1[0,1] \end{cases}$$

are satisfied. In addition assume there is a constant M, independent of λ, such that

$$|y|_1 = \max\{\sup_{[0,1]}|y(t)|,\ \sup_{(0,1)}|p(t)y'(t)|\} \le M$$

for any solution y to

$$(3.6)_\lambda \qquad \begin{cases} (py')' + \phi' py' = \lambda f(t,y,py'), & 0<t<1, \\ \lim\limits_{t\to 0^+} p(t)y'(t) = c, & \\ ay(1) + b\lim\limits_{t\to 1^-} p(t)y'(t) = d; & a>0,\ b\ge 0 \end{cases}$$

for each $\lambda\in(0,1)$. Then (3.1) has at least one solution.

PROOF. Solving $(2.5)_\lambda$ is equivalent to finding a solution $y\in C[0,1]$ with $py'\in C[0,1]$ to

$$(3.7)\qquad \begin{aligned} y(t) = {} & \frac{d}{a} - \frac{b}{a}\left(ce^{[\phi(0)-\phi(1)]} + \lambda e^{-\phi(1)}\int_0^1 e^{\phi(s)} f(s,y,py')\,ds\right) \\ & - c\int_t^1 \frac{\exp(\phi(0)-\phi(s))}{p(s)}\,ds \\ & - \lambda\int_t^1 \frac{e^{-\phi(s)}}{p(s)}\int_0^s e^{\phi(x)} f(x,y(x),p(x)y'(x))\,dx\,ds. \end{aligned}$$

To see this, notice that, if y is a solution of $(3.6)_\lambda$ then $(e^{\phi}py')' = \lambda e^{\phi} f$ almost everywhere on $[0,1]$. Integration from t_1 ($t_1>0$) to t yields

$$e^{\phi(t)}p(t)y'(t) = e^{\phi(t_1)}p(t_1)y'(t_1) + \lambda\int_{t_1}^t e^{\phi(s)} f(s,y(s),p(s)y'(s))\,ds.$$

Let $t_1\to 0^+$ to obtain

$$p(t)y'(t) = c\exp(\phi(0)-\phi(t)) + \lambda e^{-\phi(t)}\int_0^t e^{\phi(s)} f(s,y(s),p(s)y'(s))\,ds$$

with the convention that $c\exp(\phi(0)-\phi(t))=0$ if $c=0$. Notice since $e^{\phi}f$ is an L^1_ϕ-Carathéodory function then the right hand side of the previous equality is a continuous function on $[0,1]$.

Integration from t to 1 now yields

$$y(t)=-c\int_t^1\frac{\exp(\phi(0)-\phi(s))}{p(s)}\,ds - \lambda\int_t^1\frac{e^{-\phi(s)}}{p(s)}\int_0^s e^{\phi(x)}f(x,y(x),p(x)y'(x))\,dx\,ds+y(1).$$

In addition since $ay(1)+b\lim_{t\to 1^-}p(t)y'(t)=d$ we have

$$d=ay(1)+b\left(ce^{[\phi(0)-\phi(1)]}+\lambda e^{-\phi(1)}\int_0^1 e^{\phi(s)}f(s,y,py')\,ds\right)$$

and so

$$\begin{aligned}y(t)=&\frac{d}{a}-\frac{b}{a}\left(ce^{[\phi(0)-\phi(1)]}+\lambda e^{-\phi(1)}\int_0^1 e^{\phi(s)}f(s,y,py')\,ds\right)\\&-c\int_t^1\frac{\exp(\phi(0)-\phi(s))}{p(s)}\,ds\\&-\lambda\int_t^1\frac{e^{-\phi(s)}}{p(s)}\int_0^s e^{\phi(x)}f(x,y(x),p(x)y'(x))\,dx\,ds.\end{aligned}$$

Notice that since $e^{\phi}f$ is an L^1_ϕ-Carathéodory function the right hand side of the previous equality is a continuous function on $[0,1]$. Thus y is a solution of (3.7).

On the other hand if y ($y\in C[0,1]$ with $py'\in C[0,1]$) is a solution of (3.7) then since $\int_t^1\frac{e^{-\phi(s)}}{p(s)}\int_0^s e^{\phi(x)}f(x,y,py')\,dx\,ds\in AC[0,1]$ we have

$$p(t)y'(t)=c\exp(\phi(0)-\phi(t))+\lambda e^{-\phi(t)}\int_0^t e^{\phi(s)}f(s,y(s),p(s)y'(s))\,ds \tag{3.8}$$

almost everywhere on $[0,1]$. The right hand side of (3.8) is a continuous function on $[0,1]$ since $e^{\phi}f$ is an L^1_ϕ-Carathéodory function and so the equation in (3.8) holds at each $t\in[0,1]$. Thus on $[0,1]$,

$$e^{\phi(t)}p(t)y'(t)=ce^{\phi(0)}+\lambda\int_0^t e^{\phi(s)}f(s,y(s),p(s)y'(s))\,ds.$$

Also since $\int_0^t e^{\phi(x)}f(x,y,py')\,dx\in AC[0,1]$ we have

$$(e^{\phi}py')'=\lambda e^{\phi}f\quad\text{almost everywhere on }[0,1].$$

In addition, $\lim_{t\to 0^+}p(t)y'(t)=c$ and $ay(1)+b\lim_{t\to 1^-}p(t)y'(t)=d$ from (3.7) and (3.8). Thus y is a solution of $(3.6)_\lambda$.

Let $K^1 = \{u \in C[0,1],\ pu' \in C[0,1] \text{ with norm } |\cdot|_1\}$ and

$$K^1_{c,d} = \{u \in K^1 \text{ with } \lim_{t\to 0^+} p(t)u'(t) = c \text{ and } au(1) + b\lim_{t\to 1^-} p(t)u'(t) = d\}.$$

Define the operator $N_\lambda : K^1_{c,d} \to K^1_{c,d}$ by

$$\begin{aligned} N_\lambda u(t) =& \frac{d}{a} - \frac{b}{a}\left(ce^{[\phi(0)-\phi(1)]} + \lambda e^{-\phi(1)}\int_0^1 e^{\phi(s)} f(s,u,pu')\,ds\right) \\ &- c\int_t^1 \frac{\exp(\phi(0)-\phi(s))}{p(s)}\,ds \\ &- \lambda\int_t^1 \frac{e^{-\phi(s)}}{p(s)}\int_0^s e^{\phi(x)} f(x,u(x),p(x)u'(x))\,dx\,ds. \end{aligned} \tag{3.9}$$

Notice

$$\begin{aligned} p(t)(N_\lambda u)'(t) =& c\exp(\phi(0)-\phi(t)) \\ &+ \lambda e^{-\phi(t)}\int_0^t e^{\phi(s)} f(s,u(s),p(s)u'(s))\,ds. \end{aligned} \tag{3.10}$$

Now (3.7) is equivalent to the fixed point problem $y = N_\lambda y$. We *claim* that $N_\lambda : K^1_{c,d} \to K^1_{c,d}$ is continuous and completely continuous. Let $u_n \to u$ in $K^1_{c,d}$ i.e. $u_n \to u$ and $p(u_n)' \to pu'$ uniformly on $[0,1]$. The Lebesgue dominated convergence theorem (essentially the same reasoning as in theorem 2.1) implies that $N_\lambda u_n \to N_\lambda u$ and $p(N_\lambda u_n)' \to p(N_\lambda u)'$ pointwise for each $t \in [0,1]$. Next we show that the convergence is uniform. Then $N_\lambda : K^1_{c,d} \to K^1_{c,d}$ will be continuous.

Let $t, t_1 \in [0,1]$. Then

$$\begin{aligned} |N_\lambda u_n(t) - N_\lambda u_n(t_1)| \le |c|&\left|\int_t^{t_1} \frac{e^{[\phi(0)-\phi(s)]}}{p(s)}\,ds\right| \\ &+ \left|\int_t^{t_1} \frac{e^{-\phi(s)}}{p(s)}\int_0^s e^{\phi(x)} f(x,u_n,pu_n')\,dx\,ds\right|. \end{aligned}$$

Also, if $t, t_1 \in (0,1]$ with $t > t_1$ then

$$\begin{aligned} &\left|\lim_{s\to t} p(s)(N_\lambda u_n)'(s) - p(t_1)(N_\lambda u_n)'(t_1)\right| \\ &\quad\le |c|e^{\phi(0)}\left|e^{-\phi(t)} - e^{-\phi(t_1)}\right| + e^{-\phi(t_1)}\left|\int_t^{t_1} e^{\phi(s)} f(s,u_n,pu_n')\,ds\right| \\ &\qquad + \left|e^{-\phi(t_1)} - e^{-\phi(t)}\right|\left|\int_0^t e^{\phi(s)} f(s,u_n,pu_n')\,ds\right| \end{aligned}$$

whereas if $t_1 = 0$ and $t > 0$ then

$$\left|p(t)(N_\lambda u_n)'(t) - \lim_{s\to 0^+} p(s)(N_\lambda u_n)'(s)\right| \leq |c|\,|\exp(\phi(0) - \phi(t)) - 1| + \left|e^{-\phi(t)} \int_0^t e^{\phi(s)} f(s, u_n, pu_n')\, ds\right|.$$

REMARK. All inequalities above are valid at $t = 1$ since $\lim_{t\to 1^-} \phi(t)$ is either a real number or $+\infty$.

Let $\varepsilon > 0$ be given. Then there exists a $\delta > 0$ such that $t, t_1 \in [0,1]$ and $|t - t_1| < \delta$ implies

$$|N_\lambda u_n(t) - N_\lambda u_n(t_1)| < \frac{\varepsilon}{3}, \qquad |\lim_{s\to t} p(s)(N_\lambda u_n)'(s) - \lim_{s\to t_1} p(s)(N_\lambda u_n)'(s)| < \frac{\varepsilon}{3}$$

for all n and

$$|N_\lambda u(t) - N_\lambda u(t_1)| < \frac{\varepsilon}{3}, \qquad |\lim_{s\to t} p(s)(N_\lambda u)'(s) - \lim_{s\to t_1} p(s)(N_\lambda u)'(s)| < \frac{\varepsilon}{3}.$$

These inequalities together with $N_\lambda u_n \to N_\lambda u$ and $p(N_\lambda u_n)' \to p(N_\lambda u)'$ pointwise imply that the convergence is uniform.

Consequently $N_\lambda : K^1_{c,d} \to K^1_{c,d}$ is continuous. Also N_λ is completely continuous. To see this let $\Omega \subseteq K^1_{c,d}$ be bounded, i.e. there exists $r > 0$ with $\sup_{[0,1]} |v(t)| \leq r$, $\sup_{(0,1)} |p(t)v'(t)| \leq r$ for each $v \in \Omega$. Now (3.5), (3.9) and (3.10) imply that $N_\lambda\Omega$ is uniformly bounded and the equicontinuity of $N_\lambda\Omega$ on $[0,1]$ follows as above. Then $N_\lambda : K^1_{c,d} \to K^1_{c,d}$ is completely continuous by the Arzela–Ascoli theorem. Set

$$U = \left\{u \in K^1_{c,d} : |u|_1 < M + 1\right\}, \qquad K = K^1_{c,d} \text{ and } E = K^1$$

and apply theorem 1.1 with

$$p^\star = \frac{d}{a} - \frac{bc}{a} e^{[\phi(0)-\phi(1)]} - c\int_t^1 \frac{\exp(\phi(0) - \phi(s))}{p(s)}\, ds.$$

Then N_1 has a fixed point, i.e. (3.1) has a solution. □

We next establish an existence principle for the boundary value problem

$$\text{(3.11)} \qquad \begin{cases} (py')' + \phi' py' = f(t, y, py'), \quad 0 < t < 1, \\ \lim_{t\to 0^+} p(t)y'(t) = 0, \\ ay(1) + b \lim_{t\to 1^-} p(t)y'(t) = d \quad a > 0,\ b \geq 0 \end{cases}$$

without assuming that $\limsup_{t\to 0^+} \phi(t) < \infty$.

THEOREM 3.2. *Suppose f has the decomposition $f(t,u,v) = f^*(t,u,v) + \omega(t)$ and assume* (3.2) *and* (3.3) *hold. In addition suppose*

$$e^{\phi} f^* \text{ is an } L^1_\phi\text{-Carathéodory function} \tag{3.12}$$

and

$$\begin{cases} \omega : [0,1] \to \mathbf{R} \ \text{ with } \ e^{-\phi(t)} \displaystyle\int_t^1 e^{\phi(s)}\omega(s)\,ds \in C[0,1] \cap C^1(0,1), \\ \displaystyle\lim_{t\to 0^+} e^{-\phi(t)} \int_t^1 e^{\phi(s)}\omega(s)\,ds = 0 \ \text{ and } \ \dfrac{e^{-\phi(t)}}{p(s)} \displaystyle\int_t^1 e^{\phi(s)}\omega(s)\,ds \in L^1[0,1] \end{cases} \tag{3.13}$$

are satisfied. Also assume there is a constant M, independent of λ, such that

$$|y|_1 = \max\{\sup_{[0,1]} |y(t)|,\ \sup_{(0,1)} |p(t)y'(t)|\} \le M$$

for any solution $y \in C[0,1]$ with $py' \in C[0,1]$ to

$(3.14)_\lambda$

$$\begin{aligned} y(t) = \frac{d}{a} - \frac{\lambda b}{a}\left(e^{-\phi(1)} \int_0^1 e^{\phi} f^*(s,y,py')\,ds + \lim_{t\to 1^-} e^{-\phi(t)} \int_t^1 e^{\phi}\omega(s)\,ds \right) \\ -\lambda \int_t^1 \frac{e^{-\phi(s)}}{p(s)} \int_0^s e^{\phi(x)} f^*(x,y,py')\,dx\,ds + \lambda \int_t^1 \frac{e^{-\phi(s)}}{p(s)} \int_s^1 e^{\phi(x)}\omega(x)\,dx\,ds \end{aligned}$$

for each $\lambda \in (0,1)$. Then (3.11) *has at least one solution.*

PROOF. We first show that if $y \in C[0,1]$ with $py' \in C[0,1]$ is a solution to $(3.14)_\lambda$ then y is a solution of

$$\begin{cases} (py')' + \phi' py' = \lambda f(t,y,py'), & 0 < t < 1, \\ \displaystyle\lim_{t\to 0^+} p(t)y'(t) = 0, & \\ ay(1) + b \displaystyle\lim_{t\to 1^-} p(t)y'(t) = d, & a > 0,\ b \ge 0. \end{cases} \tag{3.15}_\lambda$$

To see this, notice that, if $y \in C[0,1]$ with $py' \in C[0,1]$ is a solution of $(3.14)_\lambda$ then since

$$\int_t^1 \frac{e^{-\phi(s)}}{p(s)} \int_0^s e^{\phi(x)} f^*(x,y,py')\,dx\,ds \in AC[0,1]$$

we have

$$p(t)y'(t) = \lambda e^{-\phi(t)} \int_0^t e^{\phi(s)} f^*(s,y,py')\,ds - \lambda e^{-\phi(t)} \int_t^1 e^{\phi(s)}\omega(s)\,ds \tag{3.16}$$

almost everywhere on $[0,1]$. The right hand side of (3.16) is a continuous function on $[0,1]$ since $e^{\phi} f^*$ is an L^1_ϕ-Carathéodory function and so the equation in (3.16) holds at each $t \in [0,1]$. Exactly the same reasoning as in theorem 3.1 now shows that y is a solution of $(3.15)_\lambda$.

REMARK. We note that if y is a solution of $(3.15)_\lambda$ then y need not satisfy $(3.14)_\lambda$.

Define the operator $N_\lambda : K^1_{0,d} \to K^1_{0,d}$ by

$$N_\lambda y(t) = \frac{d}{a} - \frac{\lambda b}{a}\left(e^{-\phi(1)}\int_0^1 e^{\phi(s)} f^*(s,y,py')\,ds + \lim_{t\to 1^-} e^{-\phi(t)}\int_t^1 e^{\phi(s)}\omega(s)\,ds\right)$$
$$-\lambda\int_t^1 \frac{e^{-\phi(s)}}{p(s)}\int_0^s e^{\phi(x)} f^*(x,y,py')\,dx\,ds$$
$$+\lambda\int_t^1 \frac{e^{-\phi(s)}}{p(s)}\int_s^1 e^{\phi(x)}\omega(x)\,dx\,ds.$$

Essentially the same reasoning as in theorem 3.1 implies that N_1 has a fixed point i.e. $(3.14)_1$ has a solution i.e. (3.11) has a solution. □

REMARK. Analogue versions of theorems 3.1 and 3.2 hold for the boundary value problem

$$\begin{cases} (py')' + \phi' py' = f(t,y,py'), & 0<t<1,\\ -\alpha y(0) + \beta \lim_{t\to 0^+} p(t)y'(t) = c, & \alpha>0,\ \beta\ge 0,\\ \lim_{t\to 1^-} p(t)y'(t) = d. \end{cases}$$

THEOREM 3.3. *Suppose f has the decomposition $f(t,u,v) = f^*(t,u,v) + \omega(t)$ and assume* (3.2), (3.3), (3.12) *and* (3.13) *hold. In addition suppose that*

$$(3.17)\quad \begin{cases} \text{there exist constants } \gamma_1 \text{ and } \gamma_2,\ 0\le\gamma_1,\gamma_2<1 \text{ with}\\ |f^*(t,u,v)| \le q_1(t)|u|^{\gamma_1} + q_2(t)|v|^{\gamma_2} + q_3(t) \text{ for almost all } t\in[0,1].\\ \text{Here } e^{\phi} q_j \in L^1[0,1] \text{ and } (e^{-\phi(s)}/p(s))\textstyle\int_0^s e^{\phi(x)} q_j(x)\,dx \in L^1[0,1],\\ j=1,2,3, \text{ and there exists a constant } K_0 \text{ with}\\ \sup_{[0,1]}(e^{-\phi(t)}|\textstyle\int_t^1 e^{\phi(s)}\omega(s)\,ds|) \le K_0 \end{cases}$$

is satisfied. Then (3.11) *has at least one solution.*

PROOF. By theorem 3.2 we need only establish *a priori* bounds for solutions $y \in C[0,1]$ with $py' \in C[0,1]$ to $(3.14)_\lambda$. Now for $t\in[0,1]$ we infer from (3.16) that

$$p(t)y'(t) = \lambda e^{-\phi(t)}\int_0^t e^{\phi(s)} f^*(s,y,py')\,ds - \lambda e^{-\phi(t)}\int_t^1 e^{\phi(s)}\omega(s)\,ds$$

and so (3.17) implies

$$|p(t)y'(t)| \le (|y|_0)^{\gamma_1} e^{-\phi(t)}\int_0^t e^{\phi(x)} q_1(x)\,dx + (|py'|_0)^{\gamma_2} e^{-\phi(t)}\int_0^t e^{\phi(x)} q_2(x)\,dx$$
$$+e^{-\phi(t)}\int_0^t e^{\phi(x)} q_3(x)\,dx + e^{-\phi(t)}\left|\int_t^1 e^{\phi(x)}\omega(x)\,dx\right|$$

where $|y|_0 = \sup_{[0,1]} |y(t)|$ and $|py'|_0 = \sup_{(0,1)} |p(t)y'(t)|$. Thus there exist constants A_0, A_1 and A_2 with

$$|py'|_0 \le A_0(|y|_0)^{\gamma_1} + A_1(|py'|_0)^{\gamma_2} + A_2.$$

This together with $0 \le \gamma_2 < 1$ implies that there exist constants A_3 and A_4 with

$$|py'|_0 \le A_3(|y|_0)^{\gamma_1} + A_4. \tag{3.18}$$

On the other hand from $(3.14)_\lambda$ for $t \in [0,1]$ we have

$$\begin{aligned} y(t) =& \frac{d}{a} - \frac{\lambda b}{a}\left(e^{-\phi(1)} \int_0^1 e^{\phi(s)} f^*(s,y,py')\,ds + \lim_{t\to 1^-} e^{-\phi(t)} \int_t^1 e^{\phi(s)}\omega(s)\,ds\right) \\ & - \lambda \int_t^1 \frac{e^{-\phi(s)}}{p(s)} \int_0^s e^{\phi(x)} f^*(x,y,py')\,dx\,ds + \lambda \int_t^1 \frac{e^{-\phi(s)}}{p(s)} \int_s^1 e^{\phi(x)}\omega(x)\,dx\,ds \end{aligned}$$

and so (3.17) implies that there are constants B_0, B_1 and B_2 with

$$|y|_0 \le B_0(|y|_0)^{\gamma_1} + B_1(|py'|_0)^{\gamma_2} + B_2. \tag{3.19}$$

Put (3.18) into (3.19) and deduce that there are constants C_0 and C_1 with

$$|y|_0 \le B_0(|y|_0)^{\gamma_1} + C_0(|y|_0)^{\gamma_1\gamma_2} + C_1.$$

Consequently since $\gamma_1, \gamma_2 < 1$ there exists a constant M with $|y|_0 \le M$. This together with (3.18) yields $|py'|_0 \le A_3M^{\gamma_1} + A_4 \equiv M_1$ and the result follows from theorem 3.2. □

REMARKS. (i) An analogue of theorem 3.3 may be obtained for the boundary value problem (3.1) with ϕ and f satisfying the assumptions in theorem 3.1.

(ii) We remark here as well that the Bernstein theory [1, 3–5, 8–9] could be developed for (3.1) with ϕ and f satisfying the assumptions of theorem 3.1. Since the ideas are similar to those in [1, 3] we omit the details.

Finally in this paper we obtain an existence principle for the Sturm-Liouville boundary value problem

$$\begin{cases} (py')' + \phi' py' = f(t,y,py'), & 0 < t < 1 \\ -\alpha y(0) + \beta \lim_{t\to 0^+} p(t)y'(t) = c; & \alpha > 0,\ \beta \ge 0 \\ ay(1) + b\lim_{t\to 1^-} p(t)y'(t) = d; & a > 0,\ b \ge 0. \end{cases} \tag{3.20}$$

THEOREM 3.4. *Suppose* (3.3) *holds and*

$$\begin{cases} \phi : [0,1] \to \mathbf{R} \cup \{\infty\} \text{ with } \phi \text{ differentiable almost everywhere} \\ \text{and } e^{-\phi} \text{ continuous on } [0,1]. \text{ Also, } e^{-\phi}/p \in L^1[0,1] \end{cases} \tag{3.21}$$

and

$$e^{\phi} f \text{ is an } L^1_{\phi}\text{-Carathéodory function} \tag{3.22}$$

are satisfied. In addition assume there is a constant M, independent of λ, such that $|y|_1 \leq M$ for any solution y to

$$(3.23)_\lambda \qquad \begin{cases} (py')' + \phi' py' = \lambda f(t, y, py'), & 0 < t < 1, \\ -\alpha y(0) + \beta \lim_{t \to 0+} p(t)y'(t) = c, & \alpha > 0,\ \beta \geq 0, \\ ay(1) + b \lim_{t \to 1-} p(t)y'(t) = d, & a > 0,\ b \geq 0 \end{cases}$$

for each $\lambda \in (0,1)$. Then (3.20) has at least one solution.

PROOF. Solving $(3.23)_\lambda$ is equivalent to finding a solution $y \in C[0,1]$ with $py' \in C[0,1]$ to

$$(3.24)_\lambda \qquad y(t) = B + A \int_0^t \frac{e^{-\phi(s)}}{p(s)}\, ds + \lambda \int_0^t \frac{e^{-\phi(s)}}{p(s)} \int_0^s e^{\phi(x)} f(x, y, py')\, dx\, ds$$

where

(3.25)

$$A = \frac{ca + d\alpha - \lambda\alpha a \int_0^1 \frac{e^{-\phi(s)}}{p(s)} \int_0^s e^{\phi} f(x, y, py')\, dx\, ds - \lambda b\alpha e^{-\phi(1)} \int_0^1 e^{\phi} f(x, y, py')\, dx}{a\beta e^{-\phi(0)} + \alpha a \int_0^1 \frac{e^{-\phi(s)}}{p(s)}\, ds + b\alpha e^{-\phi(1)}}$$

and

$$B = \frac{A\beta}{\alpha} e^{-\phi(0)} - \frac{c}{\alpha}. \tag{3.26}$$

To see this, notice that, if y is a solution of $(3.23)_\lambda$ then $(e^{\phi} py')' = \lambda e^{\phi} f$ almost everywhere on $[0,1]$. Integration yields

$$p(t)y'(t) = Ae^{-\phi(t)} + \lambda e^{-\phi(t)} \int_0^t e^{\phi(s)} f(s, y(s), p(s)y'(s))\, ds$$

where A is an arbitrary constant. Notice since $e^{\phi} f$ is an L^1_{ϕ}-Carathéodory function and $e^{-\phi}$ is continuous, the right hand side of the above equality is a continuous function on $[0,1]$. Another integration yields

$$y(t) = B + A \int_0^t \frac{e^{-\phi(s)}}{p(s)}\, ds + \lambda \int_0^t \frac{e^{-\phi(s)}}{p(s)} \int_0^s e^{\phi(x)} f(x, y, py')\, dx\, ds$$

where B is an arbitrary constant. Now since $-\alpha y(0) + \beta \lim_{t \to 0+} p(t)y'(t) = c$ and $ay(1) + b \lim_{t \to 1-} p(t)y'(t) = d$ we have

$$c = -\alpha B + \beta A e^{-\phi(0)} \tag{3.27}$$

and

$$d = a\left(B + A\int_0^1 \frac{e^{-\phi(s)}}{p(s)}\,ds + \lambda\int_0^1 \frac{e^{-\phi(s)}}{p(s)}\int_0^s e^{\phi(x)}f(x,y,py')\,dx\,ds\right) + b\left(Ae^{-\phi(1)} + \lambda e^{-\phi(1)}\int_0^1 e^{\phi(s)}f(s,y,py')\,ds\right). \tag{3.28}$$

Solving (3.27) and (3.28) yields (3.25) and (3.26). Thus y is a solution of $(3.24)_\lambda$.

On the other hand, if $y \in C[0,1]$ with $py' \in C[0,1]$ is a solution of $(3.24)_\lambda$ then since

$$\int_0^t \frac{e^{-\phi(s)}}{p(s)}\,ds,\ \int_0^t \frac{e^{-\phi(s)}}{p(s)}\int_0^s e^{\phi(x)}f(x,y,py')\,dx\,ds \in AC[0,1]$$

we have

$$p(t)y'(t) = Ae^{-\phi(t)} + \lambda e^{-\phi(t)}\int_0^t e^{\phi(s)}f(s,y(s),p(s)y'(s))\,ds \tag{3.29}$$

almost everywhere on $[0,1]$. Since the right hand side of (3.29) is a continuous function on $[0,1]$, the equation in (3.29) holds at each $t \in [0,1]$. Thus on $[0,1]$,

$$e^{\phi(t)}p(t)y'(t) = A + \lambda\int_0^t e^{\phi(s)}f(s,y(s),p(s)y'(s))\,ds$$

and so

$$(e^{\phi}py')' = \lambda e^{\phi}f \qquad \text{almost everywhere on } [0,1].$$

In addition, $-\alpha y(0) + \beta\lim_{t\to 0+} p(t)y'(t) = c$ and $ay(1) + b\lim_{t\to 1-} p(t)y'(t) = d$. Thus y is a solution of $(3.23)_\lambda$.

Let K^1 be as in theorem 3.1 and

$$K^1_{SL} = \{u \in K^1 : -\alpha u(0) + \beta\lim_{t\to 0+} p(t)u'(t) = c \text{ and } au(1) + b\lim_{t\to 1-} p(t)u'(t) = d\}.$$

Define the operator $N_\lambda : K^1_{SL} \to K^1_{SL}$ by

$$N_\lambda y(t) = B + A\int_0^t \frac{e^{-\phi(s)}}{p(s)}\,ds + \lambda\int_0^t \frac{e^{-\phi(s)}}{p(s)}\int_0^s e^{\phi(x)}f(x,y,py')\,dx\,ds$$

where A and B are as in (3.25) and (3.26).

Essentially the same reasoning as in theorem 3.1 implies that N_1 has a fixed point, i.e. (3.20) has a solution. □

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