TOKYO J. MATH.
VOL. 5, No. 2, 1982

Configurations and Invariant Gauss-Manin Connections of Integrals I

Kazuhiko AOMOTO

Nagoya University
(Communicated by T. Saito)

Introduction

A sequence of m polynomials $f_1, \cdots, f_m$ in $C[x_1, \cdots, x_n]$ defines a configuration of hyper-surfaces S_j: $f_j=0$ in C^n. The space of such configurations, being parametrized by the coefficients of polynomials, can be regarded as an analytic space. On this space, the integrals

$$\text{(J)} \qquad \int \exp[f_0] f_1^{\lambda_1} \cdots f_m^{\lambda_m} dx_1 \wedge \cdots \wedge dx_n$$

$$\text{(J')} \qquad \int f_0^{\lambda_0} f_1^{\lambda_1} \cdots f_m^{\lambda_m} dx_1 \wedge \cdots \wedge dx_n$$

satisfy Gauss-Manin connections or equivalently holonomic systems in the sense of S. S. K. (See [8], [16] and [18].) But generally it seems difficult to get their explicit formulae in global forms. According to the method which has been developed in [1] and [5], in this note we shall give *the Gauss-Manin connections* for the above integrals *in invariant expressions* with respect to certain algebraic groups which act on them in a natural way, *when f_0 is quadratic and $f_1, \cdots, f_m$ are all linear* (see the formulae EI_0, EII_0-EII_p, $EIII_0$, EIV_0 and EV_p). These equations generalize *the Schläfli formula* for the volume of a spherical simplex (see [2]) and *Appell's hyper-geometric functions of type* (F_4) (see [10]). In case where f_0 is linear, they have been computed in terms of logarithmic forms and simple rational 1-forms of Grassmann coordinates attached to the configuration of hyper-planes (see [3]).

In Part II of this note, we shall show from the results obtained in Part I, that in case where the exponents $\lambda_0, \lambda_1, \cdots, \lambda_m$ are all integers, the integrals can be expressed by means of logarithmic connections of basic algebraic invariants, so that they become *hyper-logarithms* in the sense

Received April 15, 1981

of [4]. This proves the Theorem 2 stated in [6]. *The integrability condition* for these Gauss-Manin connections gives certain significant relations among logarithmic forms and associated hyper-logarithms (see the formulae RI_1-RI_2, RII_1-RII_2 and also [7]). On the other hand, as V. Lakshmibai, C. Musili and C. S. Seshadri have recently shown, the classical invariant theory can be formulated in the framework of Schubert Calculus in "miniscule" flag manifolds, by using so-called "standard monomials" (see [12]). It seems interesting to study further *the structures of logarithmic forms and hyper-logarithms of standard monomials,* in relation to Schubert Calculus on the generalized flag manifolds.

As is well known, the integral (J) give some important models in mathematical physics such as *the Feynman integrals of one loop diagrams in Q. E. D.* (see [10], [15] and [17]), and *the correlation functions of the F. J. Dyson's complex systems* or *the Onsager vortex models* (see [9] and [14]). These will be discussed elsewhere.

ACKNOWLEDGMENT. The author would like to be thankful to the referee for careful reading and useful advices.

§ 1. Generalized Schläfli integrals.

First we are going to describe in an explicit way the Gauss-Manin connection for the integral, $m \geqq n$,

$$\text{(J. I}_0\text{)} \qquad \tilde{\varphi}(\lambda_1, \cdots, \lambda_m; \phi) = \int \exp(f_0) \cdot f_1^{\lambda_1} \cdot f_2^{\lambda_2} \cdots f_m^{\lambda_m} dt_1 \wedge \cdots \wedge dt_n$$

where $f_0 = (-1/2)(t_1^2 + \cdots + t_n^2)$ and $(\lambda_1, \cdots, \lambda_m) \in C^m$, by means of the fundamental $SO(n)$-invariants attached to the configuration $\langle f_1, \cdots, f_m \rangle$. It is known that $\tilde{\varphi}(\phi)$ is a meromorphic function of $\lambda_1, \cdots, \lambda_m$ satisfying the maximally overdetermined linear difference system. First we shall compute this. We shall denote by X the space of configurations $\langle f_1, \cdots, f_m \rangle$, canonically identified with $C^{(n+1)m}$.

We denote by T_j the j-th difference operators acting on the integrals defined by:

$$\text{(1.1)} \qquad T_j^{\pm} \tilde{\varphi}(\lambda_1, \cdots, \lambda_m) = \tilde{\varphi}(\lambda_1, \cdots, \lambda_j \pm 1, \cdots, \lambda_m) .$$

The followings are easily proved.

LEMMA 1. (i) $T_j \cdot T_k = T_k \cdot T_j$,
(ii) *Each* T_j *is invertible,*
(iii) *For any multiplication of a function* $\psi(\lambda)$, *we have* $T_j \circ \psi(\lambda) =$

$T_j\psi(\lambda)\circ T_j$,

(iv) *For any differential operator of constant coefficients P on*, $T_j\circ P = P\circ T_j$.

Let S_j be the hyperplane $f_j=0$, $0\leqq j\leqq m$, and $S=\bigcup_{j=1}^m S_j$ in C^n. First we assume

(H. 1) $S_0, S_1, \cdots, S_{m+1}$ are all real and normally crossing each other, where S_{m+1} denotes the hyperplane at infinity.

(H. 2) $\lambda_1, \cdots, \lambda_m$ are all real and positive, such that $\lambda_{i_1}+\cdots+\lambda_{i_p}\notin Z$ for any $1\leqq i_1<i_2<\cdots<i_p\leqq m+1$, $1\leqq p\leqq n$, where λ_{m+1} denotes $-\sum_{j=1}^m \lambda_j$.

Let M be the complement C^n-S and $H^*(M, \nabla_\omega)$ be the rational de Rham cohomology with respect to the covariant differentiation $\nabla_\omega\psi = d\psi+\omega\wedge\psi$, defined by the 1-form

$$\omega = df_0+\sum_{j=1}^m \lambda_j df_j/f_j \; . \tag{1.2}$$

This is essentially the *regularization of integrals in the sense of* Hadamard-Leray. For the general theory of this cohomology, see [1] and [8].

LEMMA 1. *$H^n(M, \nabla_\omega)$ has a basis of the following type*:

$$\varphi(i_1, \cdots, i_p) = \tau/f_{i_1}\cdots f_{i_p} \; , \tag{1.3}$$

$1\leqq i_1<\cdots<i_p\leqq m$, $0\leqq p\leqq n$, so that rk $H^n(M, \nabla_\omega)$ is equal to $\sum_{p=0}^n\binom{m}{p}$, where τ denotes the n-form $dt_1\wedge\cdots\wedge dt_n$. This is also equal to the number of connected components Δ of $R^n-R^n\cap S$.

We have the pairing:

$$\begin{aligned}\tilde\varphi(i_1, \cdots, i_p) &= \langle\varphi(i_1, \cdots, i_p), \Delta\rangle \\ &= \int \exp(f_0) f_1^{\lambda_1}\cdots f_m^{\lambda_m}\varphi(i_1, \cdots, i_p)\end{aligned} \tag{1.4}$$

between the cohomology $H^n(M, \nabla_\omega)$ and the homology of twisted cycles $H_n(M, \mathscr{S}_{-\omega})$ with the modified local system $\mathscr{S}_{-\omega}$ (see [5]), where

$$d(U\cdot\psi) = U\cdot\nabla_\omega\psi \; , \tag{1.5}$$

for the function $U=\exp(f_0)\cdot f_1^{\lambda_1}\cdots f_m^{\lambda_m}$.

PROOF. See [5].

We shall abbreviate by $\varphi(I)$ the form $\varphi(i_1, \cdots, i_p)$ for the sequence of indices $I=(i_1, \cdots, i_p)$ of length $|I|=p$. We put $f_j=\sum_{\nu=1}^n u_{j\nu}t_\nu+u_{j,0}$.

We may assume the norm of f_j, $\|f_j\|^2=\sum_{\nu=1}^n u_{j,\nu}^2$ to be equal to 1. We define the symmetric matrix $A=(a_{i,j})$, $0\leqq i, j\leqq m$ as follows:

$$A=\begin{pmatrix} a_{0,0}, a_{0,1}, \cdots, a_{0,m} \\ a_{1,0}, a_{1,1}, \cdots, a_{1,m} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ a_{m,0}, a_{m,1}, \cdots, a_{m,m} \end{pmatrix} \tag{1.6}$$

where $a_{i,j}=(f_i, f_j)$ denote the scalar product of the coefficients of f_i and f_j: $(f_i, f_j)=\sum_{\nu=1}^n u_{i\nu}u_{j\nu}$, and $a_{0,0}=1$, $a_{0,1}=a_{1,0}=u_{1,0}$, $\cdots$, $a_{m,0}=a_{0,m}=u_{m,0}$. These matrix elements invariant with respect to $O(n, C)$ parametrize the configuration space X of sequences $\langle f_1, \cdots, f_m\rangle$. Remark that A is normalized such that $a_{0,0}=a_{1,1}=\cdots=a_{m,m}=1$. As was noted in [11], this space can be identified with a Schubert variety in a big cell of a miniscule flag manifold.

NOTATION. For $1\leqq i_1<\cdots<i_p\leqq m$, $0\leqq p\leqq n$, we put $A\begin{pmatrix} i_1\cdots i_p \\ j_1\cdots j_p\end{pmatrix}$ the subdeterminant of A of the $i_1, \cdots, i_p$-th lines and $j_1, \cdots, j_p$-th columns of A and abbreviate by $A(i_1, \cdots, i_p)$ the principal one $A\begin{pmatrix} i_1\cdots i_p \\ i_1\cdots i_p\end{pmatrix}$. We shall denote by $[i_1, \cdots, i_n]$ and $[i_1, \cdots, i_{n+1}]$ the determinants of the matrices $((u_{i_\alpha,\sigma}))_{\substack{1\leqq\alpha\leqq n \\ 1\leqq\sigma\leqq n}}$ and $((u_{i_\alpha,\sigma}))_{\substack{1\leqq\alpha\leqq n+1 \\ 0\leqq\sigma\leqq n}}$ respectively. $\partial_\mu I$ and $\partial_\mu\partial_\nu I$ will represent the deleted sequences $(i_1, \cdots, i_{\mu-1}, i_{\mu+1}, \cdots, i_p)$ and $(i_1, \cdots, i_{\mu-1}, i_{\mu+1}, \cdots, i_{\nu-1}, i_{\nu+1}, \cdots, i_p)$ for $I=(i_1, \cdots, i_p)$ respectively.

PROPOSITION 1.1$_p$. (*Maximally overdetermined system of linear difference equations*). *We have, for any sequence of indices* $I=(i_1, \cdots, i_p)$, $0\leqq p\leqq n$,

(i) $T_{i_1}\tilde{\varphi}(I)=\tilde{\varphi}(\partial_1 I)$,
(*Remark that* $\tilde{\varphi}(I)$ *is symmetric with respect to* $i_1, \cdots, i_p$.)

$$\text{(D. I}_p\text{)} \quad \text{(ii)} \quad A(I)T_{i_0}\tilde{\varphi}(I)=\sum_{\mu=1}^{p}(-1)^{\mu-1}A\begin{pmatrix} I \\ i_0, \partial_\mu I\end{pmatrix}\cdot(\tilde{\varphi}(\partial_\mu I)-u_{i_\mu,0}\tilde{\varphi}(I)) + u_{i_0,0}\tilde{\varphi}(I)+\sum_{k\notin I}\lambda_k A\begin{pmatrix} i_0, I \\ k, I\end{pmatrix}\tilde{\varphi}(k, I)$$

if $i_0\notin I$.

PROOF. As the affine coordinates, we can take $y_1=f_{i_1}$, $\cdots$, $y_p=f_{i_p}$ and $y_{p+1}=f_{i_{p+1}}$, $\cdots$, $y_n=f_{i_n}$ where $(i_{p+1}, \cdots, i_n)$ are suitably chosen such that

$I\cap(i_{p+1},\cdots,i_n)=\emptyset$. Then f_0 is written as follows:

$$f_0=\frac{1}{2}\sum_{\mu,\nu=1}^{n}b_{\mu,\nu}(f_{i_\mu}-u_{i_\mu,0})(f_{i_\nu}-u_{i_\nu,0}) \,, \tag{1.7}$$

where the matrix $B=((b_{\mu,\nu}))_{1\leq\mu,\nu\leq n}$ denotes the inverse of $((a_{i_\mu,i_\nu}))_{1\leq\mu,\nu\leq n}$. Then,

$$[i_1,\cdots,i_n]\tilde{\varphi}(i_1,\cdots,i_p)=\int U\frac{df_{i_1}\wedge\cdots\wedge df_{i_n}}{f_{i_1}\cdots f_{i_p}} \tag{1.8}$$

and we have

$$\begin{aligned}0&\sim\nabla_\omega\Big(\frac{(-1)^{p+\mu-1}}{f_{i_1}\cdots f_{i_p}}df_{i_1}\wedge\cdots\wedge df_{i_{p+\mu-1}}\wedge df_{i_{p+\mu+1}}\wedge\cdots\wedge df_{i_n}\Big)\\&=-\sum_{\nu=1}^{n}\frac{b_{p+\mu,\nu}(f_{i_\nu}-u_{i_\nu,0})}{f_{i_1}\cdots f_{i_p}}df_{i_1}\wedge\cdots\wedge df_{i_n}\\&\quad+\lambda_{i_{p+\mu}}[i_1,\cdots,i_n]\varphi(I)\\&\quad+\sum_{k\notin I}\lambda_k(-1)^{p+\mu-1}[k,i_1,\cdots,\widehat{i_{p+\mu}},\cdots,i_n]\varphi(k,I)\,.\end{aligned} \tag{1.9}$$

Since the subdeterminant $B(i_{p+1},\cdots,i_n)\neq 0$ (see (H. 1)), we can solve the right hand side with respect to $T_{i_{p+\mu}}\tilde{\varphi}(I)$, $1\leq\mu\leq n-p$, and get the proposition.

As for the inverse operators T_j^{-1}, we have

Proposition 1.2$_p$. *For $0\leq p<n$ and $I=(i_1,\cdots,i_p)$,*

$$\begin{aligned}&T_{i_0}^{-1}\tilde{\varphi}(I)=\tilde{\varphi}(i_0,I)\,,\quad \text{if}\quad i_0\notin I\,,\\&(\lambda_{i_1}-1)\cdot A(I)\cdot T_{i_1}^{-1}\tilde{\varphi}(I)=\sum_{\nu=1}^{p}A\begin{pmatrix}\partial_1 I\\ \partial_\nu I\end{pmatrix}(-1)^{\nu+1}\tilde{\varphi}(\partial_\nu I)\\&\qquad-\sum_{k\notin I}\lambda_k A\begin{pmatrix}I\\ k,\ \partial_1 I\end{pmatrix}\tilde{\varphi}(k,I)-A\begin{pmatrix}I\\ 0,\ \partial_1 I\end{pmatrix}\tilde{\varphi}(I)\,.\end{aligned} \tag{D. I_p^*}$$

Proof. We have only to use the relations

$$\begin{aligned}0&\sim\nabla_\omega\Big(\frac{df_{i_2}\wedge\cdots\wedge df_{i_n}}{f_{i_1}f_{i_2}\cdots f_{i_p}}\Big)=(\lambda_{i_1}-1)[i_1,\cdots,i_n]\\&\quad\times T_{i_1}^{-1}\varphi(I)+\sum_{j\notin I}\lambda_j[j,i_2,\cdots,i_n]\varphi(j,I)\\&\quad-\sum_{\nu=1}^{n}b_{1,\nu}(f_{i_\nu}-u_{i_\nu,0})[i_1,\cdots,i_n]\varphi(I)\,,\end{aligned} \tag{1.10}$$

and to solve these with respect to $T_{i_1}^{-1}\varphi(I)$, in view of the Proposition 1.1_{p}.

For $p=n$, we have the following truncated form:

PROPOSITION 1.2_{n}. *For* $I=(i_1, \cdots, i_n)$,

(D. $\mathrm{I}_{\mathrm{n}}^*$) (i) $[i_0, i_1, \cdots, i_n]T_{i_0}^{-1}\tilde{\varphi}(I)$

$$=\sum_{\sigma=0}^{n}(-1)^{\sigma}\cdot\tilde{\varphi}(i_0, \cdots, \hat{i}_{\sigma}, \cdots, i_n)[i_0, \cdots, \hat{i}_{\sigma}, \cdots, i_n] ,$$

if $i_0 \notin I$.

(ii) $$(\lambda_{i_1}-1)T_{i_1}^{-1}\tilde{\varphi}(I)=-\sum_{k\notin I}\lambda_k\frac{[k, \partial_1 I]}{[k, I]}\tilde{\varphi}(I)$$
$$+\sum_{\nu=1}^{n}(-1)^{\nu}\frac{[k, \partial_{\nu}I][k, \partial_1 I]}{[I][k, I]}\tilde{\varphi}(k, \partial_{\nu}I)$$
$$-\sum_{\mu=1}^{n}\frac{A\begin{pmatrix}0, \partial_1 I\\ I\end{pmatrix}}{A(I)}(-1)^{\mu-1}\tilde{\varphi}(I)+\sum_{\mu=1}^{n}\frac{A\begin{pmatrix}\partial_1 I\\ \partial_{\mu}I\end{pmatrix}}{A(I)}(-1)^{\mu-1}\tilde{\varphi}(\partial_{\mu}I) .$$

PROOF. The first is simply obtained by partial fraction:

$$\frac{[i_0, i_1, \cdots, i_n]}{f_{i_0}f_{i_1}\cdots f_{i_n}}=\sum_{\sigma=0}^{n}\frac{[i_0, \cdots \hat{i}_{\sigma}, \cdots, i_n]}{f_{i_0}\cdots\hat{f}_{i_{\sigma}}\cdots f_{i_n}}(-1)^{\sigma} . \tag{1.11}$$

The second is obtained by the relation

$$0 \sim \nabla_{\omega}\left(\frac{df_{i_2}\wedge\cdots\wedge df_{i_n}}{f_{i_1}\cdot f_{i_2}\cdots f_{i_n}}\right) \tag{1.12}$$

and (D. $\mathrm{I}_{\mathrm{n}}^*$)-(1).

Propositions 1.2_{p} and 1.2_{n} imply the following:

THEOREM 1. (D. $\mathrm{I}_{\mathrm{p}}^*$), $0\leqq p\leqq n$, *define the maximally overdetermined linear difference system with respect to the basis* $\tilde{\varphi}(i_1, \cdots, i_p)$, $1\leqq i_1<\cdots<i_p\leqq m$, $0\leqq p\leqq n$. *These are also equivalent to the system* (D. I_{p}), $0\leqq p\leqq n$.

Now we have an invariant expression of the Gauss-Manin connection of the integral (J) as follows.

PROPOSITION 1.3. (*Generalization of Schläfli formula, inhomogeneous case*).

$$d\tilde{\varphi}(\phi)=\sum_{j=1}^{m}da_{j,0}\lambda_j\tilde{\varphi}(j)+\frac{1}{2}\sum_{1\leqq j\neq k\leqq m}da_{j,k}\lambda_j\lambda_k\tilde{\varphi}(jk) . \tag{E. I_0}$$

PROOF. By suitable change of coordinates we may assume that $f_1 = t_1, \cdots, f_n = t_n$ and $f_0 = -(1/2)\sum_{\mu,\nu=1}^{n} b_{\mu,\nu}(t_\mu - u_{\mu,0})(t_\nu - u_{\nu,0})$. Then we have

$$\frac{\partial\tilde{\varphi}(\phi)}{\partial u_{n+j,0}} = \lambda_{n+j}\tilde{\varphi}(n+j)\ , \tag{1.13}$$

$$\begin{aligned}\frac{\partial\tilde{\varphi}(\phi)}{\partial u_{n+j,\mu}} &= \lambda_{n+j}T_\mu(\tilde{\varphi}(n+j) - u_{\mu,0}\cdot\lambda_{n+j}\tilde{\varphi}(n+j)) \\ &= \lambda_{n+j}\left[\frac{a_{n+j,\mu}}{a_{n+j,n+j}}(\tilde{\varphi}(\phi) - u_{n+j,0}\tilde{\varphi}(n+j)) \right. \\ &\quad \left. + \sum_{k\neq n+j}\lambda_k \frac{A\begin{pmatrix} n+j, \mu \\ n+j, k\end{pmatrix}}{a_{n+j,n+j}}\tilde{\varphi}(n+j, k)\right].\end{aligned} \tag{1.14}$$

In the same way

$$\frac{\partial\tilde{\varphi}(\phi)}{\partial b_{\mu,\nu}} = -(T_\mu - u_{\mu,0})(T_\nu - u_{\nu,0})\tilde{\varphi}(\phi)\ , \quad \mu\neq\nu \tag{1.15}$$

$$\frac{\partial\tilde{\varphi}(\phi)}{\partial b_{\mu,\mu}} = -\frac{1}{2}(T_\mu - u_{\mu,0})^2\tilde{\varphi}(\phi)\ . \tag{1.16}$$

According to Proposition 1.1_p, each right hand side can be described in terms of $\tilde{\varphi}(i_1, \cdots, i_p)$ as follows: For $1\leqq\mu, \nu\leqq n$,

$$\begin{aligned}(T_\mu - u_{\mu,0})(T_\nu - u_{\nu,0})\tilde{\varphi}(\phi) &= \left(\sum_{j=1}^{m}\lambda_j\frac{a_{j,\mu}a_{j,\nu}}{a_{j,j}} + a_{\mu,\nu}\right)\tilde{\varphi}(\phi) \\ &\quad -\sum_{j=1}^{m}\lambda_j\frac{a_{j,\mu}a_{j,\nu}a_{j,0}}{a_{j,j}}\tilde{\varphi}(j) + \frac{1}{2}\sum_{\substack{j\neq k \\ 1\leqq j,k\leqq m}}\left(\frac{a_{j,\mu}A\begin{pmatrix} j, \nu \\ j, k\end{pmatrix}}{a_{j,j}} + \frac{a_{k,\mu}A\begin{pmatrix} k, \nu \\ k, j\end{pmatrix}}{a_{k,k}}\right)\tilde{\varphi}(j, k)\ ,\end{aligned} \tag{1.17}$$

$$(T_\mu - u_{\mu,0})\tilde{\varphi}(\phi) = \sum_{j=1}^{m}\lambda_j a_{j,\mu}\tilde{\varphi}(j)\ . \tag{1.18}$$

Since we have

$$\begin{aligned}d\tilde{\varphi}(\phi) &= \sum_{j=1}^{m-n} du_{n+j,0}\otimes\frac{\partial\tilde{\varphi}(\phi)}{\partial u_{n+j,0}} + \sum_{j=1}^{m-n}\sum_{\mu=1}^{n} du_{n+j,\mu}\otimes\frac{\partial\tilde{\varphi}(\phi)}{\partial u_{n+j,\mu}} \\ &\quad + \frac{1}{2}\sum_{1\leqq\mu\neq\nu\leqq n} db_{\mu,\nu}\otimes\frac{\partial\tilde{\varphi}(\phi)}{\partial b_{\mu,\nu}} + \sum_{\mu=1}^{n} db_{\mu,\mu}\otimes\frac{\partial\tilde{\varphi}(\phi)}{\partial b_{\mu,\mu}}\ ,\end{aligned} \tag{1.19}$$

the formula (E. I_0) is proved.

THEOREM 2. *In addition to* (D. I_p^*), $0\leqq p\leqq n$, (E. I_0) *gives a maximally* *overdetermined system of linear differential equations with respect to the basis* $\tilde{\varphi}(I)=\tilde{\varphi}(i_1, \cdots, i_p)$, $0\leqq p\leqq n$.

§ 2. Generalization of Schläfli formula, homogeneous case.

We assume now that f_j are all homogeneous, namely $u_{j,0}=0$, $1\leqq j\leqq m$. In stead of (H. 1) we shall put the following hypothesis:

(H. 1)* *Any n members of f_j are all linearly independent.*

LEMMA 2.1. *Under* (H. 1)* *and* (H. 2), *the cohomology* $H^n(C^n-S, \nabla_\omega)$ *has a basis*

$$\varphi(I)=\frac{dt_1\wedge\cdots\wedge dt_n}{f_{i_1}\cdots f_{i_p}} \tag{2.1}$$

for $1\leqq i_1<\cdots<i_p\leqq m$, $0\leqq p\leqq n$, *with the fundamental relations*:

$$0=\sum_{\nu=1}^{n+1}(-1)^{\nu-1}[i_1, \cdots \hat{i}_\nu, \cdots, i_{n+1}]\varphi(\partial_\nu I)\ , \tag{2.2}$$

for $I=\{i_1,\cdots,i_{n+1}\}$ *so that* $rkH^n(C^n-S,\nabla_\omega)$ *is equal to* $\sum_{\nu=0}^{n-1}\binom{m}{\nu}+\sum_{\sigma=0}^{m-n}(-1)^\sigma\binom{m}{n+\sigma}$.

RROOF. See [5].

As was proved in [5], this rank is also equal to the number of connected components Δ of $R^n-R^n\cap S$. The pairing

$$\tilde{\varphi}(I)=\langle\varphi(I), \Delta\rangle=\int_\Delta U\cdot\varphi(I) \tag{J. II$_p$}$$

defines the duality of $H^n(C^n-S, \nabla_\omega)$ and $H_n(C^n-S, \mathcal{S}_{-\omega})$ due to the well-known comparison theorem (see [8]).

As for the linear difference system, Proposition 1.1$_p$ is reduced to the following:

PROPOSITION 2.1$_p$. *For* $0\leqq p\leqq n$,

(i) $T_{i_1}\tilde{\varphi}(I)=\tilde{\varphi}(\partial_1 I)$,

$$\text{(ii)}\quad A(I)\cdot T_{i_0}\tilde{\varphi}(I)=-\sum_{\mu=1}^{p}A\begin{pmatrix} I \\ i_0,\ \partial_\mu I\end{pmatrix}(-1)^\mu\cdot\tilde{\varphi}(\partial_\mu I) + \sum_{k\notin I}\lambda_k A\begin{pmatrix} i_0, I \\ k, I\end{pmatrix}\tilde{\varphi}(k, I)\ , \tag{D. II$_p$}$$

if $i_0\notin I$.

PROOF. We have only to put $u_{j,0}=0$ in Proposition 1.1_p.

Proposition 1.2_p is transformed as follows by putting $u_{j,0}=0$.

PROPOSITION 2.2_p. *For* $0\leqq p<n$ *and* $I=(i_1, \cdots, i_p)$,

(D. II_p^*) (i) $$T_{i_0}^{-1}\tilde{\varphi}(I)=\tilde{\varphi}(i_0, I)\ ,\quad i\notin I\ ,$$

(ii) $$(\lambda_{i_1}-1)A(I)T_{i_1}^{-1}\tilde{\varphi}(I)=\sum_{\nu=1}^{p} A\begin{pmatrix}\partial_1 I\\ \partial_\nu I\end{pmatrix}(-1)^{\nu+1}\tilde{\varphi}(\partial_\nu I)-\sum_{k\notin I}\lambda_k A\begin{pmatrix} I\\ k,\ \partial_1 I\end{pmatrix}\tilde{\varphi}(k, I)\,.$$

PROOF. We have only to put $u_{j,0}=0$ in Proposition 1.2_p.

When $p=n$, we have to modify Proposition $1, 2_n$ as follows:

PROPOSITION 2.2_n. $\tilde{\varphi}(I)=\tilde{\varphi}(i_1, \cdots i_n)$ *satisfy the linear difference equations*:

(D. II_n^*) (i) $$T_{i_0}^{-1}\tilde{\varphi}(I)=\frac{-1}{2(\lambda_\infty-1)}\sum_{0\leqq\mu\neq\nu\leqq n}\frac{A(\partial_\mu\partial_\nu I)}{A\begin{pmatrix}\partial_\mu I\\ \partial_\nu I\end{pmatrix}}(-1)^{\mu+\nu}\tilde{\varphi}(\partial_\mu\partial_\nu I)\ ,$$

for $i_0\notin I$, *and*

(ii) $$\begin{aligned} T_{i_1}^{-1}\tilde{\varphi}(I)= &\sum_{\substack{i_0\notin I\\ 0\leqq\mu,\nu\leqq n,\ \mu\neq\nu,\ \nu\neq 1}}(-1)^{\mu+\nu+n+1}\cdot\operatorname{sgn}(\mu\nu)\frac{A\begin{pmatrix} i_0,\ \partial_1\partial_\nu I\\ i_0,\ \partial_\mu\partial_\nu I\end{pmatrix}}{A(i_0,\ \partial_\nu I)}\\ &\times A\begin{pmatrix} i_0,\ \partial_\nu I\\ I\end{pmatrix}\tilde{\varphi}(i_0,\ \partial_\mu\partial_\nu I)\frac{\lambda_{i_0}}{(\lambda_{i_1}-1)(\lambda_\infty-1)}\\ &-\frac{(\lambda_{i_1}+\cdots+\lambda_{i_n}-1)}{(\lambda_{i_1}-1)(\lambda_\infty-1)}\sum_{\sigma=1}^{n}(-1)^{\sigma+1}\cdot\frac{A\begin{pmatrix}\partial_1 I\\ \partial_\sigma I\end{pmatrix}}{A(I)}\tilde{\varphi}(\partial_\sigma I)\ .\end{aligned}$$

PROOF. We fix i_1. In view of the equality,

$$\begin{aligned} (2.3)\qquad 0\sim\nabla_\omega\Big(\frac{df_{i_2}\wedge\cdots\wedge df_{i_n}}{f_{i_1}f_{i_2}\cdots f_{i_n}}\Big) &=(\lambda_{i_1}-1)T_{i_1}^{-1}\varphi(I)[i_1, \cdots, i_n]+\sum_{j\notin I}\lambda_j[j,\ i_2, \cdots, i_n]\varphi(j, I)\\ &-\sum_{\nu=1}^{n}(-1)^{\nu-1}\frac{A\begin{pmatrix}\partial_1 I\\ \partial_\nu I\end{pmatrix}}{A(I)}\varphi(\partial_\nu I)[i_1, \cdots, i_n]\ .\end{aligned}$$

On the other hand we have, by partial fraction,

$$(2.4)\qquad \frac{[i_1, i_2, \cdots, i_n]}{f_{i_1}\cdots f_{i_n}}+\sum_{\nu=1}^{n}(-1)^{\nu}\frac{[j, i_1, \cdots, \hat{i}_\nu, \cdots, i_n]}{f_j f_{i_1}\cdots \hat{f}_{i_\nu}\cdots f_{i_n}}=0 .$$

Therefore the right hand side of (2.3) is described as:

$$(2.5)\qquad \Big(\lambda_{i_1}-1+\sum_{j\notin I}\lambda_j\Big)T_{i_1}^{-1}\tilde{\varphi}(I)[i_1, \cdots, i_n]$$
$$+\sum_{j\notin I}\sum_{\nu=2}^{n}(-1)^{\nu}\lambda_j T_{i_1}^{-1}\tilde{\varphi}(j, \partial_\nu I)[j, i_1, \cdots, \hat{i}_\nu, \cdots, i_n]$$
$$=Y(\partial_1 I)[i_1, \cdots, i_n] ,$$

where $Y(\partial_1 I)$ denotes

$$(2.6)\qquad Y(\partial_1 I)A(I)=\sum_{\nu=1}^{n}(-1)^{\nu-1}\cdot A\begin{pmatrix}\partial_\nu I\\ \partial_1 I\end{pmatrix}\tilde{\varphi}(\partial_\nu I) .$$

Owing to the Lemma 2.2, which will be proved later, we have the proposition.

LEMMA 2.2. *Let $X_{i_1,\dots,i_{n-1}}$ and $Y_{i_1,\dots i_{n-1}}$ be skew-symmetric with respect to $I=(i_1, \cdots, i_{n-1})$, then the linear equations*

$$(2.7)\qquad \Big(\alpha+\sum_{j\notin I}\lambda_j\Big)X_{i_1,\dots,i_{n-1}}+\sum_{j\notin I}\sum_{\nu=1}^{n-1}(-1)^{\nu}\lambda_j X_{j,i_1,\dots,\hat{i}_\nu,\dots,i_{n-1}}=Y_{i_1,\dots,i_{n-1}} ,$$

can be solved in a unique way, with respect to $X_{i_1,\dots,i_{n-1}}$ as follows:

$$(2.8)\qquad \alpha(\lambda_1+\cdots+\lambda_m)X_{i_1,\dots,i_{n-1}}=(\alpha+\lambda_{i_1}+\cdots+\lambda_{i_{n-1}})$$
$$\times Y_{i_1\cdots,i_{n-1}}+\sum_{k\notin I}\sum_{\nu=1}^{n-1}\lambda_k(-1)^{\nu-1}Y_{k,i_1,\dots,\hat{i}_\nu,\dots,i_{n-1}} .$$

PROOF. The proof is elementary, so we omit it.

PROOF OF (i). First we remark the following identity:

$$(2.9)\qquad [i_2, \cdots, i_{n+1}]\tilde{\varphi}(i_1, \cdots, i_{n+1})$$
$$=\sum_{\nu=2}^{n+1}(-1)^{\nu}[i_1, \cdots, \hat{i}_\nu, \cdots, i_{n+1}]T_{i_1}^{-1}\tilde{\varphi}(i_1, \cdots, \hat{i}_\nu, \cdots, i_{n+1}) .$$

By the substitution of the formulae (ii) for each term of the right hand side of the above formula, we have the desired formula.

Propositions $2.2_{\rm p}$ and $2.2_{\rm n}$ imply the

THEOREM 3. *The system* (D. $\mathrm{II}_{\rm p}^*$), $0\leq p\leq n$ *defines a maximally over-determined system of linear difference equations with respect to the basis $\tilde{\varphi}(i_1, \cdots, i_p)$ with the fundamental relations*:

(2.10) $$\sum_{\nu=0}^{n} \tilde{\varphi}(i_0, \cdots, \hat{i}_\nu, \cdots, i_n)[i_0, \cdots, \hat{i}_\nu, \cdots, i_n](-1)^\nu = 0 .$$

PROPOSITION 2.3. *Generalized Schläfli formula, homogeneous case.*

(E. II$_0$) $$d\tilde{\varphi}(\phi) = \frac{1}{2} \sum_{1 \leq j \neq k \leq m} \lambda_j \lambda_k \tilde{\varphi}(j, k) da_{j,k} .$$

PROOF. In Proposition 1.3 we have only to put $u_{j,0}=0$.

REMARK. This formula is just a generalization of classical Schläfli formula (see [2], (3.3)). In fact, when $m=n$, by taking the limit $\lambda_j \to 0$, we get the well-known Schläfli formula from (E. II$_0$).

THEOREM 4. *In addition to* (D, II$_p^*$), $0 \leqq p \leqq n$, *the system* (E, II$_0$) *defines a maximally overdetermined system of linear differential equations with respect to the basis* $\tilde{\varphi}(I) = \tilde{\varphi}(i_1, \cdots, i_p)$ $0 \leqq p \leqq n$, *with the fundamental relations* (2.10).

Actually we have the more explicit formula for the variation of $\tilde{\varphi}(I)$ as follows:

PROPOSITION 2.4$_p$.

(E. II$_p$) $$\begin{aligned} A(I) \cdot d\tilde{\varphi}(I) = & \frac{1}{2} \sum_{j \neq k; j, k \notin I} \left\{ dA\begin{pmatrix} I, j \\ I, k \end{pmatrix} - \frac{1}{2} d \log A(I, j) \cdot A\begin{pmatrix} I, j \\ I, k \end{pmatrix} \right. \\ & \left. - \frac{1}{2} d \log A(I, k) \cdot A\begin{pmatrix} I, j \\ I, k \end{pmatrix} \right\} \lambda_j \cdot \lambda_k \cdot \tilde{\varphi}(I, j, k) \\ & + \frac{A(I)}{2} \cdot \left\{ -d \log A(I) - \sum_{1 \leq \nu \leq p} \lambda_{i_\nu} d \log \left(\frac{A(\partial_\nu I)}{A(I)} \right) \right. \\ & \left. + \sum_{k \notin I} \lambda_k d \log \left(\frac{A(I, k)}{A(I)} \right) \right\} \tilde{\varphi}(I) \\ & + \frac{1}{2} \sum_{1 \leq \mu \neq \nu \leq p} (-1)^{\mu+\nu} \left\{ -dA\begin{pmatrix} \partial_\mu I \\ \partial_\nu I \end{pmatrix} + \frac{1}{2} A\begin{pmatrix} \partial_\mu I \\ \partial_\nu I \end{pmatrix} d \log A(\partial_\nu I) \right. \\ & \left. + \frac{1}{2} A\begin{pmatrix} \partial_\mu I \\ \partial_\nu I \end{pmatrix} d \log A(\partial_\mu I) \right\} \tilde{\varphi}(\partial_\mu \partial_\nu I) + \sum (-1)^{\mu+\nu} \lambda_k \\ & \times \left\{ dA\begin{pmatrix} k, \partial_\nu I \\ I \end{pmatrix} - \frac{1}{2} A\begin{pmatrix} k, \partial_\nu I \\ I \end{pmatrix} d \log A(\partial_\nu I) \right. \\ & \left. - \frac{1}{2} A\begin{pmatrix} k, \partial_\nu I \\ I \end{pmatrix} d \log A(k, I) \right\} \tilde{\varphi}(k, \partial_\nu I) . \end{aligned}$$

The above formula can be rewritten in the following way:

PROPOSITION 2.4$'_p$. *For* $0\leqq p\leqq n$, $p=|I|$,

$$\text{(E. II}'_p) \quad A(I)\cdot d\tilde{\varphi}(I)=\frac{1}{2}\sum_{j\neq k,\notin I} dA\begin{pmatrix} I, j\\ I, k\end{pmatrix}\lambda_j\cdot\lambda_k\cdot\tilde{\varphi}(I, j, k)$$
$$+\sum_{j\notin I} d\log A(I, j)\, W\begin{pmatrix} I\\ I, j\end{pmatrix}+\sum_{\nu=1}^{p} d\log A(\partial_\nu I)\cdot W\begin{pmatrix} I\\ \partial_\nu I\end{pmatrix}$$
$$+\frac{1}{2}\sum_{1\leqq\mu\neq\nu\leqq p} dA\begin{pmatrix}\partial_\mu I\\ \partial_\nu I\end{pmatrix}\cdot\tilde{\varphi}(\partial_\mu\partial_\nu I)$$
$$+\sum_{k\notin I}\sum_{\nu=1}^{p} dA\begin{pmatrix} k, \partial_\nu I\\ I\end{pmatrix}\cdot W\begin{pmatrix} I\\ k, \partial_\nu I\end{pmatrix}+dA(I)\cdot W(I) ,$$

where we put respectively:

$$(2.11) \quad W\begin{pmatrix} I\\ I, j\end{pmatrix}=-\frac{1}{2}\sum_{k\notin I, k\neq j} A\begin{pmatrix} I, j\\ I, k\end{pmatrix}\lambda_j\lambda_k\tilde{\varphi}(I, j, k)+\frac{1}{2}\lambda_j\cdot A(I)\cdot\tilde{\varphi}(I)$$
$$-\frac{1}{2}\sum_{\nu=1}^{p}(-1)^{p+\nu}\lambda_j A\begin{pmatrix} j, \partial_\nu I\\ I\end{pmatrix}\tilde{\varphi}(j, \partial_\nu I) ,$$

which is equal to

$$\frac{1}{2}\lambda_j A(j, I) T_j^{-1}\tilde{\varphi}(I, j)(\lambda_j-1) ,$$

and

$$(2.12) \quad W\begin{pmatrix} I\\ \partial_\mu I\end{pmatrix}=-\frac{1}{2}\lambda_{i_\mu}\tilde{\varphi}(I)+\frac{1}{2}\sum_{\nu=1}^{p}(-1)^{\mu+\nu}A\begin{pmatrix}\partial_\mu I\\ \partial_\nu I\end{pmatrix}\cdot\tilde{\varphi}(\partial_\mu\partial_\nu I)$$
$$-\frac{1}{2}\sum_{k\notin I}(-1)^{p+\mu}\lambda_k A\begin{pmatrix} k, \partial_\mu I\\ I\end{pmatrix}\tilde{\varphi}(k, \partial_\mu I) ,$$

which is equal to $-(1/2)T_{i_\mu}\tilde{\varphi}(\partial_\mu I)\cdot A(\partial_\mu I)$,

$$(2.13) \quad W\begin{pmatrix} I\\ j, \partial_\mu I\end{pmatrix}=(-1)^{p+\mu}\cdot\lambda_j\tilde{\varphi}(j, \partial_\mu I) , \quad \textit{and}$$

$$(2.14) \quad W(I)=\left\{-\frac{1}{2}+\sum_{\nu=1}^{p}\lambda_{i_\nu}-\sum_{k\notin I}\lambda_k\right\}\tilde{\varphi}(I) .$$

COROLLARY. *If* $A(I, k)$ *or* $A(\partial_\mu I)$ *vanishes, then* $W(I, k)$ *or* $W(\partial_\mu I)$ *vanishes. In this case* (E. II$'_p$) *still has a meaning.*

PROOF. According to the formulae (D. II$^*_p$), (D. II$^*_n$) and (D. II$_p$), we have

$$(2.15) \quad W\begin{pmatrix} I\\ I, j\end{pmatrix}=\frac{1}{2}\lambda_j(\lambda_j-1)\cdot A(j, I)\cdot T_j^{-1}\tilde{\varphi}(I) ,$$

$$(2.16)\qquad W\begin{pmatrix} I \\ \partial_\mu I \end{pmatrix} = -\frac{1}{2} A(\partial_\mu I)\cdot T_{i_\mu}\tilde{\varphi}(\partial_\mu I) ,$$

which immediately imply the corollary.

For the variation of $\tilde{\varphi}(I)$ when $|I|=n$, we can again apply the formula (E. II_p) by putting $p=n$, $A(I, j)=0$ and $W(I, j)=0$. Consequently we have

PROPOSITION $2.4'_\mathrm{n}$. *For $|I|=n$,*

$$(\mathrm{E.\ II}'_\mathrm{n})\qquad A(I)\cdot d\tilde{\varphi}(I) = \sum_{\nu=1}^{n} d\log A(\partial_\nu I)\cdot W\begin{pmatrix} I \\ \partial_\nu I \end{pmatrix} + \frac{1}{2}\sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{n} dA\begin{pmatrix} \partial_\mu I \\ \partial_\nu I \end{pmatrix}\cdot\tilde{\varphi}(\partial_\mu\partial_\nu I) + \sum_{k\notin I} dA\begin{pmatrix} k, \partial_\nu I \\ I \end{pmatrix}\cdot W\begin{pmatrix} I \\ k, \partial_\nu I \end{pmatrix} + dA(I)\cdot W(I) .$$

Theorem 4 can be restated as follows:

THEOREM 4′. *The system of equations* (E. II'_p), $0\leqq p\leqq n$, *defines a maximally overdetermined linear differential equations with respect to the basis $\tilde{\varphi}(I)$ with the fundamental relations* (2.10).

DEFINITION 2.1. We shall denote by $\omega\begin{pmatrix} i_1, \cdots, i_p \\ i_1, \cdots, i_p, i_{p+1}, i_{p+2} \end{pmatrix}$ the normalized logarithmic 1-form defined by

$$(2.17)\qquad \frac{1}{2\sqrt{-1}} d\log\left[\frac{-A\begin{pmatrix} i_1, \cdots, i_p, i_{p+1} \\ i_1, \cdots, i_p, i_{p+2} \end{pmatrix} + \sqrt{-A(i_1, \cdots, i_p)A(i_1, \cdots, i_{p+2})}}{-A\begin{pmatrix} i_1, \cdots, i_p, i_{p+1} \\ i_1, \cdots, i_p, i_{p+2} \end{pmatrix} - \sqrt{-A(i_1, \cdots, i_p)A(i_1, \cdots, i_{p+2})}}\right]$$

(see [2] p. 5). In particular we have

$$\omega\begin{pmatrix} \phi \\ i, j \end{pmatrix} = \frac{1}{2\sqrt{-1}} d\log\left[\frac{-a_{i,j}+\sqrt{a_{i,j}^2-1}}{-a_{i,j}-\sqrt{a_{i,j}^2-1}}\right],$$

because $a_{j,j}=1$. We shall also denote by $\omega\begin{pmatrix} i_1, \cdots, i_p \\ i_1, \cdots, i_p, j \end{pmatrix}$ the 1-form

$$(2.18)\qquad d\log[A(i_1, \cdots, i_p)/A(i_1, \cdots, i_p, j)] .$$

Then Proposition 2.4_p is expressed as follows:

PROPOSITION $2.4''_p$. *With respect to the normalized basis*

(2.19) $$\tilde{\tilde{\varphi}}(i_1, \cdots, i_p) = \sqrt{A(i_1, \cdots, i_p)}\tilde{\varphi}(i_1, \cdots, i_p) ,$$

$0 \leqq p < n$, *we have the simple formula*:

(E. II_p'') $$d\tilde{\tilde{\varphi}}(I) = \frac{1}{2} \sum_{1 \leqq j \neq k \leqq m} \lambda_j \cdot \lambda_k \omega\begin{pmatrix} I \\ I, j, k \end{pmatrix} \tilde{\tilde{\varphi}}(I, j, k) + \frac{1}{2}\left\{-\sum_{\nu=1}^{p} \lambda_{i_\nu} \omega\begin{pmatrix} \partial_\nu I \\ I \end{pmatrix} - \sum_{k \notin I} \lambda_k \omega\begin{pmatrix} I \\ I, k \end{pmatrix}\right\} \tilde{\tilde{\varphi}}(I) + \frac{1}{2} \sum_{1 \leqq \mu \neq \nu \leqq p} \omega\begin{pmatrix} \partial_\mu \partial_\nu I \\ I \end{pmatrix} \cdot \tilde{\tilde{\varphi}}(\partial_\mu \partial_\nu I) + \sum_{k \notin I} \lambda_k \left(\frac{A(k, \partial_\nu I) A(I)}{A(k, I) \cdot A(\partial_\nu I)}\right)^{-1/2} \omega\begin{pmatrix} \partial_\nu I \\ k, I \end{pmatrix} \cdot \tilde{\tilde{\varphi}}(k, \partial_\nu I) .$$

In the same manner, from Proposition $2.4_n'$ we have,

PROPOSITION $2.4_n''$. *For* $|I| = n$,

(E. II_n)″ $$d\tilde{\tilde{\varphi}}(I) = \frac{1}{2}\left\{-\sum_{\nu=1}^{n} \lambda_{i_\nu} \omega\begin{pmatrix} \partial_\nu I \\ I \end{pmatrix} - \sum_{k \notin I} \lambda_k d \log A(I)\right\} \tilde{\tilde{\varphi}}(I) + \frac{1}{2} \sum_{1 \leqq \mu \neq \nu \leqq n} \omega\begin{pmatrix} \partial_\mu \partial_\nu I \\ I \end{pmatrix} \cdot \tilde{\tilde{\varphi}}(\partial_\mu \partial_\nu I) + \frac{1}{2} \sum_{k \notin I, 1 \leqq \nu \leqq p} \lambda_k \left\{d \log A(k, \partial_\nu I) - \omega\begin{pmatrix} \partial_\nu I \\ I \end{pmatrix}\right\} \tilde{\tilde{\varphi}}(k, \partial_\nu I);$$

with the fundamental relations:

(2.10)′ $$\sum_{\nu=0}^{n} (-1)^\nu \tilde{\tilde{\varphi}}(i_0, \cdots, \hat{i}_\nu, \cdots, i_n) = 0 .$$

The integrability condition for the (E. II_p)′ gives rise to the amusing relations (see also [2]):

COROLLARY. (i) *For two sequences of indices* I, J *such that* $I \subset J$ *and* $|I| + 4 = |J|$, *we have*

(R. I_1) $$\sum_{\substack{I \subset K \subset J \\ |K| = |I|+2}} \omega\begin{pmatrix} I \\ K \end{pmatrix} \wedge \omega\begin{pmatrix} K \\ J \end{pmatrix} = 0$$

(ii) *For any* $j, k \notin I$, $j \neq k$, *we have*

(R. I_2) $$\omega\begin{pmatrix} I \\ I, j, k \end{pmatrix} \wedge \left(\omega\begin{pmatrix} I, k \\ I, j, k \end{pmatrix} - \omega\begin{pmatrix} I \\ I, j \end{pmatrix}\right) = 0 .$$

§ 3. The conformal case.

We now want to study the analytic structure of the integral

$$\text{(J. III}_0\text{)} \qquad \hat{\varphi}(\phi)=\int \hat{U}(\lambda)\cdot dx_1\wedge\cdots\wedge dx_n ,$$

for $\hat{U}(\lambda)=\hat{f}_0^{\lambda_0}\cdot\hat{f}_1^{\lambda_1}\cdots\hat{f}_m^{\lambda_m}$, where $\hat{f}_0=-(x_1^2+\cdots+x_n^2)+1$ and $\hat{f}_1,\cdots,\hat{f}_m$ are all linear functions: $\hat{f}_j=\sqrt{-1}\sum_{\nu=1}^n u_{j,\nu}x_\nu+u_{j,0}$. We denote by $\hat{S}_j$, $0\leqq j\leqq m$, the hyper-quadric $\hat{f}_0=0$ or hyper-planes $\hat{f}_j=0$. Then (J. III$_0$) can be regarded as the pairing between the twisted cohomology $H^n(C^n-\bigcup_{j=0}^m\hat{S}_j, \hat{\nabla})$ and the twisted dual homology $H_n(C^n-\bigcup_{j=0}^m\hat{S}_j, \mathscr{S}_{-\hat{\omega}})$, where $\hat{\nabla}$ denotes the covariant differentiation $\hat{\nabla}_\omega\psi=d\psi+\hat{\omega}\wedge\psi$, for $\hat{\omega}=\sum_{j=0}^m\lambda_j\, d\log\hat{f}_j$. We put $\sqrt{-1}x_j=t_j/t_0$, $1\leqq j\leqq n$ and $f_j=\sum_{\nu=0}^n u_{j,\nu}t_\nu(1\leqq j\leqq n)$, $f_0=t_0^2+\cdots+t_n^2$. Consider the homogeneous form of the integral (J. III$_0$) as follows:

$$\text{(J. I}_0\text{)}' \qquad \tilde{\varphi}(\phi)=\int\exp(-f_0/2)\cdot t_0^{\mu_0}\cdot\prod_{i=1}^m f_i^{\lambda_i}(t)dt_0\wedge\cdots\wedge dt_n .$$

By change of variables $(t_0,\cdots,t_n)\mapsto(\hat{t}=t_0^2\hat{f}_0, x_1,\cdots,x_n)$ and after the integration with respect to $\hat{t}$, this is rewritten as

$$\text{(3.1)} \qquad \begin{aligned}\tilde{\varphi}(\phi)&=\Gamma(-\lambda_0)\cdot 2^{-\lambda_0-1}(\sqrt{-1})^n\cdot\int_{\hat{\gamma}}\hat{f}_0^{\lambda_0}\prod_{i=1}^m\hat{f}_i^{\lambda_i}dx_1\wedge\cdots\wedge dx_n\\ &=\Gamma(-\lambda_0)\cdot 2^{-\lambda_0-1}\cdot\hat{\varphi}(\phi)(\sqrt{-1})^n ,\end{aligned}$$

the integration being done on a suitable twisted cycle $\hat{\gamma}$ in $C^n-\bigcup_{j=0}^m\hat{S}_j$, where λ_0 denotes $-(1/2)(\mu_0+\sum_{i=1}^m\lambda_i+n+1)$.

We denote by ρ the linear mapping defined by

$$\text{(3.2)} \qquad \rho\colon \hat{\varphi}(\phi)=\hat{\varphi}(\lambda_0,\cdots,\lambda_m;\phi)\longmapsto\tilde{\varphi}(\phi)=\varphi(\mu_0,\lambda_1,\cdots,\lambda_m;\phi) .$$

Then

LEMMA 3.1. *ρ is a monomorphism from $H^n(C^n-\bigcup_{j=0}^m\hat{S}_j, \nabla)$ into $H^{n+1}(C^{n+1}-\bigcup_{j=1}^{m+1}S_j, \nabla)$, where S_{m+1} denotes the hyperplane $t_0=0$.*

The hyper-quadric $Y=\hat{S}_0$ being fixed, we can take as coordinates of the configuration of hyper-planes $\hat{S}_1,\cdots,\hat{S}_m$, the $O(n+1, C)$-invariants $a_{i,j}=\sum_{\nu=0}^n u_{i,\nu}u_{j,\nu}$, $1\leqq i$, $j\leqq m$ and $a_{0,0}=1$, $a_{i,0}=u_{i,0}$, $1\leqq i\leqq m$.

DEFINITION 3.1. We denote by $\hat{T}_0^{\pm1}$, $\hat{T}_i^{\pm1}$ the difference operators as follows:

$$\text{(3.3)} \qquad \begin{aligned}\hat{T}_0^{\pm1}\hat{\varphi}(\lambda_0,\cdots,\lambda_m;\phi)&=\hat{\varphi}(\lambda_0\pm1,\lambda_1,\cdots,\lambda_m;\phi) ,\\ \hat{T}_i^{\pm1}\hat{\varphi}(\lambda_0,\cdots,\lambda_m;\phi)&=\hat{\varphi}(\lambda_0,\cdots,\lambda_i\pm1,\cdots,\lambda_m;\phi) .\end{aligned}$$

Then

Lemma 3.2. *The following diagrams are all commutative:*

$$(3.4)\qquad \begin{array}{ccc} & \xrightarrow{\ \rho\ } & \\ \hat{T}_i\Big\downarrow & & \Big\downarrow T_iT_0^{-1} \\ & \xleftarrow[\ \rho\ ]{} & \end{array}, \quad \begin{array}{ccc} & \xrightarrow{\ \rho\ } & \\ -2\lambda_0\hat{T}_0^{-1}\Big\downarrow & & \Big\downarrow T_0^2 \\ & \xrightarrow[\ \rho\ ]{} & \end{array}, \quad \begin{array}{ccc} & \xrightarrow{\ \rho\ } & \\ -\frac{1}{2(\lambda_0+1)}\hat{T}_0\Big\downarrow & & \Big\downarrow T_0^{-2} \\ & \xrightarrow[\ \rho\ ]{} & \end{array}.$$

We assume now

Assumption 3.1. $\hat{S}_0, \hat{S}_1, \cdots, \hat{S}_m$ are in general position among themselves in C^n, and $\hat{S}_1, \cdots, \hat{S}_m$ are all real.

The following lemma was proved in [1] (see [1], Théorème 4.3.)

Lemma 3.3. *Under the Assumption* 3.1, $H^n(C^n-\bigcup_{j=0}^m \hat{S}_j, \hat{\nabla})$ *has a basis*

$$(3.5)\qquad \varphi(i_1, \cdots, i_p)=\frac{dx_1\wedge\cdots\wedge dx_n}{\hat{f}_{i_1}\cdots\hat{f}_{i_p}}$$

for $1\leqq i_1<\cdots<i_p\leqq m$, $0\leqq p\leqq n$, *so that its dimension is equal to* $\sum_{p=0}^n\binom{m}{p}$. *The twisted homology* $H_n(C^n-\bigcup_{j=0}^m \hat{S}_j, \mathscr{S}_{-\hat{\omega}})$ *has a basis consisting of the relatively compact connected components of* $R^n-\bigcup_{j=0}^m R^n\cap\hat{S}_j$.

We denote by $\hat{\varphi}(i_1, \cdots, i_p)$ or $\hat{\varphi}_*(i_1, \cdots, i_p)$ the integrals

$$(\text{J. III}_p)\qquad \hat{\varphi}(i_1, \cdots, i_p)=\int\hat{U}(\lambda)\cdot\varphi(i_1, \cdots, i_p)=\hat{T}_{i_1}^{-1}\cdots\hat{T}_{i_p}^{-1}\hat{\varphi}(\phi)\ ,$$

$$\hat{\varphi}_*(i_1, \cdots, i_p)=\int\hat{U}(\lambda)\cdot\varphi_*(i_1, \cdots, i_p)\ ,$$

where $\varphi_*(i_1, \cdots, i_p)$ is defined to be

$$(3.6)\qquad \varphi_*(I)=\varphi(I)+\sum_{\nu=1}^p(-1)^\nu\frac{A\begin{pmatrix} I \\ 0,\ \partial_\nu I\end{pmatrix}}{A(I)}\varphi(\partial_\nu I)\ .$$

Remark that $\hat{\varphi}(\phi)=\hat{\varphi}_*(\phi)$ and $\varphi(I)$ is a linear combination of $\varphi_*(J)$ for $J\subset I$. In view of (3.1) and (3.4), according to the formulae (D. I_p) and (D. I_p^*), a direct calculation shows the following:

Proposition 3.1$_p$ (*Reduction from weight p to $p+1$*).

$$(\text{D. III}_p)\quad (\mu_0+p-1)\hat{T}_0\hat{\varphi}(I)=-2(\lambda_0+1)\frac{A(I)}{A(0,I)}\hat{\varphi}_*(I)-\sum_{k\notin I}\lambda_k\frac{A\begin{pmatrix}0,I\\k,I\end{pmatrix}}{A(0,I)}\hat{T}_0\hat{\varphi}(k,I)\,,$$

for $0\leqq|I|\leqq n$ *and, in particular*

$$(\text{D. III}_n)\quad (\mu_0+n-1)\hat{T}_0\hat{\varphi}(I)=\sum_{\nu=1}^{n}(-1)^{\nu+1}\cdot\frac{A\begin{pmatrix}k,\partial_\nu I\\0,\partial_\nu I\end{pmatrix}\cdot A\begin{pmatrix}0,k,\partial_\nu I\\0,I\end{pmatrix}}{A(0,k,\partial_\nu I)\cdot A(0,I)}\hat{\varphi}_*(k,\partial_\nu I)$$
$$+\frac{A(I)}{A(0,I)}(\mu_0+\lambda_{i_1}+\cdots+\lambda_{i_n}+n-1)\hat{\varphi}_*(I)\,,$$

for $|I|=n$.

In the same manner, according to the formula (D. I_p), we have the following difference systems.

Proposition 3.2$_p$. *For* $0\leqq p\leqq n$,

$$(\text{D. III}_p^*)\quad -2\lambda_0\cdot\hat{T}_0^{-1}\hat{\varphi}(I)=\sum_{j\notin I}\lambda_j\frac{A\begin{pmatrix}0,I\\j,I\end{pmatrix}}{A(I)}\cdot\hat{\varphi}(j,I)$$
$$+\sum_{\mu=1}^{p}\frac{A\begin{pmatrix}I\\0,\partial_\mu I\end{pmatrix}}{A(I)}(-1)^{\mu-1}(-2\lambda_0)\cdot\hat{T}_0^{-1}\hat{\varphi}(\partial_\mu I)$$
$$+(\mu_0+p+1)\frac{A(0,I)}{A(I)}\hat{\varphi}(I)\,.$$

Proposition 3.3$_p$. For $0\leqq p\leqq n$,

$$(\text{D. IV}_p^*)\quad \hat{T}_k^{-1}\hat{\varphi}(I)=\hat{\varphi}(k,I)\,,\quad \textit{for}\quad k\notin I$$

$$(\lambda_{i_\mu}-1)\hat{T}_{i_\mu}^{-1}\hat{\varphi}(I)=\sum_{k\notin I}\lambda_k\frac{A\begin{pmatrix}I\\k,\partial_\mu I\end{pmatrix}}{A(I)}(-1)^{\mu-1}\cdot\hat{\varphi}(k,I)$$
$$+(\mu_0+p+1)\frac{A\begin{pmatrix}I\\0,\partial_\mu I\end{pmatrix}}{A(I)}(-1)^{\mu}\hat{\varphi}(I)$$
$$+\sum_{\nu=1}^{p}\frac{A\begin{pmatrix}\partial_\mu I\\\partial_\nu I\end{pmatrix}}{A(I)}(-1)^{\mu+\nu}(-2\lambda_0)\hat{T}_0^{-1}\hat{\varphi}(\partial_\nu I)\,.$$

As for the differential system for $\hat{\varphi}(\phi)$, (E. II_0) and (3.1) imply the following:

LEMMA 3.4.

$$-2(\lambda_0+1)\cdot d\hat{\varphi}(\phi)=\frac{1}{2}\sum_{1\leq j\neq k\leq m}\lambda_j\lambda_k da_{j,k}\hat{T}_0\hat{\varphi}(j, k)+\sum_{k=1}^{m}\mu_0\lambda_k da_{0,k}\hat{T}_0\hat{\varphi}(k) . \tag{3.7}$$

We now want to eliminate the terms $\hat{T}_0\hat{\varphi}(j, k)$ and $\hat{T}_0\hat{\varphi}(k)$ in the right hand side, by using Proposition 3.1. Owing to Proposition 3.1_1, the above can also be written as follows:

$$d\hat{\varphi}(\phi)=\sum_{k=1}^{m}\lambda_k da_{0,k}\frac{\hat{\varphi}_*(k)}{A(0, k)}+\frac{1}{2}\sum_{1\leq j\neq k\leq m}\lambda_j\lambda_k da_{j,k}\frac{\hat{T}_0\hat{\varphi}(j, k)}{-2(\lambda_0+1)} \tag{3.8}$$
$$+\sum_{1\leq j\neq k\leq m}\lambda_j\lambda_k\frac{A\begin{pmatrix}0, k\\ k, j\end{pmatrix}}{A(0, k)}\frac{\hat{T}_0\hat{\varphi}(j, k)}{-2(\lambda_0+1)}da_{0,k} .$$

The latter can be described as follows:

$$\sum_{k=1}^{m}\theta\begin{pmatrix}\phi\\ k\end{pmatrix}\lambda_k\hat{\varphi}_*(k)+\frac{1}{2}\sum_{1\leq j\neq k\leq m}\lambda_j\lambda_k\frac{\theta\begin{pmatrix}\phi\\ j, k\end{pmatrix}\cdot\hat{T}_0\hat{\varphi}(j, k)}{-2(\lambda_0+1)} , \tag{3.9}$$

where $\theta\begin{pmatrix}\phi\\ j\end{pmatrix}$ and $\theta\begin{pmatrix}\phi\\ j, k\end{pmatrix}$ denote the 1-forms which are symmetric with respect to j, k:

$$\theta\begin{pmatrix}\phi\\ j\end{pmatrix}=da_{0,j}/A(0, j) , \tag{3.10}$$

$$\theta\begin{pmatrix}\phi\\ j, k\end{pmatrix}=\frac{1}{2}\left(da_{j,k}+\frac{A\begin{pmatrix}0, k\\ k, j\end{pmatrix}}{2A(0, k)}da_{0,k}+\frac{A\begin{pmatrix}0, j\\ j, k\end{pmatrix}}{2A(0, j)}da_{0,j}\right) . \tag{3.11}$$

More generally we define the following sequence of 1-forms

DEFINITION 3.2. We denote by $\theta\begin{pmatrix}\phi\\ j_1, \cdots, j_p\end{pmatrix}$, $3\leqq p\leqq n+1$, the 1-form which is symmetric with respect to $j_1, \cdots, j_p$ and defined by recurrence relations:

$$\theta\begin{pmatrix}\phi\\ j_1, \cdots, j_p\end{pmatrix}=\sum_{\nu=1}^{p}(-1)^\nu\cdot\theta\begin{pmatrix}\phi\\ j_1, \cdots, \hat{j}_\nu, \cdots, j_p\end{pmatrix}\cdot\frac{A\begin{pmatrix}0j_1\cdots\hat{j}_\nu\cdots j_p\\ j_1j_2\cdots j_p\end{pmatrix}}{A(j_1, \cdots, \hat{j}_\nu, \cdots, j_p)} . \tag{3.12}$$

A characterization of these forms will be given in Part II.

By applying repeatedly Proposition 3.1_p for the second term in the right hand side of (3.8), we have

PROPOSITION 3.4. *For* $\hat{\varphi}(\phi)=\hat{\varphi}_*(\phi)$,

$$(\text{E. III}_0)\qquad d\hat{\varphi}(\phi)=\sum_{s=1}^{n}\frac{1}{s!}\sum\frac{\lambda_{i_1}\cdots\lambda_{i_s}}{(\mu_0+1)\cdots(\mu_0+s-1)}\cdot\theta\begin{pmatrix}\phi\\ i_1,\cdots,i_s\end{pmatrix}\frac{A(i_1,\cdots,i_s)}{A(0,i_1,\cdots,i_s)}\hat{\varphi}_*(i_1,\cdots,i_s)$$
$$+\frac{1}{(n+1)!}\sum\frac{\lambda_{i_1}\cdots\lambda_{i_{n+1}}}{(\mu_0+1)\cdots(\mu_0+n-1)}\cdot\theta\begin{pmatrix}\phi\\ i_1,\cdots,i_{n+1}\end{pmatrix}\times\frac{\hat{T}_0\hat{\varphi}(i_1,\cdots,i_{n+1})}{-2(\lambda_0+1)},$$

where $\hat{T}_0\hat{\varphi}(i_1,\cdots,i_{n+1})$ *is given by the following lemma*:

LEMMA 3.5. *For any* $(n+1)$ *indices* $i_1,\cdots,i_{n+1}$ *we have the identity*:

$$\hat{T}_0=\sum_{1\leqq\alpha\neq\beta\leqq n+1}(-1)^{\alpha+\beta}\frac{A\begin{pmatrix}\partial_\alpha I\\ \partial_\beta I\end{pmatrix}}{A(I)}\hat{T}_{i_\alpha}\hat{T}_{i_\beta}\,. \tag{3.13}$$

Consequently,

$$\hat{T}_0\hat{\varphi}(I)=\sum_{1\leqq\alpha\neq\beta\leqq n+1}(-1)^{\alpha+\beta}\frac{A\begin{pmatrix}\partial_\alpha I)\\ \partial_\beta I\end{pmatrix}}{A(I)}\hat{\varphi}(\partial_\alpha\partial_\beta I)\,. \tag{3.14}$$

PROOF. By simple calculation, we have the identity (3.13) by replacing $\hat{T}_j$ by $\hat{f}_j$, which implies the lemma.

According to the Propositions 3.3 and 3.4, we can obtain a similar but complicated formula for $d\hat{\varphi}(i_1,\cdots,i_p)$, so that

THEOREM 5. *The difference systems* (D. III*) *and* (D. IV*) *define a maximally overdetermined one with respect to the basis* (3.5).

THEOREM 6. *The formula* (E. III_0) *together with* (D. III*) *and* (D. IV*) *defines a maximally overdetermined linear differential equations with respect to the basis* (3.5).

§ 4. Configuration of hyper-plane sections in a hyper-quadric.

As in the preceding section we make the Assumption 3.1, where we consider it *in the* $(n+1)$ *dimensional affine space* C^{n+1}. We put further,

ASSUMPTION 4.1. $\lambda_0=0$.

We denote by X or Y the complements $C^{n+1}-\bigcup_{j=1}^m \hat{S}_j$ or $\hat{S}_0-\hat{S}_0\cap\bigcup_{j=1}^m \hat{S}_j$. Then as was proved in [1], Theoreme 4.2 and [5], we have

PROPOSITION 4.1. *Let $\hat{\nabla}_0$ be the covariant differentiation defined on X or Y, by $\hat{\nabla}_0\psi=d\psi+\hat{\omega}_0\wedge\psi$ for $\hat{\omega}_0=\sum_{j=1}^m \lambda_j d\log\hat{f}_j$. We denote by $\hat{U}_0$ the corresponding function $\prod_{j=1}^m \hat{f}_j^{\lambda_j}$. Suppose that*

Assumption 4.2. $\lambda_1,\cdots,\lambda_m$ are real and positive. These are generic, in the sense that $\lambda_{i_1}+\cdots+\lambda_{i_p}\notin Z$ for any $1\leqq i_1<\cdots<i_p\leqq m+1$, $1\leqq p\leqq n$, where λ_{m+1} denotes $-\sum_{j=1}^m \lambda_j$.

Then we have

$$H^p(X,\hat{\nabla}_0)=0\,,\quad 0\leqq p\leqq n\,, \tag{4.1}$$

$$H^p(Y,\hat{\nabla}_0)=0\,,\quad 0\leqq p\leqq n-1\,. \tag{4.2}$$

Consequently the boundary homomorphism

$$\partial\colon H^{n+1}(X,\ Y;\ \hat{\nabla}_0)\longmapsto H^n(Y;\ \hat{\nabla}_0) \tag{4.3}$$

is an isomorphism. In view of the exact sequence

$$0\cong H^n(X,\hat{\nabla}_0)\longmapsto H^n(Y,\hat{\nabla}_0)\longmapsto H^{n+1}(X,\ Y;\ \hat{\nabla}_0)\longmapsto H^{n+1}(X,\hat{\nabla}_0)\longmapsto 0\,, \tag{4.4}$$

we have the isomorphism

$$H^{n+1}(X,\ Y;\ \hat{\nabla}_0)=H^n(Y;\ \hat{\nabla}_0)\oplus H^{n+1}(X;\ \hat{\nabla}_0)\,. \tag{4.5}$$

The corresponding dual basis is obtained as follows:

LEMMA 4.1. *The basis of $H_{n+1}(X,\mathcal{S}_{-\hat{\omega}_0})$ consists of the relatively compact components of $R^{n+1}-R^{n+1}\cap\bigcup_{j=1}^m S_j$. The basis of $H_{n+1}(X,\ Y;\ \mathcal{S}_{-\hat{\omega}_0})$ consists of the relatively compact components of $R^{n+1}-R^{n+1}\cap\bigcup_{j=0}^m S_j$. $H_{n+1}(X,\ Y;\ \mathcal{S}_{-\hat{\omega}_0})$ is isomorphic to a direct sum $\Xi_+\oplus\Xi_-$, such that*

$$\begin{aligned}\partial\colon \Xi_+ &\xrightarrow{\cong} H_n(Y,\mathcal{S}_{-\hat{\omega}_0})\,,\\ \partial\colon \Xi_- &\longrightarrow 0\,.\end{aligned} \tag{4.6}$$

Namely the basis of $H_n(Y;\ \mathcal{S}_{-\hat{\omega}_0})$ consists of the cycle $\Delta_+^{(1)}=\partial\Delta_+\cap Y$, where Δ_+ denotes any relatively compact connected components of $R^{n+1}-R^{n+1}\cap\bigcup_{j=0}^m S_j$.

NOTATION 4.1. According to J. Leray [12], p. 90, for any $\Omega^{n+1}(X, {}^*\bigcup_{j=1}^m \hat{S}_j)$, we shall denote by $d^k(\hat{f}_1^{\lambda_1}\cdots\hat{f}_m^{\lambda_m}\cdot\varphi)/(d\hat{f}_0/2)^{k+1}=\hat{f}_1^{\lambda_1}\cdots\hat{f}_m^{\lambda_m}\varphi^{(k+1)}$ the residue n-form on Y of k-th order $k!\ \mathrm{Res}\,(\hat{f}_1^{\lambda_1}\cdots\hat{f}_m^{\lambda_m}\varphi)/(\hat{f}_0/2)^{k+1}$.
Then the following is obvious.

LEMMA 4.2. *For any $\varphi\in\Omega^{n+1}(X, {}^*\bigcup_{j=1}^m S_j)$, $k\in Z^+$, we have*

$$\lim_{\lambda_0\to-1} -2(\lambda_0+1)\cdot\int_\gamma \hat{f}_0^{\lambda_0-k}\hat{f}_1^{\lambda_1}\cdots\hat{f}_m^{\lambda_m}\varphi = 1/k!\cdot\int_{\partial\gamma}\hat{f}_1^{\lambda_1}\cdots\hat{f}_m^{\lambda_m}d^k\varphi/(d\hat{f}_0)^{k+1}\,, \tag{4.7}$$

if $\gamma\in\Xi_+$ and the left hand side vanishes if $\gamma\in\Xi_-$.

PROOF. See J. Leray [12], pp. 90–94.

Owing to this lemma, we can compute, from the result in the preceding section, the linear difference system and the Gauss-Manin connection of the integral

$$\hat{\varphi}=\int\hat{f}_1^{\lambda_1}\cdots\hat{f}_m^{\lambda_m}\varphi^{(1)}\,, \tag{J. IV$_0$}$$

$\varphi^{(1)}\in\Omega^n(Y, {}^*\bigcup_{j=1}^m\hat{S}_j)$ on Y.

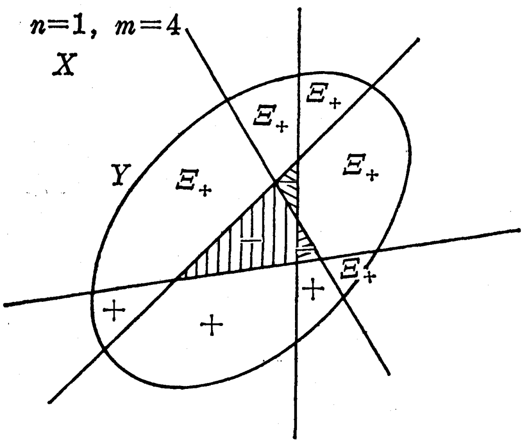

FIGURE 1

We denote by τ the standard form $dx_1\wedge dx_2\wedge\cdots\wedge dx_{n+1}$ and by $\tau^{(1)}$ $\sum_{\nu=1}^{n+1}(-1)^\nu x_\nu dx_1\wedge\cdots\wedge dx_{\nu-1}\wedge dx_{\nu+1}\wedge\cdots\wedge dx_{n+1}$, the residue n-forms

$$\varphi^{(1)}(i_1,\cdots,i_p)=\tau^{(1)}/f_{i_1}\cdots f_{i_p}=m,\ 0\leqq p\leqq n+1\,, \tag{4.8}$$

of the $(n+1)$-forms $\varphi(i_1,\cdots,i_p)=\tau/f_{i_1}\cdots f_{i_p}$, with the fundamental relations in the following way. For any sequence of indices $I=\{i_1,\cdots,i_{n+2}\}$, we have the identity:

$$\frac{1}{2}\sum_{1\leq\alpha\neq\beta\leq n+2}(-1)^{\alpha+\beta}\frac{A(0,\partial_\alpha\partial_\beta I)}{A\begin{pmatrix}0,\partial_\alpha I\\0,\partial_\beta I\end{pmatrix}}\hat{\varphi}^{(1)}(\partial_\alpha\partial_\beta I)+\sum_{\alpha=1}^{n+2}(-1)^\alpha\frac{A(\partial_\alpha I)}{A\begin{pmatrix}0,\partial_\alpha I\\I\end{pmatrix}}\hat{\varphi}^{(1)}(\partial_\alpha I)=0\,. \tag{4.9}$$

In fact, the left hand side is equal to $\hat{T}_0\hat{\varphi}(i_1,\cdots,i_{n+2})/df_0$, which is obviously equal to zero.

Proposition 4.1 follows from (4.6) and (4.7).

NOTATION. We put

$$\varphi_*^{(1)}=\varphi^{(1)}\,,$$

$$\varphi_*^{(1)}(i_1,\cdots,i_p)=\varphi^{(1)}(i_1,\cdots,i_p)+\sum_{\nu=1}^{p}(-1)^\nu\frac{A\begin{pmatrix}i_1,\cdots,i_p\\0,i_1,\cdots,\hat{i}_\nu,\cdots,i_p\end{pmatrix}}{A(i_1,\cdots,i_p)}\times\varphi^{(1)}(i_1,\cdots,\hat{i}_\nu,\cdots,i_p)\,, \tag{4.10}$$

for $0\leqq p\leqq n+1$, $1\leqq i_1<\cdots<i_p\leqq m$.

By the Lemma 4.1, we can take, as a basis of $H^n(Y,\hat{V}_0)$, the following form:

$$\varphi^{(1)}(i_1,\cdots,i_p)\,,\quad 0\leqq p\leqq n\,,\quad 1\leqq i_1<\cdots<i_p\leqq m\,,$$
$$\text{and}\quad \varphi^{(1)}(1,i_2,\cdots,i_{n+1})\,,\quad 2\leqq i_2<\cdots<i_{n+1}\leqq m\,. \tag{4.11}$$

On the other hand we know, through [3] Lemma 1, $H^{n+1}(X,\hat{V}_0)$ has a basis of the logarithmic forms

$$d\log\hat{f}_{i_1}\wedge\cdots\wedge d\log\hat{f}_{i_{n+1}} \tag{4.12}$$

for $2\leqq i_1<\cdots<i_{n+1}$. Therefore the forms

$$\varphi(i_1,\cdots,i_p)\,,\quad 0\leqq p\leqq n+1,\ 1\leqq i_1<\cdots<i_p\leqq m\,, \tag{4.13}$$

just constitute a basis of $H^{n+1}(X,Y;\hat{V}_0)$.

COROLLARY. *The dimension of* $H^n(Y,\hat{V}_0)$ *is equal to* $\sum_{p=0}^{n}\binom{m}{p}+\binom{m-1}{n}$.

We shall now represent the logarithmic forms on Y by means of the above basis. First we prove

LEMMA 4.3 (*Reduction from weight p to $p+1$*). *For $0\leqq p\leqq n$, we have the equivalence in $H^n(Y, \hat{\nabla}_0)$:*

$$(4.14)\quad (\lambda_\infty+n-p)d\log\hat{f}_{j_1}\wedge\cdots\wedge d\log\hat{f}_{j_p}\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_{n-p}}$$
$$\sim\sum_{k\notin J}\lambda_k(-1)^{\nu-1}\frac{A\begin{pmatrix}k, j_1, \cdots, j_p\\ 0, j_1, \cdots, j_p\end{pmatrix}}{A(0, j_1, \cdots, j_p, k)}\sum_{\nu=1}^{n-p}u_{k,i_\nu}$$
$$\times d\log\hat{f}_{j_1}\wedge\cdots\wedge d\log\hat{f}_{j_p}\wedge d\log\hat{f}_k\wedge dx_{i_1}\wedge\cdots\wedge\widehat{dx_{i_\nu}}\wedge\cdots\wedge dx_{i_{n-p}}$$
$$-\sum_{k\notin J}\lambda_k u\begin{bmatrix}k, j_1, \cdots, j_p\\ i_1^*, \cdots, i_{p+1}^*\end{bmatrix}\varphi_*^{(1)}(k, j_1, \cdots, j_p)$$
$$\times\operatorname{sgn}(i_1^*, \cdots, i_{p+1}^*, i_1, \cdots, i_{n-p})\,,\quad 0\leqq p\leqq n\,,$$

where $(i_1^*, \cdots, i_{p+1}^*)$ denotes the complement of $(i_1, \cdots, i_{n-p})$ in the set $(1, \cdots, n+1)$ and $u[\quad]$ the determinant

$$\begin{vmatrix}u_{k,i_1^*}\cdots u_{k,i_{p+1}^*}\\ u_{j_1,i_1^*}\cdots u_{j_1,i_{p+1}^*}\\ \\ u_{j_p,i_1^*}\cdots u_{j_p,i_{p+1}^*}\end{vmatrix}$$

In particular, for $p=n$, the logarithmic forms $d\log f_{i_1}\wedge\cdots\wedge d\log f_{i_n}$ can be expressed by means of the above basis as follows:

$$(4.15)\quad \lambda_\infty\, d\log\hat{f}_{i_1}\wedge\cdots\wedge d\log\hat{f}_{i_n}\sim$$
$$-\sum_{j\notin(i_1,\cdots,i_n)}\lambda_j[j, i_1, \cdots, i_n]\varphi_*^{(1)}(j, i_1, \cdots, i_n)\,.$$

Conversely any $\varphi_*^{(1)}(i_1, \cdots, i_{n+1})$ can be expressed as follows:

$$(4.16)\qquad [i_1, \cdots, i_{n+1}]\varphi_*^{(1)}(i_1, \cdots, i_{n+1})\sim$$
$$\sum_{\nu=1}^{n+1}(-1)^\nu d\log\hat{f}_{i_1}\wedge\cdots\wedge\widehat{d\log\hat{f}_{i_\nu}}\wedge\cdots\wedge d\log\hat{f}_{i_{n+1}}\,.$$

PROOF. An elementary computation shows that, on X for $I=(i_1, \cdots, i_p)$, $0\leqq p\leqq n$,

$$(4.17)\quad (\lambda_\infty+n-p)\hat{\nabla}_0(d\log f_{i_1}\wedge\cdots\wedge d\log f_{i_p}\wedge dx_{p+2}\wedge\cdots\wedge dx_{n+1})$$
$$=\sum_{k\notin I}\lambda_k\frac{A\begin{pmatrix}k, I\\ 0, I\end{pmatrix}}{A(0, k, I)}\sum_{\nu=p+2}^{n+1}(-1)^{\nu-p}\cdot u_{k,\nu}\cdot\hat{\nabla}_0(d\log\hat{f}_{i_1}\wedge\cdots\wedge d\log\hat{f}_{i_p}$$
$$\wedge d\log\hat{f}_k\wedge dx_{p+2}\wedge\cdots\wedge dx_{\nu-1}\wedge dx_{\nu+1}\wedge\cdots\wedge dx_{n+1})$$

$$-\sum_{k\notin I}\lambda_k u\begin{bmatrix} k, I \\ 1, \cdots, p+1\end{bmatrix}\hat{\nabla}_0\varphi_*^{(1)}(k, I)\ ,$$

in particular,

$$\lambda_\infty\hat{\nabla}_0(d\log\hat{f}_{i_1}\wedge\cdots\wedge d\log\hat{f}_{i_n})=-\sum_{k\notin I}\lambda_k[k, I]\hat{\nabla}_0\varphi_*^{(1)}(k, I)\ , \tag{4.18}$$

for $I=(i_1, \cdots, i_n)$. By applying the mapping (4.3), we get (4.14) and (4.15) respectively.

Therefore we have from (4.11)

LEMMA 4.4. *$H^n(Y, \hat{\nabla}_0)$ has a system of generators*

$$\varphi_*^{(1)}(i_1, \cdots, i_p)\ ,\quad 0\leqq p\leqq n\ , \tag{4.19}$$

and

$$d\log\hat{f}_{i_1}\wedge\cdots\wedge d\log\hat{f}_{i_n}\ ,$$

with the fundamental relations

$$\sum_{j\notin\{i_1,\cdots,i_{n-1}\}}\lambda_j d\log\hat{f}_j\wedge d\log\hat{f}_{i_1}\wedge\cdots\wedge d\log\hat{f}_{i_{n-1}}\sim 0\ , \tag{4.20}$$

for any sequence $\{i_1, \cdots, i_{n-1}\}\subset\{1, \cdots, m\}$.

Using the above basis, Proposition 3.3_p implies, through residue operation (4.7), the difference system

PROPOSITION 4.2. *For $0\leqq p\leqq n+1$ and $I=(i_1, \cdots, i_p)$,*

$$(\mathrm{D.\ V_p})\quad (\lambda_{i_\sigma}-1)\cdot\hat{T}_{i_\sigma}^{-1}\hat{\varphi}^{(1)}(I)=\sum_{k\notin I}\frac{A\begin{pmatrix} I \\ k, \partial_\sigma I\end{pmatrix}}{A(I)}(-1)^{\sigma-1}\cdot\lambda_k\cdot\hat{\varphi}^{(1)}(k, I)$$

$$+\sum_{\nu=1}^{p}\frac{A\begin{pmatrix}\partial_\sigma I \\ \partial_\nu I\end{pmatrix}}{A(I)}(-1)^{\sigma+\nu}\hat{\varphi}^{(2)}(\partial_\nu I)+(\mu_0+p+1)\frac{A\begin{pmatrix} I \\ 0, \partial_\sigma I\end{pmatrix}}{A(I)}\hat{\varphi}^{(1)}(I)(-1)^\sigma\ ,$$

$$\hat{T}_{i_0}^{-1}\hat{\varphi}^{(1)}(I)=\hat{\varphi}^{(1)}(i_0, I)\quad \text{if}\quad i_0\notin I\ ,$$

where $\varphi^{(2)}(I)$ denotes the residue form of 2-nd order: $d^2\varphi(I)/(df_0/2)^2$. From (D. III_p^*) this form itself can be expressed by recurrence formula as follows: For $0\leqq p\leqq n+1$,

$$(4.21)\quad \hat{\varphi}^{(2)}(I)=\sum_{j\notin I}\lambda_j\frac{A\begin{pmatrix}0,\ I\\ j,\ I\end{pmatrix}}{A(I)}\hat{\varphi}^{(1)}(j,\ I)$$

$$+(\mu_0+p+1)\frac{A(0,\ I)}{A(I)}\hat{\varphi}^{(1)}(I)+\sum_{\nu=1}^{p}\frac{A\begin{pmatrix}I\\ 0,\ \partial_\nu I\end{pmatrix}}{A(I)}(-1)^{\nu-1}\cdot\hat{\varphi}^{(2)}(\partial_\nu I)\ .$$

The following is proved by a direct calculation:

LEMMA 4.5.

$$(4.22)\quad \sum_{1\leqq k\leqq m}\ \sum_{1\leqq\sigma\neq\sigma'\leqq n+1}\frac{\lambda_k u_{k,\sigma}du_{k,\sigma'}}{A(0,\ k)}\operatorname{sgn}(\sigma,\ \sigma')\hat{V}_0(d\log f_k\wedge dx_1\wedge\cdots$$
$$\wedge dx_{\sigma-1}\wedge dx_{\sigma+1}\wedge\cdots dx_{\sigma'-1}\wedge dx_{\sigma'+1}\wedge\cdots\wedge dx_{n+1})(-1)^{\sigma+\sigma'}$$
$$=\lim_{\lambda_0\to-1}\sum_{1\leqq j\neq k\leqq m}\frac{1}{2}\lambda_j\lambda_k da_{j,k}\hat{T}_0\varphi(j,\ k)+\sum_{1\leqq j\neq k\leqq m}\lambda_j\lambda_k\frac{da_{0,k}A\begin{pmatrix}0,\ k\\ k,\ j\end{pmatrix}}{A(0,\ k)}\ .$$
$$\times\hat{T}_0\varphi(k,\ j)\ .$$

By this lemma, we have from (3.7) the following:

LEMMA 4.6.

$$(4.23)\quad d\hat{\varphi}^{(1)}(\phi)=\sum_{k=1}^{m}\lambda_k\frac{da_{0,k}}{A(0,\ k)}\hat{\varphi}_*^{(1)}(k)$$
$$+\sum_{k=1}^{m}\ \sum_{1\leqq\sigma<\sigma'\leqq n+1}(-1)^{\sigma+\sigma'-1}\cdot\lambda_k\frac{u_{k,\sigma'}du_{k,\sigma}-u_{k,\sigma}du_{k,\sigma'}}{A(0,\ k)}$$
$$\times\overline{d\log f_k\wedge dx_1\wedge\cdots\hat{\sigma}\cdots\hat{\sigma}'\cdots\wedge dx_{n+1}}\ .$$

PROOF. In fact

$$(4.24)\quad d\hat{\varphi}^{(1)}(\phi)=\lim_{\lambda_0\to-1}-2(\lambda_0+1)d\hat{\varphi}(\phi)$$
$$=\lim_{\lambda_0\to-1}-2(\lambda_0+1)\left[\sum_{k=1}^{m}\lambda_k\frac{da_{0,k}}{A(0,\ k)}\hat{\varphi}_*(k)+\frac{1}{2}\sum_{1\leqq j\neq k\leqq m}\theta\begin{pmatrix}\phi\\ j,\ k\end{pmatrix}\frac{\hat{T}_0\hat{\varphi}(j,\ k)}{-2(\lambda_0+1)}\right]$$
$$=\sum_{k=1}^{m}\lambda_k\frac{da_{0,k}}{A(0,\ k)}\hat{\varphi}_*^{(1)}(k)+\sum_{1\leqq k\leqq m}\frac{du_{k,\sigma}u_{k,\sigma'}}{A(0,\ k)}\lambda_k\operatorname{sgn}(\sigma,\ \sigma')$$
$$\times\int_{\Delta+}\hat{U}_0\hat{V}_0(d\log f_k\wedge dx_1\wedge\cdots\hat{\sigma}\cdots\hat{\sigma}'\cdots\wedge dx_{n+1})(-1)^{\sigma+\sigma'-1}$$
$$=\sum_{k=1}^{m}\lambda_k\frac{da_{0,k}}{A(0,\ k)}\hat{\varphi}_*^{(1)}(k)+\frac{1}{2}\sum_{k,\sigma,\sigma'}\frac{du_{k,\sigma}u_{k,\sigma'}}{A(0,\ k)}\lambda_k\operatorname{sgn}(\sigma,\ \sigma')(-1)^{\sigma+\sigma'-1}$$
$$\times\int_{\Delta_+^{(1)}}\hat{U}_0\cdot(d\log f_k\wedge dx_1\wedge\cdots\check{\sigma}\cdots\check{\sigma}'\cdots\wedge dx_{n+1})\ .$$

The lemma has been proved.

Finally we obtain the

THEOREM 7. *The variation formula for* $\hat{\varphi}_*^{(1)}(\phi)=\hat{\varphi}^{(1)}(\phi)$ *can be expressed in terms of* $\hat{\varphi}_*^{(1)}(i_1, \cdots, i_p)$ *as follows:*

$$\text{(E. IV}_0\text{)}\qquad d\hat{\varphi}_*^{(1)}(\phi)=\sum_{p=1}^{n+1}\frac{1}{p!}\sum\frac{\lambda_{i_1}\cdots\lambda_{i_p}\theta^{(1)}\begin{pmatrix}\phi\\ i_1\cdots i_p\end{pmatrix}}{(-\lambda_\infty-n+1)\cdots(-\lambda_\infty-n-1+p)}\times\hat{\varphi}_*^{(1)}(i_1, \cdots, i_p)\,,$$

where $\theta^{(1)}\begin{pmatrix}\phi\\ i_1, \cdots, i_p\end{pmatrix}$ *does not depend either on* $\lambda_1, \cdots, \lambda_m$ *or on the dimension* n.

PROOF. This follows from Lemmas 4.3, 4.4 and Lemma 4.6.

The 1-forms $\theta^{(1)}\begin{pmatrix}\phi\\ i_1, \cdots, i_p\end{pmatrix}$ will be determined later more explicitly. As a special case where $n=1$, we have the following simple formula:

$$\text{(E. IV}_0\text{)}'\ d\hat{\varphi}_*^{(1)}(\phi)=\sum_{k=1}^{m}\lambda_k\frac{da_{0,k}}{A(0, k)}\hat{\varphi}_*^{(1)}(k)$$
$$+\frac{1}{2}\sum_{1\le j\ne k\le m}\frac{A(j, k)}{A(0, j, k)}\cdot\frac{-A(0, j, k)dA\begin{pmatrix}0, j\\ 0, k\end{pmatrix}+\frac{1}{2}A\begin{pmatrix}0, j\\ 0, k\end{pmatrix}dA(0, j, k)}{A(0, j)A(0, k)}$$
$$\times\lambda_j\lambda_k\hat{\varphi}_*^{(1)}(j, k)\,.$$

The Appell's hyper-geometric function (F_4) investigated by J. Kaneko [10] is a degenerate case for $m=4$.

§ 5. Degenerate case I. Invariant connections for the diagonal general linear group $\Delta(GL_n\times GL_n)$.

Let f_0 be the quadratic form $\sum_{i=1}^n x_i\cdot y_i$ of $(x, y)\in C^n\times C^n$, and $f_1, \cdots, f_m$, $m=t+s$, be homogeneous linear functions on C^n. We consider the integral

$$\text{(J. V}_0\text{)}\quad \tilde{\varphi}(\phi)=\int\exp[-f_0(x, y)]f_1(x)^{\lambda_1}\cdots f_s(x)^{\lambda_s}\cdot f_{s+1}(y)^{\lambda_{s+1}}\cdots f_{s+t}(y)^{\lambda_{s+t}}\cdot\tau\,,$$

where τ denotes the $2n$-form $dx_1\wedge\cdots\wedge dx_n\wedge dy_1\wedge\cdots\wedge dy_n$.

Since the group $G=\Delta(GL_n\times GL_n)$ leaves f_0 invariant, the integral (J. V$_0$) admits of the action of the group G and so its Gauss-Manin connection can be expressed by means of basic algebraic invariants in the sense of H. Weyl's book (see [17], p. 45). We denote by $a_{i,j}=$

$\langle f_i, f_j\rangle=\sum_{\nu=1}^{n} u_{i,\nu}u_{j,\nu}$ the inner product of $f_i(x)=\sum_{\nu=1}^{n} u_{i,\nu}x_\nu$ and $f_i(y)=\sum_{\nu=1}^{n} u_{j,\nu}x_\nu$, $1\leqq i\leqq s$, $s+1\leqq j\leqq s+t$. These are a basic generating system of invariants with respect to G. Now we assume the following:

ASSUMPTION 5.1. (i) Arbitrary n-functions among $f_1, \cdots, f_s$ (or $f_{s+1}, \cdots, f_{s+t}$) are linearly independent, namely for any indices $(i_1, \cdots, i_n)\subset(1, \cdots, s)$ (or $\subset(s+1, \cdots, s+t)$) we have $[i_1, \cdots, i_n]\neq 0$.

(ii) For any indices $I=(i_1, \cdots, i_n)\subset(1, \cdots, s)$ and $J'=(j_1, \cdots, j_n)\subset(s+1, \cdots, s+t)$, we have $A\left(\begin{matrix} I \\ J'\end{matrix}\right)\neq 0$.

We denote by $\lambda_\infty=\sum_{j=1}^{s}\lambda_j$, and $\lambda'_\infty=\sum_{j=1}^{t}\lambda_{j+s}$.

DEFINITION 5.1. The mapping ρ. By suitable change of coordinates (x, y), we may assume that $f_1(x)$ is equal to x_1. Then by the substitution of the integral variables $x_1=x_1$, $x_j=x_1\cdot x'_j$, $2\leqq j\leqq n$ and $y_1=y_1$, $y_j=y_1\cdot y'_j$, $2\leqq j\leqq n$ respectively, (J. V_0) can be rewritten in the following way:

$$
\begin{aligned}
(\text{J. V}_0)'\ \tilde{\varphi}(\phi)=\int\exp\Big[-x_1y_1\Big(1+\sum_{j=2}^{n}x'_j\cdot y'_j\Big)\Big]&x_1^{\lambda_\infty+n-1}\cdot y_1^{\lambda'_\infty+n-1}\\
&\times f_2(1, x'_2, \cdots, x'_n)^{\lambda_2}\cdots f_s(1, x'_2, \cdots, x'_n)^{\lambda_s}f_{s+1}(1, y'_2, \cdots, y'_n)^{\lambda_{s+1}}\cdots\\
&\quad f_{s+t}(1, y'_2, \cdots, y'_n)^{\lambda_{s+t}}\\
&\times dx_1\wedge dx'_2\wedge\cdots\wedge dx'_n\wedge dy_1\wedge dy'_2\cdots\wedge dy'_n\ .
\end{aligned}
$$

In view of the fact that

$$
(5.1)\qquad \int\exp[-x_1y_1]\cdot x_1^{\lambda_\infty+n-1}y_1^{\lambda'_\infty+n-1}dx_1\wedge dy_1
$$

is equal to zero or $(2\pi i)\Gamma(\lambda_\infty+n)$ for $\lambda_\infty\neq\lambda'_\infty$ or $\lambda_\infty=\lambda'_\infty$, we have the following:

LEMMA 5.1. *$\tilde{\varphi}(\phi)$ vanishes when $\lambda'_\infty\neq\lambda_\infty$. If $\lambda'_\infty=\lambda_\infty$, then $\tilde{\varphi}(\phi)$ is equal to*

$$
\begin{aligned}
(5.2)\qquad (2\pi i)\cdot\Gamma(\lambda_\infty+n)\int&\Big(1+\sum_{j=2}^{n}x'_j\cdot y'_j\Big)^{-\lambda_\infty-n}\\
&\times f_2(1, x')^{\lambda_2}\cdots f_s(1, x')^{\lambda_s}\cdot f_{s+2}(1, x')^{\lambda_{s+2}}\cdots f_{s+t}(1, x')^{\lambda_{s+t}}dx'_2\wedge\cdots\\
&\wedge dx'_n\wedge dy'_2\wedge\cdots\wedge dy'_n
\end{aligned}
$$

for $\lambda_0=-(\lambda_\infty+n)$. We shall put $\hat{f}_0(x'y')=1+\sum_{i=2}^{n}x'_iy'_i$ and $\hat{f}_j(x')=f_j(1, x')$ respectively.

We denote by $\hat{S}_0$ the hyper-quadric $\hat{f}_0(x', y')=0$ in $C^{n-1}\times C^{n-1}$, by $\hat{S}_j$, $2\leqq j\leqq s$, the hyper-planes $\hat{f}_j(x')=0$ and by $\hat{S}_j$, $s+2\leqq j\leqq s+t$, the hyper-planes $\hat{f}_j(y')=0$. As in the preceding section, we consider the twisted cohomologies $H^n(C^n\times C^n-\bigcup_{j=1}^m S_j,\nabla)$ and $H^{2n-2}(C^{n-1}\times C^{n-1}-\bigcup_{j=0}^m \hat S_j,\hat\nabla)$ for the covariant differentiations $\nabla\psi=d\psi+\omega\wedge\psi$, and $\hat\nabla\psi=d\psi+\hat\omega\wedge\psi$, respectively, where ω and $\hat\omega$ denote $df_0+\sum_{j=1}^s\lambda_j\,d\log f_j(x)+\sum_{j=s+1}^{s+t}\lambda_j\,d\log f_j(y)$ and $d\log\hat f_0+\sum_{j=2}^s\lambda_j\,d\log\hat f_j(x')+\sum_{j=s+2}^{s+t}\lambda_j d\log\hat f_j(y')$ respectively. Then the relation (5.2) defines the linear mapping ρ from

$$H^{2n-2}\Big(C^{n-1}\times C^{n-1}-\bigcup_{\substack{j=0\\ j\neq 1,s+1}}^{m}\hat S_j,\hat\nabla\Big)\quad\text{into}\quad H^{2n}\Big(C^n\times C^n-\bigcup_{j=1}^m S_j,\nabla\Big).$$

LEMMA 5.2. *When $\lambda_\infty=\lambda'_\infty$, the mapping ρ is an isomorphism.*

PROOF. The 2-dimensional homology associated with the integral (5.1) is just equal to 1. Namely (5.1) can be integrated in a unique way. This implies the lemma.

According to Theorem 5.2 proved in [1], p. 291, the dimension of $H^{2n-2}(C^{n-1}\times C^{n-1}-\bigcup_{\substack{j=0\\ j\neq 1,s+1}}^m \hat S_j,\hat\nabla)$ is equal to the Euler number of the space $C^{n-1}\times C^{n-1}-\bigcup_{\substack{j=0\\ j\neq 1,s+1}}^m \hat S_j$, which can be computed easily by Mayer-Vietoris sequence. The result is as follows:

LEMMA 5.3. *The Euler number $\chi(C^{n-1}\times C^{n-1}-\bigcup_{j=0}^m \hat S_j)$ is equal to $\sum_{p=0}^{n-1}\binom{s-1}{p}\binom{t-1}{p}$.*

Actually this is equal to the number of a subset of connected components of the complement $R^{n-1}\times R^{n-1}-\bigcup_{j=0}^m\hat S_j$, including all the relatively compact components, where (5.2) converges (see Figure 2).

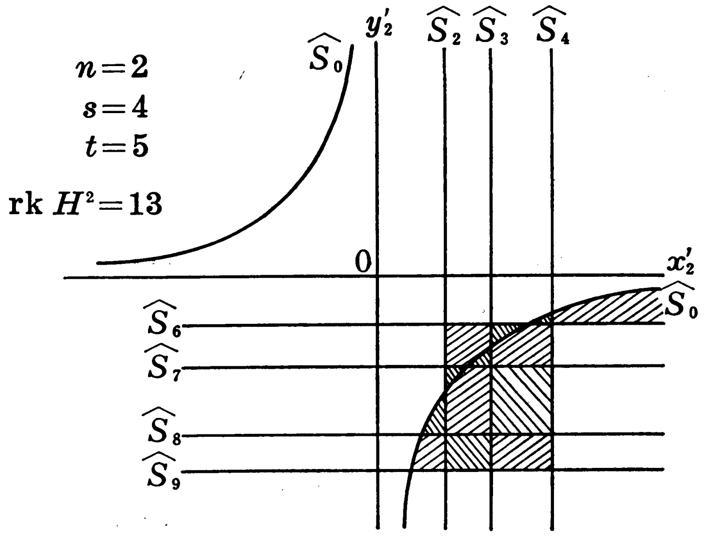

FIGURE 2

Lemma 5.2 and 5.3 imply

LEMMA 5.4. *The cohomology* $H^{2n}(C^n \times C^n - \bigcup_{j=1}^m S_j, \nabla)$ *has the dimension* $\sum_{p=0}^{n-1}\binom{s-1}{p}\binom{t-1}{p}$.

We are now in a position to prove the following:

PROPOSITION 5.1. *The cohomology* $H^{2n}(C^n \times C^n - \bigcup_{j=1}^m S_j, \nabla)$ *has a system of generators*

$$\tilde{\varphi}(I, J') = \tilde{\varphi}(i_1, \cdots, i_p, j'_1, \cdots, j'_p) = \tau / f_{i_1}(x) \cdots f_{i_p}(x) \cdot f_{j'_1}(y) \cdots f_{j'_p}(y) \tag{5.3}$$

for $I = (i_1, \cdots, i_p) \subset (1, \cdots, s)$, $J' = (j'_1, \cdots, j'_p) \subset (s+1, \cdots, s+t)$, $|I|, |J'| \leqq n$, *with the following fundamental relations*:

$$\sum_{\nu=1}^{n+1} (-1)^\nu \tilde{\tilde{\varphi}}(\partial_\nu I, J') = 0 \quad \textit{for} \quad |I| = n+1\ , \quad |J'| = n\ , \tag{5.4}$$

$$\sum_{\nu=1}^{n+1} (-1)^\nu \tilde{\tilde{\varphi}}(I; \partial_\nu J') = 0 \quad \textit{for} \quad |I| = n\ , \quad |J'| = n+1\ , \tag{5.5}$$

$$\sum_{r' \notin J'} \lambda_{r'} \tilde{\tilde{\varphi}}(I; r', J') + \sum_{\nu=1}^{|I|} (-1)^\nu \tilde{\tilde{\varphi}}(\partial_\nu I; J') = 0\ , \tag{5.6}$$

for $|I| = |J'| + 1 \leqq n$, *and*

$$\sum_{r \notin I} \lambda_r \tilde{\tilde{\varphi}}(r, I; J') + \sum_{\nu=1}^{|J'|} (-1)^\nu \tilde{\tilde{\varphi}}(I; \partial_\nu J') = 0\ , \tag{5.7}$$

for $|I| + 1 = |J'| \leqq n$, *where* $\tilde{\varphi}(I; J')$ *denotes the* $2n$-*form defined by* $A\binom{I}{J'}\tilde{\varphi}(I; J')$ (*remark that* $\tilde{\tilde{\varphi}}(I; J')$ *vanishes if* $|I| \neq |J'|$). *Therefore* H^{2n} *is spanned by the linearly independent basis* $\tilde{\tilde{\varphi}}(I; J')$, $2 \leqq i_1 < \cdots < i_p \leqq s$, $s+1 \leqq j_1 < \cdots < j_p \leqq s+t$, $0 \leqq p \leqq n-1$. *The number is just equal to* $\sum_{p=0}^{n-1}\binom{s-1}{p}\binom{t-1}{p}$.

PROOF. (5.4) and (5.5) being obvious, we have only to prove (5.6) and (5.7). The integral (J. V_0) is a special case of (J. II_0). The symmetric matrix A having parametrized the configuration of hyper-planes in §1 can be replaced here by the matrix:

$$\mathfrak{A} = \begin{pmatrix} 0 & A \\ {}^tA & 0 \end{pmatrix} = \begin{pmatrix} 0 & ((a_{i,j'})) \\ ((a_{i',j})) & 0 \end{pmatrix}. \tag{5.8}$$

In this situation we can apply the Proposition $(2.1)_p$ for $\tilde{\varphi}(I. J')$, $|I| = |J'| + 1$:

(5.9) $T_{j'_0}\tilde{\varphi}(I, J')\cdot\mathfrak{A}(I, J')$

$$=\sum_{j\notin J}\lambda_{j'}\mathfrak{A}\begin{pmatrix} I, J', j'_0 \\ I; J', j'\end{pmatrix}\cdot\tilde{\varphi}(I; J', j')+\sum_{\mu=1}^{|I|}\mathfrak{A}\begin{pmatrix} I, J' \\ \partial_\mu I; J', j'_0\end{pmatrix}(-1)^{\mu+1}\cdot\tilde{\varphi}(\partial_\mu I; J') .$$

On the other hand, we see that $\mathfrak{A}(I, J')=0$ if $|I|\neq|J'|$, and

$$\mathfrak{A}\begin{pmatrix} I; j'_0, J' \\ I; j', J'\end{pmatrix}=(-1)^{|I|}A\begin{pmatrix} I \\ j', J'\end{pmatrix}\cdot A\begin{pmatrix} j'_0, J' \\ I\end{pmatrix}, \quad \text{for} \quad |I|=|J'|+1 , \tag{5.10}$$

$$\mathfrak{A}\begin{pmatrix} I; J' \\ \partial_\mu I; j'_0, J'\end{pmatrix}=A\begin{pmatrix} I \\ j'_0, J'\end{pmatrix}\cdot A\begin{pmatrix} J' \\ \partial_\mu I\end{pmatrix}, \quad \text{for} \quad |I|=|J'|+1 . \tag{5.11}$$

By taking j'_0 such that $A\begin{pmatrix} I \\ j'_0, J'\end{pmatrix}\neq 0$, we get the (5.6). (5.7) is proved in a similar way. The last part of the Propsition 5.1 follows from the following two lemmas.

LEMMA 5.5. *The form $\tilde{\varphi}(I; J')$ for $|I|=|J'|=n$ is cohomologous to a linear combination of the forms $\tilde{\varphi}(I, J')$ for $|I|\leqq n-1$, $|J'|\leqq n-1$.*

PROOF. The relations (5.5) and (5.7) imply

$$\sum_{k\notin I}\sum_{\nu=2}^{n}\lambda_k\tilde{\tilde{\varphi}}(k, \partial_\nu I; J')(-1)^\nu+\Big(\lambda_{i_1}+\sum_{k\notin I}\lambda_k\Big)\tilde{\tilde{\varphi}}(I; J')$$
$$=\sum_{\nu=1}^{|J'|}(-1)^\nu\tilde{\tilde{\varphi}}(\partial_1 I; \partial_\nu J')+\sum_{\mu,\nu}(-1)^{\mu+\nu-1}\cdot\tilde{\tilde{\varphi}}(\partial_\mu I; \partial_\nu J') , \tag{5.12}$$

$\tilde{\tilde{\varphi}}(I; J')$ being skew-symmetric with respect to the indices $i_1, \cdots, i_n$ in I, we can apply the Lemma 2.2 and solve (5.12) with respect to $\tilde{\tilde{\varphi}}(I, J')$ as follows:

$$\tilde{\tilde{\varphi}}(I; J')\cdot\lambda_{i_1}\lambda_\infty=\sum_{k\notin I}\sum_{\substack{2\leqq\sigma\leqq n\\ 1\leqq\nu\leqq n}}\lambda_k(-1)^{\sigma+\nu}\tilde{\tilde{\varphi}}(k, \partial_1\partial_\sigma I; \partial_\nu J')$$
$$+\Big(\lambda_{i_1}+\sum_{\sigma=2}^{n}\lambda_{i_\sigma}\Big)\cdot\sum_{\nu=1}^{n}(-1)^\nu\tilde{\tilde{\varphi}}(\partial_1 I; \partial_\nu J') . \tag{5.13}$$

Taking acount of the skew-symmetry of $\tilde{\tilde{\varphi}}(I; J')$ with respect to $j'_1, \cdots, j'_n$ and (5.7) we finally have the following formula:

$$\lambda_\infty\tilde{\tilde{\varphi}}(I; J')=\sum_{\mu,\nu=1}^{n}(-1)^{\mu+\nu-1}\cdot\tilde{\tilde{\varphi}}(\partial_\mu I; \partial_\nu J') . \tag{5.14}$$

Since $\lambda_\infty\neq 0$, the lemma has been proved.

The difference system for (J. V_0) corresponding to Proposition 2.2_p can be expressed as follows:

PROPOSITION 5.2. *For* $\lambda_\infty=\lambda'_\infty+1$ *and* $I=(i_1,\cdots,i_p)$, $J'=(j_1,\cdots,i_p)$, $|I|=|J'|\leqq n-1$, *we have*

$$\text{(D. VI}_1\text{)}\qquad (\lambda_{i_1}-1)T_{i_1}^{-1}\tilde{\varphi}(I,J')=\sum_{\nu=1}^{|I|}(-1)^{1+\nu}\cdot\frac{A\begin{pmatrix}\partial_1 I\\ \partial_\nu J'\end{pmatrix}}{A\begin{pmatrix}I\\ J'\end{pmatrix}}\cdot\tilde{\varphi}(I;\partial_\nu J')-\sum_{k\notin I}\lambda_k\frac{A\begin{pmatrix}k,\partial_1 I\\ J'\end{pmatrix}}{A\begin{pmatrix}I\\ J'\end{pmatrix}}\tilde{\varphi}(k,I;J')\ ,$$

$$\text{(D. VI}_2\text{)}\qquad (\lambda'_{j_1}-1)T_{j_1'}^{-1}\tilde{\varphi}(I;J')=\sum_{\nu=1}^{|I|}\frac{A\begin{pmatrix}\partial_\nu I\\ \partial_1 J'\end{pmatrix}}{A\begin{pmatrix}I\\ J'\end{pmatrix}}(-1)^{1+\nu}\tilde{\varphi}(\partial_\nu I,J')-\sum_{j\notin J}\lambda_{j'}\frac{A\begin{pmatrix}I\\ j',\partial_1 J'\end{pmatrix}}{A\begin{pmatrix}I\\ J'\end{pmatrix}}\cdot\tilde{\varphi}(I;j',J')\ .$$

PROPOSITION 5.3. *For* $\lambda_\infty=\lambda'_\infty$ *and* $|I|=|J'|\leqq n$, *we have*

$$\text{(D. VI}_3\text{)}\qquad (\lambda_{i_1}-1)T_{i_1}^{-1}T_{i_2}\tilde{\varphi}(I;J')=\sum_{\nu=1}^{|I|-1}(-1)^{1+\nu}\cdot\frac{A\begin{pmatrix}\partial_1 I\\ \partial_\nu J'\end{pmatrix}}{A\begin{pmatrix}I\\ J'\end{pmatrix}}\tilde{\varphi}(\partial_1 I;\partial_\nu J')-\sum_{i\notin I}\lambda_i\frac{A\begin{pmatrix}i,\partial_1 I\\ J'\end{pmatrix}}{A\begin{pmatrix}I\\ J'\end{pmatrix}}\cdot\tilde{\varphi}(i,\partial_1 I;J')\ ,$$

$$\text{(D. VI}_4\text{)}\qquad (\lambda'_{j_1}-1)T_{j_1'}^{-1}T_{j_2'}\tilde{\varphi}(I;J')=\sum_{\nu=1}^{|I|-1}(-1)^{1+\nu}\frac{A\begin{pmatrix}\partial_\nu I\\ \partial_1 J'\end{pmatrix}}{A\begin{pmatrix}I\\ J'\end{pmatrix}}\tilde{\varphi}(\partial_\nu I;\partial_1 J')-\sum_{j'\notin J'}\lambda_{j'}\frac{A\begin{pmatrix}I\\ j',\partial_1 J'\end{pmatrix}}{A\begin{pmatrix}I\\ J'\end{pmatrix}}\cdot\tilde{\varphi}(I;j',\partial_1 J')\ .$$

Lemma 5.6. *The cohomology* $H^{2n}(C^n\times C^n-\bigcup_{j=1}^{m}S_j, \nabla)$ *is spanned by the basis* $\tilde{\varphi}(I; J')$ *for* $2\leqq i_1<\cdots<i_p\leqq s$, $s+2\leqq j_1<\cdots<j_p\leqq s+t$, $0\leqq p\leqq n-1$.

Proof. Lemma 5.5, Propositions 5.2 and 5.3 imply that H^{2n} is generated by the above forms $\tilde{\varphi}(I; J')$. This number is equal to $\sum_{p=0}^{n-1}\binom{s-1}{p}\binom{t-1}{p}$, which is nothing else than the dimension of H^{2n} according to the Lemma 5.4.

Combining the Propositions 5.2 and 5.3 we have

Proposition 5.4.

(D. VI$_5$)
$$(\lambda_{i_1}-1)(\lambda_{j'_1}-1)\,T_{i_1}^{-1}T_{j'_1}^{-1}A\begin{pmatrix} I \\ J'\end{pmatrix}\cdot\tilde{\varphi}(I; J')$$
$$=A\begin{pmatrix} \partial_1 I \\ \partial_1 J'\end{pmatrix}\cdot\tilde{\varphi}(I; J')+\sum_{\nu=2}^{p}(-1)^{1+\nu}\cdot A\begin{pmatrix} \partial_1 I \\ \partial_\nu J'\end{pmatrix}T_{j'_1}^{-1}\tilde{\varphi}(I'; \partial_\nu J')$$
$$-\sum_{i\notin I}\lambda_i A\begin{pmatrix} i,\ \partial_1 I \\ J'\end{pmatrix}T_{j'_1}^{-1}\tilde{\varphi}(i, I; J')$$

where $T_{j'_1}^{-1}\tilde{\varphi}(I; \partial_\nu J')$ *and* $T_{j'_1}^{-1}\tilde{\varphi}(i, I; J')$ *can be computed by the formulae* (D. VI$_4$), *choosing* $k'\notin J'$ *such that* $A\begin{pmatrix} I \\ k',\ \partial_\nu J'\end{pmatrix}\neq 0$ *and* $A\begin{pmatrix} i,\ I \\ k',\ J'\end{pmatrix}\neq 0$ *respectively.*

As a consequence we have

Theorem 8. *The difference systems* (D. VI$_1$)-(D. VI$_4$) *give a maximally overdetermined system for* (J. V$_0$).

As for the differential equations for (J. V), we have

Proposition 5.4. *The variation formula for* $\tilde{\tilde{\varphi}}(I; J')$ *can be simply expressed by means of logarithmic connections:*

(E. V$_p$)
$$d\tilde{\tilde{\varphi}}(I; J')=\sum_{i\notin I, j'\notin J'}\lambda_i\lambda_{j'}d\log A\begin{pmatrix} i,\ I \\ j',\ J'\end{pmatrix}\cdot\tilde{\tilde{\varphi}}(i, I; j'J')$$
$$+\sum_{\mu,\nu}(-1)^{\mu+\nu-1}d\log A\begin{pmatrix} \partial_\mu I \\ \partial_\nu J'\end{pmatrix}\cdot\tilde{\tilde{\varphi}}(\partial_\mu I; \partial_\nu J')$$
$$+\sum_{k\notin I}\sum_{\nu=1}^{|I|}(-1)^{\nu-1}\cdot\lambda_k d\log A\begin{pmatrix} k,\ \partial_\nu I \\ J'\end{pmatrix}\tilde{\tilde{\varphi}}(k, \partial_\nu I; J')$$
$$+\sum_{k'\notin J'}\sum_{\nu=1}^{|J'|}(-1)^{\nu-1}\lambda_{k'}d\log A\begin{pmatrix} I \\ k',\ \partial_\nu J'\end{pmatrix}\cdot\tilde{\tilde{\varphi}}(I; k', \partial_\nu J')$$

$$+\sum_{\nu=1}^{|I|}\lambda_{i_\nu} d\log A\begin{pmatrix} I \\ J'\end{pmatrix}\tilde{\tilde{\varphi}}(I;\, J')$$

$$+\sum_{\nu=1}^{|J'|}\lambda_{j'_\nu} d\log A\begin{pmatrix} I \\ J'\end{pmatrix}\cdot\tilde{\tilde{\varphi}}(I;\, J')\ ,$$

for $|I|=|J'|\leqq n-1$.

PROOF. We remark firstly the form $\omega\begin{pmatrix} & I,\, J' & \\ I, & i,\, J', & j'\end{pmatrix}$ defined in Definition 2.1 is here equal to

$$-\sqrt{-1}\left[d\log A\begin{pmatrix} I \\ J'\end{pmatrix}+d\log A\begin{pmatrix} I,\, i \\ J',\, j'\end{pmatrix}\right].$$

By applying the Proposition $2.4''_{\mathrm{p}}$ for $\tilde{\tilde{\varphi}}(I;\, J')$, we have

$$(5.15)\quad d\tilde{\tilde{\varphi}}(I;\, J')=\sum_{i\notin I,\, j'\notin J'}\lambda_i\lambda_{j'}\left\{d\log A\begin{pmatrix} I \\ J'\end{pmatrix}+d\log A\begin{pmatrix} I,\, i \\ J',\, j'\end{pmatrix}\right\}\tilde{\tilde{\varphi}}(I,\, i;\, J',\, j')$$
$$-\sum_{1\leqq\mu,\nu\leqq|I|}(-1)^{\mu+\nu}\left\{d\log A\begin{pmatrix} \partial_\mu I \\ \partial_\nu J'\end{pmatrix}+d\log A\begin{pmatrix} I \\ J'\end{pmatrix}\right\}\tilde{\tilde{\varphi}}(\partial_\mu I;\, \partial_\nu J')$$
$$+\sum_{\nu=1}^{|I|}(-1)^{\nu-1}\sum_{k\notin J}\lambda_k\left\{d\log A\begin{pmatrix} k,\, \partial_\nu I \\ J'\end{pmatrix}+d\log A\begin{pmatrix} I \\ J'\end{pmatrix}\right\}\tilde{\tilde{\varphi}}(k, \partial_\nu I;\, J')$$
$$+\sum_{k'\notin J'}\sum_{\nu=1}^{|J'|}(-1)^{\nu-1}\cdot\lambda_{k'}\left\{d\log A\begin{pmatrix} I \\ J'\end{pmatrix}+d\log A\begin{pmatrix} I \\ k,\, \partial_\nu J'\end{pmatrix}\right\}$$
$$\times\tilde{\tilde{\varphi}}(I;\, k',\, \partial_\nu J')+(\lambda_I-\lambda_{I^c}+\lambda_{J'}-\lambda_{J'^c})d\log A\begin{pmatrix} I \\ J'\end{pmatrix}\tilde{\tilde{\varphi}}(I;\, J')\ ,$$

where λ_I denotes $\sum_{k\in I}\lambda_k$ and I^c the complement of I. Taking account of the relations (5.6) and (5.7), we arrive at the conclusion. Consequently,

THEOREM 9. ($\mathrm{E.\,V_p}$) *together with* (5.14) *give a maximally over-determined system of linear differential equations.*

As an important result of the above formula, we have the following relations among the logarithmic forms $d\log A\begin{pmatrix} I \\ J'\end{pmatrix}$:

PROPOSITION 5.5. *We put, for* $|I|=|J'|\leqq n-2$,

$$(5.16)\quad \Xi\begin{pmatrix} j,\, k \\ j',\, l'\end{pmatrix}\!\!\begin{pmatrix} I \\ J'\end{pmatrix}=d\log A\begin{pmatrix} i,\, I \\ j',\, J'\end{pmatrix}\wedge d\log A\begin{pmatrix} i,\, k,\, I \\ j',\, l',\, J'\end{pmatrix}-d\log A\begin{pmatrix} k,\, I \\ j',\, J'\end{pmatrix}$$

$$\wedge d\log A\begin{pmatrix} i, k, I \\ j', l', J' \end{pmatrix} - d\log A\begin{pmatrix} i, I \\ l', J' \end{pmatrix} \wedge d\log A\begin{pmatrix} i, k, I \\ j', l', J' \end{pmatrix}$$
$$+ d\log A\begin{pmatrix} k, I \\ l', J' \end{pmatrix} \wedge d\log A\begin{pmatrix} i, k, I \\ j', l', J' \end{pmatrix}.$$

Then we have the cocycle conditions for 2-simplices i_0, i_1, i_2 and j_0, j_1, j_2 as follows:

(R. II$_1$)
$$0 = \Xi\begin{pmatrix} i_1, i_2 & I \\ *, * & J' \end{pmatrix} - \Xi\begin{pmatrix} i_0, i_2 & I \\ *, * & J' \end{pmatrix} + \Xi\begin{pmatrix} i_0, i_1 & I \\ *, * & J' \end{pmatrix},$$

(R. II$_2$)
$$0 = \Xi\begin{pmatrix} *, * & I \\ j'_1, j'_2 & J' \end{pmatrix} - \Xi\begin{pmatrix} *, * & I \\ j'_0, j'_2 & J' \end{pmatrix} + \Xi\begin{pmatrix} *, * & I \\ j'_0, j'_1 & J' \end{pmatrix}.$$

PROOF. These identities are nothing else than the integrability conditions for the Gauss-Manin connection (E. V$_0$). In fact the exterior differentiation of the left hand side of (E. V$_0$) vanishes, and so, that of does, too. By comparison of the coefficients of each monomial of $\lambda_1, \cdots,$ the right hand side λ_m we get (R. II$_1$) and (R. II$_2$).

Seeing that $A\begin{pmatrix} I \\ J' \end{pmatrix}$ for $|I| = |J'| = n$, is equal to the product of determinants $[i_1, \cdots, i_n]$ and $[j_1, \cdots, j_n]$, there are further relations among them, as was shown in [7], § 2. It seems to the author an interesting problem if these two types of relations are basic in the de Rham algebra generated by $d\log A\begin{pmatrix} I \\ J' \end{pmatrix}$.

§ 6. Degenerate case II. Invariant connections for the symplectic group.

Let f_0 be the symplectic form $\sum_{i=1}^{n} (x_i \cdot y_{i+n} - x_{i+n} \cdot y_i)$ of $(x, y) \in C^{2n} \times C^{2n}$ and $f_1, \cdots, f_s$, $s \geq 2n$, be linear homogeneous functions on C^{2n}. We shall consider the following integral:

(J. VI$_0$)
$$\tilde{\varphi}(\phi) = \int \exp[f_0(x, y)] f_1(x)^{\lambda_1} \cdots f_s(x)^{\lambda_s} f_1(y)^{\lambda_{s+1}} \cdots f_s(y)^{\lambda_{2s}} \cdot \tau,$$

where τ denotes the canonical $4n$-form $dx_1 \wedge \cdots \wedge dx_{2n} \wedge dy_1 \wedge \cdots \wedge dy_{2n}$. This can be regarded as a special case of the integral (J. V$_0$), $m = 2s$, and admit of the action of the symplectic group $S_p(n, C)$, which leaves invariant the alternating form $f_0(x, y)$. Therefore the Gauss-Manin connection of (J. VI$_0$) can be described by means of the basic algebraic symplectic invariants. For arbitrary two linear homogeneous functions $f_i = \sum_{\nu=1}^{2n} u_{i,\nu} x_\nu$ and $f_j = \sum_{\nu=1}^{2n} u_{j,\nu} x_\nu$, we denote by $a_{i,j} = [f_i, f_j]$ the bi-linear invariant $\sum_{\nu=1}^{n} (u_{i,\nu} u_{j,n+\nu} -$

$u_{i,n+\nu}u_{j,\nu})$. The matrix $A=((a_{i,j}))\ 1\leqq i,\ j\leqq s$ becomes skew-symmetric. We shall assume now:

ASSUMPTION 6.1. (i) Arbitrary n-functions among $f_1, \cdots, f_s$ are linearly independent, namely for any indices $(i_1, \cdots, i_n)\subset(1, \cdots, s)$, we have $[i_1, \cdots, i_n]\neq 0$.

(ii) For any two sequences of indices $I=(i_1, \cdots, i_p)$ and $J=(j_1, \cdots, j_p)$ for $p\leqq n$, we have $A\binom{I}{J'}\neq 0$, provided $I\neq J$ or $I=J$, $|I|=$even.

Then we have almost the same results as in § 5 with a little modification.

DEFINITION 6.1. *The mapping* ρ. By suitable change of coordinates (x, y) we may assume that $f_1(x)$ and $f_2(y)$ are equal to x_1 and y_{n+1} respectively. By the substitution of integral variables $x_1=x_1$, $x_i=x_1\cdot x'_i$ for $2\leqq i\leqq 2n$, $y_{n+1}=y_{n+1}$, $y_i=y_{n+1}\cdot y'_i$ for $1\leqq i\leqq 2n$, $i\neq n+1$, the integral (J. VI$_0$) can be written in the following way:

$$\text{(J. VI}_0\text{)}\quad \tilde{\varphi}(\phi)=\int\exp\left[x_1y_{n+1}\left(1+\sum_{i=2}^{n}x'_iy'_{i+n}-\sum_{i=1}^{n}x'_{n+i}y'_i\right)\right]x_1^{\lambda_\infty+2n-1}\cdot x'^{\lambda_2}_{n+1}$$
$$\times y_1'^{\lambda_{s+1}}\cdot y_{n+1}^{\lambda'_\infty+2n-1}\times f_3(1, x')^{\lambda_3}\cdots f_s(1, x')^{\lambda_s}f_3(1, y')^{\lambda_{s+3}}\cdots f_s(1, y')^{\lambda_{2s}}$$
$$\times dx_1\wedge dy_{n+1}\wedge dx'_2\wedge\cdots\wedge dx'_{2n}\wedge dy'_1\wedge\cdots\wedge dy'_n\wedge dy'_{n+2}$$
$$\wedge\cdots\wedge dy'_{2n}$$

which is equal to zero if $\lambda_\infty\neq\lambda'_\infty$, and otherwise equal to the following:

$$(6.1)\quad \tilde{\varphi}(\phi)=2\pi i\cdot\Gamma(\lambda_\infty+2n)\int f_0(1, x'_2, \cdots, x'_{2n}; y'_1, \cdots, y'_n, 1, y'_{n+2}, \cdots, y'_{2n})^{-\lambda_\infty-2n}$$
$$\times x'^{\lambda_2}_{n+1}y_1'^{\lambda_{1+s}}\prod_{j=3}^{s}f_j(1, x')^{\lambda_j}\cdot\prod_{j=3}^{s}f_j(1, y')^{\lambda_{s+j}}$$
$$\times dx'_2\wedge\cdots\wedge dx'_{2n}\wedge dy'_1\cdots\wedge dy'_n\wedge dy'_{n+2}\wedge\cdots\wedge dy'_{2n}\ .$$

In other words

LEMMA 6.1. *$\tilde{\varphi}(\phi)$ vanishes when $\lambda_\infty\neq\lambda'_\infty$. If $\lambda_\infty=\lambda'_\infty$, then $\tilde{\varphi}(\phi)$ is equal to* (6.1).

We denote by $\hat{S}_0$ the hyper-quadric $\hat{f}_0(x', y')=0$ in $C^{2n-1}\times C^{2n-1}$, by $\hat{S}_2$ the hyper-plane $x'_{n+1}=0$, by $\hat{S}_j$ for $3\leqq j\leqq s$, the $\hat{f}_j(x')=0$, by $\hat{S}_{s+1}$ the $y'_1=0$, and by $\hat{S}_j$ for $s+3\leqq j\leqq 2s$, the $\hat{f}_{j-s}(y')=0$. The covariant differentiations ∇ and $\hat{\nabla}$ are defined by using the 1-forms $df_0(x, y)+\sum_{j=1}^{s}\lambda_j d\log f_j(y)$, $+\sum_{j=1}^{s}\lambda_{j+s}d\log f_j(y)$, and $d\log\hat{f}_0(x', y')+\lambda_2 d\log x'_{n+1}+\lambda_{s+1}d\log y'_1+\sum_{j=3}^{s}\lambda_j\log\hat{f}_j(x')+\sum_{j=3}^{s}\lambda_{s+j}d\log\hat{f}_j(y')$ respectively. Then the relation (6.1) defines the linear mapping ρ from $H^{4n-2}(C^{2n-1}\times C^{2n-1}-\bigcup_{j=0,\ j\neq 1,s+2}^{2s}\hat{S}_j, \hat{\nabla})$ into $H^{4n}(C^{2n}\times C^{2n}-\bigcup_{j=1}^{2s}S_j, \nabla)$. Exactly by the same reason as in Lemma 6.1.

LEMMA 6.2. *When $\lambda_\infty=\lambda'_\infty$, then the mapping ρ is an isomorphism.*

LEMMA 6.3. *The Euler number $\chi(C^{2n-1}\times C^{2n-1}-\bigcup_{j=0, j\neq 1,s+2}^{2s}\hat{S}_j, C)$ is equal to $\sum_{p=0}^{2n-1}\binom{s-1}{p}^2-\sum_{p=0}^{2n-1}\binom{s-1}{p}$.*

PROOF. This can be proved by the standard argument, using the Mayer-Vietoris sequence (see the Figure 3).

According to the theorem in [1] cited in the proof of Lemma 5.4, we can conclude

LEMMA 6.4. *The dimension of $H^{4n}(C^{2n}\times C^{2n}-\bigcup_{j=1}^{2s}S_j, \nabla)$ and $H^{4n-2}(C^{2n-1}\times C^{2n-1}-\bigcup_{j=0, j\neq 1,s+2}^{2s}\hat{S}_j, \hat{\nabla})$ is equal to $\sum_{p=0}^{2n-1}\binom{s-1}{p}^2-\sum_{p=0}^{2n-1}\binom{s-1}{p}$.*

LEMMA 6.5. *The Lemma 5.5 holds, by replacing n by $2n$. The analogous formulae to (5.10) and (5.11) hold.*

PROPOSITION 6.1. *The difference systems (D. VI_1)$\sim$(D. VI_4) in Proposition 5.2 hold for $|I|=|J'|\leqq 2n-1$.*

PROPOSITION 6.2. *The cohomology $H^{4n}(C^{2n}\times C^{2n}-\bigcup_{j=1}^{2s}S_j, \nabla)$ has a system of generators*

$$\varphi(I, J')=\frac{\tau}{f_{i_1}(x)\cdots f_{i_p}(x)f_{j'_1}(y)\cdots f_{j'_p}(y)} \tag{6.2}$$

where $I\neq J$, $0=|I|=|J'|=p\leqq 2n$ or $I=J$, $|I|=|J'|=$even. These have the following fundamental relations:

$$\sum_{\nu=1}^{2n+1}(-1)^{\nu+1}\tilde{\tilde{\varphi}}(\partial_\nu I;\ J')=0\ , \tag{6.3}$$

for $|I|=2n+1$, $|J'|=2n$.

$$\sum_{\nu=1}^{2n+1}(-1)^{\nu+1}\tilde{\tilde{\varphi}}(I;\ \partial_\nu J')=0\ , \tag{6.4}$$

for $|I|=2n$, $|J'|=2n+1$.

$$0=\sum_{r\notin I}\lambda_r\tilde{\tilde{\varphi}}(r,\ I;\ J')+\sum_{\nu=1}^{|J'|}(-1)^\nu\tilde{\tilde{\varphi}}(I;\ \partial_\nu J')\ , \tag{6.5}$$

for $|I|+1=|J'|$.

$$0=\lambda_{r'}\tilde{\tilde{\varphi}}(I;\ r',\ J')+\sum_{\nu=1}^{|I|}(-1)^\nu\tilde{\tilde{\varphi}}(\partial_\nu I;\ J') \tag{6.6}$$

for $|I|=|J'|+1$, *where* $\tilde{\tilde{\varphi}}(I;\,J')$ *denotes* $A\binom{I}{J'}\cdot\tilde{\varphi}(I;\,J')$ *for* $|I|=|J'|$. *Remark that* $\tilde{\tilde{\varphi}}(I;\,I')=0$ *if* $|I|=odd$. *Therefore a basis of* H^{4n} *can be chosen as follows*:

(6.7) $\tilde{\tilde{\varphi}}(I;\,J')$, *for* $I=(i_1,\cdots,i_p)$, $J'=(j'_1,\cdots,j'_p)$, $0\leqq p\leqq 2n$,

$2\leqq i_1<\cdots<i_p\leqq s$, $2\leqq j_1<\cdots<j_p\leqq s$ *and* p *even if* $I=J$.

PROOF. In view of Lemma 6.5 and Proposition 6.1, H^{4n} turns out to be generated by $\tilde{\varphi}(I;\,J')$ for $|I|=|J'|\leqq 2n-1$. On the other hand, the formula (D. VI$_5$) in §5 shows that if $A\binom{I}{J'}=0$, and $A\binom{\partial_\mu I}{\partial_\nu J'}\neq 0$, then $\tilde{\varphi}(I;\,J')$ can be described in terms of the forms $\tilde{\varphi}\,(\partial_\mu I;\,\partial_\nu J')$, $\tilde{\varphi}(I;\,j',\,\partial_\nu J')$, $\tilde{\varphi}(i,\,\partial_\mu I;\,J')$ and $\tilde{\varphi}(i,\,I;\,j',\,J')$. Since $A\binom{I}{I'}=0$ for $|I|=$odd, $\tilde{\varphi}(I;\,I')$ is cohomologous to a linear combination of other $\tilde{\varphi}(K;\,J')$ for $K\neq J$, or $K=J$ with $|K|=$ even. Consequently H^{4n} is generated by the $\tilde{\varphi}(I;\,J')$ of (6.7). The number of linearly independent forms is therefore at most equal to $\sum_{p=0}^{2n-1}\binom{s-1}{p}^2-\sum_{p=0}^{2n-1}\binom{s-1}{p}$. On the other hand Lemma 6.3 implies that this must be equal to the dimension of H^{4n} so that the above system $\tilde{\varphi}(I;\,J')$ in (6.7) must be linearly independent. The Proposition 6.2 has now been proved.

We have finally

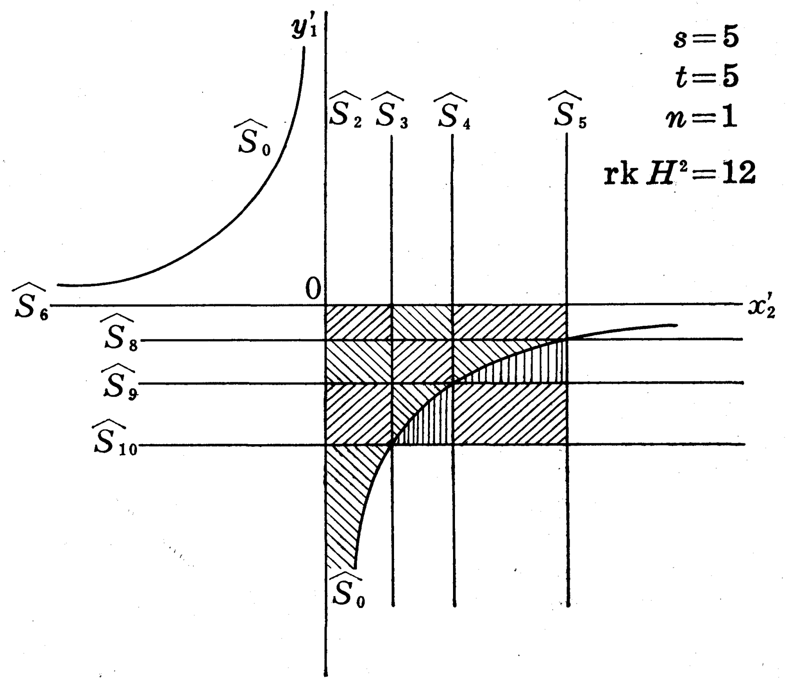

FIGURE 3

THEOREM 10. *The difference system* (D. VI_1)-(D. VI_4) *gives a maximally overdetermined system with respect to the basis* $\tilde{\tilde{\varphi}}(I; J')$ *in Proposition* 6.2.

THEOREM 11. *The equation* (E. VII_0) *together with the difference system* (D. VI) *gives a maximally overdetermined system of linear differential equations with respect to the above* $\tilde{\tilde{\varphi}}(I; J')$.

As a consequence we can obtain

PROPOSITION 6.3. *For* $|I|=|J'|\leqq 2n-2$, (R. II_1)-(R. II_2) *hold, where we must put* $d\log A\binom{K}{L'}$ *to be equal to* 0 *if* $K=L$, $|K|=$odd.

*A*DDED IN PROOF. In a recent article by M. Kashiwara and T. Kawai, it is proved in full generality that integrals of type (J) or (J') satisfy certain holonomic systems with irregular or regular singularities (see M. Kashiwara and T. Kawai, on holonomic systems of micro-differential equations III, Pub. R.I.M.S., Kyoto Univ., 17, (1981), 813-979).

References

[1] K. AOMOTO, Les équations aus différences lineaires et les intégrales des fonctions multiformes, J. Fac. Sci. Univ. of Tokyo, **22** (1975), 271-297.

[2] K. AOMOTO, Analytic structure of Schläfli function, Nagoya Math. J., **68** (1977), 1-16.

[3] K. AOMOTO, On the structure of integrals of power product of linear functions, Sci. Papers, Coll. Gen. Ed., Univ. Tokyo, **27** (1977), 49-61.

[4] K. AOMOTO, Fonctions hyper-logarithmiques et groupes de monodromie unipotents, J. Fac. Sci. Univ. of Tokyo, **25** (1978), 149-156.

[5] K. AOMOTO, Une correction et un complément à l'article "Les équations aux différences linéaires et les intégrales des fonctions multiformes", J. Fac. Sci. Univ. of Tokyo, **26** (1979), 353-523.

[6] K. AOMOTO, A generalization of Poincaré normal functions on a polarized manifold, Proc. Japan Acad., **55** (1979), 353-358.

[7] K. AOMOTO, Addition theorem of Abel type for hyper-logarithms, to appear in Nagoya Math. J.

[8] P. DELIGNE, Equations différentielles à points singuliers réguliers, Lecture Notes in Math., **163**, Springer, 1970.

[9] F. J. DYSON, Statistical theory of the energy levels of complex systems, I-III, J. Mathematical Phys., **3** (1962), 140-156, 157-165, 166-175; IV-V (with M. L. Mehta) ibid., **4** (1964), 701-712, 713-719.

[10] R. J. EDEN, P. V. LANDSHOP, D. I. OLIVE and J. C. POLKINGHORNE, The analytic S-matrix, Cambridge University Press, Cambridge, 1966.

[11] J. KANEKO, Monodromy group of Appell's system (F_4), Tokyo J. Math., **4** (1981), 35-54.

[12] V. LAKSHIMIBAI and C. S. SESHADRI, Geometry of G/P – II, Proc. Indian Acad. Sci. Sect. A, **87** (1978), 1-54.

[13] J. LERAY, Le calcul differentiel et intégral sur une variété analytique complexe (Problème de Cauchy III), Bull. Sci. Math., **87** (1959), 81-180.

[14] T. S. LUNDGREN and Y. B. POINTIN, Statistical mechanics of two dimensional vortices, J. Statist. Phys., **17** (1977), 323-354.

[15] N. NAKANISHI, Quantized Field Theory, Baifukan, Tokyo, 1975 (in Japanese).

[16] F. PHAM, Singularités des Systèmes Différentielles de Gauss-Manin, Birkhäuser, Boston, 1979.

[17] T. E. REGGE, Feynman integrals and S-matrix, Actes, Congrès Intern. Math., Nice, Tome 3, 1970, 9-14.

[18] M. SATO, T. KAWAI and M. KASHIWARA, Micro-functions and pseudo-differential equations, Hyper-functions and Pseudo-Differential Equations, Lecture Notes in Math., **287**, Springer, 1971.

[19] H. WEYL, The Classical Groups, Princeton University Press, Princeton, 1946.

[20] M. KITA and M. NOUMI, On the structure of cohomology groups attached to integrals of certain many-valued analytic functions, to appear in Japanese J. Math.

Present Address:
Mathematical Institute
Faculty of Sciences
Nagoya University
Chikusa-ku, Nagoya