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Paracomplex Structures and Affine Symmetric Spaces

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(Dedicated to the memory of Prof. Mikao Moriya)

Introduction

Let M be a smooth manifold. A splitting of the tangent bundle $T(M)$ of M into the Whitney sum of two subbundles $T^{\pm}(M)$ is called an *almost paracomplex structure* on M, if $T^{\pm}(M)$ have the same fiber dimension. This is characterized by a (1, 1)-tensor field I satisfying the conditions: $I^2=\mathrm{id}$, and ± 1-eigenspaces of I_p $(p\in M)$ are the fibers of $T^{\pm}(M)$ over p. If the distributions on M defined by $T^{\pm}(M)$ are both completely integrable, then the almost paracomplex structure is called a *paracomplex structure*. These two structures were originally introduced by P. Libermann in 1952 ([5], [6]), in analogy with almost complex or complex structures. Libermann also introduced, although in somewhat vague fashion, the notions of *parahermitian metrics* and *parakähler metrics*, which are the paracomplex analogues of hermitian and Kähler metrics. It should be noted that a parakähler manifold has naturally a symplectic structure. The main interest is thus to what extent one can develop the theory of paracomplex manifolds in parallel with the theory of complex manifolds.

In this article, we introduce a class of affine symmetric spaces, called *parahermitian symmetric spaces*, a paracomplex analogue of hermitian symmetric spaces. §1 is devoted to some definitions and basic properties on paracomplex structures. In §2 we give the definition of parahermitian symmetric spaces and include Lie algebraic considerations. In §3 we give a group-theoretic characterization for an affine symmetric coset space G/H with G semisimple to be parahermitian symmetric (cf. Theorems 3.6 and 3.7). In §4 we consider a relation between parahermitian symmetric spaces of semisimple Lie groups and symmetric R-spaces (cf. Proposition 4.1 and Theorem 4.3). Finally we give the infinitesimal classification of parahermitian symmetric spaces with semisimple automorphism groups,

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up to paraholomorphic equivalence.

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NOTATIONS. The following notations will be used throughout this paper:

G^0 the identity component of a Lie group G,
Lie G the Lie algebra of a Lie group G
$C_G(Z)=\{a\in G: (\mathrm{Ad}\,a)Z=Z\}$ the centralizer of $Z\in \mathrm{Lie}\,G$ in G,
$\boldsymbol{H}$ the quaternion number field,
$\boldsymbol{O}$ the octanion number field,
$\mathfrak{g}^C$ the complexification of a Lie algebra $\mathfrak{g}$,
G_σ the totality of the elements in a group G left fixed by an automorphism σ,
$\mathfrak{c}_{\mathfrak{g}}(Z)$ the centralizer of the element Z, in $\mathfrak{g}$,
id the identity mapping.

§1. Paracomplex structures.

Let M be a $2n$-dimensional real smooth manifold and $\mathfrak{X}(M)$ be the Lie algebra of smooth vector fields on M. A smooth (1, 1)-tensor field I on M is called an *almost paracomplex structure* ([6]), if the following conditions are satisfied:

AP i) $I^2=\mathrm{id}$,

AP ii) for each point $p\in M$, the ± 1-eigenspaces $T_p^{\pm}(M)$ of I_p(= the value of I at p) are both n-dimensional subspaces of the tangent space $T_p(M)$ at p.

In this case, the pair (M, I) is called an *almost paracomplex manifold*. On an almost paracomplex manifold (M, I) we define the tensor field T, called the *torsion tensor field* of I, by putting ([13])

$$T(X, Y)=[IX, IY]-I[IX, Y]-I[X, IY]+[X, Y] \tag{1.1}$$

for $X, Y\in\mathfrak{X}(M)$. If T vanishes identically on M, then I is called a *paracomplex structure* ([6]), and (M, I) is called a *paracomplex manifold*. The following proposition was proved by Yano [13] and Walker [12] for almost product structures, making use of tensor calculus.

PROPOSITION 1.1. *Let (M, I) be an almost paracomplex manifold. Then I is paracomplex if and only if the distribution $\mathscr{T}^{\pm}$: $p\mapsto T_p^{\pm}(M)$, $p\in M$ are both completely integrable.*

The proof of 1.1 can be done by applying the Frobenius theorem to vector fields of the form $X \pm IX$ $(X \in \mathfrak{X}(M))$, which belong to the distributions $\mathscr{T}^{\pm}$.

PROPOSITION 1.2. *Let (M, I) be a $2n$-dimensional paracomplex manifold. Then M has an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ with U_α open and $\varphi_\alpha = (x_1^\alpha, \cdots, x_n^\alpha, y_1^\alpha, \cdots, y_n^\alpha)$ coordinate map satisfying the condition; if $U_\alpha \cap U_\beta \neq \varnothing$ then the following para-Cauchy-Riemann equation*

$$
\left\{
\begin{aligned}
\frac{\partial x_i^\beta}{\partial x_j^\alpha} &= \frac{\partial y_i^\beta}{\partial y_j^\alpha} \\
\frac{\partial x_i^\beta}{\partial y_j^\alpha} &= \frac{\partial y_i^\beta}{\partial x_j^\alpha}
\end{aligned}
\right.
\qquad (1 \leqq i,\ j \leqq n;\ \alpha,\ \beta \in A) \tag{1.2}
$$

is satisfied. In this case, on each U_α, I is given by

$$
I\left(\frac{\partial}{\partial x_i^\alpha}\right) = \frac{\partial}{\partial y_i^\alpha}\,, \qquad I\left(\frac{\partial}{\partial y_i^\alpha}\right) = \frac{\partial}{\partial x_i^\alpha}\,. \tag{1.3}
$$

Conversely, suppose that M has an atlas $\{(U_\alpha, \varphi_\alpha)\}$ satisfying (1.2). *Then, if we define I on U_α by* (1.3), *then I is globally defined on M, and (M, I) is a paracomplex manifold.*

PROOF. Suppose that (M, I) is a paracomplex manifold. Then M is covered by coordinate neighborhoods U_α $(\alpha \in A)$ with coordinate map $\psi_\alpha = (u_1^\alpha, \cdots, u_n^\alpha, v_1^\alpha, \cdots, v_n^\alpha)$ such that the distribution $\mathscr{T}^+$ is spanned by $(\partial/\partial u_1^\alpha), \cdots, (\partial/\partial u_n^\alpha)$ and that $\mathscr{T}^-$ is spanned by $(\partial/\partial v_1^\alpha), \cdots, (\partial/\partial v_n^\alpha)$. The coordinate system $(x_1^\alpha, \cdots, x_n^\alpha, y_1^\alpha, \cdots, y_n^\alpha)$ on U_α defined by $x_i^\alpha = u_i^\alpha + v_i^\alpha$, $y_i^\alpha = u_i^\alpha - v_i^\alpha$ is the desired one. The proposition is proved by straightforward computations.

Let (M, I) and (M', I') be (almost) paracomplex manifolds. Then a smooth map f of M to M' is called a *(almost) paraholomorphic map*, if the relation

$$
f_{*p} I_p = I'_{f(p)} f_{*p} \tag{1.4}
$$

is satisfied for each point $p \in M$, where f_{*p} is the differential of f at p. If there is a paraholomorphic diffeomorphism of M onto M', then (M, I) and (M', I') are said to be *paraholomorphically equivalent*. A paraholomorphic diffeomorphism of M onto itself is called a *paraholomorphic transformation* of M. The distributions $\mathscr{T}^{\pm}$ are invariant by paraholomorphic transformations. We denote by $\mathrm{Aut}\,(M, I)$ the group of all paraholomorphic transformations of (M, I).

EXAMPLE. Let $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ be the natural coordinate on $\boldsymbol{R}^{2n}$. Let us consider the following two kinds of foliations;

$$x_i+y_i=\text{const} \qquad 1\leqq i\leqq n\ ,$$
$$x_i-y_i=\text{const} \qquad 1\leqq i\leqq n\ ,$$

which define a paracomplex structure on $\boldsymbol{R}^{2n}$. Then the above foliations are both invariant under the translations by the lattice $\boldsymbol{Z}^{2n}$ of all integral points in $\boldsymbol{R}^{2n}$. So, they naturally induce a paracomplex structure on the torus $\boldsymbol{R}^{2n}/\boldsymbol{Z}^{2n}$. The canonical projection π of $\boldsymbol{R}^{2n}$ onto the torus is paraholomorphic.

Now let (M, I) be a (almost) paracomplex manifold, and g be a pseudo-Riemannian metric of M. If the equality

$$g(IX, Y)+g(X, IY)=0 \qquad X, Y\in \mathfrak{X}(M) \tag{1.5}$$

is satisfied, then g is called a *(almost) parahermitian metric* ([6]), and (M, I, g) is called a *(almost) parahermitian manifold.* In this case we can consider a 2-form ω defined by

$$\omega(X, Y)=g(X, IY) \qquad X, Y\in \mathfrak{X}(M)\ . \tag{1.6}$$

Note that an almost parahermitian manifold is orientable. If ω is closed, then g is called a *(almost) parakähler metric* ([6]), and (M, I, g) is then called a *(almost) parakähler manifold.* It should be noted that if (M, I, g) is almost parakähler, then ω is a symplectic 2-form on M, and consequently M has an almost complex structure. Given a connected (almost) parahermitian or parakähler manifold (M, I, g) and denoting by $I(M, g)$ the isometry group of M with respect to g, the *automorphism group* of (M, I, g) is defined by putting

$$\mathrm{Aut}\,(M, I, g)=\mathrm{Aut}\,(M, I)\cap I(M, g)\ ,$$

which is a closed subgroup of $I(M, g)$, and consequently a Lie transformation group of M. If $\mathrm{Aut}\,(M, I, g)$ acts transitively on M, then (M, I, g) is called a *homogeneous (almost) parahermitian* or *parakähler manifold,* according as it is (almost) parahermitian or parakähler. Note that a homogeneous almost parakähler manifold is a homogeneous symplectic manifold with respect to ω and $\mathrm{Aut}\,(M, I, g)$.

REMARK. In defining parahermitian metrics, one need not to assume AP ii). From (1.5) it follows that the eigenspaces $T_p^{\pm}(M)$ of I_p are maximal totally isotropic subspaces with respect to g_p. This can be shown without using the assumption AP ii). The decomposition $T_p(M)=$

$T_p^+(M)+T_p^-(M)$ is then the Witt decomposition, which implies that two maximal totally isotropic subspaces $T_p^\pm(M)$ are equidimensional. We have then that the signature of g is (n, n) and that M must be even dimensional.

PROPOSITION 1.3. *Let (M, I, g) be an almost parahermitian manifold, and ∇ be the Levi-Civita connection of g. Then we have, for $X, Y, Z \in \mathfrak{X}(M)$,*

$$(1.7)\qquad \begin{aligned} &2g((\nabla_X I)Y, Z)+3d\omega(X, IY, IZ)+3d\omega(X, Y, Z) \\ &\quad =-g(IX, T(Y, Z)) . \end{aligned}$$

PROOF. We proceed in the similar way as in the complex structure (cf. Kobayashi-Nomizu [3]). We have

$$\begin{aligned} 2g((\nabla_X I)Y, Z) &= 2g(\nabla_X(IY), Z)+2g(\nabla_X Y, IZ) \\ &= IY(g(X, Z))-Z(g(X, IY))+Y(g(X, IZ))-IZ(g(X, Y)) \\ &\quad +g([X, IY], Z)+g([Z, X], IY)-g(IX, I[Z, IY]) \\ &\quad +g([X, Y], IZ)+g([IZ, X], Y)-g(IX, I[IZ, Y]) , \end{aligned}$$

$$\begin{aligned} 3d\omega(X, IY, IZ) &= X(g(IY, Z))-IY(g(X, Z))+IZ(g(X, Y)) \\ &\quad -g([X, IY], Z)+g([X, IZ], Y)-g([IY, IZ], IX) , \end{aligned}$$

$$\begin{aligned} 3d\omega(X, Y, Z) &= X(g(Y, IZ))-Y(g(X, IZ))+Z(g(X, IY)) \\ &\quad -g([X, Y], IZ)+g([X, Z], IY)-g([Y, Z], IX) . \end{aligned}$$

From these equalities we get (1.7) by direct calculations.

COROLLARY 1.4. *Let (M, I, g) and ∇ be as in 1.3. Then (M, I, g) is parakähler if and only if $\nabla I=0$.*

PROOF. The equality $\nabla I=0$ implies $\nabla\omega=0$. This means $d\omega=0$, since ∇ is torsionfree. So, by (1.7) and 1.1, we conclude that I is paracomplex, and (M, I, g) is parakähler. The converse is also easily seen by 1.3.

LEMMA 1.5. *Let (M, I, g) be parakähler and R be the curvature tensor field of the Levi-Civita connection. Then we have*

$$R(IX, Y)+R(X, IY)=0 \qquad X, Y \in \mathfrak{X}(M) .$$

PROOF. By 1.4, we see $[\nabla_X, I]=0$; so, from the definition of the curvature tensor field, we have $[R(X, Y), I]=0$. Hence it follows that, for $X, Y, U, V \in \mathfrak{X}(M)$,

$$\begin{aligned} g(R(IX, Y)V, U) &= g(R(U, V)Y, IX) \\ &= -g(IR(U, V)Y, X) = -g(R(U, V)IY, X) \\ &= -g(R(X, IY)V, U) , \end{aligned}$$

which proves the lemma.

§2. Parahermitian symmetric spaces.

DEFINITION 2.1. A connected almost parahermitian manifold (M, I, g) is called a *parahermitian symmetric space*, if for each point $p \in M$ there exists an almost paraholomorphic isometry $s_p \in \mathrm{Aut}(M, I, g)$, called the symmetry at p, such that

i) $s_p^2 = \mathrm{id}$,

ii) p is an isolated fixed point of s_p.

LEMMA 2.1. *Let (M, I, g) be a parahermitian symmetric space. Then* $\mathrm{Aut}(M, I, g)$ *acts transitively on M.*

PROOF. M is affine symmetric with respect to the Levi-Civita connection; so, the proof is done by the same arguments as in Kobayashi-Nomizu [3, II, p. 223].

PROPOSITION 2.2. *A parahermitian symmetric space (M, I, g) is homogeneous parakähler, and hence homogeneous symplectic.*

PROOF. We denote by G the identity component of $\mathrm{Aut}(M, I, g)$. Take a point $o \in M$. Then the mapping σ sending $a \in G$ to $s_o a s_o$ is an involutive automorphism of G. Let us denote by G_σ the subgroup of all σ-fixed elements in G, and let H be the isotropy subgroup of G at o. Then we have

$$G_\sigma^0 \subset H \subset G_\sigma , \tag{2.1}$$

where G_σ^0 is the identity component of G_σ. And $M = G/H$ is a symmetric homogeneous space and the Levi-Civita connection coincides with the canonical connection (cf. Nomizu [8]). I being G-invariant, we have $\nabla I = 0$ ([8]). So, by 1.4 and 2.1, (M, I, g) is homogeneous parakähler.

DEFINITION 2.2. Let G be a connected Lie group and H be a closed subgroup of G. The coset space $M = G/H$ is called a *parahermitian symmetric coset space* if the following three conditions are satisfied:

i) (G, H, σ) is a *symmetric triple*, that is, σ is an involutive automorphism satisfying (2.1),

ii) there exist a (1, 1)-tensor field I and a pseudo-Riemannian metric g on M such that (M, I, g) is almost parahermitian,

iii) I and g are both G-invariant.

LEMMA 2.3. *A parahermitian symmetric space can be represented as a parahermitian symmetric coset space. Conversely a parahermitian symmetric coset space is a parahermitian symmetric space.*

PROOF. The first assertion was proved in the proof of 2.2. Let $(G/H, I, g)$ be a parahermitian symmetric coset space, and π be the projection of G onto G/H. The symmetry s_o at o is given by

$$s_o(\pi(a))=\pi(\sigma(a)) , \quad a\in G .$$

Then, by the same way as in the hermitian symmetric case, one can deduce that s_o is an almost paraholomorphic isometry (cf. Helgason [1]). So the symmetry s_p at any point $p\in G/H$ is also an almost paraholomorphic isometry.

Later on we shall be mainly concerned with parahermitian symmetric coset spaces. Let $\mathfrak{g}$ be a real Lie algebra, $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$ and σ be an involutive automorphism of $\mathfrak{g}$. If $\mathfrak{h}$ is the fixed set in $\mathfrak{g}$ by σ, then $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ is called a *symmetric triple*. Let G be a connected Lie group with $\operatorname{Lie} G=\mathfrak{g}$, and H be a closed subgroup of G with $\operatorname{Lie} H=\mathfrak{h}$. Then we say that the coset space G/H is *associated with* $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ if σ can be extended to an involutive automorphism (denoted by the same letter σ) of G and if (2.1) is satisfied.

PROPOSITION 2.4. *Let* $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ *be a symmetric triple, and*

$$\mathfrak{g}=\mathfrak{h}+\mathfrak{m} \tag{2.2}$$

be the eigenspace decomposition by σ. *Suppose that a coset space* G/H *is associated with* $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$. *Then* G/H *is a parahermitian symmetric coset space, if and only if the following condition* (C_1) *is satisfied*:

(C_1) *There exists a linear endomorphism* I_o *on* $\mathfrak{m}$ *and a non-degenerate symmetric bilinear form* $\langle\ ,\ \rangle$ *on* $\mathfrak{m}$ *such that*

(C_1-1) $I_o^2=\mathrm{id}$,

(C_1-2) $[I_o, \mathrm{Ad}_{\mathfrak{m}} H]=0$,

(C_1-3) $\langle I_oX, Y\rangle+\langle X, I_oY\rangle=0$, $X, Y\in\mathfrak{m}$,

(C_1-4) $\langle(\mathrm{Ad}_{\mathfrak{m}} h)X, (\mathrm{Ad}_{\mathfrak{m}} h)Y\rangle=\langle X, Y\rangle$, $X, Y\in\mathfrak{m}$, $h\in H$.

PROOF. Suppose that G/H is parahermitian symmetric. Let us identify $\mathfrak{m}$ with the tangent space $T_o(M)$ at the origin o(=the coset H), and choose the symmetric bilinear form $\langle\ ,\ \rangle$ to be $g_o(\ ,\ )$. Then (C_1) is satisfied. Conversely, suppose that (C_1) holds. Then the quite analogous procedure as in the hermitian symmetric case shows that I_o and $\langle\ ,\ \rangle$ can be extended to the G-invariant tensor fields on the whole M. So, the lemma follows.

DEFINITION 2.3. Let $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ be a symmetric triple and $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ be the eigenspace decomposition by σ. Suppose that $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ satisfies the following condition (C_2):

(C_2) There exist a linear endomorphism I_o on $\mathfrak{m}$ and a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{m}$ such that

$(C_1\text{-}1)$ $I_o^2=\mathrm{id}$,

$(C_2\text{-}2)$ $[I_o, \mathrm{ad}_{\mathfrak{m}}\,\mathfrak{h}]=0$,

$(C_1\text{-}3)$ $\langle I_oX, Y\rangle+\langle X, I_oY\rangle=0$, $X, Y\in\mathfrak{m}$,

$(C_2\text{-}4)$ $\langle(\mathrm{ad}\,X)Y_1, Y_2\rangle+\langle Y_1, (\mathrm{ad}\,X)Y_2\rangle=0$, $X\in\mathfrak{h}$, $Y_1, Y_2\in\mathfrak{m}$.

Then $\{\mathfrak{g}, \mathfrak{h}, \sigma, I_o, \langle\ ,\ \rangle\}$ is called a *parahermitian symmetric system*. Furthermore if the pair $\{\mathfrak{g}, \mathfrak{h}\}$ is effective, then it is called an *effective* parahermitian symmetric system.

LEMMA 2.5. *Let $\{\mathfrak{g}, \mathfrak{h}, \sigma, I_o, \langle\ ,\ \rangle\}$ be an effective parahermitian symmetric system, and let us define a linear endomorphism $\tilde{I}_o$ of $\mathfrak{g}$ by putting $\tilde{I}_o|_{\mathfrak{h}}=0$, and $\tilde{I}_o|_{\mathfrak{m}}=I_o$. Then $\tilde{I}_o$ is a derivation of $\mathfrak{g}$.*

PROOF. We have to prove the equality

$$\tilde{I}_o[X, Y]=[\tilde{I}_oX, Y]+[X, \tilde{I}_oY] \quad X, Y\in\mathfrak{g}\,. \tag{2.3}$$

For the case where $X\in\mathfrak{h}$, $Y\in\mathfrak{m}$, the condition $(C_2\text{-}2)$ implies (2.3). Let us consider the case where $X, Y\in\mathfrak{m}$. The left-hand side of (2.3) is zero in this case, since $[\mathfrak{m}, \mathfrak{m}]\subset\mathfrak{h}$. For $X, Y\in\mathfrak{h}$, (2.3) is trivially satisfied. Let G/H be a coset space associated with $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$. Then, from 2.4, it is seen that G/H^0 is a parahermitian symmetric coset space corresponding to $\{\mathfrak{g}, \mathfrak{h}, \sigma, I_o, \langle\ ,\ \rangle\}$. Applying 1.5 to G/H^0, we get

$$\begin{aligned}\mathrm{ad}_{\mathfrak{m}}[I_oX, Y]&=-R_o(I_oX, Y)=R_o(X, I_oY)\\&=-\mathrm{ad}_{\mathfrak{m}}[X, I_oY]\,.\end{aligned}$$

The linear isotropy representation of $\mathfrak{h}$ on $\mathfrak{m}$ is faithful for the almost effective affine symmetric space G/H^0. Hence we obtain $[I_oX, Y]=-[X, I_oY]$, which proves the proposition.

The following lemma is an easy consequence of 2.5 and $(C_2\text{-}2)$.

LEMMA 2.6. *Let $\{\mathfrak{g}, \mathfrak{h}, \sigma, I_o, \langle\ ,\ \rangle\}$ be an effective parahermitian sym-*

metric system, and $\mathfrak{m}^{\pm}$ *be the* ± 1*-eigenspaces of* I_o *in* $\mathfrak{m}$. *Put* $\mathfrak{g}_{\pm 1}=\mathfrak{m}^{\pm}$, $\mathfrak{g}_0=\mathfrak{h}$. *Then the decomposition* $\mathfrak{g}=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1$ *satisfies*

$$[\mathfrak{g}_{\lambda}, \mathfrak{g}_{\mu}]\subset\mathfrak{g}_{\lambda+\mu}\,.$$

REMARK. For an almost effective parahermitian symmetric coset space G/H, the distribution $\mathscr{T}^{\pm}$ defined by I coincides with the G-invariant distributions obtained from $\mathfrak{m}^{\pm}$. The leaves of $\mathscr{T}^{\pm}$ are flat totally geodesic submanifolds of G/H.

§3. Semisimple parahermitian symmetric coset spaces.

LEMMA 3.1. *Let* $\{\mathfrak{g}, \mathfrak{h}, \sigma, I_o, \langle\ ,\ \rangle\}$ *be an effective semisimple (that is,* $\mathfrak{g}$ *semisimple) parahermitian symmetric system. Then there exists a unique element* $Z^0\in\mathfrak{h}$ *such that* $\mathfrak{h}$ *is the centralizer* $\mathfrak{c}(Z^0)$ *of* Z^0 *in* $\mathfrak{g}$ *and that* $I_o=\mathrm{ad}_{\mathfrak{m}} Z^0$.

PROOF. Let $\tilde{I}_o$ be as in 2.5. It follows from 2.5 that there exists an element $Z^0\in\mathfrak{g}$ such that $\tilde{I}_o=\mathrm{ad}\, Z^0$, since $\mathfrak{g}$ is semisimple. $\mathfrak{h}$ is contained in $\mathfrak{c}(Z^0)$, since $\tilde{I}_o|_{\mathfrak{h}}=0$. Take $X\in\mathfrak{c}(Z^0)$ and put $X=X_1+X_2$, where $X_1\in\mathfrak{h}$, $X_2\in\mathfrak{m}$. Then we have $0=[Z^0, X]=[Z^0, X_2]$, and so $X_2=I_o^2X_2=[Z^0, [Z^0, X_2]]=0$, which implies $X\in\mathfrak{h}$ and $\mathfrak{h}=\mathfrak{c}(Z^0)$. $\{\mathfrak{g}, \mathfrak{h}\}$ being effective, $\mathrm{ad}_{\mathfrak{m}} X=0$ $(X\in\mathfrak{h})$ implies $X=0$. Hence we get the uniqueness of the element Z^0.

Let $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ be an effective semisimple symmetric triple. We need here the following assumption (C_3).

(C_3) There exists an element $Z\in\mathfrak{g}$ such that $\mathrm{ad}\, Z$ is a semisimple operator having real eigenvalues only and that $\mathfrak{h}=\mathfrak{c}_{\mathfrak{g}}(Z)$.

PROPOSITION 3.2. *Let* $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ *be an effective semisimple symmetric triple, and* $\mathfrak{g}=\mathfrak{g}_1+\cdots+\mathfrak{g}_s$ *be the decomposition into simple ideals. Suppose that* $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ *satisfies* (C_3). *Then* i) $\mathfrak{g}_i$ $(1\leqq i\leqq s)$ *is stable under* σ, ii) *denoting by* σ_i *the restriction of* σ *to* $\mathfrak{g}_i$ *and putting* $\mathfrak{h}_i=\mathfrak{g}_i\cap\mathfrak{h}$, *the triple* $\{\mathfrak{g}_i, \mathfrak{h}_i, \sigma_i\}$ *is an effective simple (that is,* $\mathfrak{g}_i$ *simple) symmetric triple satisfying* (C_3). *In other words,* $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ *is the direct sum of* $\{\mathfrak{g}_i, \mathfrak{h}_i, \sigma_i\}$ *satisfying* (C_3).

PROOF. We write Z in the form $Z=Z_1+\cdots+Z_s$, $Z_i\in\mathfrak{g}_i$. Then it is easily seen that $\mathfrak{h}_i$ coincides with the centralizer $\mathfrak{c}_{\mathfrak{g}_i}(Z_i)$ of Z_i in $\mathfrak{g}_i$, since $\mathfrak{h}=\mathfrak{c}_{\mathfrak{g}}(Z)$ holds. Hence we have $\mathfrak{h}=\mathfrak{h}_1+\cdots+\mathfrak{h}_s$. Each $Z_i (1\leqq i\leqq s)$ is not zero. In fact, if $Z_i=0$, then we get $\mathfrak{h}_i=\mathfrak{g}_i$, which contradicts to the effectivity of $\{\mathfrak{g}, \mathfrak{h}\}$. $\sigma(\mathfrak{g}_i)$ is a simple ideal of $\mathfrak{g}$ and so, there exists a $j (1\leqq j\leqq s)$ such that $\sigma(\mathfrak{g}_i)=\mathfrak{g}_j$. Suppose that $j\neq i$. Then we have $\mathfrak{g}_j\supset$

$\sigma(\mathfrak{h}_i)=\mathfrak{h}_i\subset\mathfrak{g}_i$, which is a contradiction. So $\mathfrak{g}_i$ is stable under σ. Now it is easily seen that $\{\mathfrak{g}_i, \mathfrak{h}_i, \sigma_i\}$ is an effective symmetric triple. The operator $\operatorname{ad} Z$ is the direct sum of the operators $\operatorname{ad} Z_i$: hence the triple $\{\mathfrak{g}_i, \mathfrak{h}_i, \sigma_i\}$ satisfies (C_3) with respect to $Z_i\in\mathfrak{g}_i$.

Let $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ be an effective semisimple symmetric triple. Then, from 3.1 it follows that (C_2) implies (C_3). The following proposition shows the converse implication: $(C_3)\Rightarrow(C_2)$. Let τ be a Cartan involution of $\mathfrak{g}$ which commutes with σ, and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition with respect to τ, where $\mathfrak{k}$ and $\mathfrak{p}$ are $+1$ and -1 eigenspaces of τ, respectively. Then we have

PROPOSITION 3.3. *Let $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ be an effective semisimple symmetric triple, satisfying (C_3). Then there exists an element $Z^0\in\mathfrak{h}\cap\mathfrak{p}$ such that $\{\mathfrak{g}, \mathfrak{h}, \sigma, I_0, \langle\ ,\ \rangle\}$ is a parahermitian symmetric system, where $I_0=\operatorname{ad}_{\mathfrak{m}} Z^0$ and $\langle\ ,\ \rangle$ denotes the restriction of the Killing form of $\mathfrak{g}$ to the -1-eigenspace $\mathfrak{m}$ of σ in $\mathfrak{g}$.*

PROOF. We will preserve the notations in the proof of 3.2. Putting $\mathfrak{m}_i=\mathfrak{m}\cap\mathfrak{g}_i$, we have the decomposition $\mathfrak{g}_i=\mathfrak{h}_i+\mathfrak{m}_i$, which is also the ±1 eigenspace decomposition under σ_i. It is easily seen that each $\mathfrak{g}_i$ is stable under τ. We denote by τ_i the restriction of τ to $\mathfrak{g}_i$. Putting $\mathfrak{k}_i=\mathfrak{k}\cap\mathfrak{g}_i$ and $\mathfrak{p}_i=\mathfrak{p}\cap\mathfrak{g}_i$, we have $\mathfrak{g}_i=\mathfrak{k}_i+\mathfrak{p}_i$, which is the ±1 eigenspace decomposition by τ_i. By the choice of τ, we have $\tau_i\sigma_i=\sigma_i\tau_i$. So $\mathfrak{h}_i$ and hence the center $\mathfrak{z}(\mathfrak{h}_i)$ of $\mathfrak{h}_i$ are stable under τ_i. Hence we have

$$\mathfrak{z}(\mathfrak{h}_i)=\mathfrak{z}(\mathfrak{h}_i)\cap\mathfrak{k}_i+\mathfrak{z}(\mathfrak{h}_i)\cap\mathfrak{p}_i\ .$$

Since Z_i is in $\mathfrak{z}(\mathfrak{h}_i)$, one can write Z_i in the form $Z_i=Z_i'+Z_i''(1\leqq i\leqq s)$, where $Z_i'\in\mathfrak{z}(\mathfrak{h}_i)\cap\mathfrak{k}_i$, $Z_i''\in\mathfrak{z}(\mathfrak{h}_i)\cap\mathfrak{p}_i$. Then we have $\operatorname{ad} Z_i=\operatorname{ad} Z_i'+\operatorname{ad} Z_i''$. Eigenvalues of the semisimple operators $\operatorname{ad} Z_i'$ and $\operatorname{ad} Z_i''$ are all purely imaginary and all real, respectively. On the other hand, the operator $\operatorname{ad} Z_i$ is semisimple with real eigenvalues only. So we get $\operatorname{ad} Z_i'=0$ and consequently $Z_i'=0$, which implies $Z_i\in\mathfrak{z}(\mathfrak{h}_i)\cap\mathfrak{p}_i$. Hence, by a result of S. Koh [4], the eigenvalues of $\operatorname{ad}_{\mathfrak{m}_i} Z_i$ should be $\pm\lambda_i$ $(\lambda_i>0)$. Let us put $Z^0=\lambda_1^{-1}Z_1+\cdots+\lambda_s^{-1}Z_s\in\mathfrak{h}\cap\mathfrak{p}$. Then $\operatorname{ad} Z^0$ is a semisimple operator; one has $\operatorname{ad} Z^0|_{\mathfrak{h}}=0$ and the eigenvalues of $\operatorname{ad} Z^0|_{\mathfrak{m}}$ are ±1. Therefore $I_0=\operatorname{ad}_{\mathfrak{m}} Z^0$ satisfies $(C_1\text{-}1)$ and $(C_2\text{-}2)$. Since $\langle\ ,\ \rangle$ is the restriction of the Killing form of $\mathfrak{g}$, it is non-degenerate and the conditions $(C_1\text{-}3)$, $(C_2\text{-}4)$ are satisfied.

LEMMA 3.4. *Let G be a connected Lie group and $C(Z)$ be the* *centralizer in G of an element $Z \in \mathfrak{g}(=\mathrm{Lie}\, G)$. Let $\tilde{G}$ be a covering group of G and $\tilde{C}(Z)$ be the centralizer of Z in $\tilde{G}$. Then we have $G/C(Z)=\tilde{G}/\tilde{C}(Z)$ as coset spaces.*

The proof is easy and so omitted.

Let $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ be an effective semisimple symmetric triple satisfying (C_3), $\mathfrak{g}^C$ be the complexification of $\mathfrak{g}$, and $\hat{G}^C$ be the simply connected Lie group with $\mathrm{Lie}\, \hat{G}^C=\mathfrak{g}^C$. σ is extended to an involutive automorphism (denoted by the same letter) of $\mathfrak{g}^C$. Let us denote by θ the conjugation of $\mathfrak{g}^C$ with repect to $\mathfrak{g}$. σ and θ can be extended to involutive automorphisms of $\hat{G}^C$, which are denoted by the same letters.

LEMMA 3.5. *Let $\hat{G}$ be the analytic subgroup of $\hat{G}^C$ with* $\mathrm{Lie}\, \hat{G}=\mathfrak{g}$, *$\hat{C}(Z)$ the centralizer of Z in $\hat{G}$, and $\hat{G}_\sigma$ be the set of fixed points of σ in $\hat{G}$. Then we have $\hat{G}_\sigma=\hat{C}(Z)$.*

PROOF. The set of θ-fixed points in $\hat{G}^C$, denoted by $(\hat{G}^C)_\theta$, is connected, since $\hat{G}^C$ is simply connected (cf. Koh [4]). Hence we get $(\hat{G}^C)_\theta=\hat{G}$. If we denote by $\hat{C}_C(Z)$ (resp. $\hat{C}_C(iZ)$) the centralizer of Z (resp. iZ) in $\hat{G}^C$, then we have

$$\hat{C}_C(Z)=\hat{C}_C(iZ)\ ;$$

the group in the right-hand side is connected by R. A. Shapiro [9]. Noting that $\mathfrak{h}^C=\mathfrak{c}_\mathfrak{g}(Z)^C=\mathfrak{c}_{\mathfrak{g}^C}(Z)$, it follows that $\hat{C}_C(Z)$ is the analytic subgroup of $\hat{G}^C$ corresponding to the complexification $\mathfrak{h}^C$. And $\mathfrak{h}^C$ is the σ-fixed set in $\mathfrak{g}^C$. So, denoting by $(\hat{G}^C)_\sigma$ the σ-fixed set in $\hat{G}^C$, we have $(\hat{G}^C)_\sigma=\hat{C}_C(Z)$, since $\hat{G}^C$ is simply connected ([4]). Therefore we have

$$(\hat{C}_C(Z))_\theta=((\hat{G}^C)_\sigma)_\theta=(\hat{G}^C)_\sigma \cap \hat{G}=\hat{G}_\sigma\ .$$

On the other hand, one has $(\hat{C}_C(Z))_\theta=\hat{C}_C(Z)\cap\hat{G}=\hat{C}(Z)$. q.e.d.

THEOREM 3.6. *Let $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ be an effective semisimple symmetric triple, and G/H be a coset space associated with the triple. Suppose that G/H is a parahermitian symmetric coset space. Then $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ satisfies (C_3), and H is an open subgroup of the centralizer $C(Z)$ in G.*

PROOF. By 2.4 and 3.1, $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ satisfies (C_3). The identity components of H and of $C(Z)$ coincide. We will next show $H\subset C(Z)$. Take an element $h\in H$. Then, by $(\mathrm{C}_1$-$2)$ and 3.1 we have $[\mathrm{Ad}_\mathfrak{m} h, \mathrm{ad}_\mathfrak{m} Z]=0$, which implies $[\mathrm{Ad}\, h, \mathrm{ad}\, Z]=0$ on $\mathfrak{g}$. So, for an element $X\in\mathfrak{g}$,

$$[Z, (\mathrm{Ad}\, h)X]=(\mathrm{Ad}\, h)[Z, X]=[(\mathrm{Ad}\, h)Z, (\mathrm{Ad}\, h)X]$$

is valid. This shows $(\operatorname{Ad} h)Z=Z$, since $\mathfrak{g}$ is semisimple.

Conversely we have

THEOREM 3.7. *Let* $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ *be an effective semisimple symmetric triple satisfying* (C_3). *Then there exists a connected Lie group* G *with* $\operatorname{Lie} G=\mathfrak{g}$ *such that the coset space* $G/C(Z)$ *is associated with* $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$, *where* $C(Z)$ *is the centralizer of* Z *in* G. *Furthermore, for an arbitrary open subgroup* H *of* $C(Z)$, *the coset space* G/H *is a parahermitian symmetric space.*

PROOF. If we regard $\hat{G}$ in 3.5 as G here, the first assertion follows from 3.5. We will use the notations in the proof of 3.2 and 3.3. From 3.3, $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ has the structure of a parahermitian symmetric system $\{\mathfrak{g}, \mathfrak{h}, \sigma, \operatorname{ad} Z^0, \langle\ ,\ \rangle\}$. In order to apply 2.4, one has only to verify (C_1-2), (C_1-4). Take an element $h \in H$. Then $(\operatorname{Ad} h)Z=Z$ holds. We write Z in the form $Z=Z_1+\cdots+Z_s$, where $Z_i \in \mathfrak{g}_i$. So we have $(\operatorname{Ad} h)Z_i=Z_i$; this implies $(\operatorname{Ad} h)Z^0=Z^0$. Hence we have $[\operatorname{Ad} h, \operatorname{ad} Z^0]=0$. So we see that $I_o=\operatorname{ad}_{\mathfrak{m}} Z^0$ satisfies (C_1-2). (C_1-4) is also satisfied, since $\langle\ ,\ \rangle$ is the restriction of the Killing form of $\mathfrak{g}$. Thus, by 2.4, G/H is a parahermitian symmetric coset space.

EXAMPLE. Let $\boldsymbol{F}$ be the field $\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$, and let us consider the Lie algebra $\mathfrak{g}=\mathfrak{sl}(p+q, \boldsymbol{F})$, and the subalgebra $\mathfrak{h}=\mathfrak{s}(\mathfrak{gl}(p, \boldsymbol{F})+\mathfrak{gl}(q, \boldsymbol{F}))$ consisting of all elements of the form

$$\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$$

in $\mathfrak{g}$, where $X_1 \in \mathfrak{gl}(p, \boldsymbol{F})$ and $X_2 \in \mathfrak{gl}(q, \boldsymbol{F})$. Define the involutive automorphism σ of $\mathfrak{g}$ by putting

$$\sigma(X)={}^tI_{p,q}XI_{p,q}\,, \qquad X \in \mathfrak{g}\,,$$

where $I_{p,q}=\operatorname{diag}(E_p, -E_q)$, E_j being the unit matrix of degree j. Then it is easily verified that $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ is an effective symmetric triple. We define the element $Z \in \mathfrak{g}$ to be

$$Z=\begin{pmatrix} aE_p & 0 \\ 0 & bE_q \end{pmatrix}, \qquad (a=-q/(p+q),\ b=p/(p+q))\,.$$

Then it follows that $\mathfrak{g}$ is decomposed by $\operatorname{ad} Z$ into eigenspaces corresponding to the eigenvalues 0, ± 1, and that $\mathfrak{c}_{\mathfrak{g}}(Z)=\mathfrak{h}$. Therefore ($C_3$) is valid in this case. The simply connected Lie group $\hat{G}^C$ generated by $\mathfrak{g}^C$ is

$SL(p+q, \boldsymbol{C})$, $SL(p+q, \boldsymbol{C})\times SL(p+q, \boldsymbol{C})$ or $SL(2p+2q, \boldsymbol{C})$ according as $\boldsymbol{F}=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$, respectively. The analytic subgroup $\hat{G}$ of $\hat{G}^c$ corresponding to $\mathfrak{g}$ is $SL(p+q, \boldsymbol{F})$. And the centralizer $\hat{C}(Z)$ of Z in $\hat{G}$ is given by

$$\begin{aligned}\hat{C}(Z) &= S(GL(p, \boldsymbol{F})\times GL(q, \boldsymbol{F}))\\ &= \left\{\begin{pmatrix} A & 0\\ 0 & B\end{pmatrix}:\ A\in GL(p, \boldsymbol{F}),\ B\in GL(q, \boldsymbol{F})\quad (\det A)(\det B)=1\right\}\end{aligned}$$

It can be verified by direct calculations or by 3.5 that $\hat{G}_\sigma=\hat{C}(Z)$. So $\hat{G}/\hat{C}(Z)=SL(p+q, \boldsymbol{F})/S(GL(p, \boldsymbol{F})\times GL(q, \boldsymbol{F}))$ is a parahermitian symmetric coset space. $\hat{C}(Z)$ has two connected components if $\boldsymbol{F}=\boldsymbol{R}$; otherwise it is connected.

i) The case $\boldsymbol{F}=\boldsymbol{R}$.

If $(p, q)\neq(1, 1)$, then we have $\pi_1(\hat{G}/\hat{C}(Z))\cong\boldsymbol{Z}_2$. The coset space $\hat{G}/\hat{C}(Z)$ (resp. $\hat{G}/\hat{C}^0(Z)$) is the cotangent bundle of the real (resp. oriented real) Grassmann manifold (cf. §4). If $(p, q)=(1, 1)$, then we have

$$\hat{G}/\hat{C}(Z)=SL(2, \boldsymbol{R})/\boldsymbol{R}^\times=SO^0(2, 1)/SO^0(1, 1)\ ,$$

which is realized as the hyperboloid H^2 of one sheet

$$\{(x_1, x_2, x_3)\in R^3:\ x_1^2+x_2^2-x_3^2=1\}\ .$$

The distributions $\mathscr{T}^\pm$ on $SO^0(2, 1)/SO^0(1, 1)$ are realized on H^2 as two families of generating lines through each point of H^2.

ii) The case $\boldsymbol{F}=\boldsymbol{C}$ or $\boldsymbol{H}$.

In this case, we have $\hat{G}=SL(p+q, \boldsymbol{C})$ or $SL(p+q, \boldsymbol{H})$, which is simply connected. Consequently, $\hat{G}/\hat{C}(Z)$ is simply connected, and is the cotangent bundle of the complex or quaternion Grassmann manifold, according as $\boldsymbol{F}=\boldsymbol{C}$ or $\boldsymbol{H}$.

§4. Locally isomorphic parahermitian symmetric spaces.

DEFINITION 4.1. Let $\{\mathfrak{g}, \mathfrak{h}, \sigma, I_o, \langle\ ,\ \rangle\}$ and $\{\mathfrak{g}', \mathfrak{h}', \sigma', I_o', \langle\ ,\ \rangle\}$ be two effective semisimple parahermitian symmetric systems. Then they are said to be *isomorphic* if there exists an isomorphism f of $\mathfrak{g}$ onto $\mathfrak{g}'$ such that $f\cdot\sigma=\sigma'\cdot f$ and $f\cdot I_o=I_o'\cdot f$ are valid.

DEFINITION 4.2. Let $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ and $\{\mathfrak{g}', \mathfrak{h}', \sigma'\}$ be effective semisimple symmetric triples satisfying (C_3), and $Z^0\in\mathfrak{h}$, $Z'^0\in\mathfrak{h}'$ be the elements in 3.3. Then $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ and $\{\mathfrak{g}', \mathfrak{h}', \sigma'\}$ are said to be *isomorphic* if there exists an isomorphism f of $\mathfrak{g}$ onto $\mathfrak{g}'$ such that $f\cdot\sigma=\sigma'\cdot f$ and $f(Z^0)=Z'^0$.

Let $\mathfrak{g}$ be a real semisimple Lie algebra and τ be a Cartan involution of $\mathfrak{g}$. Then the pair $\{\mathfrak{g}, \tau\}$ is called a *positive definite symmetric graded Lie algebra* if the following conditions are satisfied (cf. Takeuchi [11]): i) $\mathfrak{g}$ is a graded Lie algebra $\mathfrak{g}=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1$ with $\mathfrak{g}_{-1}\neq(0)$, ii) the representation $\mathfrak{g}_0\rightarrow \mathrm{ad}_{\mathfrak{g}_{-1}}\mathfrak{g}_0$ is faithful, iii) $\tau(\mathfrak{g}_\lambda)=\mathfrak{g}_{-\lambda}$ ($\lambda=0, \pm1$). Two positive definite symmetric graded Lie algebras are said to be *isomorphic* if they are isomorphic as graded Lie algebras.

PROPOSITION 4.1. *There exist one-to-one correspondences between the following three objects*:
a) *an effective semisimple symmetric triple satisfying* (C_2),
b) *an effective semisimple symmetric triple satisfying* (C_3),
c) *a positive definite symmetric graded Lie algebra.*

PROOF. For an effective semisimple symmetric triple $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$, we consider the following three conditions: i) $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ satisfies (C_2): ii) $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ satisfies (C_3): iii) $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ has a structure of a positive definite symmetric graded Lie algebra. We will first prove that these three conditions are mutually equivalent. By 3.1 and 3.3, i) and ii) are equivalent. We will show that ii) implies iii). Let us take $Z^0\in\mathfrak{h}\cap\mathfrak{p}$ in 3.3 instead of Z. Let $\mathfrak{g}_0$ be $\mathfrak{h}$, and $\mathfrak{g}_{\pm1}$ be the ±1-eigenspaces of $\mathrm{ad}\, Z^0$ in the -1-eigenspace $\mathfrak{m}$ of σ in $\mathfrak{g}$. Then we have $\mathfrak{g}=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1$, which is the decomposition into graded subspaces. Since τ sends Z^0 to $-Z^0$, we have $\tau(\mathfrak{g}_\lambda)=\mathfrak{g}_{-\lambda}$ ($\lambda=0, \pm1$). Take an element $X\in\mathfrak{g}_0$, and suppose $[X, \mathfrak{g}_{-1}]=0$. Then, by using the fact that the Killing form of $\mathfrak{g}$ is non-degenerate on $\mathfrak{g}_{-1}\times\mathfrak{g}_1$, we can easily see that $[X, \mathfrak{g}_1]=0$. So we have $[X, \mathfrak{m}]=0$. Therefore, by the effectivity of $\{\mathfrak{g}, \mathfrak{h}\}$, we conclude $X=0$. Thus we have proved that ii) implies iii). In order to show that iii) implies i), we first define σ to be

$$\sigma=\begin{cases}\mathrm{id} & \text{on } \mathfrak{h}=\mathfrak{g}_0\\ -\mathrm{id} & \text{on } \mathfrak{m}=\mathfrak{g}_{-1}+\mathfrak{g}_1\,.\end{cases}$$

Then σ is an involutive automorphism of $\mathfrak{g}$, which commutes with τ. Let $\mathfrak{n}$ be an ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$, and take an element $X\in\mathfrak{n}$. Then we have $[X, \mathfrak{m}]\subset\mathfrak{m}\cap\mathfrak{n}=(0)$, that is, $\mathrm{ad}_{\mathfrak{m}}X=0$. Since the representation $\mathrm{ad}_{\mathfrak{m}}$ of $\mathfrak{h}$ is faithful, we get $X=0$. Thus $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ is an effective semisimple symmetric triple. Next we define the linear endomorphism $\tilde{I}_o$ on $\mathfrak{g}$ by puting $\tilde{I}_o=\lambda(\mathrm{id})$ on $\mathfrak{g}_\lambda$, $\lambda=0, \pm1$. Then there exists an element $Z^0\in\mathfrak{h}$ such that $\tilde{I}_o=\mathrm{ad}\, Z^0$ (cf. 3.1 and 3.3). Put $I_o=\mathrm{ad}_{\mathfrak{m}} Z^0$, and let $\langle\ ,\ \rangle$ be the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{m}$. The condition (C_2) is then satisfied.

From these arguments it follows that the three objects in 4.1 are in one-to-one correspondence. It is easy to see that the above correspondences are compatible with isomorphisms between respective objects. q.e.d.

Let $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ be an effective semisimple symmetric triple with (C_3), and Z^0 be the element of $\mathfrak{h}$ given in 3.3. And let G^* be the adjoint group corresponding to $\mathfrak{g}$, $U^*=N^*(\mathfrak{u})$ be the normalizer in G^* of $\mathfrak{u}=\mathfrak{h}+\mathfrak{m}^+=\mathfrak{g}_0+\mathfrak{g}_1$, and K^* be the analytic subgroup of G^* corresponding to the maximal compact subalgebra $\mathfrak{k}$. Put $K_0^*=U^*\cap K^*$. Then the followings are known ([7], [10]):

(A) The coset space $M_0^*=G^*/U^*=K^*/K_0^*$ is a symmetric R-space,

(B) $\mathfrak{u}$ is the Lie algebra of U^*,

(C) the subgroup $M^+=\exp\mathfrak{m}^+$ of G^* is simply connected,

(D) $U^*=M^+\cdot C^*(Z^0)$ (semidirect), where $C^*(Z^0)$ being the centralizer of Z^0 in G^*,

(E) $K_0^*=C^*(Z^0)\cap K^*$,

(F) $M^*=G^*/C^*(Z^0)$ is diffeomorphic to the cotangent bundle of M_0^*.

By using these facts, we can prove the following theorem about the structure of semisimple parahermitian symmetric spaces.

THEOREM 4.3. *Let $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$ be an effective semisimple symmetric triple satisfying (C_3), and let $M=G/H$ be a parahermitian symmetric coset space associated with $\{\mathfrak{g}, \mathfrak{h}, \sigma\}$. Then there exists a covering manifold M_0 of the symmetric R-space $M_0^*=G^*/U^*$ such that M is diffeomorphic to the cotangent bundle $T^*(M_0)$ of M_0.*

PROOF. Let U be the normalizer of $\mathfrak{u}$ in G, and $C(Z)$ be the centralizer of Z in G. Note that $C(Z)=C(Z^0)$ (cf. the proof of 3.3). Then H is an open subgroup of $C(Z)$. From (F) we have $U=M^+\cdot C(Z)$ (semidirect), where $M^+=\exp_G\mathfrak{m}^+$. Furthermore we have (cf. 3.4 and (B)) $M^*=G^*/C^*(Z)=G/C(Z)$ and $M_0^*=G^*/U^*=G/U=G/M^+\cdot C(Z)$. We define $M_0=G/M^+\cdot H$. Then we have the following commutative diagram of the natural mappings:

$$\begin{array}{ccc} M=G/H & \longrightarrow & M^*=G/C(Z) \\ \downarrow & & \downarrow \\ M_0=G/M^+\cdot H & \longrightarrow & M_0^*=G/M^+\cdot C(Z) \end{array}$$

Let K be the analytic subgroup of G corresponding to $\mathfrak{k}$, and $\tilde{K}$ be the universal covering group of K. Since M_0^* is represented as the coset space of $\tilde{K}$, so is M_0. From this, it follows that K acts transitively on M_0. Putting $K_0=K\cap M^+\cdot H$, we have $M_0=K/K_0$. Also we have

$K_0 = K \cap H$. In fact, let π be the natural projection of G onto G^*, and take an element $k = mh \in K_0$, where $m \in M^+$, $h \in H$. We have then $\pi(k) = \pi(m)\pi(h)$, where $\pi(k) \in K^*$, $\pi(m) \in M^+$ and $\pi(h) \in C^*(Z)$. So, by the arguments in [10], $\pi(m)$ is the identity, which implies that m is the identity (cf. (C)). Hence we have $k = h \in H$. Using this and (E), we can prove by the same arguments as in the proof of (G) (cf. [7], [10]) that the fiber bundle $M \to M_0$ is the associated bundle of the principal bundle $K \to K/K_0$ with Ad K_0-module $\mathfrak{m}^+$ as the standard fiber, which is diffeomorphic to the cotangent bundle of M_0.

PROPOSITION 4.4. *Let G/H be an affine symmetric coset space. Suppose that G is simple. Then G-invariant paracomplex structures are unique up to sign.*

PROOF. The natural G-invariant paracomplex structure I_o on G/H is given by $\mathrm{ad}_{\mathfrak{m}} Z^0$, where Z^0 is the one given in 3.3. Let I be an arbitrary G-invariant paracomplex structure on G/H. Then we have

$$[\mathrm{Ad}_{\mathfrak{m}} H, I] = 0 \ . \tag{4.1}$$

From this it follows that $[\mathrm{Ad}_{\mathfrak{m}} \exp tZ^0, I] = 0$, and consequently $[\mathrm{ad}_{\mathfrak{m}} Z^0, I] = 0$. Hence ± 1-eigenspaces $\mathfrak{m}^\pm$ of I_o are stable by I. On the other hand, $\mathfrak{m}^\pm$ are irreducible invariant subspaces under $\mathrm{Ad}_{\mathfrak{m}} H$ (cf. Koh [4]). Therefore, by (4.1), the operators $I|_{\mathfrak{m}^\pm}$ commute with each operator in $\mathrm{Ad}_{\mathfrak{m}^\pm} H$. So, by Kobayashi-Nomizu [3], we conclude that $I|_{\mathfrak{m}^\pm}$ are scalar matrices or the matrices of the form $\lambda(\mathrm{id}) + \mu J$, where $J^2 = -\mathrm{id}$, $\lambda, \mu \in \boldsymbol{R}$. From this it follows that $I|_{\mathfrak{m}^\pm} = \pm(\mathrm{id})$ or $\mp(\mathrm{id})$.

Now we would like to consider the problem: Classify all parahermitian symmetric spaces (M, I, g) with $\mathrm{Aut}(M, I, g)$ semisimple, up to paraholomorphic equivalence. Denoting by G the identity component of $\mathrm{Aut}(M, I, g)$, M is represented as the semisimple affine symmetric coset space G/H. It can be proved that there exists a one-to-one correspondence between parahermitian symmetric spaces (M, I, g) with $G = \mathrm{Aut}^0(M, I, g)$ semisimple and effective semisimple parahermitian symmetric coset spaces. Therefore, in view of 3.6 and 3.7, in order to classify parahermitian symmetric spaces with G semisimple up to local paraholomorphic equivalence, it is enough to classify effective semisimple symmetric triples satisfying (C_3). And, by 3.2, this problem is reduced to classifying simple symmetric triples satisfying (C_3). Furthermore the last problem is equivalent to classifying simple positive definite symmetric graded Lie algebras (cf. 4.1). This has been done by Kobayashi-Nagano

[2]. Taking 4.4 into account, that is enough.

The following is the list of simple symmetric Lie algebras $\{\mathfrak{g}, \mathfrak{h}\}$, which correspond to local paraholomorphic equivalence classes of all parahermitian symmetric spaces with G simple. These are taken from Kobayashi-Nagano [2] and Takeuchi [11].

$\{\mathfrak{g}, \mathfrak{h}\}$	M_0^*
$(\mathfrak{sl}(p+q, \boldsymbol{R}), \mathfrak{sl}(p, \boldsymbol{R})+\mathfrak{sl}(q, \boldsymbol{R})+\boldsymbol{R})$	$G_{p,q}(\boldsymbol{R})$
$(\mathfrak{su}^*(2p+2q), \mathfrak{su}^*(2p)+\mathfrak{su}^*(2q)+\boldsymbol{R})$	$G_{p,q}(\boldsymbol{H})$
$(\mathfrak{sl}(p+q, \boldsymbol{C}), \mathfrak{sl}(p, \boldsymbol{C})+\mathfrak{sl}(q, \boldsymbol{C})+\boldsymbol{C})$	$G_{p,q}(\boldsymbol{C})$
$(\mathfrak{su}(n, n), \mathfrak{sl}(n, \boldsymbol{C})+\boldsymbol{R})$	$U(n)$
$(\mathfrak{so}(n, n), \mathfrak{sl}(n, \boldsymbol{R})+\boldsymbol{R})$	$SO(n)$
$(\mathfrak{so}^*(4n), \mathfrak{su}^*(2n)+\boldsymbol{R})$	$U(2n)/Sp(n)$
$(\mathfrak{so}(2n, \boldsymbol{C}), \mathfrak{sl}(n, \boldsymbol{C})+\boldsymbol{C})$	$SO(2n)/U(n)$
$(\mathfrak{so}(p+1, q+1), \mathfrak{so}(p, q)+\boldsymbol{R})$	$Q_{p+1,q+1}(\boldsymbol{R})$
$(\mathfrak{so}(n+2, \boldsymbol{C}), \mathfrak{so}(n, \boldsymbol{C})+\boldsymbol{C})$	$Q_n(\boldsymbol{C})$
$(\mathfrak{sp}(n, \boldsymbol{R}), \mathfrak{sl}(n, \boldsymbol{R})+\boldsymbol{R})$	$U(n)/O(n)$
$(\mathfrak{sp}(n, n), \mathfrak{su}^*(2n)+\boldsymbol{R})$	$Sp(n)$
$(\mathfrak{sp}(n, \boldsymbol{C}), \mathfrak{sl}(n, \boldsymbol{C})+\boldsymbol{C})$	$Sp(n)/U(n)$
$(E_6^1, \mathfrak{so}(5, 5)+\boldsymbol{R})$	$G_{2,2}(\boldsymbol{H})/\boldsymbol{Z}_2$
$(E_6^4, \mathfrak{so}(1, 9)+\boldsymbol{R})$	$P_2(\boldsymbol{O})$
$(E_6^C, \mathfrak{so}(10, \boldsymbol{C})+\boldsymbol{C})$	$E_6/Spin(10)\cdot T^1$
$(E_7^1, E_6^1+\boldsymbol{R})$	$SU(8)/Sp(4)\cdot \boldsymbol{Z}_2$
$(E_7^3, E_6^4+\boldsymbol{R})$	$T^1\cdot E_6/F_4$
$(E_7^C, E_6^C+\boldsymbol{C})$	$E_7/E_6\cdot T^1$

In the above list, $G_{p,q}(\boldsymbol{F})$ denotes the Grassmann manifold of p-planes in $\boldsymbol{F}^{p+q}$, where $\boldsymbol{F}=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$. $Q_{p,q}(\boldsymbol{R})$ denotes the real quadric in $P_{p+q-1}(\boldsymbol{R})$ defined by the quadratic form of signature (p, q). $Q_n(\boldsymbol{C})$ denotes the complex quadric in $P_{n+1}(\boldsymbol{C})$. $P_2(\boldsymbol{O})$ denotes the octanion projective plane.

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