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A Construction of the Groups of Units of Some Number Fields from Certain Subgroups

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Introduction

All number fields we consider are in the complex number field. The symbol $\langle S\rangle$ denotes a multiplicative group generated by an element or a set S. For any complex number x, a j-th root of x, which is taken to be positive real if x is positive real, is denoted by $\sqrt[j]{x}$.

0.1. For a finite algebraic number field k, let E_k be the group of units of k and W_k be the torsion part of E_k. Then E_k is generated by W_k and by a set $\{\varepsilon_j \mid j=1, \cdots, r\}$ of fundamental units of k. The number r is called the *Dirichlet number* of k. In general, some geometrical calculation is necessary to obtain fundamental units of k (see [1] or Chap. 2, §5.3 of [2]). Those methods are very complicated when r is large. If k is a real abelian number field, there is an effective method, which requires no geometrical calculation, to obtain fundamental units of k (see [5]). Our main interest is, in case k is not galois or galois but not abelian over $\boldsymbol{Q}$, to construct $\{\varepsilon_j \mid j=1, \cdots, r\}$ from certain subgroups of E_k without any geometrical calculation. Let E'_k be the subgroup of E_k generated by W_k and the units of all proper subfields of k. If the index $(E_k: E'_k)$ is finite, we may construct E_k from E'_k. Such a problem is treated in some cases when k is galois over $\boldsymbol{Q}$, see [6] and [7] for example. If $(E_k: E'_k)$ is not finite, we consider the following subgroup H_k, the group of *relative units* of k, in addition to E'_k:

$$H_k=\{\varepsilon \in E_k \mid N_{k/k_1}(\varepsilon) \in W_k \text{ for any proper subfield } k_1 \text{ of } k\} .$$

The object of the present article is to show a way how E_k is constructed from E'_k and H_k in some cases. Our main tool is Proposition 1 in §1, which can be applied to k of types as in 1.2. To explain our actual calculation, we take for k a subfield of a dihedral extension of

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degree 8 or 12 over $\boldsymbol{Q}$. Most of the results have been announced in our previous note [12] without proof.

0.2. Throughout the paper we suppose $n=2$ or 3. Let L be a dihedral extension of degree $4n$ over $\boldsymbol{Q}$ with the galois group G:

$$G=\langle\sigma,\tau\rangle\,;\qquad \sigma^{2n}=\tau^2=(\sigma\tau)^2=1\,.$$

Let K, F and Ω be the invariant subfield of $\langle\tau\rangle$, $\langle\sigma^3\tau\rangle$ and $\langle\sigma^n\rangle$ respectively. The quadratic subfields K_2 and F_2 of K and F are the invariant subfields of $\langle\sigma^2,\tau\rangle$ and $\langle\sigma^3\tau,\sigma^2\rangle$ respectively. When $n=3$, the cubic subfield $K_3=K\cap F$ is the invariant subfield of $\langle\sigma^3,\tau\rangle$. The quartic subfield $\Lambda=K_2\cdot F_2$ is the invariant subfield of $\langle\sigma^2\rangle$ and is the maximal abelian subfield of L. Another quadratic subfield Λ_2, which is the invariant subfield of $\langle\sigma\rangle$, is contained in Λ. Note that $\Lambda=\Omega$ when $n=2$, and that $W_L=W_\Lambda$ in any case. We shall investigate the groups E_K, E_F and E_L. The fields K and F are pure number fields of degree $2n$ if and only if Λ_2 is the $2n$-th cyclotomic field, and then

$$K=\boldsymbol{Q}(\sqrt[2n]{d})\quad\text{and}\quad F=\boldsymbol{Q}(\sqrt[2n]{-n^n d})$$

with a natural number d.

T. Nagell [10], in case $n=2$, and H.-J. Stender [16], in case $n=3$ and K is pure, have given a classification of K and F in terms of the structure of $E_K/E'_K\cdot H_K$ and $E_F/E'_F\cdot H_F$. We can complete such a classification as a corollary of Proposition 1. Moreover, Proposition 1 has another application to the field of type $\boldsymbol{Q}(\sqrt[p]{f},\sqrt[p]{g})$ as in Proposition 4 in 1.5.

0.3. Let $k/\boldsymbol{Q}$ be a finite galois extension with the galois group G'. A. Brumer [4], L. Bouvier-J. Payan [3] and N. Moser [8] have investigated the structure of E_k as a $\boldsymbol{Z}[G']$-module when G' is a certain cyclic or dihedral group, and have given several conditions for k to have a *Minkowski unit* (a unit which forms, together with some of its conjugates, a set of fundamental units of k). But in our case when $k=L$ and $G'=G$ given as above, there seems to be no literature concerning Minkowski units. Let us assume $L\cap\boldsymbol{R}=K$. Then, by the result in §1, we see

$$(1)\qquad H_K=\langle -1\rangle\times\langle\varepsilon_1\rangle\quad\text{with}\quad \varepsilon_1>1\,.$$

Let $\eta_2(>1)$ be the fundamental unit of K_2. Further let η_3 (>1) be the fundamental unit of K_3 when $n=3$. Then we can prove the following theorem by considering the relation between E_K and E_F and by constructing E_L from E'_L.

THEOREM 1. *Let assumptions be as above.* (i) *In case $n=2$, there is a real Minkowski unit for L if and only if the following holds:*

$$E_K=H_K\times\langle\varepsilon_2\rangle \quad with \quad \varepsilon_2=\sqrt{\varepsilon_1\eta_2}\,,$$
$$E_A=E'_A \quad and \quad K\neq K_2(\sqrt{2\eta_2})\,, \qquad \neq Q(\sqrt[4]{2})\,.$$

Moreover, if this condition is satisfied, the unit ε_2 is a Minkowski unit for L. (ii) *In case $n=3$, there is a real Minkowski unit for L if and only if the following holds:*

$$E_K=H_K\times\langle\varepsilon_2\rangle\times\langle\varepsilon_3\rangle \quad with \quad \varepsilon_2=\sqrt[8]{\varepsilon_1\eta_2^{\pm1}}\,, \qquad \varepsilon_3=\sqrt{\varepsilon_1\eta_3}\,,$$
$$E_\Omega=E'_\Omega \quad and \quad E_A=E'_A\,.$$

Moreover, if this condition is satisfied, the unit $\varepsilon_2^{-1}\varepsilon_3$ is a Minkowski unit for L.

0.4. Stender [15], [16] and [17] have given a series of K with explicit fundamental units when K is a pure number field. The method used there is to construct E_K from H_K and E'_K. By the same method, we obtain a new series of K with explicit fundamental units.

THEOREM 2. *Let d be a square free integer greater than 1 and put $\theta=\sqrt[2n]{d}$. Assume $K=Q(\theta)$ has a binomial unit $a-b\theta(a,b\in N)$ such that*

$$a\geqq b^{2n}-1\,.$$

Then a set of fundamental units of K is given as follows:

$$\xi_1=a-b\theta\,, \qquad \xi_2=a+b\theta \qquad\qquad in\ case \quad n=2\,;$$
$$\xi_1=a-b\theta\,, \qquad \xi_2=a+b\theta\,, \qquad \xi_3=a^2+ab\theta+b^2\theta^2 \qquad in\ case \quad n=3\,.$$

Combining this theorem with Theorem 1, we obtain examples of L with explicit real Minkowski units.

THEOREM 3. *Assumptions being the same as in Theorem 2, the unit ξ_1 is a real Minkowski unit for the galois closure L of K/Q unless $d=2$.*

We also give examples of L with no Minkowski unit in case $n=2$, see Propositions 7 and 8.

0.5. In §1, we prove Proposion 1 and apply it to the groups of units of some number fields, especially to E_K and E_F. (In order to construct E_K and E_F, it is the most important to find a finite index subgroup of H_K or H_F. When $L\cap R=K$, we can compute ε_1 in (1) from a so called "elliptic unit", and then we have an effective method to compute fundamental units and the class numbers of K and F, see [13] and [14].) In §2, we study the relation between E_K and E_F. The proof of Theorem 1 is given in §3 and the proofs of Theorems 2 and 3 are given in §4.

§1. A property of a free abelian group.

1.1. Let E be a free abelian group with finite rank r, and E_i be subgroups of E with rank r_i $(1\leqq i\leqq m)$. Assume that natural numbers n_i and homomorphisms $f_i\colon E\to E_i$ $(1\leqq i\leqq m)$ are given and satisfy

$$f_i(x)=x^{\delta_{ij}n_i} \quad \text{for} \quad x\in E_j \ (1\leqq i,\ j\leqq m)\,,$$

where δ_{ij} is Kronecker's delta. Put

$$E_0=\bigcap_{i=1}^{r} \mathrm{Ker}\,(f_i)$$

and let r_0 be the rank of E_0.

PROPOSITION 1 (Lemma 1 of [12]). *Notations being as above, the following holds:*

(i) $\langle E_i\,|\,0\leqq i\leqq m\rangle=E_0\times E_1\times\cdots\times E_m$ *(direct product);*

(ii) *The product map* $f=f_1\times\cdots\times f_m\colon E\to E_1\times\cdots\times E_m$ *induces the isomorphism*

$$E/(E_0\times\cdots\times E_m)\cong f(E)/(E_1^{n_1}\times\cdots\times E_m^{n_m})\,,$$

and thus

$$r=r_0+\cdots+r_m\,,\qquad (E\colon E_0\times\cdots\times E_m)\,|\,n_1^{r_1}\cdots n_m^{r_m}\,;$$

(iii) *If* x_i $(1\leqq i\leqq r-r_0)$ *are elements of* E *such that* $f(E)=\langle f(x_i)\,|\,1\leqq i\leqq r-r_0\rangle$, *we have*

$$E=E_0\times\langle x_1\rangle\times\cdots\times\langle x_m\rangle \quad \textit{(direct product)}\,;$$

(iv) *If* n_i $(1\leqq i\leqq m)$ *are pairwise relatively prime,*

$$f(E)=f_1(E)\times\cdots\times f_m(E) \quad \textit{(direct product)}\,.$$

PROOF. For $i=0,1,\cdots,m-1$, let

$$x_i=x_{i+1}\cdots x_m \quad \text{with} \quad x_j\in E_j \ (i\leqq j\leqq m)\,.$$

If $i \geqq 1$, then

$$x_i^{n_i} = f_i(x_i) = f_i(x_{i+1}) \cdots f_i(x_m) = 1 ,$$

hence $x_i = 1$. If $i=0$, then, for $j=1, \cdots, m$,

$$1 = f_j(x_0) = f_j(x_1) \cdots f_j(x_m) = x_j^{n_j} ,$$

hence $x_j = 1$, and so $x_0 = 1$. Therefore $E_i \cap (E_{i+1} \cdots E_m) = 1$ for $i=0, 1, \cdots, m-1$. Thus (i) is proved. To see (ii), it is enough to prove

$$f^{-1}(E_1^{n_1} \times \cdots \times E_m^{n_m}) = E_0 \times \cdots \times E_m . \tag{2}$$

The right side of (2) is obviously contained in the left side. Let x be an element of the left side of (2), then $f_i(x) = x_i^{n_i}$ with $x_i \in E_i$ $(1 \leqq i \leqq m)$, so $x(x_1 \cdots x_m)^{-1} \in \operatorname{Ker}(f) = E_0$. Hence (2) is proved. It is easy to see (iii) from the exact sequence

$$1 \longrightarrow E_0 \xrightarrow{\text{inc.}} E \xrightarrow{f} f(E) \longrightarrow 1$$

which splits via a natural homomorphism from $f(E)$ to E. Let $n_1, \cdots, n_m$ be pairwise relatively prime. Since $f(E)$ is always contained in $f_1(E) \cdots f_m(E)$, it is enough to show

$$f_i(E) \subset f(E) \qquad (1 \leqq i \leqq m) \tag{3}$$

in order to prove (iv). Take $\mu, \nu \in Z$ such that

$$\mu n_1 + \nu a = 1 , \qquad a := n_2 \cdots n_m .$$

For $x \in E$, let

$$y := x^{\nu a} f_1(x)^{\mu} \Big(\prod_{i=2}^{m} f_i(x)^{a_i} \Big)^{-\nu} , \qquad a_i := a/n_i ,$$

then $f(y) = f_1(x)$, hence $f_1(E) \subset f(E)$. Similarly, (3) holds for $2 \leqq i \leqq m$. Thus we complete the proof of Proposition 1.

1.2. We can apply Proposition 1 to the following finite algebraic number field k;

A. the field k is an extension of relative degree n_1 over a proper subfield k_1 of k;

B. the field k is the composite of two subfields k_1 and k_2, which are linearly disjoint over Q, with $[k: k_i] = n_i$ $(i=1, 2)$;

C. the field k contains M subfields k_i $(1 \leqq i \leqq M)$ which are pairwise linearly disjoint over Q and $[k: k_i] = [k_i: Q] = N$ $(1 \leqq i \leqq M)$.

In all these cases, let $E:=E_k/W_k$, $E_i:=E_{k_i}\cdot W_k/W_k(\simeq E_{k_i}/W_{k_i})$ and $f_i:=N_{k/k_i}$. Then $f_i(W_k)\subset W_{k_i}\subset W_k$, so f_i is regarded as a homomorphism from E to E_i. Let r and r_i be the Dirichlet numbers of k and k_i respectively. Then the assumptions of Proposition 1 are satisfied with

$$\begin{aligned} m&=1 \quad \text{in case A}\,,\\ m&=2 \quad \text{in case B}\,,\\ m&=M \quad \text{in case C}\,. \end{aligned}$$

Define the groups of *relative units with respect to* k/k_i by

$$H_i:=\{\varepsilon\in E_k\mid N_{k/k_i}(\varepsilon)\in W_{k_i}\}\,.$$

Then W_k is contained in H_i, and

$$E_0=\Big(\bigcap_{i=1}^{m}H_i\Big)\Big/W_k\,.$$

In the rest of this section, typical examples of each case are studied.

1.3. Notations being as in 0.2, let $n=2$. The fields K and F belong to the case A of 1.2. Note that $E_K'=E_{K_2}$ and $E_F'=E_{F_2}$, because $W_K=W_{K_2}$ and $W_F=W_{F_2}$.

Assume L is imaginary and the maximal real subfield of L is galois over $\boldsymbol{Q}$, i.e.,

$$L\cap\boldsymbol{R}=\Omega\,.$$

Then E_K and E_F have the same situation, so we may treat E_K. Applying Proposition 1 as in the case A of 1.2, we see $(E_K\colon E_{K_2})=1$ or 2. More precisely, we have the following.

PROPOSITION 2 (Nagell). *If* $n=2$ *and* $L\cap\boldsymbol{R}=\Omega$, *then* $E_K=E_{K_2}$.

PROOF. See §2.11 of [10], p. 359.

Assume L is imaginary and the maximal real subfield of L is not galois over $\boldsymbol{Q}$. Then we may assume

$$L\cap\boldsymbol{R}=K\,.$$

In this case, we have

$$E_{K_2}=\langle -1\rangle\times\langle\eta_2\rangle \quad\text{with}\quad \eta_2>1\,,\qquad E_{F_2}=W_{F_2}\,. \tag{4}$$

Applying Proposition 1 as in the case A of 1.2, we see

$$(5)\qquad H_K=\langle -1\rangle\times\langle\varepsilon_1\rangle \quad\text{with}\quad \varepsilon_1>1\,, \qquad E_F=H_F=W_{F_2}\times\langle\varepsilon_0\rangle\,,$$

and

$$(6)\qquad E_K=H_K\times\langle\varepsilon_2\rangle\,,$$

where ε_2 can be chosen as one and only one of the following forms:

$$(7)\qquad \varepsilon_2=\begin{cases}\eta_2 & \text{if}\quad \pm\eta_2\notin N_{K/K_2}(E_K)\\ \sqrt{\eta_2}\quad\text{or}\quad\sqrt{\varepsilon_1\eta_2} & \text{otherwise}\,.\end{cases}$$

REMARK 1. Since $N_{K/K_2}(\alpha)^\sigma=|\alpha^\sigma|^2\geqq 0$ for $\alpha\in K$, it is obtained that

$$H_K=\{\varepsilon\in E_K\,|\,N_{K/K_2}(\varepsilon)=1\}$$

and that

$$\pm\eta_2\notin N_{K/K_2}(E_K)\Longleftrightarrow \eta_2^\sigma\notin N_{K/K_2}(E_K)\,.$$

If $\varepsilon_2=\sqrt{\eta_2}$ in (7), then $K=K_2(\sqrt{\eta_2})$, and so $\eta_2^\sigma<0$, thus

$$\varepsilon_2=\sqrt{\eta_2}\quad\text{in (7)}\Longrightarrow N_{K_2/Q}(\eta_2)=-1\,.$$

Assume L is real, then we may treat only E_K. Similarly as above we have

$$(8)\qquad E_{K_2}=\langle -1\rangle\times\langle\eta_2\rangle\quad\text{with}\quad \eta_2>1\,,$$

and see

$$(9)\qquad H_K=\langle -1\rangle\times\langle\varepsilon_0\rangle\times\langle\varepsilon_1\rangle\quad\text{with}\quad \varepsilon_0,\,\varepsilon_1>1\,,$$

$$(10)\qquad E_K=H_K\times\langle\varepsilon_2\rangle\,,$$

where ε_2 can be chosen as one and only one of the following forms:

$$(11)\qquad \varepsilon_2=\begin{cases}\eta_2 & \text{if}\quad \pm\eta_2\notin N_{K/K_2}(E_K)\\ \sqrt{\eta_2},\ \sqrt{\varepsilon_0\eta_2},\ \sqrt{\varepsilon_1\eta_2}\quad\text{or}\quad\sqrt{\varepsilon_0\varepsilon_1\eta_2} & \text{otherwise}\,.\end{cases}$$

REMARK 2. As in Remark 1, we have

$$\varepsilon_2=\sqrt{\eta_2}\quad\text{in (11)}\Longrightarrow N_{K_2/Q}(\eta_2)=1\,.$$

1.4. Notations being as in 0.2, let $n=3$. The fields K and F belong to the case B of 1.2. Note that $E'_K=E_{K_2}\cdot E_{K_3}$ and $E'_F=E_{F_2}\cdot E_{K_3}$, because $W_K=W_{K_2}$ and $W_F=W_{F_2}$.

Assume L is imaginary and the maximal real subfield of L is galois over Q, i.e.,

$$L \cap R = \Omega .$$

It is sufficient to consider only E_K.

PROPOSITION 3. *If $n=3$ and $L \cap R = \Omega$, then $E_K = W_{K_2} \cdot E_{K_3}$.*

PROOF. First we mention that $E_K^{1-\sigma^3}$ is contained in W_{K_2}. Because, for $\varepsilon \in E_K$ and $i \in Z$,

$$|\varepsilon^{(1-\sigma^3)\sigma^i}|^2 = N_{L/\Omega}(\varepsilon^{(1-\sigma^3)\sigma^i}) = 1 ,$$

so every conjugate of $\varepsilon^{1-\sigma^3}$ has its absolute value 1, hence $\varepsilon^{1-\sigma^3}$ is a root of unity in K. Suppose $E_K \neq E'_K$. Then, since $\lambda^{1-\sigma^3} = \lambda^2$ for $\lambda \in W_{K_2}$, there exists $\varepsilon \in E_K$ such that

$$\varepsilon^{\sigma^3} = \delta\varepsilon \quad \text{with} \quad \delta = \begin{cases} \sqrt{-1} & \text{if } K_2 = Q(\sqrt{-1}) \\ -1 & \text{otherwise ,} \end{cases}$$

on account of Proposition 1 applied as in the case B of 1.2. Now let $K_2 = Q(\sqrt{-d})$ with a squarefree natural number d, and write $\varepsilon = \alpha + \beta\sqrt{-d}$ with $\alpha, \beta \in K_3$. Then it follows that

$$\alpha - \beta\sqrt{-d} = \delta(\alpha + \beta\sqrt{-d}) ,$$

which implies that

$$\varepsilon^{1+\sigma^3} = \begin{cases} 2\beta^2 & \text{if } d=1 \\ d\beta^2 & \text{otherwise .} \end{cases}$$

This is a contradiction because the ideal (2) or $(d) \neq (1)$ cannot be a square in K_3. Hence $E_K = E'_K = E_{K_3} \cdot E_{K_2} = E_{K_3} \cdot W_{K_2}$.

Assume L is imaginary and the maximal real subfield of L is not galois over Q. Then we may suppose

$$L \cap R = K .$$

In this case, we have

$$(12) \qquad E_{K_2} = \langle -1 \rangle \times \langle \eta_2 \rangle , \quad E_{K_3} = \langle -1 \rangle \times \langle \eta_3 \rangle \quad \text{with} \quad \eta_2, \eta_3 > 1 , \quad E_{F_2} = W_{F_2} = \langle \rho \rangle .$$

Applying Proposition 1 as in the case B of 1.2, we see

$$(13) \qquad H_K = \langle -1 \rangle \times \langle \varepsilon_1 \rangle \quad \text{with} \quad \varepsilon_1 > 1 , \quad H_F = W_{F_2} \times \langle \varepsilon_0 \rangle$$

and

(14) $$E_K=H_K\times\langle\varepsilon_2\rangle\times\langle\varepsilon_3\rangle\ ,\qquad E_F=H_F\times\langle\varepsilon_3'\rangle\ ,$$

where ε_2, ε_3 or ε_3' can be taken as one and only one of the following forms:

(15) $$\varepsilon_2=\begin{cases}\eta_2 & \text{if}\quad \pm\eta_2\notin N_{K/K_2}(E_K)\\ \sqrt[3]{\eta_2},\quad \sqrt[3]{\varepsilon_1\eta_2}\quad\text{or}\quad \sqrt[3]{\varepsilon_1\eta_2^{-1}} & \text{otherwise}\ ,\end{cases}$$

(16) $$\varepsilon_3=\begin{cases}\eta_3 & \text{if}\quad \pm\eta_3\notin N_{K/K_3}(E_K)\\ \sqrt{\eta_3}\quad\text{or}\quad \sqrt{\varepsilon_1\eta_3} & \text{otherwise}\ ;\end{cases}$$

(17) $$\varepsilon_3'=\begin{cases}\eta_3 & \text{if}\quad \pm\eta_3\notin N_{F/K_3}(E_F)\\ \sqrt{\rho\eta_3},\quad \sqrt{\varepsilon_0\eta_3}\quad\text{or}\quad \sqrt{\rho\varepsilon_0\eta_3} & \text{otherwise}\ .\end{cases}$$

REMARK 3. If ε is an element of H_K, we have $N_{K/Q}(\varepsilon)=1$, and so $N_{K/K_3}(\varepsilon)=1$, hence

$$H_K=\{\varepsilon\in E_K\mid N_{K/K_2}(\varepsilon)=\pm1,\ N_{K/K_3}(\varepsilon)=1\}\ .$$

Further $N_{K/K_2}(\varepsilon)=\operatorname{sgn}(\varepsilon)$ for $\varepsilon\in H_K$, therefore

$$\langle\varepsilon_1\rangle=\{\varepsilon\in E_K\mid N_{K/K_2}(\varepsilon)=N_{K/K_3}(\varepsilon)=1\}\ .$$

As -1 belongs to $N_{K/K_2}(E_K)$, it holds that

$$\pm\eta_2\notin N_{K/K_2}(E_K)\Longleftrightarrow\eta_2\notin N_{K/K_2}(E_K)\ .$$

If $-\eta_3$ belongs to $N_{K/K_3}(E_K)$, we see that -1 also belongs to $N_{K/K_3}(E_K)$ and that

$$\pm\eta_3\notin N_{K/K_3}(E_K)\Longleftrightarrow\eta_3\notin N_{K/K_3}(E_K)\ .$$

Since every element of $N_{F/K_3}(E_F)$ is positive, it follows that

$$H_F=\{\varepsilon\in E_F\mid N_{F/K_3}(\varepsilon)=1\}$$

and that

$$\pm\eta_3\notin N_{F/K_3}(E_F)\Longleftrightarrow\eta_3\notin N_{F/K_3}(E_F)\ .$$

We shall see later in Proposition 6 that $\varepsilon_3=\sqrt{\eta_3}$ never occurs in (16) and that $\varepsilon_3'=\sqrt{\rho\eta_3}$ never occurs in (17).

Assume L is real, then we may treat only E_K. We can classify E_K similarly be using Proposition 1, though the statements are more complicated. As it is not used in the rest of this paper, we shall omit the explicit formulation of the classification, which is found in Corollary 2. (ii) of [12].

1.5. As an example of the case C in 1.2, we study the following number field. Let p be an odd prime number, f and g be natural numbers such that all $f^i g$ $(i=1, 2, \cdots, p)$ and f are not perfect p-th power in $\boldsymbol{Q}$, and put

$$k:=\boldsymbol{Q}(\sqrt[p]{f}, \sqrt[p]{g}) , \qquad k_i:=\boldsymbol{Q}(\sqrt[p]{f^i g})\ (i=1, 2, \cdots, p)\ , k_{p+1}:=\boldsymbol{Q}(\sqrt[p]{f}) .$$

Then the conditions of C in 1.2 are satisfied with $N=p$ and $M=p+1$, and we have $r_i=(p-1)/2$ as the Dirichlet numbers of k_i $(i=1, 2, \cdots, p+1)$. On the other hand, the Dirichlet number of k is given by $r=(p^2-1)/2$. Therefore $r=r_1+\cdots+r_{p+1}$ follows. Since there is no proper subfield other than k_i $(i=1, 2, \cdots, p+1)$, the following proposition is proved by Proposition 1.

PROPOSITION 4. *Notations being as above, the group E_k/E'_k is an elementary abelian p-group with p-rank 0 to $(p^2-1)/2$.*

§2. Relations between E_K and E_F.

2.1. Let $n=2$ and keep notations in 0.2. The following multiplicative homomorphisms are useful to study the relations between E_K and F_F:

$$\phi\colon K^\times \longrightarrow F^\times;\ x \longmapsto x^{1+\sigma}\ ;$$
$$\psi\colon F^\times \longrightarrow K^\times;\ y \longmapsto y^{1+\sigma^3}\ .$$

LEMMA 1. *Let ϕ and ψ be as above, then it holds that*

$$N_{F/F_2}(\phi(x))=N_{K/Q}(x)\ , \qquad N_{K/K_2}(\psi(y))=N_{F/Q}(y)\ ,$$
$$\psi\circ\phi(x)=x^2N_{K/K_2}(x)^\sigma\ , \qquad \phi\circ\psi(y)=y^2N_{F/F_2}(y)^\sigma\ ,$$

for $x\in K^\times$ and $y\in F^\times$.

PROOF. Every formula is easily checked by direct calculation.

The next proposition tells us that generators of E_K can be utilized to determine generators of E_F, and vice versa.

PROPOSITION 5. *Let ϕ and ψ be as above, then we have*

$$(H_K\colon W_K\cdot\psi(H_F))(H_F\colon W_F\cdot\phi(H_K))=2 \quad or \quad 4$$

respectively when $L\cap\boldsymbol{R}=K$ or L is real. Especially when $L\cap\boldsymbol{R}=K$, it holds that

$$E_F=W_F\times\langle\phi(\varepsilon_2)\rangle\ , \qquad H_K=W_K\times\langle\psi(\varepsilon_0)\rangle$$

if $\varepsilon_2=\sqrt{\varepsilon_1\eta_2}$ in (7). *Here* ε_0, ε_1 *and* η_2 *are given as in* (4) *and* (5).

PROOF. By Lemma 1, ϕ and ψ induce the homomorphisms

$$\tilde{\phi}: H_K/W_K \longrightarrow H_F/W_F \quad \text{and} \quad \tilde{\psi}: H_F/W_F \longrightarrow H_K/W_K ,$$

which satisfy that $\tilde{\psi}\circ\tilde{\phi}$ and $\tilde{\phi}\circ\tilde{\psi}$ are the squaring endomorphisms of H_K/W_K and H_F/W_F. Since H_K/W_K is torsion free, $\tilde{\phi}$ is injective, and so $\tilde{\phi}^{-1}((H_F/W_F)^2)=\tilde{\psi}(H_F/W_F)$. Therefore

$$(H_K/W_K)/\tilde{\psi}(H_F/W_F)\simeq\tilde{\phi}(H_K/W_K)/(H_F/W_F)^2 .$$

Thus

$$((H_K/W_K):\tilde{\psi}(H_F/W_F))((H_F/W_F):\tilde{\phi}(H_K/W_K))=((H_F/W_F):(H_F/W_F)^2) ,$$

which proves the former part of the proposition on account of (5) and (9). Assume $L\cap R=K$ and $\varepsilon_2=\sqrt{\varepsilon_1\eta_2}$ in (7), then $\phi(\varepsilon_2)$ belongs to H_F by Lemma 1 and $\phi(\varepsilon_2)^2=\pm\phi(\varepsilon_1)$. This proves the latter by virtue of the former statement.

2.2. Let $n=3$ and keep notations in 0.2. The following multiplicative homomorphisms are useful to study the relations between E_K and E_F:

$$\phi: K^\times \longrightarrow F^\times ; \qquad x \longrightarrow x^{\sigma+\sigma^2} ;$$
$$\psi: F^\times \longrightarrow K^\times ; \qquad y \longrightarrow y^{\sigma+\sigma^2} .$$

Similarly as in 2.1, we have

LEMMA 2. *Let* ϕ *and* ψ *be as above, then it holds that*

$$N_{F/F_2}(\phi(x))=N_{K/Q}(x) , \qquad N_{K/K_2}(\psi(y))=N_{F/Q}(y) ,$$
$$N_{F/K_3}(\phi(x))=N_{K/Q}(x)/N_{K/K_3}(x) , \qquad N_{K/K_3}(\psi(y))=N_{F/Q}(y)/N_{F/K_3}(y) ,$$
$$\psi\circ\phi(x)=x^{-3}N_{K/K_2}(x)N_{K/K_3}(x^2) , \qquad \phi\circ\psi(y)=y^{-3}N_{F/F_2}(y)N_{F/K_3}(y^2) ,$$

for $x\in K^\times$ *and* $y\in F^\times$.

PROOF. The formulas are obtained by direct calculation.

PROPOSITION 6. *Let* ϕ *and* ψ *be as above, then we have*

$$(H_K: W_K\cdot\psi(H_F))(H_F: W_F\cdot\phi(H_K))=3 \quad or \quad 9$$

respectively when $L\cap R=K$ *or* L *is real. Especially when* $L\cap R=K$, *it holds that*

$$H_F = W_F \times \langle \phi(\varepsilon_2) \rangle\ , \qquad H_K = W_K \times \langle \psi(\varepsilon_0) \rangle$$

if $\varepsilon_2 = \sqrt[3]{\varepsilon_1 \eta_2^{\pm 1}}$ *in* (15); the case $\varepsilon_3 = \sqrt{\eta_3}$ *in* (16) *or* $\varepsilon_3' = \sqrt{\eta_3 \rho}$ *in* (17) *never occurs; and the case* $\varepsilon_2 = \eta_3$ *in* (16) *occurs if and only if* $\varepsilon_3' = \eta_3$ *in* (17). *Here* ρ, ε_0, ε_1, η_2 *and* η_3 *are given as in* (12) *and* (13).

PROOF. The first statement is proved by Lemma 2 similarly as in the proof of Proposition 5. Let $L \cap \boldsymbol{R} = K$. Assume $\varepsilon_2 = \sqrt[3]{\varepsilon_1 \eta_2^{\pm 1}}$ in (15), then $\phi(\varepsilon_2)$ belongs to H_F by Lemma 2 and $\phi(\varepsilon_2)^3 = \pm\phi(\varepsilon_1)$, therefore $H_F = W_F \times \langle \phi(\varepsilon_2) \rangle$ and $H_K = W_K \times \langle \psi(\varepsilon_0) \rangle$ on account of the first statement of this proposition. Assume $\varepsilon_3 = \sqrt{\eta_3}$ in (16), then $\phi(\varepsilon_2)^2 = \eta_3^{-1} > 0$, which is a contradiction because $F \cap \boldsymbol{R} = K_3$ and $\phi(\varepsilon_3)$ cannot belong to K_3. Hence $\sqrt{\eta_3}$ does not belong to K. Assume $\varepsilon_3' = \sqrt{\rho\eta_3}$ in (17), then $\psi(\varepsilon_3')^2 = \eta_3^{-1}$ which is impossible since $\sqrt{\eta_3}$ does not belong to K. Hence $\sqrt{\rho\eta_3}$ is not an element of F. If there is a unit ε of K such that $N_{K/K_3}(\varepsilon) = \eta_3$, see Remark 3, then the third formula of Lemma 2 tells that $N_{F/K_3}(\phi(\varepsilon^{-1})) = \eta_3$. From the fourth formula of Lemma 2, it is derived that η_3 belongs to $N_{K/K_3}(E_K)$ if η_3 is an element of $N_{F/K_3}(E_F)$. Thus the proof is complete.

COROLLARY. *When* $L \cap \boldsymbol{R} = K$, *we have*

$$E_F = H_F \times \langle \phi(\varepsilon_3) \rangle\ , \qquad E_K = H_K \times \langle \varepsilon_2 \rangle \times \langle \psi(\varepsilon_3') \rangle\ ,$$

where ε_2, ε_3 *and* ε_3' *are as in* (15), (16) *and* (17). *Especially*

$$E_F = W_F \cdot \phi(E_K)$$

if $\varepsilon_2 = \sqrt[3]{\varepsilon_1 \eta_2^{\pm 1}}$ *in* (15).

PROOF. Combining the first four formulas of Lemma 2 with Proposition 1.(iii), we see the former. Then the latter is clear from Proposition 6.

§3. Minkowski units.

3.1. Notations being as in 0.2, we assume $n=2$ and $L \cap \boldsymbol{R} = K$.

PROOF OF THEOREM 2 IN CASE $n=2$. Assume ε is a Minkowski unit for L which belongs to K, then

$$\varepsilon^{1+\sigma^2} = N_{L/\Lambda}(\varepsilon) = N_{K/K_2}(\varepsilon)\ ,$$

which is not any ν-th power modulo W_L in E_L for $\nu \geqq 2$. Therefore $\varepsilon^{1+\sigma^2}$ is a fundamental unit of both K_2 and Λ, is a norm from E_K and is not a square modulo ± 1 in E_K. Hence $E_\Lambda = W_\Lambda \times \langle \eta_2 \rangle$ and $\varepsilon_2 = \sqrt{\varepsilon_1 \eta_2}$ in (7). Put $\alpha := \sqrt{2\eta_2}$ and suppose $K = K_2(\alpha)$, then $\phi(\alpha)^2 = -4$ since η_2 cannot be totally positive, where ϕ is given as in 2.1. Therefore $\sqrt{-1}$ is an element of L and $(\alpha/(1-\sqrt{-1}))^2 = \sqrt{-1}\eta_2$. This implies that $\varepsilon^{1+\sigma^2}$ is a square modulo W_L, and a contradiction. Thus $K \neq K_2(\sqrt{2\eta_2})$. Suppose $K = \boldsymbol{Q}(\sqrt[4]{2})$, then $((1+\omega)/\sqrt[4]{2})^2 = \omega\eta_2$ with $\omega := (1+\sqrt{-1})/\sqrt{2}$ and $\eta_2 = 1+\sqrt{2}$. This is also a contradiction. Hence we have proved the "only if" part of the theorem.

Assume $E_\Lambda = W_\Lambda \times \langle \eta_2 \rangle$ and $\varepsilon_2 = \sqrt{\varepsilon_1 \eta_2}$, then we see

$$E_K = \langle -1, \varepsilon_2, \varepsilon_2^{\sigma^2} \rangle \quad \text{and} \quad E_\Lambda = W_\Lambda \times \langle \varepsilon_2^{1+\sigma^2} \rangle ,$$

and so it follows from Proposition 5 that

$$E'_L = W_L \times \langle \varepsilon_2 \rangle \times \langle \varepsilon_2^{\sigma} \rangle \times \langle \varepsilon_2^{\sigma^2} \rangle . \tag{18}$$

Therefore, since E_L^2 is contained in E'_L, for every ξ in E_L,

$$\xi^2 \equiv \varepsilon_2^{\nu_0} \varepsilon_2^{\nu_1 \sigma} \varepsilon_2^{\nu_2 \sigma^2} \pmod{W_L} \tag{19}$$

with ν_i in $\boldsymbol{Z}$ $(i=0, 1, 2)$. Operating $1+\tau$ and $1+\sigma^2$ on the both sides of (19), it is obtained that

$$\xi^{2(1+\tau)} \equiv \varepsilon_2^{2\nu_2-\nu_1} \varepsilon_2^{(2\nu_2-\nu_1)\sigma^2} \pmod{W_K} ,$$
$$\xi^{2(1+\sigma^2)} \equiv \varepsilon_2^{(\nu_0+\nu_2-\nu_1)(1+\sigma^2)} \pmod{W_\Lambda} .$$

So the congruences

$$\nu_1 \equiv \nu_0 + \nu_2 \equiv 0 \pmod 2 \tag{20}$$

are derived. If $\nu_0 \equiv \nu_2 \equiv 1 \pmod 2$, it follows from (19) that there exists a unit λ of L such that

$$\lambda^2 \equiv \varepsilon_2^{1+\sigma^2} \equiv \eta_2 \pmod{W_L} .$$

If $\lambda^2 = \eta_2$, we see that λ is an element of $K = L \cap \boldsymbol{R}$, however it contradicts to $\varepsilon_2 = \sqrt{\varepsilon_1 \eta_2}$. Therefore we may assume

$$\lambda^2 = \begin{cases} -\eta_2 & \text{if } \sqrt{-1} \notin L \\ \omega\eta_2 \quad \text{with} \quad \omega := (1+\sqrt{-1})/\sqrt{2} & \text{if } \Lambda = \boldsymbol{Q}(\sqrt{2}, \sqrt{-1}) \\ \sqrt{-1}\eta_2 & \text{otherwise} . \end{cases}$$

When $\sqrt{-1}$ does not belong to L, then, since λ is neither in Λ nor in K, we see λ is in K^σ and λ^σ is in K, so $\lambda^{2\sigma} = -\eta_2^\sigma > 0$. Hence $\eta_2 = -\eta_2^{-\sigma} = (\lambda^{-\sigma})^2$ with $\lambda^{-\sigma}$ in K, which also contradicts to $\varepsilon_2 = \sqrt{\varepsilon_1 \eta_2}$. When $\Lambda =$

$Q(\sqrt{2},\sqrt{-1})$, put $\theta:=(1+\omega)/\lambda$, then $\theta^2=\sqrt{2}$ and $K=Q(\theta)=Q(\sqrt[4]{2})$. Otherwise, $\sqrt{-1}$ belongs to L and $\Lambda\neq Q(\sqrt{2},\sqrt{-1})$, and then $(\lambda+\lambda^\tau)^2=2\eta_2$, while $2\eta_2$ is not a square in K_2 by Satz 13 of [7] since $K_2\neq Q(\sqrt{2})$ and $K_2\neq Q(\sqrt{3})$. Hence $K=K_2(\lambda+\lambda^\tau)=K_2(\sqrt{2\eta_2})$. Thus we have shown that $\nu_0\equiv\nu_1\equiv\nu_2\equiv 0 \pmod 2$ on account of (20) if $K\neq Q(\sqrt[4]{2})$ and $K\neq K_2(\sqrt{2\eta_2})$. So ξ^2 has already been a square in E'_L modulo W_L by (19), which implies that $E_L=E'_L$ and that, by (18), ε_2 is a Minkowski unit for L. This completes the proof of Theorem 1 in case $n=2$.

3.2. Notations being as in 0.2, we assume $n=3$ and $L\cap R=K$.

Proof of Theorem 2 in case $n=3$. Assume ε is a Minkowski unit for L which belongs to K, then

$$\varepsilon^{1+\sigma^3}=N_{L/\Omega}(\varepsilon)=N_{K/K_3}(\varepsilon)\ ,$$

which is not any ν-th power modulo W_L in E_L for $\nu\geqq 2$. Therefore $\varepsilon^{1+\sigma^3}$ is a fundamental unit of K_3 and is a norm from E_K, hence $\varepsilon_3=\sqrt{\varepsilon_1\eta_3}$ in (16) by Proposition 6. Further, since $\varepsilon^{1+\sigma^3}$ and $\varepsilon^{\sigma+\sigma^4}$ are units of Ω which, together with W_L, generate a maximal subgroup with free rank 2 in E_L, it follows that $E_\Omega=W_\Omega\times\langle\eta_3\rangle\times\langle\eta_3^\sigma\rangle$. Similarly, from

$$\varepsilon^{1+\sigma^2+\sigma^4}=N_{L/\Lambda}(\varepsilon)=N_{K/K_2}(\varepsilon)\ ,$$

we obtain that $E_\Lambda=W_\Lambda\times\langle\eta_2\rangle$ and that $\varepsilon_2=\sqrt[3]{\varepsilon_1\eta_2^{\pm 1}}$ or $\sqrt[3]{\eta_2}$. Here $\varepsilon_2\neq\sqrt[3]{\eta_2}$ because $\varepsilon^{1+\sigma^2+\sigma^4}$ cannot be a cube in E_L modulo W_L. Thus the "only if" part of the theorem is shown.

Assume $\varepsilon_2=\sqrt[3]{\varepsilon_1\eta_2^{\pm 1}}$ in (15) and $\varepsilon_3=\sqrt{\varepsilon_1\eta_3}$ in (16), and put $\varepsilon:=\varepsilon_3/\varepsilon_2$. Then it is easy to see that

$$E_K=W_K\times\langle\varepsilon\rangle\times\langle\varepsilon^{\sigma^3}\rangle\times\langle\varepsilon^{\sigma^2+\sigma^4}\rangle\ .$$

If we further assume $E_\Lambda=W_\Lambda\times\langle\eta_2\rangle$ and $E_\Omega=W_\Omega\times\langle\eta_3\rangle\times\langle\eta_3^\sigma\rangle$, then

$$E_\Omega=W_\Omega\times\langle\varepsilon^{1+\sigma^3}\rangle\times\langle\varepsilon^{\sigma+\sigma^4}\rangle\ ,$$
$$E_\Lambda=W_\Lambda\times\langle\varepsilon^{1+\sigma^2+\sigma^4}\rangle\ .$$

Now, by the corollary of Proposition 6, we see

$$F_F=W_F\times\langle\varepsilon^{\sigma+\sigma^2}\rangle\times\langle\varepsilon^{\sigma^4+\sigma^5}\rangle\ .$$

Therefore

$$E'_L=W_L\times\langle\varepsilon\rangle\times\langle\varepsilon^{\sigma}\rangle\times\langle\varepsilon^{\sigma^2}\rangle\times\langle\varepsilon^{\sigma^3}\rangle\times\langle\varepsilon^{\sigma^4}\rangle\ , \tag{21}$$

and, since E_L^3 is contained in E'_L, for every ξ in E_L,

$$\xi^3 \equiv \varepsilon^\gamma \pmod{W_L}, \qquad \gamma := \nu_0 + \nu_1\sigma + \cdots + \nu_4\sigma^4$$

with ν_i in Z $(i=0, 1, \cdots, 4)$. Operating $1+\tau$, $1+\sigma^3\tau$ and $1+\sigma^3$, the congruences

$$\nu_i \equiv 0 \pmod 3 \qquad (i=0, 1, \cdots, 4)$$

are derived similarly as in case $n=2$. Thus ξ^3 is a cube in E'_L modulo W_L, which means $E_L = E'_L$, and hence, by (21), ε is a Minkowski unit for L. This completely proves Theorem 1 in case $n=3$.

3.3. Notations being as in 0.2, let $n=2$ and $L \cap R = K$. It seems to be a little more complicated to see whether L has a Minkowski unit ε which is not necessarily real, e.g. $E_L = W_L \times \langle\varepsilon\rangle \times \langle\varepsilon^\sigma\rangle \times \langle\varepsilon^\tau\rangle$. The following proposition gives a necessary condition for L to have a Minkowski unit.

PROPOSITION 7. *Assumptions being as above, let η_2 be as in* (4). *Then there is no Minkowski unit for L if $\varepsilon_2 = \eta_2$ in* (7) *and $E_A \neq W_A \times \langle\eta_2\rangle$.*

PROOF. Suppose that there is a Minkowski unit ε for L. Then one of the following sets is a system of fundamental units of L:

$$(22) \qquad \begin{cases} \{\varepsilon, \varepsilon^\sigma, \varepsilon^{\sigma^2}\}, & \{\varepsilon, \varepsilon^\sigma, \varepsilon^\tau\}, \quad \{\varepsilon, \varepsilon^\sigma, \varepsilon^{\sigma\tau}\}, \quad \{\varepsilon, \varepsilon^\sigma, \varepsilon^{\sigma^2\tau}\}, \\ \{\varepsilon, \varepsilon^\sigma, \varepsilon^{\sigma^3\tau}\}, & \{\varepsilon, \varepsilon^{\sigma^2}, \varepsilon^\tau\}, \quad \{\varepsilon, \varepsilon^{\sigma^2}, \varepsilon^{\sigma\tau}\}, \end{cases}$$

Since $\{\varepsilon, \varepsilon^\sigma, \varepsilon^{\sigma\tau}\}^\tau = \{\varepsilon^{\sigma\tau}, (\varepsilon^{\sigma\tau})^\sigma, (\varepsilon^{\sigma\tau})^\tau\}$, $\{\varepsilon, \varepsilon^\sigma, \varepsilon^{\sigma^2\tau}\}^\sigma = \{\varepsilon^\sigma, (\varepsilon^\sigma)^\sigma, (\varepsilon^\sigma)^\tau\}$ and $\{\varepsilon, \varepsilon^\sigma, \varepsilon^{\sigma^3\tau}\}^{\sigma\tau} = \{\varepsilon^{\sigma^2\tau}, (\varepsilon^{\sigma^2\tau})^\sigma, (\varepsilon^{\sigma^2\tau})^\tau\}$, we may treat the first two and the last two sets of (22). If the first set of (22) is a system of fundamental units of L, we have

$$E_A = W_A \times \langle\varepsilon^{1+\sigma^2}\rangle .$$

If $\varepsilon^{1+\sigma^2}$ is not a fundamental unit of K_2, i.e., $E_A \neq W_A \times \langle\eta_2\rangle$, then $\varepsilon^{(1+\sigma^2)(1+\tau)}$ is a fundamental unit of K_2. From

$$\varepsilon^{(1+\sigma^2)(1+\tau)} = \varepsilon^{(1+\tau)(1+\sigma^2)} ,$$

it follows that η_2 is element of $N_{K/K_2}(E_K)$ and that $\varepsilon_2 \neq \eta_2$ in (7). Therefore $E_A = W_A \times \langle\eta_2\rangle$ or $\varepsilon_2 \neq \eta_2$ in (7) if the first set of (22) is a system of fundamental units of L. Similarly, if either of the last two sets of (22) is a system of fundamental units of L, we have $E_A = W_A \times \langle\eta_2\rangle$ or $\varepsilon_2 \neq \eta_2$

in (7). If the second set of (22) is a system of fundamental units of L, it holds that

$$\varepsilon^{\sigma\tau} \equiv \varepsilon^{\nu_0}\varepsilon^{\nu_1\sigma}\varepsilon^{\nu_2\tau} \pmod{W_L}$$

with ν_i in $\boldsymbol{Z}$ $(i=0, 1, 2)$. Then, operating τ on the both sides, we obtain that

$$\varepsilon^{\sigma} \equiv \varepsilon^{\nu_0\nu_1+\nu_2}\varepsilon^{\nu_1^2\sigma}\varepsilon^{(\nu_0+\nu_1\nu_2)\tau} \pmod{W_L}$$

and that

$$\varepsilon^{\sigma\tau} \equiv \varepsilon^{\nu_0}\varepsilon^{\nu_1\sigma}\varepsilon^{-\nu_1\nu_0\tau} \pmod{W_L}$$

with $\nu_1=\pm 1$. Therefore

$$\varepsilon^{\sigma(1+\tau)} \equiv \varepsilon^{\nu_0(1-\nu_1\tau)}\varepsilon^{(1+\nu_1)\sigma} \pmod{W_L} .$$

Since $N_{L/K}(E_L)$ is a finite index subgroup of E_K, $\varepsilon^{1+\tau}$ and $\varepsilon^{\sigma(1+\tau)}$ are independent units of K, so $\nu_1=1$ and

(23) $$\varepsilon^{\sigma\tau} \equiv \varepsilon^{\nu_0}\varepsilon^{\sigma}\varepsilon^{-\nu_0\tau} \pmod{W_L} .$$

Now, for ξ in E_K, let

(24) $$\xi \equiv \varepsilon^{\mu_0}\varepsilon^{\mu_1\sigma}\varepsilon^{\mu_2\tau} \pmod{W_L}$$

with μ_i in $\boldsymbol{Z}$ $(i=0, 1, 2)$, then, on account of $\xi=\xi^{\tau}$, it follows from (23) that

(25) $$\xi \equiv \varepsilon^{\mu_0(1+\tau)}\varepsilon^{\mu_1(\sigma-\nu_0\tau)} \pmod{W_L} .$$

If $E_A \neq W_A \times \langle \eta_2 \rangle$, then η_2 is a square modulo W_A in E_A by Satz 12 of [7]. Therefore, when $\xi=\eta_2$ in (24), all the μ_i $(i=0, 1, 2)$ should be even, hence, by (25), η_2 is a square modulo W_L in $E_K \cdot W_L$. Thus, by $E_K \cdot W_L/W_L \simeq E_K/W_K$, η_2 is a square modulo W_K in E_K, i.e. $\varepsilon_2=\sqrt{\eta_2}$ in (7), and the proof is complete.

§4. Binomial units.

4.1. For $n=2$ or 3, let $K=\boldsymbol{Q}(\theta)$, $\theta=\sqrt[2n]{d}$ with $2n$-th power free d in $\boldsymbol{N}$, be a real pure number field of degree $2n$, and ζ be a primitive $2n$-th root of unity. Then the galois closure L of $K/\boldsymbol{Q}$ and the galois group G of $L/\boldsymbol{Q}$ are given by

$$L=\boldsymbol{Q}(\theta, \zeta)=K(\zeta) ; \quad G=\langle \sigma, \tau \rangle , \quad \theta^{\sigma}=\zeta\theta , \quad \theta^{\tau}=\theta , \quad \zeta^{\sigma}=\zeta , \quad \zeta^{\tau}=\zeta^{-1} .$$

The group G is a dihedral group of order $4n$. Using the notations in 0.2, we see that

$$L\cap R=K\,,\qquad F=Q(\sqrt[2n]{-n^n d})\,.$$

We determine fundamental units of K, F and L explicitly in a certain case, assuming that K has a binomial unit. Before we prove Theorems 2 and 3, we make a few remarks.

REMARK 4. Let S be the set of K which has a unit of type $a-b\theta$ with a, b in Z, $a>0$, $b>1$. Then, by Ljunggren's theorem (Satz 3 and Satz 7 of [17]), there is only one unit of type $a-b\theta$ in K for each K in S, and the fields K differ if the pairs (a, b) differ. Therefore, if we put

$$S_b:=\{K\in S\,|\,a\in N,\ a-b\theta\in E_K\}$$

for each $b>1$, we see

$$S=\bigcup_{b=2}^{\infty} S_b \qquad \text{(disjoint union)}\,.$$

So we fix $b>1$ and consider K it S_b. Then, by a similar manner as in H. Yokoi [18], we see that $a-b\theta$ is a unit of K if and only if

$$a=b^{2n}c+a_0$$
$$d=\begin{cases} c(b^4c+2a_0)(b^8c^2+2a_0b^4c+2a_0^2)+d_0 & \text{for} \quad n=2 \\ c(b^6c+2a_0)(b^{12}c^2+a_0b^6c+a_0^2)(b^{12}c^2+3a_0b^6c+3a_0^2)+d_0 & \text{for} \quad n=3 \end{cases}$$

with a certain natural number c and a rational integer a_0 such that

$$a_0^{2n}\equiv\pm1 \pmod{b^{2n}}\,,\qquad \frac{1\pm1}{2}-b^{2n}<a_0\leqq\frac{1\pm1}{2}\,. \tag{26}$$

Here the rational integer d_0 is given by $a_0^{2n}=\pm1+b^{2n}d_0$. If $c>1$, we see $a>b^{2n}$, so the assumption $a\geqq b^{2n}-1$ removes only finitely many fields in S_b. Especially in the trivial case in (26), $a_0=\pm1$,

$$d=\begin{cases} c(b^4c\pm2)(b^8c^2\pm2b^4c+2) & \text{for} \quad n=2\,, \\ c(b^6c\pm2)(b^{12}c^2\pm b^6c+1)(b^{12}c^2\pm3b^6c+3) & \text{for} \quad n=3\,. \end{cases} \tag{27}$$

By [9], we can prove, for any fixed b in N, odd in case $n=2$, there are infinitely many square free d of the form (27). Therefore there are infinitely many fields in the set S_b (b: odd if $n=2$) which satisfy the assumption of Theorem 2. In the case d is as in (27), the condition

$a \geqq b^{2n}-1$ is satisfied even if $c=1$. The same fact holds for S_b when b is even and $n=2$.

We note that d becomes very large since d is a polynomial of degree $2n(2n-1)$ in b and of degree $2n$ in c.

REMARK 5. By using Satz 3, Satz 7, Satz 16 and Satz 22 of [17], we see that the field K in Theorem 2 is different from those in Stender [17] unless $b=1$. Of course, Theorem 3 is a new result including the case $b=1$.

4.2.

PROOF OF THEOREM 2. (i) Let $n=2$. When $b=1$, the proof is seen in [15]. Assume $b>1$. Then the inequalities

$$a \geqq 15\,, \qquad \mathrm{Max}\,(a, b\theta) < a+1\,, \qquad \theta^{-1} < \sqrt[4]{a+1}\,/(a-1) \tag{28}$$

are obtained easily. Let $\delta := a^4 - b^4 d = N_{K/Q}(\xi_1)$, and η_2, ε_1 be as in (4), (5). Then

$$\eta_2 = \delta \xi_1^{-1} \xi_2^{-1} = \delta N_{K/K_2}(\xi_2^{-1}) = a^2 + b^2\theta^2 \tag{29}$$

and

$$\varepsilon_1 = \xi_2^2 \eta_2 = \delta \xi_1^{-1} \xi_2 = (a^2 + b^2\theta^2)(a+b\theta)^2 \tag{30}$$

are proved as follows. In (29), only the first equality is non-trivial. Suppose $a^2+b^2\theta^2=\eta_2^\nu$ with $\nu \geqq 2$, then

$$1 \leqq \theta^{-2}(\eta_2 + |\eta_2^\sigma|) < \theta^{-2}(\sqrt{a^2+b^2\theta^2}+1)$$

since $0 \neq \theta^{-2}(\eta_2 - \eta_2^\sigma) \in \boldsymbol{Z}$ and $1 < \eta_2 \leqq \sqrt{a^2+b^2\theta^2}$. On the other hand

$$\theta^{-2}(\sqrt{a^2+b^2\theta^2}+1) < 1$$

follows from (28). This is a contradiction, and so (29) is proved. In (30), only the first equality is non-trivial. Note that $\sqrt{\eta_2}$ does not belong to K because, if it belongs to K, we have $\phi(\sqrt{\eta_2})^2 = -1$, though $F_2 \neq \boldsymbol{Q}(\sqrt{-1})$, where ϕ is given as in 2.1. Suppose $\xi_2^2\eta_2 = \varepsilon_1^\nu$ with $\nu \geqq 3$. Then, by (3), (4), we see $1 < \theta^{-3}(3+\sqrt[3]{\xi_2^2\eta_2})$, applying Hilfssatz 1 of [15]. While $\theta^{-3}(3+\sqrt[3]{\xi_2^2\eta_2}) < 1$ follows from (28). This is a contradiction, and hence (30) is proved. Thus, by (29), (30), and by (6), (7), Theorem 2 is completely proved in case $n=2$.

(ii) Let $n=3$. When $b=1$, the proof is seen in [16]. So we assume $b>1$. Then the inequalities

(31) $\qquad a \geqq 63\ , \qquad \mathrm{Max}\,(a, b\theta) < a+1\ , \qquad \theta^{-1} < \sqrt[6]{a+1}/(a-1)$

are proved easily. Similarly as in case $n=2$, we obtain

(32) $\qquad \eta_2 = \delta\xi_1^{-1}\xi_3^{-1} = \delta N_{K/K_2}(\xi_1^{-1}) = a^3 + b^3\theta^3\ ,$

(33) $\qquad \eta_3 = \delta\xi_1^{-1}\xi_2^{-1} = \delta N_{K/K_3}(\xi_1^{-1}) = a^4 + a^2b^2\theta^2 + b^4\theta^4\ ,$

(34) $\qquad \varepsilon_1 = \xi_1^{-6}\eta_2^{-2}\eta_3^{-3} = \delta\xi_1^{-1}\xi_2^3\xi_3^2 = (a+b\theta)^3(a^2+ab\theta+b^2\theta^2)^2(a^5+a^4b\theta+\cdots+b^5\theta^5)$

by using (31) and the results in [16]. Here $\delta = a^6 - b^6d = N_{K/Q}(\xi_1)$, and ε_1, η_2 and η_3 are given as in (12) and (13). The detailed proofs of (32), (33) and (34) are done in the same way as in [11], so they are omitted. Now we see, on account of (32), (33), (34), that $\varepsilon_2 = \sqrt[3]{\varepsilon_1\eta_2^{-1}} = \xi_2\xi_3$ in (15), $\varepsilon_3 = \sqrt{\varepsilon_1\eta_3} = \delta\xi_1^{-1}\xi_2\xi_3$ in (16), and that ξ_1, ξ_2, ξ_3 form a set of fundamental units of K. Thus the proof of Theorem 2 is complete in case $n=3$.

4.3.

Proof of Theorem 3. (i) Let $n=2$. Then $K \neq K_2(\sqrt{2\eta_2})$, because $F_2 \neq Q(\sqrt{-1})$. Therefore, on account of Theorem 1, it is sufficient to show

(35) $$E_A = W_A \times \langle\eta_2\rangle\ ,$$

since $\sqrt{\varepsilon_1\eta_2} = \delta\xi_1^{-1} \in K$ has already been shown in the proof of Theorem 2. We mention that $d>3$ by (29) and by the assumption. Then we can apply Satz 13 of [7] to prove (35). Assume that the ideal (2) is a square of a principal ideal in K_2. Then $2\eta_2 = ((x+y\theta^2)/2)^2$ with x, y in Z, $x \equiv y \pmod 2$. From this and (29), $4b^2 = xy$ and $x \equiv y \equiv 0 \pmod 2$ follow. Therefore

$$2\eta_2 = (s+t\theta^2)^2$$

with s, t in Z. Taking the norms of the both sides, we have $4 = (s^2 - t^2d)^2$. Hence

$$2a^2 = s^2 + t^2d\ , \qquad \pm 2 = s^2 - t^2d\ ,$$

and so $a^2 \pm 1 = s^2$. This is not the case and (35) follows from Satz 13 of [7] since $d>3$. The proof is complete in case $n=2$.

(ii) Let $n=3$. It is sufficient to show

(36) $$E_A = W_A \times \langle\eta_2\rangle \quad \text{and} \quad E_\Omega = W_\Omega \times \langle\eta_3\rangle \times \langle\eta_3^\sigma\rangle$$

on account of Theorem 1, because $\varepsilon_2 = \sqrt[3]{\varepsilon_1\eta_2^{-1}}$ in (15), $\varepsilon_3 = \sqrt{\varepsilon_1\eta_3}$ in (16) and $\varepsilon_3/\varepsilon_2=\delta/\xi_1$ have been shown in the proof of Theorem 2. Mention that $d>3$ by (32) and by the assumption. Assume $E_A\neq W_A\times\langle\eta_2\rangle$, then, applying Satz 14 of [7] similarly as in the proof of Proposition 4.(i) of [11], we have

$$3a^3=x^2+y^2d\ ,\qquad \pm 3=x^2-y^2d\ ,\qquad 3b^3=2xy\ ,$$

with x, y in $\boldsymbol{Z}$. From this, it follows that $3|x$, $(x/3, y)=1$ and that $x=3z^3$ or $x=12z^3$ with z in $\boldsymbol{Z}$. Therefore

$$a^3\pm 1=6(z^2)^3 \quad\text{or}\quad 12(2z^2)^3$$

with z in $\boldsymbol{Z}$, which is impossible, hence $E_A=W_A\times\langle\eta_2\rangle$. Assume $E_\Omega\neq W_\Omega\times\langle\eta_3\rangle\times\langle\eta_3^\sigma\rangle$, then, applying Proposition 1.2 of [6] similarly as in the proof of Proposition 4.(ii) of [11], we have

$$3(a^4+\delta a^2+1)=(x+y)^2-3xy$$

with x, y in $\boldsymbol{Z}$ such that $a^4+\delta a^2+1$ divides $b^4(x+y)$. Note here that $b^6d=(a^2-\delta)(a^4+\delta a^2+1)$, $(a^2-\delta)^2+3\delta a^2=a^4+\delta a^2+1$ and that d is square-free. Therefore every prime divisor of $a^4+\delta a^2+1$ divides b, hence

$$a^4+\delta a^2+1\leqq b^7\ .$$

On the other hand, we have

$$a^4+\delta a^2+1\geqq(b^6-1)^4-(b^6-1)^2+1\ .$$

This is impossible except for $b=1$, $\delta=-1$ and $a=1$, and then $d=2$, which is the case removed by the assumption. Therefore $E_\Omega=W_\Omega\times\langle\eta_3\rangle\times\langle\eta_3^\sigma\rangle$. Thus (36) is shown and the proof is complete in case $n=3$.

4.4. Lastly, we give an example of L which has no Minkowski unit, real or not, in case $n=2$.

PROPOSITION 8. *The field $L=\boldsymbol{Q}(\sqrt[4]{3g^2}, \sqrt{-1})$ has no Minkowski unit if g is a square free natural number prime to* 3.

PROOF. We apply Proposition 7. Notations being as in 0.2, we see $\Lambda=\boldsymbol{Q}(\sqrt{3}, \sqrt{-1})$. Let η_2 be as in (4), then $\eta_2=2+\sqrt{3}$ and

$$E_A\neq W_A\times\langle\eta_2\rangle\ .$$

Put $\theta:=\sqrt[4]{3g^2}$, then, for ε in E_K, it follows from [15] that

$$2g\varepsilon\theta=gx_0+gx_1\theta+x_2\theta^2+x_3\theta^3$$

with rational integers x_i $(i=0, 1, 2, 3)$. If $N_{K/K_2}(\varepsilon)=\eta_2=2+\sqrt{3}$,

$$\begin{cases} x_0^2+3x_2^2-6gx_1x_3=-12g \\ 2x_0x_2-gx_1^2-3gx_3^2=-8g \end{cases}$$

are derived. But this is impossible, hence $\varepsilon_2=\eta_2$ in (7), see also Remark 1. Therefore the assumptions of Proposition 7 are satisfied, and L has no Minkowski unit.

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