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Omega Theorems for Divisor Functions

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Introduction

In what follows, ε denotes any positive number and c, with or without suffix, denotes a positive absolute constant, not necessarily the same at each occurence, unless otherwise specified.

Let $B_k(x)$ denote the k-th Bernoulli polynomial, $[x]$ the integral part of x, $P_k(x):=B_k(x-[x])$ the k-th periodic Bernoulli polynomial, $\sigma_r(n):=\sum_{d|n} d^r$ the sum of r-th powers of divisors of n, and define the basic functions $G_{a,k}(x)$ by

$$G_{a,k}(x):=\sum_{n\leq x^{1/2}} n^a P_k\left(\frac{x}{n}\right)$$

for real a and $k\in \boldsymbol{N}$.

As is well known, the summatory function of the divisor function $d(n):=\sigma_0(n)$ admits the asymptotic formula

$$\sum_{n\leq x} d(n)=x\log x+(2\gamma-1)x+\Delta(x)\ ,$$

where $\gamma=0.5772\ldots$ is the Euler(-Mascheroni) constant and the estimate $\Delta(x)=O(x^{1/2})$ is due to Dirichlet. The problem of estimating the error term $\Delta(x)$ carries the name of the Dirichlet divisor problem, the best known estimate being

$$\Delta(x)=O(x^{35/108+\varepsilon})$$

due to Kolesnik [16], and there is a conjecture that

$$\Delta(x)=O(x^{1/4+\varepsilon})\ . \tag{0.1}$$

In view of the well-known asymptotic relation (see e.g. MacLeod [19])

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(0.2) $$\Delta(x)=-2G_{0,1}(x)+O(1)\ ,$$

(0.1) is equivalent to

(0.3) $$G_{0,1}(x)=O(x^{1/4+\varepsilon})\ .$$

As a generalization of (0.3), Chowla and Walum [3] conjectured that for $0\leqq a\in \boldsymbol{Z}$, as $x\to\infty$,

(0.4) $$G_{a,k}(x)=O(x^{a/2+1/4+\varepsilon})$$

holds, and proved the special case $a=1, k=2$ of (0.4) without ε-factor. Another special case $a=0, k=2$, which is much harder than the former one, was again stated by Chowla in his book [4].

After Chowla and Walum's work [3], several attempts have been made toward conjecture (0.4) (see [12]-[15], [19]-[22], [25]), and the best known estimate is

(0.5) $$G_{a,k}(x)=\begin{cases} O(x^{a/2+1/4}) & \text{for } a>\dfrac{1}{2}\ , \\ O(x^{1/2}\log x) & \text{for } a=\dfrac{1}{2}\ , \\ O(x^{4a/3+5/18}(\log x)^{\delta_{a,1/5}}) & \text{for } \dfrac{1}{5}\leqq a<\dfrac{1}{2}\ , \\ O(x^{17/42+2/7}(\log x)^{1-20a/11}) & \text{for } 0\leqq a<\dfrac{1}{5}\ , \end{cases}$$

the first two being due to the author and Sita Rama Chandra Rao [12], and the second, due to Nowak [21].

On the other hand, regarding the estimation from below, that is, regarding Ω-results, the following has just been announced ([14], Theorem 2):

(0.6) $$G_{a,2}(x)=\begin{cases} \Omega_{+}(x^{a/2+1/4}(\log x)^{1/4-a/2}) & \text{for } 0\leqq a<\dfrac{1}{2}\ , \\ \Omega_{-}\left(x^{a/2+1/4}\exp\left\{c\dfrac{(\log\log x)^{1/4-a/2}}{(\log\log\log x)^{3/4-a/2}}\right\}\right) & \text{for } 0<a<\dfrac{1}{2}\ , \end{cases}$$

(0.7) $$\liminf_{x\to\infty} x^{-1/4}G_{0,2}(x)=-\infty\ ,$$

(0.8) $$G_{1/2,2}(x)=\Omega_{+}(x^{1/2}\log\log x)\ ,$$

(0.9) $$G_{a,2}(x)+x^{a-1}G_{2-a,2}(x)=\Omega_{\pm}(x^{a/2+1/4}) \quad \text{for } \frac{1}{2}<a<\frac{3}{2}\ , \quad a\neq 1\ .$$

Of these, (0.9) follows from a theorem of Chandrasekharan and Narasimhan [2] together with our Theorem 1 below. If we could prove $x^{a-1}G_{2-a,2}(x)=o(x^{a/2+1/4})$, then (0.9) would provide a complete solution of conjecture (0.4) in the case $k=2$, $1/2<a<3/2$, $a\neq 1$. However, we have only $x^{a-1}G_{2-a,2}(x)=O(x^{a/2+1/4})$ and so it is still an open problem to get an omega result for $G_{a,2}(x)$ in the case $a>1/2$. The Ω_+-results in (0.6) and (0.8) are consequences of a general theorems of Hafner [8] combined with Theorem 1, which also give Ω_+-results for $G_{a,2}(x)$ in the case $-1/2<a<0$. Finally, (0.7) is a consequence of a theorem of Berndt [1] and the series representation proved in [13].

The first object of this paper is to prove Theorem 1 in section 2, giving an asymptotic relation between $R(x, r)$ (for this, see (0.13)) and $G_{1-r,2}(x)$ valid for $-1/2\leqq r<3$, $r\neq 0, 1$, and analogous to (0.2), which yields, with the aid of (0.5), improvements on the so far known best results due to Landau [17] and Wilson [30] (which are improvements of Ramanujan's results [23]).

The second (and main) objective is to establish a general Ω-theorem for the Riesz sums of arithmetical functions whose generating Dirichlet series satisfy the functional equation of the type studied by Chandrasekharan and Narasimhan [2], Berndt [1], Hafner [7], [8], et al. As one of the corollaries to this theorem we deduce Ω-results for $R(x, r)$, which then lead us to Ω_+-results for $G_{a,2}(x)$ when $-2<a<-1/2$ as well as the Ω_--result for $G_{a,2}(x)$ stated in (0.6). All of these will be done in section 3.

In closing this section we shall introduce further notation which will be used throughout in what follows.

For $b>0$, $\rho\geqq 0$, put

$$P^{\rho}(x, r, b)=\frac{1}{\Gamma(\rho+1)}\sum_{n\leq x}{}' (x^b-n^b)^{\rho}\sigma_{-r}(n)-S^{\rho}((\pi x)^b, r, b)\ , \tag{0.10}$$

$$R(x, r)=P^1(x, r, r)\ , \tag{0.11}$$

where the prime on the summation sign means that if $\rho=0$ and $n=x$, the term $\sigma_{-r}(n)$ is to be multiplied by 1/2, and

$$S^{\rho}(x, r, b)=\sum_{\xi}\operatorname*{Res}_{s=\xi}\frac{\zeta(s)\pi^{-bs}\zeta(bs)\Gamma(bs+r)}{\Gamma(s+\rho+1)}x^{s+\rho}\ , \tag{0.12}$$

wherein ξ runs through all the poles of the function described above in the half-plane $\operatorname{Re}\xi>-\rho-1-k$, and k is such that

$$k>\left|\frac{-r+1/2}{2b}\right|\ ,$$

so that

$$(0.13)\qquad R(x, r)=\sum_{n\leq x}(x^r-n^r)\sigma_{-r}(n)-\left\{\frac{r}{1-r}\zeta(1-r)x+\frac{r}{1+r}\zeta(1+r)x^{r+1}-\frac{\zeta(r)}{2}x^r+\frac{\zeta(-r)}{2}\right\}.$$

Actually, we apply Corollary 1 to Theorem 3, a specialized form of Theorem 2, to $P^\rho(x, r, b)$ and obtain Ω-results for it under some restrictions (Corollay 2). Then we incorporate Ω-results for $R(x, r)$ obtainable from the theorems of Berndt [1], Hafner [8], Steinig [27] with ours to get Corollary 3.

§ 1. A relation between $R(x, r)$ and $G_{1-r,2}(x)$.

The main result in this section is the following

THEOREM 1. *Let* $-1/2\leq r<3$ *and* $r\neq 0, 1$. *Then*

$$R(x, r)=-\frac{r}{2}x^{r-1}G_{1-r,2}(x)+O(x^{(2r-1)/4}(\log x)^{\delta_{r,-1/2}}).$$

For the proof of this theorem we need some lemmas.

LEMMA 1. *For complex* $s\neq -1$ *we have*

$$\sum_{n\leq x}n^s=\frac{x^{s+1}}{s+1}-P_1(x)x^s+\zeta(-s)+\frac{s}{2}P_2(x)x^{s-1}+O(x^{\sigma-2}).$$

PROOF. An application of the Euler-Maclaurin sum formula shows that for $s\neq -1$

$$(1.1)\qquad \sum_{n\leq x}n^s=\frac{x^{s+1}}{s+1}-P_1(x)x^s+\frac{s}{2}P_2(x)x^{s-1}+\begin{cases}c(s)+O(x^{\sigma-2}) & \text{if } \sigma<2\\ O(x^{\sigma-2}) & \text{if } \sigma\geq 2,\end{cases}$$

where for $\sigma<2$, $s\neq -1$

$$(1.2)\qquad c(s)=-\frac{1}{s+1}+\frac{1}{2}-\frac{s}{12}+\frac{s(s-1)(s-2)}{3!}\int_1^\infty\frac{P_3(t)}{t^{3-s}}dt.$$

Hence it suffices to prove the lemma in the case $\sigma<2$. But, then, from (1.2) we see that $c(s)$ is analytic in $\{s\in \boldsymbol{C}\,|\,\sigma<2, s\neq -1\}$, where, from (1.1) we have $c(s)=\zeta(-s)$ for $\sigma<-1$. Hence, by analytic continuation, the conclusion follows.

LEMMA 2. *Let* $F(x, r)=\sum_{n\leq x}\sigma_r(n)$. *Then for* $r\neq 0, -1$ *we have*

$$F(x, r)=\frac{\zeta(r+1)}{r+1}x^{r+1}-x^rG_{-r,1}(x)+\frac{r}{2}x^{r-1}G_{1-r,2}(x)+\zeta(1-r)x$$
$$-G_{r,1}(x)-\{P_2(x^{1/2})+P_1^2(x^{1/2})\}x^{r/2}-\frac{1}{2}\zeta(-r)$$
$$+O(x^{(r-1)}+x^{r-2}(\log x)^{\delta(r)}) ,$$

where $\delta(r)=1$ *or* 0 *according as* $r=3$ *or not, and the error estimate is meaningful only if* $r\leqq 4$.

PROOF. Since

$$F(x, r)=\sum_{ab\leqq x} a^r=\sum_{a\leqq x^{1/2}} a^r\left[\frac{x}{a}\right]+\sum_{b\leqq x^{1/2}}\sum_{x^{1/2}<a\leqq x/b} a^r$$
$$=x\sum_{a\leqq x^{1/2}} a^{r-1}-G_{r,1}(x)-\{x^{1/2}-P_1(x^{1/2})\}\sum_{a\leqq x^{1/2}} a^r+\sum_{b\leqq x^{1/2}}\sum_{a\leqq x/b} a^r ,$$

we may apply Lemma 1 repeatedly to compute the sums of the form $\sum a^r$. After simplification, we get the assertion of the lemma.

REMARK 0. In [19, Theorem 6] MacLeod obtains an asymptotic formula for the sum $\sum_{n\leqq x} n^t\sigma_a(n)$ for integers $t\geqq 0, a\geqq 1$, which overlaps with our Lemma 2 in some cases.

PROOF OF THEOREM 1. Since $x^rF(x,-r)-F(x, r)=\sum_{n\leqq x}(x^r-n^r)\sigma_{-r}(n)$, we find by Lemma 2 that for $r\neq 0, \pm 1, |r|<3$

$$R(x, r)=-\frac{r}{2}x^{r-1}G_{1-r,2}(x)-\frac{r}{2}x^{-1}G_{r+1,2}(x)+O(x^{(r-1)/2}) , \tag{1.3}$$

whence follows the theorem on appealing to (0.5).

COROLLARY 1. *We have*

$$R(x, r)=\begin{cases} O(x^{(2r-1)/4}) & \text{if } 0<r<\frac{1}{2} ,\\ O(\log x) & \text{if } r=\frac{1}{2} ,\\ O(x^{5(2r-1)/18}(\log x)^{\delta_{r,4/5}}) & \text{if } \frac{1}{2}<r\leqq\frac{4}{5} ,\\ O(x^{(29-17r)/42}(\log x)^{(20r-9)/11}) & \text{if } \frac{4}{5}<r\leqq 1 .\end{cases}$$

PROOF. This follows immediately from Theorem 1 and (0.5).
Note that Corollary 1 improves in all cases the so far known best

results

$$R(x, r)=\begin{cases} O(x^{(2r-1)/4}) & \text{if } 0<r<\dfrac{1}{2}, \\ O(\log^2 x) & \text{if } r=\dfrac{1}{2}, \\ O(x^{(2r-1)(r+1)/(2r+3)}) & \text{if } \dfrac{1}{2}<r<1, \end{cases}$$

the first two being due to Wilson [30] and the third due to Landau [17].

§ 2. Ω-theorems.

Before stating the main theorem we must introduce some notation essentially due to Chandrasekharan and Narasimhan [2].

DEFINITION. Let $\{a_n\}$, $\{b_n\}$ be two sequences of complex numbers not all the terms of which are zero. Let $\{\lambda_n\}$, $\{\mu_n\}$ be two sequences of positive numbers, strictly increasing to ∞. Let

$$(2.1) \qquad \Delta(s)=\prod_{\nu=1}^{N}\Gamma(\alpha_\nu s+\beta_\nu)\ ,$$

where $N\in \boldsymbol{N}$, β_ν is an arbitrary complex number, $\alpha_\nu>0$, and $A:=\sum_{\nu=1}^{N}\alpha_\nu>1/2$. Suppose that

$$\phi(s)=\sum_{n=1}^{\infty}a_n\lambda_n^{-s} \quad \text{and} \quad \psi(s)=\sum_{n=1}^{\infty}b_n\mu_n^{-s}\ ,$$

each of which converges in some half-plane with finite abscissa of absolute convergence σ_a^* and σ_b^*, respectively. Then $\phi(s)$ and $\psi(s)$ are said to satisfy the functional equation

$$(2.2) \qquad \phi(s)\Delta(s)=\psi(\delta-s)\Delta(\delta-s)$$

if there exists in the s-plane a domain $\mathscr{D}$, which is the exterior of a bounded, closed set $\mathscr{S}$, in which there exists a holomorphic function $\chi(s)$ with the property

$$(2.3) \qquad \lim_{|t|\to\infty}\chi(\sigma+it)=0\ ,$$

uniformly in any finite interval $-\infty<\sigma_1\leqq\sigma\leqq\sigma_2<\infty$, and

$$\chi(s)=\phi(s)\Delta(s)\ , \quad \text{for} \quad \sigma>\sigma_a^*\ ,$$
$$\chi(s)=\psi(\delta-s)\Delta(\delta-s)\ , \quad \text{for} \quad \sigma<\delta-\sigma_b^*\ .$$

For $\rho \geqq 0$ we form the Riesz sum of a_n:

$$A^\rho(x):=\frac{1}{\Gamma(\rho+1)}\sum_{\lambda_n\leqq x}{}'(x-\lambda_n)^\rho a_n=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)\phi(s)}{\Gamma(s+\rho+1)}x^{s+\rho}ds\ ,$$

where $c>0$, $c>\sigma_a^*$, and the prime on the summation sign means as before that if $\rho=0$, $\lambda_n=x$ then we add only $a_n/2$. Furthermore, define the main term (sum of the residues) $S^\rho(x)$ as in [2], [11] by

$$S^\rho(x)=\frac{1}{2\pi i}\int_{\mathscr{C}}\frac{\Gamma(s)\phi(s)}{\Gamma(s+\rho+1)}x^{s+\rho}ds\ , \tag{2.4}$$

where $\mathscr{C}$ is the rectangle with vertices at c_1-iR, c_1+iR, c_2+iR, c_2-iR. Herein $c_1>0$, $c_1>\sigma_a^*$; R is so large that the integrand is regular for $|t|\geqq R$. Moreover, $c_2=-(m_0+1/2)$, $0\leqq m_0\in \boldsymbol{Z}$. m_0 is chosen so large that $\mathscr{C}$ encloses all the singularities of the integrand to the right of $\sigma=-\rho-1-k$, where k is such that $k>|\delta/2-1/4A|$, and all the singularities of $\phi(s)$ lie in $\sigma>-k$ (so that if ρ is integral, $\mathscr{C}$ encloses all the singularities of $\phi(s)$ and the poles $0, -1, \cdots, -\rho$ of Γ), and that

$$\begin{cases}\delta+\dfrac{1}{2}+m_0>\sigma_\delta^*,\ \delta+\dfrac{1}{2}+m_0>\operatorname{Re}\left(-\dfrac{\beta_\nu}{\alpha_\nu}\right),\ 1\leqq\nu\leqq N\ ,\\ \dfrac{\delta}{2}+\dfrac{1}{2}+m_0>\max\left\{\dfrac{3}{2A},\ \dfrac{\rho+1/2}{2A}\right\},\ \dfrac{1}{2}+m_0>\operatorname{Re}\left(\dfrac{\beta_\nu-1}{\alpha_\nu}\right),\ 1\leqq\nu\leqq N\ .\end{cases} \tag{2.5}$$

Then we shall consider the error term $P^\rho(x)$ defined by

$$P^\rho(x):=A^\rho(x)-S^\rho(x)=\frac{1}{2\pi i}\int_{\mathscr{C}_1}\frac{\Gamma(s)\phi(s)}{\Gamma(s+\rho+1)}x^{s+\rho}ds\ , \tag{2.6}$$

where $\mathscr{C}_1$ is a contour made up of lines $c_1-i\infty$, c_1-iR, c_2-iR, c_2+iR, c_1+iR, $c_1+i\infty$.

We define as in [11] the Laplace transform $g_{r,\rho}(s)$ of $P^\rho(x)$ for $\sigma>0$ and

$$r>2A\left(m_0+\frac{1}{2}\right)-(2A-1)\rho \tag{2.7}$$

by

$$g_{r,\rho}(s)=\int_0^\infty x^r e^{-sx}P^\rho(x^{2A})dx\ , \tag{2.8}$$

where the integral on the right-hand side of (2.8) converges absolutely. This can be seen as follows: Since $A^\rho(x)=0$ for $x<\lambda_1$, and $S^\rho(x)=$

$O(x^{-m_0-1/2+\rho})$, $x\to 0$, by the definition of $S^\rho(x)$, we have

$$P^\rho(x)=O(x^{-m_0-1/2+\rho})\ ,\quad x<\lambda_1\ . \tag{2.9}$$

For $\rho>0$, the integral (2.6) is absolutely convergent, so that

$$P^\rho(x)=O(x^{c_1+\rho})\ ,\quad x\geqq\lambda_1\ . \tag{2.10}$$

We have, however, $S^0(x)=O(x^{c_1})$ for $x\to\infty$, since $A^0(x)=O(x^{c_1})$ because $\sum|a_n|\lambda_n^{-c_1}<+\infty$. Hence (2.10) holds for $\rho\geqq 0$. By (2.7), (2.9) and (2.10) the integral (2.8) is absolutely convergent.

We need the following results proved by Joris:

LEMMA 3 ([11], Lemma 4). *Let*

$$H=\sum_{\nu=1}^{N}\alpha_\nu\log\alpha_\nu\ ,\quad B=\sum_{\nu=1}^{N}\beta_\nu\ ,\quad E=2A\cdot e^{-H/A}\ , \tag{2.11}$$

$$D=-\rho/2-3/4-A\delta/2-B+N/2\ ,$$

$$\alpha=r+(2A-1)\rho+1/2+A\delta\ . \tag{2.12}$$

Suppose that there exists a number γ *such that*

$$|\mu_n-\mu_{n\pm1}|^{-1}=O(\mu_n^\gamma) \tag{2.13}$$

for $n\geqq 2$. *Then for* $\mu_1\leqq\mu_m<Y$, $0<\sigma<(1/2)E\mu_1^{1/2A}$ *we have*

$$\sigma^\alpha g_{r,\rho}(\sigma\pm iE\mu_m^{1/2A})=B\exp(\pm\pi iD)b_m\mu_m^{-(\delta/2+1/4A+\rho/2A)}+O(\sigma Y^F)$$

with a constant $B>0$ *and a real number* F.

LEMMA 4 ([11], Lemma 5). *Let* $0<\omega<E\mu_1^{1/2A}$. *Notation being the same as in Lemma 3, we have for* $\sigma>0$ *and* $|s\pm iE\mu_n^{1/2A}|\geqq\omega$, $n\in\boldsymbol{N}$

$$g_{r,\rho}(s)=O(\omega^{-\alpha}|s|^F)+O(|s|^{-r-1-2A\rho+2A(m_0+1/2)})\ ,$$

where $F>0$ *is a constant.*

Now we can state our Theorem 2 whose proof is in principle in the spirit of Ingham [9], Gangadharan [6], Corrádi and Kátai [5], and Redmond [24], using the above two lemmas due to Joris [11].

Suppose that the following conditions are satisfied. First, for each $x\geqq 1$, the set $\{\mu_n\leqq x\}$ contains a subset $Q=Q_x:=\{\mu_{n_k}\leqq x|k=1,\cdots,N=N(x)\}$ such that no number $\mu_n^{1/2A}$ is representable as a linear combination of the numbers $\mu_{n_k}^{1/2A}$ with coefficients ±1, unless $\mu_n^{1/2A}=\mu_{n_r}^{1/2A}$ for some r, in which case $\mu_n^{1/2A}$ has no other representation, and

(2.14) $$\sum_{\mu_{n_k}\in Q_x} |\mathrm{Re}\, b_{n_k}| \mu_{n_k}^{-(A\delta+\rho+1/2)/2A} \geqq c_3 L(x)\ , \quad c_3>0\ ,$$

for all $x\geqq 1$, where $L(x)$ is an increasing slowly varying function (for the details of the theory of slowly varying functions, see Seneta [26]). Let

$$S_x:=\left\{\eta=\left|\mu_m'^{1/2A}+\sum_{k=1}^{N(x)} r_k\mu_{n_k}^{1/2A}\right| \,\middle|\, \mu_m'=0,\ \mu_1,\ \mu_2,\ \cdots;\ r_k=0,\ \pm1,\ \sum_{k=1}^{N(x)} r_k^2\geqq 1\right\}\ .$$

Suppose there exists an $\eta\in S_x$ such that $\eta<1$. Then the function

$$\tilde{\eta}(x):=\inf S_x=\min S_x$$

satisfies the inequalities

(2.15) $$0<\tilde{\eta}(x)<1\ ,$$

and so we may put

(2.16) $$q(x):=-\log\tilde{\eta}(x) \quad (>0)\ ,$$

on which we impose the following restrictions:

(2.17) $$c_4 x\leqq q(x)\leqq \exp\left(c_5\frac{x}{\log x}\right)\ , \quad c_4,\ c_5>0\ .$$

Moreover, suppose that

$$N(x)\leqq \exp\left(B_2'\frac{x}{\log x}\right)\ , \quad B_2'>0\ ,$$

(2.18) $$\max \mu_{n_k}=o\left(\exp\exp\left(c\frac{x}{\log x}\right)\right)\ , \quad c>0\ ,$$

and put

(2.19) $$c_6:=\max(c_5,\ 2B_2')\ .$$

Then

(2.20) $$Q(x):=\exp\left(c_6\frac{x}{\log x}\right)$$

satisfies

(2.21) (i) $c_6>0$,

(ii) $q(x)\leqq Q(x)$,

(iii) $Q(x)/x$ is increasing,

(iv) $N(x)=o(Q(x))$.

Note that from (iii) it follows that $Q(x)$ is strictly increasing.
Finally, suppose that

$$\operatorname{Re} b_n \neq 0 \tag{2.22}$$

for at least one value of n.

THEOREM 2. *Under the conditions* (2.1)-(2.22) *we have* ($c_7>0$)

$$\operatorname{Re} P^\rho(x)=\Omega_\pm\{x^{\delta/2-1/4A+(1-1/2A)\rho}L(c_7 \log\log x\cdot\log\log\log x)\} .$$

If in assumptions (2.14) *and* (2.22) *we replace* $\operatorname{Re} b_{n_k}$ *by* $\operatorname{Im} b_{n_k}$, $\operatorname{Re} b_n$ *by* $\operatorname{Im} b_n$, *respectively, then in conclusion we should have* $\operatorname{Im} P^\rho(x)$ *instead of* $\operatorname{Re} P^\rho(x)$.

PROOF. For a sufficiently large x we let

$$\theta_x=A\delta-\frac{1}{2}+(2A-1)\rho+\frac{1}{Q(x)}\ ,\quad \gamma_x=\sup_{u\geq 1}\operatorname{Re} P^\rho(u^{2A})u^{-\theta_x} . \tag{2.23}$$

In view of (2.22) we may apply Chandrasekharan and Narasimhan's result ([2], Theorem 3.2) to obtain

$$\gamma_x>0 .$$

Also, if $\gamma_x=+\infty$, then

$$\operatorname{Re} P^\rho(u)=\Omega_+(u^{\theta_x/2A})=\Omega_+(u^{\delta/2-1/4A+(1-1/2A)\rho+(1/2A)Q(x)}) ,$$

which is much stronger than the result claimed in our theorem. Thus we may suppose that

$$0<\gamma_x<+\infty .$$

Hence we may put

$$\phi_x(u)=\gamma_x u^{\theta_x}-\operatorname{Re} P^\rho(u^{2A}) . \tag{2.24}$$

Then $\phi_x(u)\geqq 0$ for $u\geqq 1$.
Moreover, for $x\geqq x_0$, t real, $u\geqq 0$, let

$$\sigma_x=\exp(-2Q(x)) ,$$

$$V(t)=\frac{1}{2}(2+e^{it}+e^{-it})=2\left(\cos\frac{t}{2}\right)^2\geqq 0 \quad \text{(Fejér kernel of order 2)} , \tag{2.25}$$

$$T_x=T_x(u)=\prod_{\mu_{n_k}\in Q_x} V(E\mu_{n_k}^{1/2A}u+\rho_k)\geqq 0 ,$$

where ρ_k are real numbers to be precisely specified later.

By (2.7), (2.9) and (iv) of (2.21), we see that

$$\sigma_x^\alpha\int_0^1 u^r P^\rho(u^{2A})T_x(u)\exp(-\sigma_x u)du=O(\sigma_x^\alpha 2^{N(x')})=O[\exp\{-2\alpha Q(x)+o(Q(x))\}]$$
$$=o(1)$$

as $x\to\infty$, on noting $\alpha>1$ in view of (2.7) and (2.12). Hence

$$(2.26)\quad \sigma_x^\alpha\int_0^\infty u^r\phi_x(u)T_x(u)\exp(-\sigma_x u)du$$
$$=\sigma_x^\alpha\int_1^\infty u^r\phi_x(u)T_x(u)\exp(-\sigma_x u)du+\sigma_x^\alpha\gamma_x\int_0^1 u^{r+\theta_x}T_x(u)\exp(-\sigma_x u)du$$
$$-\sigma_x^\alpha\int_0^1\mathrm{Re}\,P^\rho(u^{2A})u^r T_x(u)\exp(-\sigma_x u)du$$
$$\geqq 0+0+o(1)=o(1)\ .$$

Now, for any trigonometric polynomial

$$T(u)=\sum_\nu k_\nu\exp(-it_\nu u)\ ,$$

where k_ν are complex and t_ν are real and distinct, and any holomorphic function $U(s)$, we introduce the notation $T\wedge U(s)$ as in Gangadharan [6] (each of the subsequent authors Corrádi and Kátai [5], Joris [10] and Redmond [24] uses this same notation) to mean

$$(2.27)\qquad (T\wedge U)(s)=\sum_\nu k_\nu U(s+it_\nu)\ .$$

Moreover, we put for $\sigma>0$

$$(2.28)\qquad I_\theta(s)=s^{-\theta}\ .$$

Using (2.8), (2.27) and (2.28), we may rewrite the left-hand side of (2.26) as

$$\sigma_x^\alpha\gamma_x\Gamma(r+\theta_x+1)T_x\wedge I_{r+\theta_x+1}(\sigma_x)-\sigma_x^\alpha\,\mathrm{Re}\,(T_x\wedge g_{r,\rho}(\sigma_x))\ ,$$

thus obtaining the fundamental inequality

$$(2.29)\qquad o(1)\leqq\sigma_x^\alpha\gamma_x\Gamma(r+\theta_x+1)T_x\wedge I_{r+\theta_x+1}(\sigma_x)-\sigma_x^\alpha\,\mathrm{Re}\,(T_x\wedge g_{r,\rho}(\sigma_x))\ ,$$

as $x\to\infty$.

Now we are in a position to prove an analogue of Lemma 3 in Redmond [24] which in turn is a generalization of the corresponding results in the papers of Gangadharan [6], Corrádi and Kátai [5] and Joris [10].

LEMMA 5. (1) *If $\theta \geqq 0$ and $\sigma > 0$, then as $x \to \infty$, we have*

$$(T_x \wedge I_\theta)(\sigma) = \sigma^{-\theta} + O(3^{N(x)} e^{\theta Q(x)}) .$$

(2) *With θ_x and σ_x as defined by* (2.23) *and* (2.25), *respectively, we have, as $x \to \infty$*

$$\sigma_x^{\alpha}\{T_x \wedge I_{r+\theta_x+1}(\sigma_x)\} = e^2 + o(1) .$$

(3) *As $x \to \infty$,*

$$\mathrm{Re}\, \sigma_x^{\alpha}\{T_x \wedge g_{r,\rho}(\sigma_x)\} = B \sum_{\mu_{n_k} \in Q_x} \frac{|\mathrm{Re}\, b_{n_k}|}{\mu_{n_k}^{(\delta/2+1/4A+\rho/2A)}} + o(1) ,$$

where B is the constant appearing in Lemma 3.

PROOF. We write

$$T_x(u) = T_0(u) + T_1(u) + \bar{T}_1(u) + T_2(u) , \tag{2.30}$$

with

$$\begin{cases} T_0(u) = 1 , \\ T_1(u) = \dfrac{1}{2} \displaystyle\sum_{\mu_{n_k} \in Q_x} \exp(i\rho_k + iE\mu_{n_k}^{1/2A} u) , \\ T_1(u) = \overline{T_1(u)} \\ T_2(u) = \displaystyle\sum_m h_m \exp(-E\mu_m u) , \end{cases} \tag{2.31}$$

where m ranges over $3^N - 2N - 1$ integers, $|h_m| \leqq 1/4$ and the η_m are real and are distinct numbers of the form

$$\sum_{\mu_{n_k} \in Q} r_k \mu_{n_k}^{1/2A} ,$$

where $r_k \in \{0, \pm 1\}$ with $\sum_{\mu_{n_k} \in Q} r_k^2 \geqq 2$. From the definition of $\tilde{\eta}(x)$ (2.16), and (ii) of (2.21), we see that

$$E|\eta_j \pm \mu_m'^{1/2A}| \geqq E\tilde{\eta}(x) \geqq Ee^{-Q(x)} \tag{2.32}$$

for $\mu_m' = 0, \mu_1, \mu_2, \cdots$, and every j, $1 \leqq j \leqq N$.

(1) By (2.27) we have

$$T_0 \wedge I_\theta(\sigma) = \sigma^{-\theta} .$$

Next,

$$|T_1 \wedge I_\theta(\sigma)| = \frac{1}{2}\left|\sum_{\mu_{n_k}\in Q} \exp(i\rho_k)(\sigma - iE\mu_{n_k}^{1/2A})^{-\theta}\right|$$

$$\leqq \frac{1}{2}E^{-\theta}\sum_{\mu_{n_k}\in Q}\mu_{n_k}^{-\theta/2A} \ll \sum_{\mu_{n_k}\in Q} 1 = N$$

as $x \to \infty$. Likewise

$$|\bar{T}_1 \wedge I_\theta(\sigma)| \ll N \quad \text{as} \quad x \to \infty .$$

Since $|h_m| \leqq 1/4$, we have by (2.32) with $\mu'_m = 0$, as $x \to \infty$,

$$|T_2 \wedge I_\theta(\sigma)| = \left|\sum_m h_m(\sigma + iE\eta_m)^{-\theta}\right| \ll 3^N e^{\theta Q(x)} .$$

Combining these results gives (1) by (2.31).

(2) Since

$$r + \theta_x + 1 = r + A\delta + \frac{1}{2} + (2A-1)\rho + \frac{1}{Q(x)} = \alpha + \frac{1}{Q(x)} > 1 \tag{2.33}$$

by (2.7), (2.23) and (2.12), we may apply the result in (1) to get

$$\sigma_x^\alpha\{T_x \wedge I_{r+\theta_x+1}(\sigma_x)\} = \sigma_x^{-r-\theta_x-1+\alpha} + \sigma_x^\alpha\{T_x \wedge I_{r+\theta_x+1}(\sigma_x) - \sigma_x^{-r-\theta_x-1}\}$$
$$= \sigma_x^{-1/Q(x)} + O(\sigma_x^\alpha 3^N e^{(r+\theta_x+1)Q(x)}) ,$$

which is seen to be identical with

$$e^2 + o(1)$$

by successive applications of (2.25), (2.23) and (iv) of (2.21). Thus (2) is proved.

(3) We have by Lemma 4 and (2.12)

$$\sigma_x^\alpha\{T_0 \wedge g_{r,\rho}(\sigma_x)\} = O(\sigma_x^{2A(m_0+1/2+\delta/2)-\rho-1/2}) = o(1) \quad \text{as} \quad x \to \infty \tag{2.34}$$

in view of the third inequality in (2.5). By (2.27), (2.31) and Lemma 3 with $Y = \mu_{n_N}^{1/2A}$

$$\sigma_x^\alpha\{T_1 \wedge g_{r,\rho}(\sigma_x)\} = \frac{1}{2}\sum_{\mu_{n_k}\in Q} e^{-\rho_k}\sigma_x^\alpha g_{r,\rho}(\sigma_x - iE\mu_{n_k}^{1/2A})$$
$$= \frac{B}{2}\sum_{\mu_{n_k}\in Q}\exp(i(\rho_k - \pi D))b_{n_k}\mu_{n_k}^{-(\delta/2+1/4A+\rho/2A)}$$
$$+ O(\sigma_x \mu_{n_N}^F \sum_{\mu_{n_k}\in Q} 1)$$
$$= \frac{B}{2}\sum_{\mu_{n_k}\in Q} \operatorname{Re} b_{n_k}\cos(\rho_k - \pi D)\mu_{n_k}^{-(\delta/2+1/4A+\rho/2A)}$$
$$- \operatorname{Im} b_{n_k}\sin(\rho_k - \pi D)\mu_{n_k}^{-(\delta/2+1/4A+\rho/2A)} + O(\sigma_x N\mu_{n_N}^F) .$$

Hence, choosing

$$\rho_k=\begin{cases}\pi D & \text{if} \quad \mathrm{Re}\, b_{n_k}\geqq 0\\ \pi(D+1) & \text{if} \quad \mathrm{Re}\, b_{n_k}<0\ ,\end{cases}$$

we have

$$\begin{aligned}(2.35)\qquad \mathrm{Re}\,\sigma_x^\alpha\{T_1\wedge g_{r,\rho}(\sigma_x)\}&=\frac{B}{2}\sum_{\mu_{n_k}\in Q}|\mathrm{Re}\, b_{n_k}|\mu_{n_k}^{-(3/2+1/4A+\rho/2A)}\\ &\quad+O\Big(\exp\Big\{-2\exp\Big(c_6\frac{x}{\log x}\Big)(1+o(1))+B_2'\frac{x}{\log x}\Big\}\Big)\\ &=\frac{B}{2}\sum_{\mu_{n_k}\in Q}|\mathrm{Re}\, b_{n_k}|\mu_{n_k}^{-(3/2+1/4A+\rho/2A)}+o(1)\end{aligned}$$

by (2.25), (2.20). In a similar way, we have

$$(2.36)\qquad \mathrm{Re}\,\sigma_x^\alpha\{\bar{T}_1\wedge g_{r,\rho}(\sigma_x)\}=\frac{B}{2}\sum_{\mu_{n_k}\in Q}|\mathrm{Re}\, b_{n_k}|\mu_{n_k}^{-(3/2+1/4A+\rho/2A)}+o(1)\ .$$

Finally, consider

$$\sigma_x^\alpha\{T_2\wedge g_{r,\rho}(\sigma_x)\}=\sigma_x^\alpha\sum_m h_m g_{r,\rho}(\sigma_x+iE\eta_m)\ .$$

We have

$$|\sigma_x+i(E\pm\mu_n'^{1/2A})|\geqq E|\eta_m\pm\mu_n'^{1/2A}|\geqq Ee^{-Q(x)}$$

for $\mu_n'=0, \mu_1, \mu_2, \cdots$ by (2.32). Also, by the definition of η_m we have for x sufficiently large

$$|\sigma_x+iE\eta_m|\leqq\sigma_x+EN\mu_{n_N}^{1/2A}\leqq 2E\mu_{n_N}^{1/2A}\ .$$

Thus by Lemma 4 with $\omega=Ee^{-Q(x)}$, we have

$$\begin{aligned}(2.37)\qquad \sigma_x^\alpha\{T_2\wedge g_{r,\rho}(\sigma_x)\}&\ll\sigma_x^\alpha e^{\alpha Q(x)}(N\mu_{n_N}^{1/2A})^F 3^N+\sigma_x^\alpha 3^N e^{(r+1+2A\rho-2A(m_0+1/2))Q(x)}\\ &\leqq\exp\Big\{-\alpha Q(x)+O\Big(\frac{x}{\log x}\Big)+o\Big(\exp\Big(c\frac{x}{\log x}\Big)\Big)\Big\}\\ &\quad+\exp\Big\{(r+1+2A\rho-2A(m_0+1/2)-2\alpha)Q(x)\\ &\quad+o\Big(\exp\Big(c\frac{x}{\log x}\Big)\Big)\Big\}=o(1)\ ,\end{aligned}$$

as $x\to\infty$ by (2.18), (iv) of (2.21), (2.25) and the first inequality of (2.5) and the fact that $\alpha>1$.

Combining the results (2.34)-(2.37) gives (3) by (2.15). This completes the proof of Lemma 5.

If we combine (2.29) with the results of Lemma 5, we have as $x\to\infty$

$$(e^2+o(1))\gamma_x\Gamma(r+\theta_x+1)\geqq B\sum_{\mu_{n_k}\in Q}|\mathrm{Re}\, b_{n_k}|\mu_{n_k}^{-(\delta/2+1/4A+\rho/2A)}+o(1)\ . \tag{2.38}$$

Now $\alpha+(1/Q(x))$ is positive and bounded away from zero, and $1/Q(x)\to 0$ as $x\to\infty$ by (iii) of (2.21), we have, as $x\to\infty$

$$\Gamma\left(\alpha+\frac{1}{Q(x)}\right)=\Gamma(\alpha)+o(1)\ .$$

From this and (2.38) we conclude that

$$\gamma_x\geqq B'\sum_{\mu_{n_k}\in Q}|\mathrm{Re}\, b_{n_k}|\mu_{n_k}^{-(\delta/2+1/4A+\rho/2A)}+o(1)\ ,$$

as $x\to\infty$, where $B'=B/(\Gamma(\alpha)e^2)>0$. Hence by the definition (2.23) of γ_x, there exists a sequence $\{u_x\}$ tending to infinity as $x\to\infty$ such that

$$\begin{aligned}\mathrm{Re}\, P^{\rho}(u_x^{2A})u_x^{-\theta_x}&\geqq B'\sum_{\mu_{n_k}\in Q}|\mathrm{Re}\, b_{n_k}|\mu_{n_k}^{-(\delta/2+1/4A+\rho/2A)}\\&\geqq B'c_8L(x)\ ,\end{aligned}$$

by (2.14), i.e.

$$u_x^{-\theta_x}\,\mathrm{Re}\, P^{\rho}(u_x^{2A})\geqq c_8L(x)\ ,\quad c_8>0\ . \tag{2.39}$$

We now proceed as in Redmond [24]: Let $v_x=u_x^{1/Q(x)}$. Then $(2\log u_x)/Q(x)=2\log v_x$. If $2\log v_x\leqq 1$, then $2\log u_x\leqq Q(x)$. Since $Q(x)$ is strictly increasing by (iii) of (2.21), we have

$$Q^{-1}(2\log u_x)\leqq x\ ,$$

where Q^{-1} denotes the functional inverse of Q. If $2\log v_x\geqq 1$, then by (iii) of (2.21),

$$\frac{Q(x)}{x}\cdot\frac{Q(2x\log v_x)}{2\log u_x}=\frac{Q(2x\log v_x)}{2x\log v_x}\geqq\frac{Q(x)}{x}\ .$$

Therefore $Q(2x\log v_x)\geqq 2\log u_x$, for x sufficiently large and hence $Q^{-1}(2\log u_x)\leqq 2x\log v_x$. If we set $w_x=\max\{1, 2\log v_x\}$, we may put

$$Q^{-1}(2\log u_x)\leqq xw_x\ . \tag{2.40}$$

For x sufficiently large, we have

$$w_x\leqq c_9v_x^{\varepsilon}\ ,\quad (c_9>0)\ , \tag{2.41}$$

for any $\varepsilon>0$. Hence by (2.23), (2.40) and (2.41) we have

$$\frac{\operatorname{Re} P^{\rho}(u_x^{2A})}{u_x^{A\delta-1/2+(2A-1)\rho}} \cdot \frac{1}{L\{Q^{-1}(2\log u_x)\}}$$
$$=\frac{\operatorname{Re} P^{\rho}(u_x^{2A})}{u_x^{\theta_x}} \cdot \frac{u_x^{1/Q(x)}}{L\{Q^{-1}(2\log u_x)\}} c_8 L(x) \cdot \frac{v_x}{L(c_9 x v_x^{\varepsilon})} .$$

Since $L(x)$ is slowly varying, it admits the representation

$$L(x)=\exp\left\{\eta(x)+\int_{x_0}^{\infty}\frac{\varepsilon(t)}{t}dt\right\}$$

with some constant $x_0>0$, where η is a bounded measurable function on $[x_0, \infty)$ such that $\eta(x)\to C$ ($|C|<\infty$), and ε is a continuous function on $[x_0, \infty)$ such that $\varepsilon(x)\to 0$ as $x\to\infty$. Hence, if x is sufficiently large, we have for any $\delta>0$,

$$|\varepsilon(t)|<\delta \qquad \text{for} \quad t>x$$

and

$$|\eta(x)-\eta(xw_x)|<|\eta(x)-C|+|\eta(xw_x)-C|<2\delta$$

in view of $w_x\geqq 1$. Therefore

$$0\leqq\log\frac{L(xw_x)}{L(x)}=\eta(xw_x)-\eta(x)+\int_x^{xw_x}\frac{\varepsilon(t)}{t}dt<2\delta+\delta\log w_x$$
$$<2\delta+\delta\log c_9+\varepsilon\delta\log v_x<a\log v_x$$

with some a, $0<a<1$, by (2.41), so that

$$c_8\frac{L(x)}{L(c_9 x v_x^{\varepsilon})}v_x>c_8 v_x^{1-a}>c_8 ,$$

since $v_x\geqq 1$. This gives

(2.42) $$\operatorname{Re} P^{\rho}(x)=\Omega_+(x^{\delta/2-1/4A+(1-1/2A)\rho}L\{Q^{-1}(\log x)\}) .$$

Since

$$Q(c_6^{-1}\log x\log\log x)=\exp\{\log x+o(\log x)\} ,$$

we see that there exist positive c_{10}, c_{11} such that, for x sufficiently large, we have

(2.43) $$c_{10}\log x\log\log x<Q^{-1}(x)<c_{11}\log x\log\log x .$$

Combining (2.42) and (2.43) gives

$$\operatorname{Re} P^{\rho}(x)=\Omega_{+}\{x^{\delta/2-1/4A+(1-1/2A)\rho} L(c_7 \log\log x \log\log\log x)\}\ .$$

The Ω_{-}-result is proved similarly.
This completes the proof of Theorem 2.

THEOREM 3. *Under the suppositions of Theorem 2 we put* $\mu_n = A_1 n^b$, *where* $A_1>0$, $b>0$ *are absolute constants, suppose that* $2A/b$ *is a natural number, and take* Q *as follows:* $Q=Q_x=\{q^b \mid q \in Q'\}$, *where* $Q'=Q'_x$ *is the set of all square free integers made up of all the primes in* P, *written in increasing order of magnitude, where* $P=P_x$ *is a set of prime numbers satisfying the estimates*

$$B_1 \frac{x}{\log x} \leqq \sum_{p \in P} 1 \leqq B_2 \frac{x}{\log x} \tag{2.44}$$

for all $x>1$ *and some positive constants* B_1 *and* B_2; *take as* $L(x)$ *the function*

$$\exp\left(c_{12} \frac{x^{\lambda}}{\log x}\right), \tag{2.45}$$

where $0<\lambda<1$, *and* $c_{12}>0$ *is an absolute constant. Then we have*

$$\operatorname{Re} P^{\rho}(x)=\Omega_{\pm}\left\{x^{\delta/2-1/4A+(1-1/2A)\rho} \exp\left(c_{13} \frac{(\log\log x)^{\lambda}}{(\log\log\log x)^{1-\lambda}}\right)\right\},$$

where $c_{13}>0$ *is an absolute constant.*

If the condition (2.14) *holds for* $\operatorname{Im} b_{n_k}$ *instead of* $\operatorname{Re} b_{n_k}$, *with the function in* (2.45) *as* $L(x)$, *then we have a corresponding result for* $\operatorname{Im} P^{\rho}(x)$ *also.*

PROOF. The proof goes on the same lines as that of Theorem 2 save for the necessary modifications of the values of constants appearing in the argument, in view of the presence of the superfluous factor A_1, which does not make its appearance in the proof of Theorem 2. The essential point in the proof is the application of Joris' or Besicovitch's result on the linear independence of fractional powers of integers over Q (see H. Joris [11]).

COROLLARY 1. *If in addition to the conditions of Theorem 2 we suppose that* $\operatorname{Re} b_n = \operatorname{Re} b(n)$ *is a multiplicative function of* n *which satisfies*

$$|\operatorname{Re} b(p)| \geqq c_{14} p^{a} \tag{2.46}$$

for all $p \in P$ *and some* a *such that*

$$a > \delta/2 + 1/4A + \rho/2A - 1/b\ , \tag{2.47}$$

then we have

$$\operatorname{Re} P^{\rho}(x) = \Omega_{\pm}\left\{x^{\delta/2-1/4A+(1-1/2A)\rho} \exp\left(c_{15} \frac{(\log\log x)^{1+b(a-\delta/2-1/4A-\rho/2A)}}{(\log\log\log x)^{b(\delta/2+1/4A+\rho/2A-a)}}\right)\right\}$$

with an absolute constant $c_{15} > 0$, *where the same proviso as that of Theorem 2 holds here also.*

PROOF. Follows as in Redmond [24] by virtue of the left-hand side inequality of (2.43).

COROLLARY 2. *Under the notation of section* 1 *we have if*

$$r \geqq 0,\ \rho < r + 1/2,\ m_0 + 1/2 > \max\left(\frac{r}{b}, \frac{r+2}{2b}, \frac{r+\rho-1/2}{2b}\right), \tag{2.48}$$

then

$$P^{\rho}(x, r, b) = \Omega_{\pm}\left\{x^{-r/2+1/4+(b-1/2)\rho} \exp\left(c_{16} \frac{(\log\log x)^{(1/2+r-\rho)/2}}{(\log\log\log x)^{(-r+3/2+\rho)/2}}\right)\right\}, \tag{2.49}$$

where $c_{16} > 0$ *and we have more explicitly than in section* 1,

$$P^{\rho}(x, r, b) = \frac{1}{\Gamma(\rho+1)} \sum_{n \leq x}{}' (x^b - n^b)^{\rho} \sigma_{-r}(n) - \left\{ \frac{\Gamma\left(\frac{1-r}{b}\right)\zeta(1-r)x^{1-r+b\rho}}{b\Gamma\left(\frac{1-r}{b}+\rho+1\right)} \right. \tag{2.50}$$

$$\left. + \frac{\zeta(1+r)\Gamma\left(\frac{1}{b}\right)x^{1+b\rho}}{b\Gamma\left(\frac{1}{b}+\rho+1\right)} + \sum_{n=0}^{m_0} \frac{(-1)^n}{n!} \frac{\zeta(-bn)\zeta(r-bn)}{\Gamma(-n+\rho+1)} x^{b(\rho-n)} \right\}.$$

PROOF. We take

$$\phi(s) = \psi(s) = \sum_{n=1}^{\infty} \sigma_{r}(n)(\pi n)^{-bs} = \zeta(bs)\zeta(bs+r)\pi^{-bs}\ , \tag{2.51}$$

so that $A_1 = \pi^b$. Then ϕ, ψ satisfy the functional equation with

$$\sigma_a^* = \sigma_b^* = \max\left\{\frac{1-r}{b}, \frac{1}{b}\right\},\ \delta = \frac{1-r}{b},\ \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \frac{b}{2},\ A = b\ .$$

Moreover, we have $\sigma_{-r}(p) \leqq 1$, so that $a = 0$. Since (2.50) assures that the conditions in (2.5) are satisfied, we may apply Corollary 1 to obtain the assertion of our Corollary 2.

Incorporating the results deducible from the theorems of Berndt [1], Chandrasekharan and Narasimhan [2], Hafner [8], Steinig [27] and ours in this paper, we now state

COROLLARY 3. *There exist positive constants* c_{17}, c_{18} *such that*

$$R(x, r)=\begin{cases}\Omega_{\pm}\left\{x^{r/2-1/4}\exp\left(c_{17}\dfrac{(\log\log x)^{r/2-1/4}}{(\log\log\log x)^{5/2-r/2}}\right)\right\} & \\ \qquad\qquad \text{(Corollary 2 of this paper)} & r\geqq\dfrac{3}{2}\\ O(x^{r/2}) \quad \text{(trivial)} & \end{cases}$$

$$R(x, r)=\begin{cases}\Omega_{\pm}\{x(\log x)^{r/2-1/4}\} \quad \text{(Berndt, Hafner, Steinig)} & 1<r<\dfrac{3}{2}\\ O(x^{r/2}) \quad \text{(trivial)} & \end{cases}$$

$$R(x, r)=\begin{cases}\Omega_{-}\{(x\log x)^{r/2-1/4}\} \quad \text{(Berndt, Hafner, Steinig)} & \\ \Omega_{+}\left\{x^{r/2-1/4}\exp\left(c_{18}\dfrac{(\log\log x)^{r/2-1/4}}{(\log\log\log x)^{5/2-r/2}}\right)\right\} & \dfrac{1}{2}<r\leqq 1\\ \qquad\qquad \text{(Corollary 2 of this paper)} & \\ O(x^{3(2r-1)/10}) & \end{cases}$$

$$R\left(x, \frac{1}{2}\right)=\begin{cases}\Omega_{-}(\log\log x) & \text{(Hafner)}\\ O(\log x) & \text{(Corollary 1 of this paper)}\end{cases}$$

$$R(x, r)=\begin{cases}\Omega_{\pm}(x^{r/2-1/4}) \\ O(x^{r/2-1/4})\end{cases} \text{(Chandrasekharan and Narasimhan)} \quad |r|<\frac{1}{2}.$$

PROOF. Our results follow from Corollary 2 on taking $\rho=1, b=r>(1/2)$, $m_0=2$ since the contribution from $S^1(x, r, r)$ comes only from those terms corresponding to $n=0, 1$.

REMARK 1. Although we applied theorems of Berndt [1], Hafner [8] and Steinig [27] in their simplest form, they are applicable to $P^{\rho}(x, r, b)$ (Steinig's theorem is applicable in case $b=1$) and give non-trivial Ω-results.

REMARK 2. As Theorem 1 suggests, it is very likely that there might be more general relations between $G_{a,k}(x)$ and $P^{\rho}(x, r, b)$ than that given there, although we have not succeeded in finding such relations yet. Such relations, if any, would provide the O-estimates and Ω-results with each other just as has been done in this paper.

REMARK 3. Once one notices the fact that the function in (2.51) satisfies the functional equation, one could obtain a series representation for $P^{\rho}(x, r, b)$ on appealing to a theorem of Hafner [7], in particular, a series representation for $R(x, r)$ would follow, which, then, yields a series representation for $G_{1-r,2}(x)$ in view of Theorem 1. We retained, however our original proof in [13], since for one thing it is self-contained and elementary, and another thing it can be used in another setting, e.g. in the investigation of the logarithmic Riesz sum of $\sigma_r(n)$, which is relevant, more or less, to Chowla and Walum's conjecture. The details will appear in a forthcoming paper.

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