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Non-Expansive Attractors with Specification

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Introduction

Let f: $Y \to Y$ be a continuous surjection of a compact metric space Y. The inverse limit of f induces a compact metric space $\bar{Y}$ and a homeomorphism $\bar{f}$ of $\bar{Y}$. $(\bar{Y}, \bar{f})$ is called the natural extension of f. As R. Williams proved in [12], if a 1-dimensional branched manifold Y admits an expanding immersion g: $Y \to Y$, then Y has no endpoints. Moreover $(\bar{Y}, \bar{g})$ is topologically conjugate to an attractor of some Axiom A diffeomorphism. But some attractors, as Hénon's attractors, resemble the natural extension $(\bar{I}, \bar{f})$ of a continuous surjection f of an interval I with endpoints. It is a problem whether there exist any diffeomorphisms which have an attractor topologically conjugate to $(\bar{I}, \bar{f})$.

For the continuous surjection $f(x)=1-|2x-1|$ on the interval $I=[0, 1]$, we show in this paper that there exists a diffeomorphism of the 3-sphere which has an attractor topologically conjugate to $(\bar{I}, \bar{f})$. Furthermore we show that $(\bar{I}, \bar{f})$ satisfies not expansiveness but specification (these properties have been used in papers [1, 2], [3, 4], [5], [8] and [10] on ergodic theory). To realize the attractor in the 3-sphere, our key ingregient is in constructing a fine foliation of a closed 3-ball.

§1. Definitions and results.

Let $X=(X, d)$ be a compact metric space and σ a homeomorphism of X (i.e. from X onto itself). By $\boldsymbol{R}$, $\boldsymbol{Z}$ and $\boldsymbol{N}$ we denote the set of real numbers, the set of integers and the set of positive integers respectively. (X, σ) is *expansive* if there exists a $\delta>0$ such that, for every pair of distinct points $x, y \in X$, there is an $n \in \boldsymbol{Z}$ with $d(\sigma^n x, \sigma^n y)>\delta$. (X, σ) is said to satisfy *specification* if the following holds; for every $\varepsilon>0$ there exists an integer $K=K(\varepsilon)>0$ such that, for every $k \geqq 1$, for every

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k points $x_1, \cdots, x_k \in X$, for every integers

$$a_1 \leqq b_1 < a_2 \leqq b_2 < \cdots < a_k \leqq b_k$$

with

$$a_{i+1} - b_i \geqq K \qquad (1 \leqq i \leqq k-1)$$

and for every integer p with $p \geqq b_k - a_1 + K$, there exists a point $x \in X$ with $\sigma^p x = x$ such that

$$d(\sigma^n x, \sigma^n x_i) < \varepsilon \qquad \text{for} \quad a_i \leqq n \leqq b_i,\ 1 \leqq i \leqq k\ .$$

(X, σ) is said to be *topologically transitive* if $\{\sigma^n x : n \in \boldsymbol{Z}\}$ is dense in X for some $x \in X$. If (X, σ) satisfies specification, then it is clearly topologically transitive. Let σ_1 be a homeomorphism of a compact metric space X_1. (X, σ) and (X_1, σ_1) are said to be *topologically conjugate* to each other if there exists a homeomorphism φ from X onto X_1 such that $\varphi \circ \sigma = \sigma_1 \circ \varphi$. The topological conjugacy is an equivalent relation under which specification, topological transitivity and expansiveness are preserved.

Let $Y = (Y, d)$ be a compact metric space and f: $Y \to Y$ a continuous surjection. (Y, f) is said to satisfy *positive specification* if it satisfies the condition of specification for $a_1 \geqq 0$. We define the metric $\bar{d}$ of the direct product space Y^N by $\bar{d}(\bar{x}, \bar{y}) = \sum_{i=1}^{\infty} 2^{-i} d(x_i, y_i)$ for $\bar{x} = (x_i)_1^\infty$ and $\bar{y} = (y_i)_1^\infty$ in Y^N. The compact subset X of Y^N is defined by

$$X = \{\bar{x} \in Y^N : f(x_{i+1}) = x_i,\ i \in N\}\ .$$

Let $\sigma: X \to X$ be the homeomorphism defined by $\sigma(\bar{x}) = (fx_1, fx_2, fx_3, \cdots) = (fx_1, x_1, x_2, \cdots)$ for $\bar{x} = (x_1, x_2, \cdots) \in X$. (X, σ) is called *the natural extension* of (Y, f), and it is denoted by $(X, \sigma) = \underleftarrow{\lim}\,(Y, f)$.

Let g be a diffeomorphism of a compact manifold M. A g-invariant subset Λ of M is said an *attractor* of g if there exists a closed neighborhood W of Λ such that

(i) $g(W) \subset \operatorname{int}(W)$,

(ii) $\Lambda = \bigcap_{n \geqq 0} g^n(W)$ and

(iii) $g|\Lambda: \Lambda \to \Lambda$ is topologically transitive.

We denote by (Λ, g) the restriction of (M, g) to an attractor Λ. Our main results are stated in the theorems below:

THEOREM 1. *Let $I = [1, 0]$ be a compact interval with the euclidian metric, and $f: I \to I$ a continuous surjection defined by $f(x) = 1 - |2x - 1|$ $(x \in I)$. Let (X, σ) be the natural extension of (I, f). Then the following* *holds*:

(A) (X, σ) *is not expansive*,

(B) (X, σ) *satisfies specification*

and

(C) *each point of* X *has a neighborhood which is homeomorphic to the product of a compact interval and a Cantor set*.

THEOREM 2. *Let* (X, σ) *be as in Theorem* 1. *Then there exists a* C^1-*diffeomorphism* g *of the* 3-*sphere* S^3 *which has an attractor* Λ *such that* (Λ, g) *is topologically conjugate to* (X, σ).

§2. Proof of Theorem 1.

We denote by d the euclidian metric of I; i.e. $d(x, y)=|x-y|$ for $x, y \in I$.

(I) PROOF OF (A). Let $1/2>\varepsilon>0$ be given. For each $i\geqq 1$, we put $x_i=2^{-i}(1-\varepsilon)$ and $y_i=2^{-i}(1+\varepsilon)$. Then $\bar{x}=(x_1, x_2, \cdots)$ and $\bar{y}=(y_1, y_2, \cdots)$ are distinct points of X, because $x_1\neq y_1$, $f(x_{i+1})=x_i$ and $f(y_{i+1})=y_i$ for each $i\geqq 1$. To prove (A), it is enough to show that $d(\sigma^n\bar{x}, \sigma^n\bar{y})\leqq\varepsilon$ for every $n\in \boldsymbol{Z}$. Let $n\in \boldsymbol{Z}$ be given. If $n\geqq 0$, using the fact that $f^i(x_1)=f^i(y_1)$ for every $i\geqq 1$, we have

$$\begin{aligned}\bar{d}(\sigma^n\bar{x}, \sigma^n\bar{y}) &= \bar{d}((f^n x_1, f^{n-1}x_1, \cdots, fx_1, x_1, x_2, \cdots),\\ &\qquad (f^n y_1, f^{n-1}y_1, \cdots, fy_1, y_1, y_2, \cdots))\\ &= \sum_{i=1}^{\infty} 2^{-(n+i)} d(x_i, y_i)\\ &= 2^{-n+1}\sum_{i=1}^{\infty} 2^{-2i}\\ &= 2^{-n+1}\varepsilon/3<\varepsilon .\end{aligned}$$

If $n<0$, we have

$$\begin{aligned}\bar{d}(\sigma^n\bar{x}, \sigma^n\bar{y}) &= \bar{d}((x_{1-n}, x_{2-n}, \cdots), (y_{1-n}, y_{2-n}, \cdots))\\ &= \sum_{i=1}^{\infty} 2^{-i} d(x_{i-n}, y_{i-n})\\ &= 2^{n+1}\varepsilon/3<\varepsilon .\end{aligned}$$

Therefore (X, σ) is not expansive.

(II) PROOF OF (B). To prove (B), it is enough to prove the next two propositions.

PROPOSITION 2.1. *If* (I, f) *satisfies positive specification, then*

$(X, \sigma)=\varprojlim (I, f)$ *satisfies specification.*

PROPOSITION 2.2. *(I, f) satisfies positive specification.*

PROOF OF PROPOSITION 2.1. Assume that (I, f) satisfies positive specification. Let $\varepsilon>0$ be given. Choose a positive integer N such that $2^{-N}<\varepsilon/2$. Let $K'=K'(\varepsilon/2)>0$ be as in the definition of positive specification. Put $K=K'+N$ and take any integer $k\geqq 1$. Let $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k \in X$ be given, as well as integers $a_1\leqq b_1<a_2\leqq b_2<\cdots<a_k\leqq b_k$ and p with $a_{i+1}-b_i\geqq K$ $(1\leqq i\leqq k-1)$ and $p\geqq b_k-a_1+K$. We have to show that there exists a $\bar{y}\in X$ with $\sigma^p\bar{y}=\bar{y}$ such that $\bar{d}(\sigma^n\bar{y}, \sigma^n\bar{x}_i)<\varepsilon$ for every $a_i\leqq n\leqq b_i$ and $1\leqq i\leqq k$. To do this we consider two cases separately.

Case (i): $a_1\geqq 0$. For each $1\leqq i\leqq k$, the point $\bar{x}_i$ is expressed by $\bar{x}_i=(x_1^i, x_2^i, \cdots)$ where $x_j^i\in I$ $(j\in \boldsymbol{N})$. Note that $a_{i+1}-(b_i+N)\geqq K'$ $(1\leqq i\leqq k-1)$ and $p\geqq(b_k+N)-a_1+K'$. Since (I, f) satisfies positive specification, for $x_N^i\in I$ $(1\leqq i\leqq k)$, for $a_1\leqq b_1+N<a_2\leqq b_2+N<\cdots<a_k\leqq b_k+N$ and for p, there exists $y\in I$ with $f^p y=y$ such that $d(f^n y, f^n x_N^i)<\varepsilon/2$ for every $a_i\leqq n\leqq b_i+N$ and $1\leqq i\leqq k$. Define $\bar{y}\in X$ by

$$\bar{y}=(f^{N-1}y, f^{N-2}y, \cdots, fy, y, f^{p-1}y, f^{p-2}y, \cdots, fy, y, f^{p-1}y, \cdots) .$$

Then $\bar{y}$ satisfies $\sigma^p\bar{y}=\bar{y}$. For each $1\leqq i\leqq k$, since $x_j^i=f(x_{j+1}^i)$ for every $j\in \boldsymbol{N}$, $\bar{x}_i$ is expressed by

$$\bar{x}_i=(f^{N-1}x_N^i, \cdots, fx_N^i, x_N^i, x_{N+1}^i, \cdots) .$$

Since $\operatorname{diam}(I)=1$, we have, for every $a_i\leqq n\leqq b_i$,

$$\begin{aligned}\bar{d}(\sigma^n\bar{y}, \sigma^n\bar{x}_i)&=\bar{d}((f^{n+N-1}y, \cdots, f^{n+1}y, f^n y, f^{n+p-1}y, \cdots),\\ &\qquad (f^{n+N-1}x_N^i, \cdots, f^{n+1}x_N^i, f^n x_N^i, f^n x_{N+1}^i, \cdots))\\ &\leqq\sum_{j=1}^{N}2^{-j}d(f^{n+N-j}y, f^{n+N-j}x_N^i)+\sum_{j=N+1}^{\infty}2^{-j}\\ &<\varepsilon/2+1/2^N<\varepsilon .\end{aligned}$$

Case (ii): $a_1<0$. Put $\bar{x}_i'=\sigma^{a_1}\bar{x}_i$, $a_i'=a_i-a_1$ and $b_i'=b_i-a_1$ $(1\leqq i\leqq k)$. Note that $a_{i+1}'-(b_i'+N)\geqq K'$ and $p\geqq(b_k'+N)-a_1'+K'$. Apply the case (i) to $\bar{x}_i'\in X$ $(1\leqq i\leqq k)$, $0=a_1'\leqq b_1'+N<a_2'\leqq b_2'+N<\cdots<a_k'\leqq b_k'+N$ and p. Then we get $\bar{y}'\in X$ with $\sigma^p\bar{y}'=\bar{y}'$ such that $\bar{d}(\sigma^n\bar{y}', \sigma^n\bar{x}_i')<\varepsilon$ for $a_i'\leqq n\leqq b_i'$, $1\leqq i\leqq k$. Put $\bar{y}=\sigma^{-a_1}\bar{y}'$, then this is a required point. Proposition 2.1 is proved.

To prove Proposition 2.2, we prepare two lemmas.

LEMMA 2.3. *Let Y be a compact interval and ξ: $Y\rightarrow \boldsymbol{R}$ a continuous* *map. Let a closed interval* $J\subset\xi(Y)$ *be given. Then there exists a closed interval* $J'\subset Y$ *such that* $\xi(J')=J$.

PROOF. Put $J=[a, b]$. If $a=b$, the assertion is trivial. Suppose $a<b$. Then there are $c, d\in Y$ such that $\xi(c)=a$ and $\xi(d)=b$. If $c<d$, put $q=\inf\{x\in[c, d]: \xi(x)=b\}$ and $p=\sup\{x\in[c, q]: \xi(x)=a\}$. Otherwise, put $p=\sup\{x\in[d, c]: \xi(x)=b\}$ and $q=\inf\{x\in[p, c]: \xi(x)=a\}$. In any case, by the intermediate-value theorem, we have $\xi([p, q])=J$.

For $x\in I$ and $\varepsilon>0$, define $I(x, \varepsilon)=\{y\in I: d(x, y)\leqq\varepsilon\}$.

LEMMA 2.4. *Let* $\varepsilon>0$ *be given.*

(i) *For every* $x\in I$ *and* $n\geqq 0$, *it follows that*

$$f^n(I(x, \varepsilon/2^n))=I(f^n x, \varepsilon)$$

and

$$d(f^i x, f^i y)\leqq\varepsilon \quad \textit{for every} \quad 0\leqq i\leqq n \quad \textit{and} \quad y\in I(x, \varepsilon/2^n)\,.$$

(ii) *There exists an integer* $K=K(\varepsilon)>0$ *such that the following holds: for every* $x\in I$, *for every closed interval* $I'\subset I$ *and for every* $n\geqq K$, *there is a closed interval* $J\subset I(x, \varepsilon)$ *such that* $f^n(J)=I'$.

PROOF. By the definition of f, one has $f(I(x, \varepsilon))=I(fx, 2\varepsilon)$ for every $x\in I$ and every $\varepsilon>0$ (not necessary $\varepsilon<1$). Applying this to $I(x, \varepsilon/2^n)$ repeatedly, we get (i). To see (ii), choose $K>0$ such that $2^{-K}<\varepsilon$. Then, since $2^K\varepsilon\geqq 1$, it follows that $f^n(I(x, \varepsilon))=I(f^n x, 2^n\varepsilon)=I$ for every $x\in I$ and every $n\geqq K$. Replacing ξ in Lemma 2.3 by f^n, we get (ii).

PROOF OF PROPOSITION 2.2. Let $\varepsilon>0$ be given. Choose a number ε' with $0<\varepsilon'<\varepsilon$. Let $k=k(\varepsilon')>0$ be an integer as in Lemma 2.4 (ii). Take any $k\geqq 1$. Let $x_1, \cdots, x_k\in X$ be given, as well as integers $0\leqq a_1\leqq b_1<a_2\leqq b_2<\cdots<a_k\leqq b_k$ and p with $a_{i+1}-b_i\geqq K$ $(1\leqq i\leqq k-1)$ and $p\geqq b_k-a_1+K$. Put $a_{k+1}=p+a_1$.

In order to find an interval $I_1\subset I\subset(f^{a_1}x_1, \varepsilon')$ such that $f^p(I_1)\supset I_1$, put $I_{k+1}=I(f^{a_1}x_1, \varepsilon')$. Then I_i $(i\leqq k)$ is determined recursively as follows. By Lemma 2.4 (ii), there is an interval $J_i\subset I(f^{b_i}x_i, \varepsilon')$ such that $f^{a_{i+1}-b_i}(J_i)=I_{i+1}$. Since $f^{b_i-a_i}(I(f^{a_i}x_i, \varepsilon'/2^{b_i-a_i}))=I(f^{b_i}x_i, \varepsilon')$ (by Lemma 2.4 (i)), there exists an interval $I_i\subset I(f^{a_i}x_i, \varepsilon'/2^{b_i-a_i})$ such that $f^{b_i-a_i}(I_i)=J_i$ (by Lemma 2.3).

Since $f^{a_{i+1}-a_i}(I_i)=I_{i+1}$ for $1\leqq i\leqq k$, one has $I_{k+1}=f^{a_{k+1}-a_1}(I_1)=f^p(I_1)$. Note that $I_1\subset I(f^{a_1}x_1, \varepsilon')=I_{k+1}$. By the intermediate-value theorem, theer exists a $y\in I_1$ such that $f^p y=y$. Put $x=f^{p-a_1}y$. Clearly $f^p x=x$ holds.

For every $1\leqq i\leqq k$ and $a_i\leqq n\leqq b_i$, one has $f^n x=f^{n-a_1}y\in f^{n-a_1}(I_1)=f^{n-a_i}(I_i)\subset f^{n-a_i}I(f^{a_i}x_i, \varepsilon'/2^{b_i-a_i})\subset I(f^n x_i, \varepsilon')$; i.e., $d(f^n x, f^n x_i)\leqq\varepsilon'<\varepsilon$. This means that (I, f) satisfies positive specification. The proof is completed.

(III) PROOF OF (C). Let $\bar{a}=(a_1, a_2, \cdots)\in X$ be given. Denote by J_1 the 1/4-closed neighborhood of a_1 in I. If $J\subset I$ is an interval such that $\operatorname{diam}(J)\leqq 1/2$, then $f^{-2}(J)$ has at least two connected components and the diameter of each connected component of $f^{-2}(J)$ is not greater than $(1/2)\operatorname{diam}(J)$. Hence, for $J_n=f^{-2(n-1)}(J_1)$ $(n\geqq 1)$, there exists a homeomorphism $\psi_n: J_n\to I\times F_n$ where F_n is a finite set with $\operatorname{card}(F_n)\geqq 2^{n-1}$. Put $V_0=\{\bar{x}\in X: x_1\in J_1\}$. Clearly V_0 is a neighborhood of $\bar{a}$, and this is expressed by the inverse limit of the sequence

$$J_1 \underset{f^2}{\longleftarrow} J_2 \underset{f^2}{\longleftarrow} J_3 \underset{f^2}{\longleftarrow} \cdots .$$

Therefore V_0 is homeomorphic to the inverse limit of the sequence

$$I\times F_1 \underset{\Psi_1}{\longleftarrow} I\times F_2 \underset{\Psi_2}{\longleftarrow} I\times F_3 \underset{\Psi_3}{\longleftarrow} \cdots ,$$

where $\Psi_n=\psi_n\circ f^2\circ\psi_{n+1}^{-1}$ $(n\geqq 1)$. This implies that V_0 is homeomorphic to the product of I and a Cantor set. The proof is completed.

§3. Proof of Theorem 2.

Let $f(x)=1-|2x-1|$ as before. Define the continuous map $h: (-1/2, 3/2)\to \boldsymbol{R}$ by

$$h(x)=\begin{cases} f(x)-(2\pi)^{-1}\sin(2\pi x) & (-1/2<x\leqq 1/2) \\ f(x)+(2\pi)^{-1}\sin(2\pi x) & (1/2<x<3/2) . \end{cases}$$

Clearly h satisfies the following.

(L.1) (i) $h(0)=0$, $h(1/2)=1$ and $h(x)=h(1-x)$ for $-1/2<x<3/2$.
(ii) $h(-x)=-h(x)$ for $-1/2<x<1/2$.
(iii) $h'(0)=1$ and $h'(x)>1$ for $x\in(-1/2, 1/2)-\{0\}$.
(iv) $h(x)=x+o(x^2)$.

Here h' denotes the derivative of h, and $o(t)$ means a function such that $o(t)/t\to 0$ as $t\to 0$.

Note that the restriction of h to I is a continuous map from I onto itself. Let $(X_h, \sigma_h)=\varprojlim(I, h)$ and $(X, \sigma)=\varprojlim(I, f)$. Clearly Theorem 2 is obtained from the next two propositions.

PROPOSITION 3.1. *(X_h, σ_h) is topologically conjugate to (X, σ).*

PROPOSITION 3.2. *There exists $g: S^3 \to S^3$, a C^1-diffeomorphism of the 3-sphere which has an attractor Λ such that (Λ, g) is topologically conjugate to (X_h, σ_h).*

(I) PROOF OF PROPOSITION 3.1. We have to show that there exists a homeomorphism φ_0 from X onto X_h such that $\varphi_0 \circ \sigma = \sigma_h \circ \varphi_0$. To do this we need several Lemmas.

Let $T^1 = \boldsymbol{R}/\boldsymbol{Z}$ and denote the natural projection by $\pi_0: \boldsymbol{R} \to T^1$. For each $x \in T^1$ there is a unique $t_x \in [0, 1]$ with $\pi_0(t_x) = x$. Hence the continuous map $p_0: T^1 \to \boldsymbol{I}$ is well defined by $p_0(x) = 1 - |2t_x - 1|$. Consider the continuous map $\bar{\eta}(x) = 2x - (2\pi)^{-1} \sin(2\pi x)$ $(x \in \boldsymbol{R})$ and denote by $\eta: T^1 \to T^1$ the factor of $\bar{\eta}$ under π_0. Let ζ denote the endomorphism of T^1 defined by $\zeta(x) = 2x$ $(x \in T^1)$.

(L.2) (i) p_0 is an open map. (ii) $p_0(x) = p_0(-x)$ $(x \in T^1)$. (iii) $p_0 \circ \zeta = f \circ p_0$. (iv) $p_0 \circ \eta = h \circ p_0$. (v) $\bar{\eta}(x) + \bar{\eta}(1-x) = 2$ $(x \in \boldsymbol{R})$. (vi) For every nonempty open set U in T^1, there exists an integer $N > 0$ such that $\eta^N(U) = T^1$.

PROOF. (i)~(v) are easy. (vi) follows from the fact that $\bar{\eta}'(x) > 1$ for every $x \in \boldsymbol{R} - \boldsymbol{Z}$.

We denote by $C^0(Y)$ the set of all continuous maps from a topological space Y to itself. For each $\alpha \in C^0(T^1)$, we denote by $\bar{\alpha} \in C^0(\boldsymbol{R})$ a lift of α. Then it is well known (P. 64 of [9]) that, for every $x \in \boldsymbol{R}$ and $n \in \boldsymbol{Z}$ with $n \neq 0$, the number $(1/n)(\bar{\alpha}(x+n) - \bar{\alpha}(x))$ is an integer, and that this integer is independent of the choice of x and n. Such an integer is called the *degree* of α and denoted by $\deg(\alpha)$. A map $\alpha \in C^0(T^1)$ is said to be *monotone* if a lift $\bar{\alpha}$ satisfies $\bar{\alpha}(x_1) \geqq \bar{\alpha}(x_2)$ for every $x_1, x_2 \in \boldsymbol{R}$ with $x_1 \geqq x_2$ (this definition is obviously independent of the choice of $\bar{\alpha}$).

(L.3) (i) $\deg(\zeta) = 2$.
(ii) $\deg(\eta) = 2$.
(iii) η is monotone.

PROOF. Obvious.

(L.4) There exists a homeomorphism $\alpha \in C^0(T^1)$ satisfying
(i) $\alpha(x) + \alpha(-x) = 0$ $(x \in T^1)$
and
(ii) $\alpha \circ \eta = \zeta \circ \alpha$.

PROOF. Define

$$H = \{\alpha \in C^0(T^1): \alpha \text{ is monotone and satisfies } \deg(\alpha) = 1\}$$

and

$$V=\{\bar{\alpha}\in C^0(\boldsymbol{R})\colon \bar{\alpha} \text{ is a lift of some } \alpha\in H.\ \ \bar{\alpha}(x)+\bar{\alpha}(1-x)=1\ \ (x\in\boldsymbol{R})\}\ .$$

Since $\alpha\in H$ is degree-one, the metric function D of V is defined by

$$D(\bar{\alpha},\bar{\beta})=\max\{d(\bar{\alpha}(x),\bar{\beta}(x))\colon x\in[0,1]\}\qquad \text{for}\quad \bar{\alpha},\bar{\beta}\in V\ ,$$

where d denotes the euclidian metric of $\boldsymbol{R}$.

We claim that V is a complete metric space. Indeed, if $\{\bar{\alpha}_i\}$ is a Cauchy sequence with respect to D, then $\{\bar{\alpha}_i\}$ uniformly converges to some $\bar{\alpha}\in C^0(\boldsymbol{R})$. Since a uniform limit of lifts of degree-one maps is itself a lift, $\bar{\alpha}$ is a lift of some $\alpha_0\in C^0(T^1)$. As $\{\bar{\alpha}_i\}\subset V$, α_0 is monotone and degree-one. Also $\bar{\alpha}$ satisfies $\bar{\alpha}(x)+\bar{\alpha}(1-x)=1$ $(x\in\boldsymbol{R})$. Hence $\bar{\alpha}$ belongs to V, i.e. V is complete.

Let $\bar{\zeta}$ be the lift of ζ defined by $\bar{\zeta}(x)=2x$ $(x\in\boldsymbol{R})$. Define the map $T\colon V\to C^0(\boldsymbol{R})$ by $T(\bar{\alpha})=\bar{\zeta}^{-1}\circ\bar{\alpha}\circ\bar{\eta}$. We claim that T is a contraction map on V. Let $\bar{\alpha}\in V$ be given. Since $\deg(\eta)=2$ and $\deg(\alpha)=1$, we have $T(\bar{\alpha})(n+x)-T(\bar{\alpha})(x)=(1/2)(\bar{\alpha}(2n+\bar{\eta}(x))-\bar{\alpha}\bar{\eta}(x))=n$ for every $x\in\boldsymbol{R}$ and $n\in\boldsymbol{Z}$. So $T(\bar{\alpha})$ is a lift of some $\alpha'\in H$. Using (L.2(v)) and the equation $\bar{\alpha}(x)+\bar{\alpha}(1-x)=1$, we have

$$\begin{aligned}T(\bar{\alpha})(x)+T(\bar{\alpha})(1-x)&=(1/2)\bar{\alpha}\bar{\eta}(x)+(1/2)\bar{\alpha}(2-\bar{\eta}(x))\\&=(1/2)(\bar{\alpha}\bar{\eta}(x)+\bar{\alpha}(1-\bar{\eta}(x))+1)=1\ ,\end{aligned}$$

so that $T(\bar{\alpha})\in V$. This means $T(V)\subset V$. For every $\bar{\alpha},\bar{\beta}\in V$, we have

$$\begin{aligned}D(T(\bar{\alpha}),T(\bar{\beta}))&=\max\{d(\bar{\zeta}^{-1}\bar{\alpha}\bar{\eta}(x),\bar{\zeta}^{-1}\bar{\beta}\bar{\eta}(x))\colon x\in[0,1]\}\\&=(1/2)\max\{d(\bar{\alpha}(y),\bar{\beta}(y))\colon y=\bar{\eta}(x)\in[0,2]\}\\&=(1/2)D(\bar{\alpha},\bar{\beta})\ .\end{aligned}$$

Therefore T is a contraction map on V.

Since V is complete, T has a unique fixed point $\bar{\alpha}$ in V; i.e. $\bar{\alpha}\circ\bar{\eta}=\bar{\zeta}\circ\bar{\alpha}$. Denote by α the factor of $\bar{\alpha}$ under π_0. It is easy to see that $\alpha\circ\eta=\zeta\circ\alpha$ and $\alpha(x)+\alpha(-x)=0$ $(x\in T^1)$. To complete the proof of (L.4), it only remains to show that α is one-to-one. Assume that $x\neq y$ and $\alpha(x)=\alpha(y)$ for some $x,y\in T^1$. Then there is a nonempty open interval $U\subset T^1$ with $\alpha(U)=\alpha(x)$, because α is monotone and degree-one. By (L.2(vi)) one has $\eta^N(U)=T^1$ for some $N>0$. Hence $T^1=\alpha\eta^N(U)=\zeta^N\alpha(U)=\zeta^N\alpha(x)$, which is a contradiction.

(L.5) There is a homeomorphism $\beta\colon I\to I$ such that $\beta\circ h=f\circ\beta$.

PROOF. Let α and p_0 be as in (L.4) and (L.2) respectively. Suppose

$p_0(x)=p_0(y)$ and $x\neq y$. Then one has $x=-y$, so that $p_0\alpha(x)=p_0\alpha(-y)=p_0(-\alpha(y))=p_0\alpha(y)$ by (L.4(i)) and (L.2(ii)). Hence there is a map $\beta: I\to I$ such that $\beta\circ p_0=p_0\circ\alpha$. By (L.2(i)), β is continuous. Similarly, since α is a homeomorphism, there is a continuous map $\beta': I\to I$ such that $\beta'\circ p_0=p_0\circ\alpha^{-1}$. Then one has $\beta\circ\beta'\circ p_0=\beta\circ p_0\circ\alpha^{-1}=p_0\circ\alpha\circ\alpha^{-1}=p_0$, and also $\beta'\circ\beta\circ p_0=p_0$. Since p_0 is surjective, we have $\beta\circ\beta'=\beta'\circ\beta=\mathrm{id}$; i.e. β is a homeomorphism. By (L.2(iv)), (L.4(ii)) and (L.2(iii)), it follows that $\beta\circ h\circ p_0=f\circ\beta\circ p_0$. Using $p_0(T^1)=I$, we get $\beta\circ h=f\circ\beta$.

Now we complete the proof of Proposition 3.1. Let β be as in (L.5). Define the continuous map $\varphi_0: X\to I^N$ by $\varphi_0((x_i)_{i\geq 1})=(\beta^{-1}x_i)_{i\geq 1}$ for $(x_i)_{i\geq 1}\in X$. Since $h(\beta^{-1}x_{i+1})=\beta^{-1}f(x_{i+1})=\beta^{-1}(x_i)$ for every $(x_i)_{i\geq 1}\in X$, one has $\varphi_0(X)\subset X_h$. Since $\beta^{-1}: I\to I$ is a homeomorphism, φ_0 is a homeomorphism from X onto X_h. Using the equation $\beta^{-1}\circ f=h\circ\beta^{-1}$, we have

$$\varphi_0\sigma((x_i)_{i\geq 1})=(\beta^{-1}f(x_i))_{i\geq 1}=(h\beta^{-1}(x_i))_{i\geq 1}=\sigma_h\varphi_0((x_i)_{i\geq 1})$$

for every $(x_i)_{i\geq 1}\in X$. Therefore (X_h, σ_h) is topologically conjugate to (X, σ). The proof is completed.

(II) PROOF OF PROPOSITION 3.2. First of all we prepare some notation. Let $\kappa=\sinh^{-1}(2)$ (≈ 1.44). Define

$$M=\{(x, y, z)\in \boldsymbol{R}^3: |x|\leqq\kappa,\ y\in[0, \pi],\ |z|\leqq\kappa\}$$

and

$$U_0=\bigcup_{v\in M}\{u\in \boldsymbol{R}^3: d(u, v)<1/2\},$$

where d denotes the enclidian metric of $\boldsymbol{R}^3$. Then there exists a C^∞-map $\Phi: U_0\to \boldsymbol{R}^3$ such that

$$\Phi(x, y, z)=\begin{cases}(\sinh(x),\ -\cos(y)\cosh(z),\ \sin(y)\sinh(z)) \\ \qquad\qquad \text{for}\quad (x, y, z)\in U_0 \quad\text{with}\quad y\leqq\pi/4, \\ (\sin(y)\sinh(x),\ -\cos(y)\cosh(x),\ \sinh(z)) \\ \qquad\qquad \text{for}\quad (x, y, z)\in U_0 \quad\text{with}\quad y\geqq 3\pi/4,\end{cases}$$

and $\Phi|M': M'\to\Phi(M')$ is a C^∞-diffeomorphism, where

$$M'=\{(x, y, z)\in M: \pi/4\leqq y\leqq 3\pi/4\}.$$

Indeed, as such a C^∞-map we can choose

$$\Phi(x, y, z)=\chi_0(y)(\sinh(x),\ -\cos(y)\cosh(z),\ \sin(y)\sinh(z)) \\ +\bar{\chi}_0(y)(\sin(y)\sinh(x),\ -\cos(y)\cosh(x),\ \sinh(z))$$

where χ_0: $\boldsymbol{R} \to \boldsymbol{R}$ is a monotone decreasing C^∞-function such that $\chi_0(y)=1$ $(y \leqq \pi/4)$ and $\chi_0(y)=0$ $(y \geqq 3\pi/4)$, and $\bar{\chi}_0$ is defined by $\bar{\chi}_0(y)=1-\chi_0(y)$ $(y \in \boldsymbol{R})$.
As an easy corollary the following holds.

(L.6) (i) Let $M(t)$ $(t \in [0, \pi])$ be the leaf of foliation defined by $M(t)=\{(x, y, z) \in M: y=t\}$, then Φ is one-to-one on $M-(M(0) \cup M(\pi))$.

(ii) Φ is a C^∞-local diffeomorphism on $M-(\{(x, 0, 0) \in M(0)\} \cup \{(0, \pi, z) \in M(\pi)\})$.

(iii) There is a number $c>0$ such that $d(\Phi(u), \Phi(v)) \leqq c d(u, v)$ for every $u, v \in M$.

(iv) $\Phi(x, 0, z)=\Phi(x, 0, -z)$ for $(x, 0, z) \in M(0)$, and

$$\Phi(x, \pi, z)=\Phi(-x, \pi, z) \quad \text{for} \quad (x, \pi, z) \in M(\pi) .$$

(v) Φ is an open map.

(vi) Put $W=\Phi(M)$ (this is illustrated in Figure 1), then $\Phi(M_0)=\operatorname{int}(W)$ where $M_0=\{(x, y, z) \in M: |x|<\kappa, |z|<\kappa\}$.

(vii) Put $W(t)=\Phi(M(t))$ $(t \in [0, \pi])$. For each $u \in W$ there is a unique $t_u \in [0, \pi]$ with $u \in W(t_u)$. Then the map p: $W \to I$ defined by $p(u)=t_u/\pi$ is continuous.

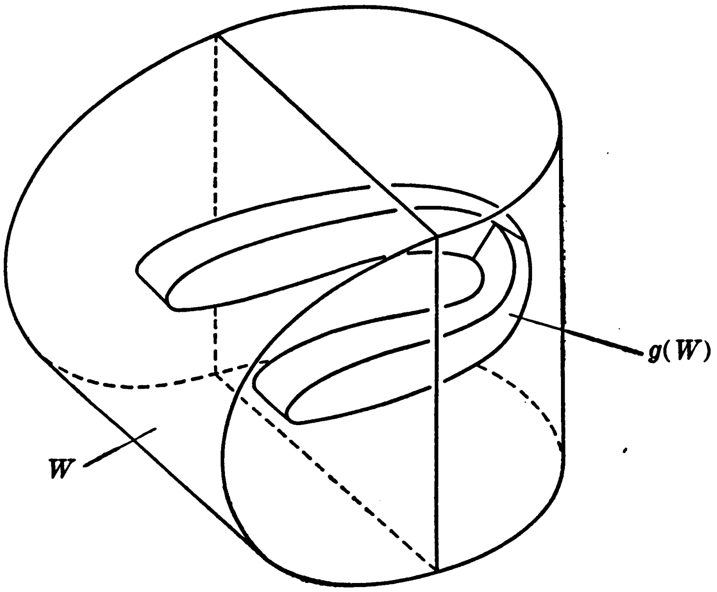

FIGURE 1

PROPOSITION 3.3. *Let W and $\{W(t): t \in [0, \pi]\}$ be as above. Then there exists a continuous map g: $W \to W$ which satisfies the following conditions;*

(1) *let h be a map as in* (L.1) *and define h_1: $(-\pi/2, 3\pi/2) \to \boldsymbol{R}$ by $h_1(t)=\pi h(t/\pi)$, then $g(W(t)) \subset W(h_1(t))$ for every $t \in [0, \pi]$,*

(2) $g(W) \subset \mathrm{int}(W)$,

(3) $\max_{t\in[0,\pi]} \mathrm{diam}\, g^n(W(t)) \to 0$ *as* $n \to \infty$,

(4) g *is one-to-one*,

(5) g *is a* C^∞*-local diffeomorphism on* W-$(L_1 \cup L_2)$ *where*

$$L_1 = \Phi(\{(x, 0, 0) \in M(0)\}) \quad and \quad L_2 = \Phi(\{(0, \pi, z) \in M(\pi)\}) ,$$

(6) g *is a* C^1*-local diffeomorphism on* $L_1 \cup L_2$,

(7) g *is isotopic to the identity map of* W.

If Proposition 3.3 holds, then Proposition 3.2 is proved as follows. Let g be the continuous map as in Proposition 3.3. From (4), (5) and (6) it follows that $g: W \to W$ is a C^1-diffeomorphism. We consider W to be $W \subset \boldsymbol{R}^3 \subset S^3$. By the isotopy extension theorem (P. 180 of [7]), g is extended to a C^1-diffeomorphism from S^3 onto itself. Denote the extended diffeomorphism by the same symbol g. Then $\Lambda = \bigcap_{n\geq 0} g^n(W)$ is a g-invariant compact set.

To show that (Λ, g) is topologically conjugate to (X_h, σ_h), let $p: W \to I$ be the continuous map as in (L.6(vii)). Then one has $h \circ p = p \circ g$ by (1). Since $hpg^{-(i+1)}(u) = pg^{-i}(u)$ for every $u \in \Lambda$ and $i \geqq 0$, the continuous map $\varphi: \Lambda \to X_h$ is well defined by $\varphi(u) = (p(u), pg^{-1}(u), pg^{-2}(u), \cdots)$. We claim that φ is one-to-one and onto; i.e. a homeomorphism. Indeed, if $pg^{-i}(u) = pg^{-i}(u')$ for every $i \geqq 0$, then there are $t_i \in [0, \pi]$ $(i \geqq 0)$ such that $u, u' \in g^i(W(t_i))$. By (3) one has $u = u'$; i.e. φ is one-to-one. To see $\varphi(\Lambda) = X_h$, let $(y_i)_{i\geq 1} \in X_h$ be given. It is easy to see that $\pi y_i = h_1(\pi y_{i+1})$ for each $i \geqq 1$. Hence one has $g^i(W(\pi y_{i+1})) \subset g^{i-1}(W(\pi y_i))$ $(i \geqq 1)$ by (1). By (3) there is $u_y \in \Lambda$ with $\{u_y\} = \bigcap_{i\geq 1} g^i(W(\pi y_{i+1}))$. Since $\varphi(u_y) = (pg^{-i+1}(u_y))_{i\geq 1} = (y_i)_{i\geq 1} \in \varphi(\Lambda)$, φ is onto. Since $\sigma\varphi(u) = (hpg^{-i+1}(u))_{i\geq 1} = (pg^{-i+2}(u))_{i\geq 1} = \varphi g(u)$ for every $u \in \Lambda$, (Λ, g) is topologically conjugate to (X_h, σ_h) under φ.

(Λ, g) satisfies specification since so does (X_h, σ_h) (by combining Theorem 1(B) and Proposition 3.1). Obviously (Λ, g) is topologically transitive. Hence Λ is an attractor of g by (2). This prove Proposition 3.2.

It remains only to prove Proposition 3.3.

(III) Proof of Proposition 3.3. We must construct a continuous map g satisfying the conditions (1)~(7). To do this we define several functions.

(L.7) Let $h_2: \boldsymbol{R} \to \boldsymbol{R}$ be a C^∞-function such that

(i) $h_2(-t) = -h_2(t)$ $(t \in \boldsymbol{R})$, (ii) $h_2(\kappa) = \kappa/3$,

(iii) $h_2'(0) = 1$ and $0 < h_2'(t) < 1$ $(t \neq 0)$,

(iv) $h_2''(t)<0$ $(t>0)$,

(v) $\sqrt{(h_2(t))^2+(h_2(s))^2} \leqq \sqrt{2}\, h_2(\sqrt{t^2+s^2}/\sqrt{2})$ $((t, s) \in \boldsymbol{R}^2)$.

(As such a function, we can choose $h_2(t)=\lambda \tan^{-1}(t/\lambda)$ where λ is the root of $\tan(\kappa/(3\lambda))=\kappa/\lambda$ with $0<\lambda<\pi/2$; $\lambda \approx 0.306$.) Then one obtains

(vi) $h_2(t)=t+o(t^2)$, (vii) $h_2(t_1)<h_2(t_2)$ $(t_1<t_2)$,

(viii) $\lim_{n\to\infty} h_2^n(t)=0$ $(t \in \boldsymbol{R})$, (ix) $|h_2(t)| \geqq |t|/3$ $(|t| \leqq \kappa)$,

(x) $|h_2(t)-h_2(t')| \leqq 2h_2(|t-t'|/2)$ $(t, t' \in \boldsymbol{R})$.

Let h_0 define by $h_0(y)=2y-(1/2)\sin(2y)$ $(y \in \boldsymbol{R})$. Recall the map h_1 as in (1). We remark that $h_1(y)=h_0(y)$ on $(-\pi/2, \pi/2]$ and $h_1(y)=2\pi-h_0(y)$ on $[\pi/2, 3\pi/2)$. Choose a constant $\alpha>0$ such that $h_0((\pi/2)-\alpha)>3\pi/4$. Put $M_1=\bigcup_{t\in[0,\pi/2]} M(t)$ and $M_2=\bigcup_{t\in[\pi/2,\pi]} M(t)$. We denote by U_i the α-open neighborhood of M_i in $\boldsymbol{R}^3$ $(i=1, 2)$. Take a monotone decreasing C^∞-function $\chi_1: \boldsymbol{R}\to\boldsymbol{R}$ such that

$$\chi_1(y)=1 \quad (y \leqq \pi/4) \quad \text{and} \quad \chi_1(y)=0 \quad (y \geqq (\pi/2)-\alpha)\ ,$$

and a monotone increasing C^∞-function $\chi_2: \boldsymbol{R}\to\boldsymbol{R}$ such that

$$\chi_2(y)=0 \quad (y \leqq (\pi/2)+\alpha) \quad \text{and} \quad \chi_2(y)=1 \quad (y \geqq 3\pi/4)\ .$$

Put $\bar{\chi}_i(y)=1-\chi_i(y)$ $(i=1, 2)$. We define two C^∞-diffeomorphisms $G_i: U_i\to\boldsymbol{R}^3$ $(i=1, 2)$ by

$$G_1(x, y, z)=\chi_1(y)\Big(\frac{x}{3}-\frac{\kappa}{2}, h_0(y), h_2(z)\Big) + \bar{\chi}_1(y)\Big(\frac{1}{3\sqrt{2}}(x-z)-\frac{\kappa}{2}, h_0(y), \frac{1}{3\sqrt{2}}(x+z)\Big)$$

and

$$G_2(x, y, z)=\chi_2(y)\Big(\frac{z}{3}+\frac{\kappa}{2}, 2\pi-h_0(y), h_2(x)\Big) + \bar{\chi}_2(y)\Big(\frac{1}{3\sqrt{2}}(z-x)+\frac{\kappa}{2}, 2\pi-h_0(y), \frac{1}{3\sqrt{2}}(z+x)\Big)\ .$$

By the definitions of G_i and M_i one has $G_i(M_i)\subset M$ for $i=1, 2$. Take an open neighborhood $U_i'\subset U_i$ of M_i such that $G_i(U_i')\subset U_0$ $(i=1, 2)$. We define the map $G: U_1'\cup U_2'\to\boldsymbol{R}^3$ by

$$G=G_1 \quad \text{on} \quad \{(x, y, z)\in U_1': y \leqq \pi/2\}$$

and

$$G=G_2 \quad \text{on} \quad \{(x, y, z)\in U_2': y>\pi/2\}\ .$$

Notice that G is not continuous at $(x, \pi/2, z) \in U_1' \cap U_2'$. Nevertheless, the composition $\Phi \circ G$: $U_1' \cup U_2' \to \boldsymbol{R}^3$ is a C^∞-map. Because, for $(x, y, z) \in U_1' \cap U_2'$, taking account of the inequalities

$$|y-(\pi/2)|<\alpha\ , \qquad 3\pi/4<h_0(y)<\pi+\frac{1}{2} \quad \text{and} \quad 3\pi/4<2\pi-h_0(y)<\pi+\frac{1}{2}\ ,$$

one can easily verify that the definitions of G_1, G_2 and Φ imply the relation

$$\Phi G_1(x, y, z)=\Phi G_2(x, y, z)=\Big(\sin(h_0(y))\sinh\Big(\frac{1}{3\sqrt{2}}(x-z)-\frac{\kappa}{2}\Big)\ ,$$
$$-\cos(h_0(y))\cosh\Big(\frac{1}{3\sqrt{2}}(x-z)-\frac{\kappa}{2}\Big),\ \sinh\Big(\frac{1}{3\sqrt{2}}(x+z)\Big)\Big)\ .$$

(L.8) (i) $G(M(t))\subset\{(x, h_1(t), z)\in M\colon |x|<\kappa,\ |z|<\kappa\}$ for $t\in[0, \pi]$.
(ii) $\Phi\circ G$ is one-to-one on $M-(M(0)\cup M(\pi))$.
(iii) $\Phi\circ G$ is a C^∞-local diffeomorphism on

$$M-(\{(x, 0, 0)\in M(0)\}\cup\{(0, \pi, z)\in M(\pi)\})\ .$$

PROOF. (i) follows from $h_1(t)=h_0(t)$ $(t\leqq\pi/2)$ and $h_1(t)=2\pi-h_0(t)$ $(t>\pi/2)$. (ii) and (iii) follow immediately from the definitions of Φ and G.

Now we show the existence of a map $g\colon W\to W$ with $g\circ\Phi=\Phi\circ G$. Suppose that $\Phi(x, y, z)=\Phi(x', y', z')$ and $(x, y, z)\neq(x', y', z')$. By (L.6(i)) we have either $x=x'$, $y=y'=0$ and $z=z'$, or $x=-x'$, $y=y'=\pi$ and $z=z'$. Hence, by (L.6(iv)) and (L.7(i)), we have

$$\Phi G(x, 0, z)=\Phi\Big(\frac{x}{3}-\frac{\kappa}{2},\ 0,\ h_2(z)\Big)=\Phi\Big(\frac{x}{3}-\frac{\kappa}{2},\ 0,\ h_2(-z)\Big)=\Phi G(x, 0, -z)\ .$$

Similarly $\Phi G(x, \pi, z)=\Phi G(-x, \pi, z)$ holds. Consequently we have $\Phi G(x, y, z)=\Phi G(x', y', z')$. This implies that there exists a map g such that $g\circ\Phi=\Phi\circ G$. The image $g(W)$ is illustrated in Figure 1.

In order to prove Proposition 3.3, it remains only to show that g is continuous and satisfies the conditions (1)$\sim$(7).

(L.9) (i) g is continuous.
(ii) g satisfies the conditions (1)$\sim$(5).

PROOF. (i) follows from (L.6(v)). g satisfies (1) by (L.8(i)), (2) by (L.6(vi)) and (L.8(i)), and (5) by (L.8(iii)). We prove that g satisfies (3). Let $y\in[0, \pi]$ be given. Suppose $y\leqq\pi/2$. Then, for every $u=(x, y, z)$

and $u'=(x', y, z')$ in $M(y)$, we have

$$d(G(u), G(u'))\leqq\chi_1(y)\left\|\left(\frac{x-x'}{3}, 0, h_2(z)-h_2(z')\right)\right\|+\bar{\chi}_1(y)\left\|\left(\frac{x-x'}{3}, 0, \frac{z-z'}{3}\right)\right\|$$
$$\leqq 2\left\|\left(h_2\left(\frac{x-x'}{3}\right), 0, h_2\left(\frac{z-z'}{2}\right)\right)\right\| \quad \text{(by (ix) and (x) in (L.7))}$$
$$\leqq 2^{3/2}h_2(2^{-3/2}d(u, u')) \quad \text{(by (L.7(v)))} .$$

Similarly, for $y>\pi/2$, we have $d(G(u), G(u'))\leqq 2^{3/2}h_2(2^{-3/2}d(u, u'))$ for every $u, u'\in M(y)$. Hence it follows that

$$2^{-3/2}\operatorname{diam} G^n(M(y))\leqq h_2(2^{-3/2}\operatorname{diam} G^{n-1}(M(y)))\leqq\cdots$$
$$\leqq h_2^n(2^{-3/2}\operatorname{diam} M(y))=h_2^n(\kappa)$$

for every $y\in[0, \pi]$ and $n>0$. From this we get

$$\max_{y\in[0,\pi]}\operatorname{diam} g^n(W(y))=\max_{y\in[0,\pi]}\operatorname{diam}\Phi\circ G^n(M(y))$$
$$\leqq c\cdot\max_{y\in[0,\pi]}\operatorname{diam} G^n(M(y)) \quad \text{(by (L.6(iii)))}$$
$$\leqq 2^{3/2}ch_2^n(\kappa)\longrightarrow 0 \text{ (as } n\longrightarrow\infty) \quad \text{(by (L.7(viii)))} ;$$

i.e. g satisfies (3).

We prove that g satisfies (4). By (L.6(i)) and (L.8(ii)), g is one-to-one on $W-(W(0)\cup W(\pi))$. Let $(r, s, 0), (r', s', 0)\in W(0)$ satisfy $g(r, s, 0)=g(r', s', 0)$. There exist $(x, 0, z)$ and $(x', 0, z')$ in $M(0)$ such that $\Phi(x, 0, z)=(r, s, 0)$ and $\Phi(x', 0, z')=(r', s', 0)$. Since $g\circ\Phi=\Phi\circ G$, we have

$$\left(\sinh\left(\frac{x}{3}-\frac{\kappa}{2}\right), -\cosh(h_2(z)), 0\right)=\left(\sinh\left(\frac{x'}{3}-\frac{\kappa}{2}\right), -\cosh(h_2(z')), 0\right) .$$

By (L.7(i)) we get eigher $x=x'$ and $z=z'$, or $x=x'$ and $z=-z'$. In any case, $\Phi(x, 0, z)=\Phi(x', 0, z')$; i.e. $(r, s, 0)=(r', s', 0)$. Hence g is one-to-one on $W(0)$. Similarly it follows that g is one-to-one on $W(\pi)$. Since $g(W-(W(0)\cup W(\pi)))\cap g(W(0)\cup W(\pi))=\varnothing$, g is one-to-one on W; i.e. g satisfies (4).

(L.10) g satisfies (6) and (7).

Proof. First we prove that g is a C^1-local diffeomorphism on L_1. Let $v_0=(r_0, -1, 0)$ be a point in L_1 and $v=(r, s, t)$ a point sufficiently near v_0 with $v\neq v_0$. Take a point $u_0=(x_0, 0, 0)$ such that $\Phi(u_0)=v_0$. There is a point $u=(x, y, z)$ in U_0 such that $\Phi(u)=v$. Since u is also sufficiently near u_0 by (L.6(v)), we may asume that $-\pi/4\leqq y\leqq\pi/4$. Then we have

$$(r, s, t)=(\sinh (x),\ -\cos (y) \cosh (z),\ \sin (y) \sinh (z))$$

and

$$\begin{aligned} g(v)&=(g_1, g_2, g_3) \\ &=\left(\sin\left(\frac{x}{3}-\frac{\kappa}{2}\right),\ -\cos (h_1(y)) \cosh (h_2(z)),\ \sin (h_1(y)) \sinh (h_2(z))\right) . \end{aligned}$$

Hence

$$\frac{dg_1}{dr}=\frac{dg_1}{dx}\Big/\frac{dr}{dx}\longrightarrow a_0 \qquad (\text{as } x\longrightarrow x_0;\ \text{i.e. } r\longrightarrow r_0)$$

where $a_0=\{\cosh ((x_0/3)-(\kappa/2))\}/\{3\cosh(x_0)\}>0$. Using (L.1(iv)) and (L.7(vi)), we get

$$\begin{aligned} &\lim_{(s,t)\to(-1,0)} \frac{\|(g_2(v), g_3(v))-(g_2(v_0), g_3(v_0))-(s+1, t)\|}{\|(s+1, t)\|} \\ &= \lim_{(y,z)\to(0,0)} \left[\frac{\{o(z^2) \cos (y)+o(y^2) \cosh (z)+o(y^2)o(z^2)\}^2}{\{\cosh (z)-\cos (y)\}^2}\right. \\ &\quad \left.+\frac{\{o(z^2) \sin (y)+o(y^2) \sinh (z)+o(y^2)o(z^2)\}^2}{\{\cosh (z)-\cos (y)\}^2}\right]^{1/2} \\ &=0 . \end{aligned}$$

Therefore g is differentiable at $(r_0, -1, 0)$ and one has

$$Dg(r_0, -1, 0)=\begin{bmatrix} a_0 & & 0 \\ & 1 & \\ 0 & & 1\end{bmatrix} \quad \text{and} \quad Jg(r_0, -1, 0)=a_0>0 .$$

From an easy calculation it follows that

$$\frac{\partial(g_1, g_2, g_3)}{\partial(r, s, t)}=\frac{\partial(g_1, g_2, g_3)}{\partial(x, y, z)}\cdot\left[\frac{\partial(r, s, t)}{\partial(x, y, z)}\right]^{-1}\longrightarrow\begin{bmatrix} a_0 & & 0 \\ & 1 & \\ 0 & & 1\end{bmatrix} \quad (\text{as } u\longrightarrow u_0) .$$

This implies that g is a C^1-local diffeomorphism on L_1. Similarly we can prove that g is a C^1-local diffeomorphism on L_2. Therefore g satisfies (6).

From (4), (5) and (6), g is a C^1-diffeomorphism from W into $\boldsymbol{R}^3$. Since W is a closed ball in $\boldsymbol{R}^3$ and $Jg(u)>0$ holds at $u=(r_0, -1, 0)\in W$, g is orientation preserving. Therefore g is isotopic to the identity map (P. 117 of [7]); i.e. g satisfies (7). The proof is completed.

References

[1] N. Aoki and M. Dateyama, The relationship between algebraic number and expansiveness of group automorphisms, to appear in Fund. Math..

[2] N. Aoki, M. Dateyama and M. Komuro, Solenoidal automorphisms with specifications, Monatsh. Math., **93** (1982), 79–110.

[3] R. Bowen, Periodic points and measures for axiom A diffeomorphisms, Trans. Amer. Math. Soc., **154** (1971), 377–397.

[4] R. Bowen, Some systems with unique equilibrium state, Math. Systems Theory, **8** (1974), 193–202.

[5] M. Denker, C. Grillenberger and K. Sigmund, Ergodic Theory on Compact Spaces, Lecture Notes in Math., **527**, Springer, 1976.

[6] M. Hénon, A two-dimensional mapping with a strange attractor, Comm. Math. Phps., **50** (1976), 69–77.

[7] M. W. Hirsch, Differential Topology, Graduate Texts in Math., **33**, Springer, 1976.

[8] D. Lind, Ergodic group automorphisms and specification, Lecture Notes Math., **729** (edited by Denker and Jacobs) Ergodic Theory, Springer, 1979, 93–104.

[9] Z. Nitecki, Differentiable Dynamics, The MIT Press, Cambridge Ma., 1971.

[10] K. Sigmund, On dynamical systems with the specification property, Trans. Amer. Math. Soc., **190** (1974), 285–299.

[11] S. Smale, Differentiable dynamical systems, Bull. Amer. Math. Soc., **73** (1967), 747–817.

[12] R. F. Williams, One-dimensional non-wandering sets, Topology, **6** (1967), 473–487.

[13] R. F. Williams, Classification of one-dimensional attractors, Global Analysis, Proceedings of Symposia in Pure Math., **14** (1970), Amer. Math. Soc., Providence, 341–361.

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