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Applications of an Inverse Abel Transform for Jacobi Analysis: Weak-L^1 Estimates and the Kunze-Stein Phenomenon

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Abstract. For the Jacobi hypergroup $(\mathbf{R}_+, \Delta, *)$, the weak-$L^1$ estimate of the Hardy-Littlewood maximal operator was obtained by W. Bloom and Z. Xu, later by J. Liu, and the endpoint estimate for the Kunze-Stein phenomenon was obtained by J. Liu. In this paper we shall give alternative proofs based on the inverse Abel transform for the Jacobi hypergroup. The point is that the Abel transform reduces the convolution $*$ to the Euclidean convolution. More generally, let T be the Hardy-Littlewood maximal operator, the Poisson maximal operator or the Littlewood-Paley g-function for the Jacobi hypergroup, which are defined by using $*$. Then we shall give a standard shape of Tf for $f \in L^1(\Delta)$, from which its weak-L^1 estimate follows. Concerning the endpoint estimate of the Kunze-Stein phenomenon, though Liu used the explicit form of the kernel of the convolution, we shall give a proof without using the kernel form.

1. Introduction

Let $w(x)$ be a positive measurable function on $\mathbf{R}_+$. We denote by $L^p(w)$ the space of measurable functions f on $\mathbf{R}_+$ with finite L^p-norm $\|f\|_{L^p(w)}$ with respect to $w(x)\,dx$. For $1 \le p, q \le \infty$ we define the Lorentz space $L^{p,q}(w)$ on $\mathbf{R}_+$ with respect to $w(x)\,dx$ by the usual way (see [8]) and denote its quasi-norm by $\|\cdot\|_{L^{p,q}(w)}$. We see that $L^{p,q}(w) \subset L^{p,q'}(w)$ if $q \le q'$, $L^{p,p}(w) = L^p(w)$ and $L^{p,\infty}(w)$ coincides with the space consisting of all weak-$L^p(w)$ functions on $\mathbf{R}_+$. We denote $L^p(1)$ by $L^p(\mathbf{R}_+)$. We often regard functions f on $\mathbf{R}_+$ as even functions on $\mathbf{R}$, which are denoted by the same symbol f.

Let $\alpha \ge \beta \ge -1/2$ and $\Delta(x) = \Delta_{\alpha,\beta}(x)$ be a weight function on $\mathbf{R}_+$ defined by $(2\sinh x)^{2\alpha+1}(2\cosh x)^{2\beta+1}$. For $\alpha > -\frac{1}{2}$ we denote by $(\mathbf{R}_+, \Delta, *)$ the Jacobi hypergroup with the convolution structure $*$ (see §2). Roughly speaking, Jacobi analysis is a harmonic analysis on $\mathbf{R}_+$ with a convolution and a weight measure having an exponential growth order. As in the Euclidean case, by using the convolution $*$, we can introduce the Hardy-Littlewood maximal operator M_{HL}, the Poisson maximal operator M_{P} and the Littlewood-Paley g-function for $(\mathbf{R}_+, \Delta, *)$ as follows.

$$M_{\mathrm{HL}} f(x) = \sup_{t>0} |f| * \widetilde{\chi}_t(x)\,,$$

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$$M_{\mathrm{P}} f(x) = \sup_{t>0} |f * p_t(x)|\,, \tag{1}$$

$$g(f)(x) = \Big(\int_0^\infty \Big| f * t\frac{dp_t}{dt}(x)\Big|^2 \frac{dt}{t}\Big)^{\frac{1}{2}}\,.$$

See §6 for the definitions of the normalized characteristic function $\widetilde{\chi}_t$ and the Poisson kernel p_t. In particular, when α, β are specialized integers or half-integers, the Jacobi analysis coincides with harmonic analysis on real rank one noncompact semisimple Lie groups and the convolution $*$ coincides with one defined from the group structure.

For a noncompact semisimple Lie group G with general rank, the strong type L^p estimate of the Hardy-Littlewood maximal operator M_{HL} on G was proved by Clerc and Stein [6] for $p > 1$ and later, the weak type L^1 estimate was obtained by Strömberg [15]. His proof was improved in [11] and was applied to the weak-L^1 estimates of the Poisson maximal operator M_{P} and the Littlewood-Paley g-function on G (see [1]).

On the other hand, Kunze and Stein proved that functions on $SL(2, \mathbf{R})$ satisfy

$$\|f * g\|_{L^2(\Delta)} \le c_p \|f\|_{L^p(\Delta)} \|g\|_{L^2(\Delta)}$$

for $1 \le p < 2$. Cowling [4] extended this inequality for all noncompact semisimple Lie groups G and, if G is of real rank one, he deduced the Lorentz space version:

$$\|f * g\|_{L^{p,w}(\Delta)} \le c \|f\|_{L^{p,u}(\Delta)} \|g\|_{L^{p,v}(\Delta)}\,,$$

where $1 \le p < 2$, $1 \le u, v, w \le \infty$, and $1 + \frac{1}{w} = \frac{1}{u} + \frac{1}{v}$ (see [5]). Then Ionescu [9] obtained that

$$\|f * g\|_{L^{2,\infty}(\Delta)} \le c \|f\|_{L^{2,1}(\Delta)} \|g\|_{L^{2,1}(\Delta)}$$

at the endpoint $p = 2$, which covers Cowling's result by interpolation.

As mentioned above, since harmonic analysis on G of real rank one corresponds to the Jacobi analysis, these weak-$L^1(\Delta)$ estimates and the Kunze-Stein phenomena hold for the Jacobi hypergroup with special α, β. Furthermore, we can easily generalize their proofs to general α, β. Hence, we can deduce the following:

THEOREM 1. *Let $f \in L^1(\Delta)$ and $\lambda > 0$. Then*

$$|\{x \in \mathbf{R}_+ \mid M_{\mathrm{HL}} f(x) > \lambda\}| \le c \frac{\|f\|_{L^1(\Delta)}}{\lambda}\,,$$

where $|S|$ denotes the volume of S with respect to $\Delta(x)\,dx$. Moreover, this inequality holds if M_{HL} is replaced by M_{P} and the g-function.

THEOREM 2. *Let $f \in L^p(\Delta)$, $1 \le p < 2$, and $g \in L^2(\Delta)$. Then*

$$\|f * g\|_{L^2(\Delta)} \le c_p \|f\|_{L^p(\Delta)} \|g\|_{L^2(\Delta)}\,,$$

where c_p does not depend on f or g.

THEOREM 3. *Let* $f, g \in L^{2,1}(\Delta)$. *Then*

$$\|f * g\|_{L^{2,\infty}(\Delta)} \le c\|f\|_{L^{2,1}(\Delta)}\|g\|_{L^{2,1}(\Delta)},$$

where c *does not depend on* f *or* g.

Later, J. Liu gave, respectively in [13] and [14], quite simple proofs of Theorem 1 and Theorem 3 based on a kernel form of the convolution structure (see (8)). The aim of this paper is to give other simple proofs based on the inversion formula for the Abel transform obtained in [10] (see (13)). Since the kernel can be expressed by using the inverse Abel transform (see Remark 1 in §7), our approach corresponds to a transfer of Liu's one. However, we can deduce a standard shape of the maximal function, from which the weak-L^1 estimate appeared in Theorem 1 follows easily (see §6).

2. Jacobi and Abel transforms

We recall the basic properties of the Jacobi hypergroup $(\mathbf{R}_+, \Delta_{\alpha,\beta}, *)$. We refer to [7] and [12] for the details of content stated below. We denote $\Delta = \Delta_{\alpha,\beta}$ and put $\rho = \alpha + \beta + 1$. For $\lambda \in \mathbf{C}$ the solutions of the differential equation

$$\Delta(x)^{-1}\frac{d}{dx}\Big(\Delta(x)\frac{du}{dx}\Big) = -(\lambda^2 + \rho^2)u(x)$$

with $u(0) = 1$ and $u'(0) = 0$ are given as the Jacobi functions of the first kind with order (α, β):

$$\phi_\lambda(x) = {}_2F_1\big(\tfrac{1}{2}(\rho + i\lambda), \tfrac{1}{2}(\rho - i\lambda); \alpha + 1; -(\sinh x)^2\big),$$

where ${}_2F_1$ denotes the hypergeometric function. We note that

$$\phi_\lambda(x) = O(1 + x)e^{-\rho x}. \tag{2}$$

For $f \in L^1(\Delta)$, the Jacobi transform $\hat{f}(\lambda)$, $\lambda \in \mathbf{R}$, is defined by

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(x)\phi_\lambda(x)\Delta(x)\,dx. \tag{3}$$

The Paley-Wiener theorem asserts that the map $f \to \hat{f}$ is a bijection of the space of compactly supported smooth even functions on $\mathbf{R}$ onto the space of entire holomorphic even functions of exponential type, and the inverse transform is given by

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \hat{f}(\lambda)\phi_\lambda(x)|C(\lambda)|^{-2}d\lambda,$$

where $C(\lambda)$ is Harish-Chandra's C-function. Moreover, the Plancherel theorem asserts that the map $f \to \hat{f}$ extends to an isometry of $L^2(\Delta)$ onto $L^2(\mathbf{R}_+, |C(\lambda)|^{-2}d\lambda)$:

$$\int_0^\infty |f(x)|^2\Delta(x)dx = \int_0^\infty |\hat{f}(\lambda)|^2|C(\lambda)|^{-2}\,d\lambda. \tag{4}$$

When $\alpha > -\frac{1}{2}$, as a function of λ, $\phi_\lambda(x)$ is the Fourier Cosine transform of a bounded function $A(\cdot, x)$, which is compactly supported on $[0, x]$:

$$\Delta(x)\phi_\lambda(x) = \int_0^x \cos\lambda y A(y, x)\, dy\,. \tag{5}$$

Then the Abel transform $W_+(f)$ of f is defined by

$$W_+(f)(x) = \int_x^\infty f(y)A(x, y)\, dy \tag{6}$$

for $x \in \mathbf{R}_+$.

EXAMPLE 1. (i) When $(\alpha, \beta) = (-\frac{1}{2}, -\frac{1}{2})$, $\Delta = 1$, $\rho = 0$, $\phi_\lambda(x) = \cos(\lambda x)$ and $C(\lambda) = 1$. Hence the Jacobi transform is nothing but the Fourier Cosine transform.

(ii) When $(\alpha, \beta) = (\frac{1}{2}, -\frac{1}{2})$ and $G = SO(3, 1)$, $\Delta(x) = 4(\sinh x)^2$, $\rho = 1$,

$$\phi_\lambda(x) = \frac{\sin\lambda x}{\lambda\sinh x} = \frac{1}{i\lambda}\frac{e^{i\lambda x}}{2\sinh x} + \frac{1}{-i\lambda}\frac{e^{-i\lambda x}}{2\sinh x}$$

and $C(\lambda) = (i\lambda)^{-1}$. Hence $A(y, x)$ in (5) is given by

$$A(x, y) = 4\sinh x \cdot \chi_{[0,x]}(y)\,.$$

Substituting $\phi_\lambda(x)\Delta(x)$ in (3) with (5) and changing the order of integrations, we see that for $f \in L^1(\Delta)$,

$$\hat{f}(\lambda) = \mathcal{F}_C(W_+(f))(\lambda)\,, \tag{7}$$

where $\mathcal{F}_C$ is the Fourier Cosine transform on $\mathbf{R}$.

We define the kernel function $K(x, y, z)$ as

$$\phi_\lambda(x)\phi_\lambda(y) = \int_0^\infty \phi_\lambda(z)K(x, y, z)\Delta(z)\, dz\,.$$

Then the generalized translation $T_x f$ of f is defined by

$$T_x f(y) = \int_0^\infty f(z)K(x, y, z)\Delta(z)\, dz$$

and the convolution of $f, g \in L^1(\Delta)$ is given by

$$f * g(x) = \int_0^\infty f(y)T_x g(y)\Delta(y)dy = \int_0^\infty\int_0^\infty f(y)g(z)K(x, y, z)\Delta(y)\Delta(z)\, dydz\,. \tag{8}$$

Similarly as the Euclidean Fourier transform, it follows that

$$\widehat{T_x f(\lambda)} = \phi_\lambda(x)\hat{f}(\lambda), \quad \widehat{f * g}(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda)\,.$$

Hence, it follows from (7) and the fact that $\mathcal{F}(e^{\rho x} f)(\lambda) = \mathcal{F}(f)(\lambda + i\rho)$ that

$$W_+(f * g) = W_+(f) \otimes W_+(g)\,,$$

$$e^{\rho x} W_+(f * g) = (e^{\rho x} W_+(f)) \otimes (e^{\rho x} W_+(g)), \tag{9}$$

where we regard each function as an even function on $\mathbf{R}$ and denote by $\otimes$ the Euclidean convolution on $\mathbf{R}$. To analyze the Abel transform W_+, Koornwinder [12] generalizes the classical Weyl type fractional operator as follows: Let $\sigma > 0$, $n = 0, 1, 2, \dots$ and $\mu > -n$. For a function F on $\mathbf{R}_+$, $W_\mu^\sigma(F)$ is defined by

$$W_\mu^\sigma(F)(s) = c_{\mu,n} \int_s^\infty \Big(\frac{d}{d(\cosh \sigma t)}\Big)^n F(t)(\cosh \sigma t - \cosh \sigma s)^{\mu+n-1} d(\cosh \sigma t), \tag{10}$$

where $c_{\mu,n} = \frac{(-1)^n}{\Gamma(\mu+1)}$. By using generalized Weyl type fractional operators the Abel transform $W_+(f)$ is given as a composition of W_μ^σ:

$$F = W_+(f) = 2^{3\alpha+\frac{3}{2}} W_{\alpha-\beta}^1 \circ W_{\beta+1/2}^2(f). \tag{11}$$

Therefore, the inverse transform W_- of W_+ is given by

$$f = W_-(F) = 2^{-(3\alpha+\frac{3}{2})} W_{-(\beta+1/2)}^2 \circ W_{-(\alpha-\beta)}^1(F).$$

3. A version of the inverse Abel transform

In [10] we compare W_μ^σ (see (10)) with the classical Weyl type fractional operator $W_\mu^{\mathbf{R}}$:

$$W_\mu^{\mathbf{R}}(F)(s) = c_{\mu,n} \int_s^\infty \Big(\frac{d}{dt}\Big)^n F(t)(t-s)^{\mu+n-1}\, dt$$

and we can rewrite the inverse Abel transform W_- in terms of $W_\mu^{\mathbf{R}}$. Actually, by letting $\nu = \alpha + \frac{1}{2}$ and $\nu' = \alpha - \beta$ in [10], Theorem 3.6 or by replacing F with $e^{\rho x} F$ in [10], Corollary 3.7, we can deduce the following formula: Let $\beta + \frac{1}{2} = n + \mu$ and $\alpha - \beta = n' + \mu'$, where $n, n' \in \mathbf{Z}$ and $0 \le \mu, \mu' < 1$, and put

$$\begin{aligned} \Gamma_0 &= \{k + \mu + \mu' | k \in \mathbf{Z}, 1_{n+n'} \le k \le n + n'\}, \\ \Gamma_1 &= \{k, k+\mu, k+\mu' | k \in \mathbf{Z}, 1_{n+n'} \le k \le n + n'\}, \end{aligned} \tag{12}$$

where $1_n = 1$ if $n \ge 1$ and $1_n = 0$ if $n = 0$. Then for $F = W_+(f)$,

$$\begin{aligned} f(x) \sim \frac{e^{\rho x}}{\Delta(x)} \Big(&\sum_{\gamma \in \Gamma_0} (\tanh x)^\gamma W_{-\gamma}^{\mathbf{R}}(F)(x) \\ &+ \sum_{\gamma \in \Gamma_1} (\tanh x)^\gamma \int_x^\infty W_{-\gamma}^{\mathbf{R}}(F)(s) A_\gamma(x,s)\, ds \Big), \end{aligned} \tag{13}$$

where $A_\gamma(x,s)$ is of the form $A_\gamma(x,s) = Q_\gamma(x,s) Z_\gamma(s-x)$ and $A_\gamma, Q_\gamma, Z_\gamma$ satisfy the following estimates: For $\gamma \in \Gamma_1$, there exists $0 < \xi_\gamma < 1$ such that

(i) $|Z_\gamma(u)| \le c(\tanh u) u^{-(1+\xi_\gamma)}$ for $u > 0$,

$$\text{(ii)}\quad |Q^{\sigma}_{\gamma}(x,s)| \le c\frac{(\tanh x)^{\xi_\gamma}}{\tanh s} \quad \text{for } s > x\,,$$

$$\text{(iii)}\quad \int_0^s |A_\gamma(x,s)|\,dx \le c \text{ for all } s > 0\,, \tag{14}$$

$$\text{(iv)}\quad \int_x^\infty |A_\gamma(x,s)|\,ds \le c \quad \text{for all } x > 0\,.$$

Moreover, checking the process for the proof of (13) (see [10]), we may suppose that ξ_γ satisfies $\gamma + \xi_\gamma \le \alpha + \frac{1}{2}$.

PROPOSITION 1. *Let notations be as above. The integral term in* (13) *can be rewritten as*

$$\int_x^\infty W^{\mathbf{R}}_{-\gamma}(F)(s)A_\gamma(x,s)\,ds = \int_x^\infty W^{\mathbf{R}}_{-(\gamma+\xi_\gamma)}(F)(s)\widetilde{A}_\gamma(x,s)\,ds\,,$$

where

$$|\widetilde{A}_\gamma(x,s)| \le c(\tanh x)^{\xi_\gamma - 1} \quad \textit{for } x \le s\,.$$

PROOF. Clearly, $\widetilde{A}_\gamma(x,s)$ is given by $\widetilde{W}^{\mathbf{R}}_{\xi_\gamma}(\chi^c_x A_\gamma(x,\cdot))(s)$, where χ^c_x is the characteristic function of $[x,\infty)$ and $\widetilde{W}^{\mathbf{R}}_{\xi_\gamma}$ is the dual operator of $W^{\mathbf{R}}_{\xi_\gamma}$. Since

$$(\tanh x)x^{-(1+\xi_\gamma)} \le x^{-\xi_\gamma}\,,$$

it follows that

$$\begin{aligned}|\widetilde{W}^{\mathbf{R}}_{\xi_\gamma}(\chi^c_x A_\gamma(x,\cdot))(s)| &\le c\int_x^s \frac{(\tanh x)^{\xi_\gamma}}{\tanh t}(t-x)^{-\xi_\gamma}(s-t)^{\xi_\gamma-1}\,dt \\ &\sim (\tanh x)^{\xi_\gamma-1}\int_x^s (t-x)^{-\xi_\gamma}(s-t)^{\xi_\gamma-1}\,dt\,.\end{aligned}$$

Since the last integral is bounded, the desired estimate follows. □

PROPOSITION 2. *Let the notations be as above, and suppose that f is supported on* $\{x \ge 1\}$. *Then*

(i) *If $\alpha > \frac{1}{2}$ or $(\alpha,\beta) = (\frac{1}{2}, \pm\frac{1}{2})$, then we may suppose that $\gamma \ge 1$ for all $\gamma \in \varGamma_0 \cup \varGamma_1$ in* (13).

(ii) *If $\alpha = \frac{1}{2}$ and $-\frac{1}{2} < \beta < \frac{1}{2}$, then* (13) *can be rewritten as*

$$f(x) \sim \frac{e^{\rho x}}{\varDelta(x)}\Big(F'(x) + \int_x^\infty F''(s)B(x,s)\,ds\Big)\,, \tag{15}$$

where $|B(x,s)| \le c$ for all $x \ge 1$.

(iii) *If* $-\frac{1}{2} < \alpha < \frac{1}{2}$, *then* (13) *can be rewritten as*

$$f(x) \sim \frac{e^{\rho x}}{\Delta(x)}\Big(W^{\mathbf{R}}_{-(\alpha+\frac{1}{2})}(F)(x) + \int_x^\infty W^{\mathbf{R}}_{-2\alpha}F(s)C(x,s)\,ds\Big), \tag{16}$$

where $\int_x^\infty |C(x,s)|ds \le c$ *for all* $x \ge 1$.

PROOF. Since $f(x)$ is supported on $\{x \ge 1\}$, we may ignore the Q^σ_μ-functions in (14) in the following arguments.

(i) We assume that $\alpha > \frac{1}{2}$ or $(\alpha,\beta) = \left(\frac{1}{2}, \pm\frac{1}{2}\right)$. The domain of (α,β) is a union of the following sets of (α,β): (1) $\beta + \frac{1}{2} \ge 1$, (2) $\alpha - \beta \ge 1$, (3) $\alpha > \frac{1}{2}, \beta + \frac{1}{2} < 1, \alpha - \beta < 1$. Hence, if we denote $\beta + \frac{1}{2} = n + \mu$ and $\alpha - \beta = n' + \mu'$, where $n, n' = 0, 1, 2, \ldots$ and $0 \le \mu, \mu' < 1$, the above conditions are equivalent to (1) $n \ge 1$, (2) $n' \ge 1$, (3) $\mu + \mu' > 1$ if $n = n' = 0$. When $n = n' = 0$, we can deduce that the integral terms in (13) consist of

$$\int_x^\infty F(s)A_0(x,s)\,ds\,,$$

$$\int_x^\infty W^{\mathbf{R}}_{-\gamma}(F)(s)A_\gamma(x,s)\,ds,\ \gamma = \mu, \mu'\,, \tag{17}$$

where the Z-functions in (14) satisfy, by letting $\gamma' = \mu'$ if $\gamma = \mu$ and $\gamma' = \mu$ if $\gamma = \mu'$,

$$|Z_0(x)| \le c(\tanh x)x^{-1-(\mu+\mu')}\,,$$

$$|Z_\gamma(x)| \le c(\tanh x)x^{-(1+\gamma')}\,.$$

We rewrite the above integrals as

$$\int_x^\infty F'(s)W^{\mathbf{R}}_1(A_0(x,\cdot))(s)\,ds\,,$$

$$\int_x^\infty F'(s)W^{\mathbf{R}}_{1-\gamma}(A_\gamma(x,\cdot))(s)\,ds,\ \gamma = \mu, \mu'\,.$$

Then the renewed Z-functions are dominated by $(\tanh x)x^{-(\mu+\mu')}$. Since $1 < \mu + \mu' < 2$, $W^{\mathbf{R}}_1(A_0(x,\cdot))(s)$ and $W^{\mathbf{R}}_{1-\gamma}(A_\gamma(x,\cdot))(s)$, $\gamma = \mu, \mu'$, still satisfy (14). Therefore, we may suppose that $\gamma \ge 1$ for $\gamma \in \Gamma_0 \cup \Gamma_1$.

(ii) We note that $\mu + \mu' = \alpha + \frac{1}{2} = 1$. We recall the process in [10], §3 to deduce (13). Then integration by parts yields the desired result.

(iii) We note that $\mu + \mu' = \alpha + \frac{1}{2} < 1$. We rewrite the integral term (17) as

$$\int_x^\infty W^{\mathbf{R}}_{-2\alpha}F(s)\widetilde{W}^{\mathbf{R}}_{2\alpha-\gamma}(A_\gamma(x,\cdot))(s)\,ds, \quad \gamma = \mu, \mu'\,.$$

Then the corresponding Z-functions are dominated by $(\tanh x)x^{-1+(\alpha-\frac{1}{2})}$. Since $-1 < \alpha - \frac{1}{2} < 0$, $\widetilde{W}^{\mathbf{R}}_{2\alpha-\gamma}(A_\gamma(x,\cdot))(s)$ satisfies the desired condition. □

4. Key estimates

We shall obtain some L^p estimates of $W^{\mathbf{R}}_{-\gamma}(F)$, which will be used in the proofs of Theorems 1, 2 and 3 (see §6 and §7).

PROPOSITION 3. *Let $F = W_+(f)$ for $f \in L^1(\Delta)$. Then*

(i) *If $0 \le \gamma < \alpha + \frac{1}{2}$, then*

$$\|W^{\mathbf{R}}_{-\gamma}(F)\|_{L^\infty(\mathbf{R}_+)} \le c\|f\|_{L^1((\tanh x)^{2\alpha-\gamma}e^{\rho x})}\,.$$

(ii) *We have*

$$\|W^{\mathbf{R}}_{-(\alpha+\frac{1}{2})}(F)\|_{L^\infty(\mathbf{R}_+)} \le c\|f\|_{L^1((\tanh x)^{\alpha+\frac{3}{2}}e^{\rho x})} + \|f'\|_{L^1((\tanh x)^{\alpha+\frac{1}{2}}e^{\rho x})}\,.$$

(iii) *If $\gamma > 0$, then*

$$\|W^{\mathbf{R}}_{\gamma}(F)\|_{L^\infty(\mathbf{R}_+)} \le c\|f\|_{L^1((\tanh x)^{2\alpha}x^\gamma e^{\rho x})}\,.$$

PROOF. We use (6) to obtain these estimates. The explicit form of $A(x,y)$ is given by [12], (2.19), from which $A(x,y)$ satisfies

$$A(x,y) = R(x,y)\sinh y(\cosh y - \cosh x)^{\alpha-\frac{1}{2}} \tag{18}$$

and for $k = 0, 1$,

$$\left|\left(\frac{d}{dy}\right)^k R(x,y)\right| \le ce^{(\rho-(\alpha+\frac{1}{2}))y}(\tanh y)^k$$

for $y \ge x$. Then, it follows that

$$\begin{aligned} F(x) &= \int_x^\infty f(y)R(x,y)\sinh y(\cosh y - \cosh x)^{\alpha-\frac{1}{2}}dy \\ &= c\int_x^\infty \frac{d}{dy}\big(f(y)R(x,y)\big)(\cosh y - \cosh x)^{\alpha+\frac{1}{2}}dy\,. \end{aligned}$$

We note that $\cosh y - \cosh x \sim e^y \tanh(y^2 - x^2)$. Therefore, if $0 \le \gamma < \alpha + \frac{1}{2}$, then it follows that

$$\begin{aligned} |W^{\mathbf{R}}_{-\gamma}(F)| &= c\left|\int_x^\infty f(y)R(x,y)\frac{d}{dy}W^{\mathbf{R}}_{-\gamma}\big((\cosh y - \cosh\cdot)^{\alpha+\frac{1}{2}}\big)(x)\,dy\right| \\ &\le c\int_x^\infty |f(y)||R(x,y)|e^{(\alpha+\frac{1}{2})y}(\tanh y)^{2\alpha-\gamma}\,dy \\ &\le c\int_0^\infty |f(y)|(\tanh y)^{2\alpha-\gamma}e^{\rho y}\,dy\,. \end{aligned}$$

Moreover, if $\gamma = \alpha + \frac{1}{2}$, then

$$|W^{\mathbf{R}}_{-\gamma}(F)| = c\Big|\int_x^\infty \frac{d}{dy}\Big(f(y)R(x,y)\Big)e^{(\alpha+\frac{1}{2})y}(\tanh y)^{\alpha+\frac{1}{2}}\,dy\Big|$$
$$\leq c\int_0^\infty \big(|f'(y)|(\tanh y)^{\alpha+\frac{1}{2}} + |f(y)|(\tanh y)^{\alpha+\frac{3}{2}}\big)e^{\rho y}\,dy\,.$$

Hence (i) and (ii) follow. As for (iii), since $|A(x,y)| \leq ce^{\rho y}(\tanh y)^{2\alpha}$, it follows that for $\gamma > 0$,

$$|W^{\mathbf{R}}_{\gamma}(F)(x)| \leq \int_x^\infty \Big(\int_t^\infty |f(s)|A(t,s)ds\Big)(t-x)^{\gamma-1}dt$$
$$= \int_x^\infty |f(s)|\Big(\int_x^s A(t,s)(t-x)^{\gamma-1}dt\Big)\,ds$$
$$\leq \int_x^\infty |f(s)|e^{\rho s}(\tanh s)^{2\alpha}\Big(\int_x^s (t-x)^{\gamma-1}dt\Big)\,ds\,.$$

Hence the desired result follows. □

To estimate the L^1-norm of $W^{\mathbf{R}}_{-\gamma}(F)$, we first prepare the following lemma.

LEMMA 1. *Let notations be as above and put* $\gamma = \mu + n$, $0 \leq \mu < 1$, $n = 0, 1, 2, \ldots$.
(i) *Let* $\delta > 0$ *and* ε *satisfy*

$$\begin{cases} \varepsilon > -1, & \mu = 0\,, \\ \varepsilon > 0, & \mu > 0\,. \end{cases}$$

Then

$$\|W^{\mathbf{R}}_{-\gamma} \circ W^{\sigma}_{\gamma}(f)\|_{L^1((\tanh x)^{\varepsilon}e^{\delta x})} \leq c\|f\|_{L^1((\tanh x)^{\varepsilon+\gamma}e^{(\sigma\gamma+\delta)x})}\,.$$

(ii) *Let* $-\sigma\gamma + \delta > 0$ *and* ε *satisfy*

$$\begin{cases} \varepsilon > -1, & \gamma = 0\,, \\ \varepsilon > \gamma, & 0 < \gamma < 1\,, \\ \varepsilon > 2\gamma - 2, & 0 < \gamma,\ \mu = 0\,, \\ \varepsilon > 2\gamma - 1 - \mu, & 0 < \gamma,\ \mu > 0\,. \end{cases}$$

Then

$$\|W^{\sigma}_{-\gamma} \circ W^{\mathbf{R}}_{\gamma}(f)\|_{L^1((\tanh x)^{\varepsilon}e^{\delta x})} \leq c\|f\|_{L^1((\tanh x)^{\varepsilon-\gamma}e^{(-\sigma\gamma+\delta)x})}\,.$$

PROOF. We may suppose that $\alpha + \frac{1}{2} > 0$. When γ is an integer (i.e., $\mu = 0$), the lemma is obvious from the definitions of $W^{\mathbf{R}}_{\pm\gamma}$ and $W^{\sigma}_{\pm\gamma}$. Next, let us suppose that $0 < \gamma < 1$. As for (i), we note that $W^{\mathbf{R}}_{-\gamma}(f)(x)$ can be rewritten as

$$\begin{aligned}
&\int_x^\infty f'(s)(s-x)^{-\gamma}\,ds\\
&=\int_x^\infty f'(s)\Big(\frac{\cosh\sigma s-\cosh\sigma x}{s-x}\Big)^{\gamma}(\cosh\sigma s-\cosh\sigma x)^{-\gamma}\,ds\\
&=\int_x^\infty f'(s)(\sigma\sinh\sigma x)^{\gamma}(\cosh\sigma s-\cosh\sigma x)^{-\gamma}\,ds\\
&\quad+\int_x^\infty f'(s)\Big(\Big(\frac{\cosh\sigma s-\cosh\sigma x}{s-x}\Big)^{\gamma}-(\sigma\sinh\sigma x)^{\gamma}\Big)(\cosh\sigma s-\cosh\sigma x)^{-\gamma}\,ds\\
&=(\sigma\sinh\sigma x)^{\gamma}W^{\sigma}_{-\gamma}(f)(x)+\int_x^\infty f(s)B_\gamma(x,s)\,ds\,,
\end{aligned}$$

where, by letting

$$H_x(s)=\frac{\cosh\sigma s-\cosh\sigma x}{s-x}\,,$$

we put

$$B_\gamma(x,s)=\frac{d}{ds}\big((H_x(s)^{\gamma}-H_x(x)^{\gamma})H_x(s)^{-\gamma}(s-x)^{-\gamma}\big)\,.$$

Here we used the fact that the inside of the parentheses in $B_\gamma(x,s)$ is equal to 0 at $s=x$. Then it follows that

$$\begin{aligned}
W^{\mathbf{R}}_{-\gamma}(W^{\sigma}_{\gamma}(f))(x)&=(\sigma\sinh\sigma x)^{\gamma}f(x)+\int_x^\infty W^{\sigma}_{\gamma}(f)(s)B_\gamma(x,s)\,ds\\
&=(\sigma\sinh\sigma x)^{\gamma}f(x)+\int_x^\infty f(t)(\widetilde{W}^{\sigma}_{\gamma,2}B_\gamma)(x,t)\,dt\,,
\end{aligned}$$

where 2 in the bottom suffix of $\widetilde{W}^{\sigma}_{\gamma,2}$ implies that we apply $\widetilde{W}^{\sigma}_{\gamma}$ to the second variable, that is,

$$\begin{aligned}
(\widetilde{W}^{\sigma}_{\gamma,2}B_\gamma)(x,t)&=\widetilde{W}^{\sigma}_{\gamma}(B_\gamma(x,\cdot))(t)\\
&=\sinh\sigma t\int_x^t B_\gamma(x,s)(\cosh\sigma t-\cosh\sigma s)^{\gamma-1}\,ds\,.
\end{aligned}$$

Since

$$|B_\gamma(x,s)|\le cH'_x(s)H_x(s)^{-1}(s-x)^{-\gamma}\le c(s-x)^{-\gamma}(\tanh x)^{-1}$$

for $s>x$, $(\widetilde{W}^{\sigma}_{\gamma,2}B_\gamma)(x,t)$ is dominated by

$$\begin{aligned}
&c(\sinh\sigma t)(\tanh x)^{-1}\int_x^t(s-x)^{-\gamma}(\cosh\sigma t-\cosh\sigma s)^{\gamma-1}\,ds\\
&\le c(\sinh\sigma t)(\tanh x)^{-1}e^{\sigma(\gamma-1)t}(\tanh t)^{\gamma-1}\int_x^t(s-x)^{-\gamma}(t-s)^{\gamma-1}\,ds
\end{aligned}$$

$$\leq c(\tanh t)^{\gamma}(\tanh x)^{-1}e^{\sigma\gamma t}\,.$$

Therefore, since $\varepsilon - 1 > -1$, it follows that

$$\begin{aligned}
&\int_0^\infty |W^{\mathbf{R}}_{-\gamma}(W^{\sigma}_{\gamma}(f))(x)|(\tanh x)^{\varepsilon}e^{\delta x}\,dx\\
&\leq c\int_0^\infty |f(x)|(\tanh x)^{\varepsilon+\gamma}e^{(\sigma\gamma+\delta)x}\,dx\\
&\qquad\qquad + c\int_0^\infty |f(t)|(\tanh t)^{\gamma}e^{\gamma\sigma t}\Big(\int_0^t(\tanh x)^{\varepsilon-1}e^{\delta x}\,dx\Big)\,dt\\
&\leq c\int_0^\infty |f(x)|(\tanh x)^{\varepsilon+\gamma}e^{(\sigma\gamma+\delta)x}\,dx\,.
\end{aligned}$$

As for (ii), similarly as above, we can deduce that

$$W^{\sigma}_{-\gamma}(f)(x) = (\sigma\sinh\sigma x)^{-\gamma}W^{\mathbf{R}}_{-\gamma}(f)(x) + \int_x^\infty f(s)C_{\gamma}(x,s)\,ds\,,$$

where we put

$$C_{\gamma}(x,s) = \frac{d}{ds}\big((H_x^{-\gamma}(s) - H_x^{-\gamma}(x))H_x^{\gamma}(s)(\cosh\sigma s - \cosh\sigma x)^{-\gamma}\big)\,.$$

Hence it follows that

$$\begin{aligned}
W^{\sigma}_{-\gamma}(W^{\mathbf{R}}_{\gamma}(f))(x) &= (\sigma\sinh\sigma x)^{-\gamma}f(x) + \int_x^\infty W^{\mathbf{R}}_{\gamma}(f)(s)C_{\gamma}(x,s)\,ds\\
&= (\sigma\sinh\sigma x)^{-\gamma}f(x) + \int_x^\infty f(t)(\widetilde{W}^{\mathbf{R}}_{\gamma,2}C_{\gamma})(x,t)\,dt\,,
\end{aligned}$$

where

$$(\widetilde{W}^{\mathbf{R}}_{\gamma,2}C_{\gamma})(x,t) = \widetilde{W}^{\mathbf{R}}_{\gamma}(C_{\gamma}(x,\cdot))(t) = \int_x^t C_{\gamma}(x,s)(t-s)^{\gamma-1}\,ds\,.$$

Since

$$|C_{\gamma}(x,s)| \leq c(\tanh x)^{-1}(\cosh\sigma s - \cosh\sigma x)^{-\gamma}$$

for $s > x$, $(\widetilde{W}^{\mathbf{R}}_{\gamma,2}C_{\gamma})(x,t)$ is dominated by

$$(\tanh x)^{-1}\int_x^t(\cosh\sigma s - \cosh\sigma x)^{-\gamma}(t-s)^{\gamma-1}\,ds \leq c(\tanh x)^{-1-\gamma}e^{-\sigma\gamma x}\,.$$

Therefore, since $\varepsilon - 1 - \gamma > -1$ and $-\sigma\gamma + \delta > 0$, it follows that

$$\int_0^\infty \big|W^{\sigma}_{-\gamma}(W^{\mathbf{R}}_{\gamma}(f))(x)\big|(\tanh x)^{\varepsilon}e^{\delta x}\,dx$$

$$\leq c\int_0^\infty |f(x)|(\tanh x)^{\varepsilon-\gamma}e^{(-\sigma\gamma+\delta)x}dx$$
$$+ c\int_0^\infty |f(t)|\Big(\int_0^t (\tanh x)^{\varepsilon-1-\gamma}e^{(-\sigma\gamma+\delta)x}\,dx\Big)dt$$
$$\leq c\int_0^\infty |f(x)|(\tanh x)^{\varepsilon-\gamma}e^{(-\sigma\gamma+\delta)x}\,dx\,.$$

Hence we obtain the desired estimate for $0<\gamma<1$.

Finally let $\gamma=n+\mu>1$ and $\mu>0$. As for (i), we note that

$$\begin{aligned}
&W^{\mathbf{R}}_{-\gamma}\circ W^\sigma_\gamma(f)(x)\\
&=W^{\mathbf{R}}_{-\mu}\circ W^{\mathbf{R}}_{-n}\circ W^\sigma_n\circ W^\sigma_\mu(f)(x)\\
&=W^{\mathbf{R}}_{-\mu}\circ W^{\mathbf{R}}_{-n}\Big(\int_x^\infty W^\sigma_\mu(f)(s)(\cosh\sigma s-\cosh\sigma x)^{n-1}\,d\cosh\sigma s\Big)\\
&=W^{\mathbf{R}}_{-\mu}\Big(cW^\sigma_\mu(f)(x)(\sinh\sigma x)^n+c\int_x^\infty W^\sigma_\mu(f)(s)B(x,s)\,ds\Big)\,,
\end{aligned} \tag{19}$$

where

$$B(s,x)=\Big(\frac{d}{dx}\Big)^n(\cosh\sigma s-\cosh\sigma x)^{n-1}\sinh\sigma s\,.$$

The first term in (19) can be written as

$$\begin{aligned}
&W^{\mathbf{R}}_{-\mu}(W^\sigma_\mu(f)(x)(\sinh\sigma x)^n)\\
&=c\int_x^\infty (W^\sigma_\mu(f)(s)(\sinh\sigma s)^n)'(s-x)^{-\mu}ds\\
&=c\int_x^\infty \Big(\frac{d}{ds}W^\sigma_\mu(f)(s)(\sinh\sigma x)^n\\
&\qquad+\frac{d}{ds}(W^\sigma_\mu(f)(s)(\sinh\sigma s)^n-(\sinh\sigma x)^n)\Big)(s-x)^{-\mu}\,ds\\
&=cW^{\mathbf{R}}_{-\mu}\circ W^\sigma_\mu(f)(x)(\sinh\sigma x)^n\\
&\qquad+c\int_x^\infty W^\sigma_\mu(f)(s)((\sinh\sigma s)^n-(\sinh\sigma x)^n)(s-x)^{-\mu-1}\,ds\\
&=I_{11}(x)+I_{12}(x)
\end{aligned}$$

Clearly, it follows from the previous result for $0<\gamma<1$ that for $\varepsilon,\delta>0$,

$$\int_0^\infty |I_{11}(x)|(\tanh x)^\varepsilon e^{\delta x}dx\leq c\int_0^\infty |f(x)|(\tanh x)^{\varepsilon+\gamma}e^{(\sigma\gamma+\delta)x}dx\,.$$

As for I_{12}, we note that I_{12} is estimated as

$$\int_x^\infty |f(t)|\Big(\int_x^t (\cosh\sigma t-\cosh\sigma s)^{\mu-1} \times \frac{(\sinh\sigma s)^n-(\sinh\sigma x)^n}{s-x}(s-x)^{-\mu}\,ds\Big)\,d\cosh\sigma t$$
$$\le \int_x^\infty |f(t)|e^{\sigma\gamma t}(\tanh t)^{\gamma-1}\Big(\int_x^t (t-s)^{\mu-1}(s-x)^{-\mu}\,ds\Big)\,dt$$
$$\le \int_x^\infty |f(t)|e^{\sigma\gamma t}(\tanh t)^{\gamma-1}\,dt\,.$$

Hence for $\varepsilon,\delta>0$,

$$\int_0^\infty |I_{12}(x)|(\tanh x)^{\varepsilon}e^{\delta x}\,dx \le c\int_0^\infty |f(x)|(\tanh x)^{\varepsilon+\gamma}e^{(\sigma\gamma+\delta)x}\,dx\,.$$

Then the second term in (19) can be written as

$$W^{\mathbf{R}}_{-\mu}\Big(\int_x^\infty W^\sigma_\mu(f)(s)B(x,s)\,ds\Big)$$
$$=\int_x^\infty \frac{d}{dt}\Big(\int_t^\infty W^\sigma_\mu(f)(s)B(t,s)ds\Big)(t-x)^{-\mu}\,dt$$
$$=\int_x^\infty \Big(W^\sigma_\mu(f)(t)B(t,t)+\int_t^\infty W^\sigma_\mu(f)(s)\frac{d}{dt}B(t,s)ds\Big)(t-x)^{-\mu}\,dt$$
$$=c\int_x^\infty W^\sigma_\mu(f)(t)B(t,t)(t-x)^{-\mu}dt + c\int_x^\infty\Big(\int_t^\infty W^\sigma_\mu(f)(s)\frac{d}{dt}B(t,s)ds\Big)(t-x)^{-\mu}\,dt$$
$$=I_{21}(x)+I_{22}(x)\,.$$

Since $B(t,t)=c(\sinh t)^{n-1}\cosh t$, I_{21} is dominated by

$$\int_x^\infty |f(s)|\Big(\int_x^s (\sinh t)^{n-1}\cosh t \times(\cosh\sigma s-\cosh\sigma t)^{\mu-1}(t-x)^{-\mu}dt\Big)\,d\cosh\sigma s$$
$$\le c\int_x^\infty |f(s)|e^{\sigma\gamma s}(\tanh s)^{\gamma-1}\Big(\int_x^s (s-t)^{\mu-1}(t-x)^{-\mu}dt\Big)\,ds$$
$$\le c\int_x^\infty |f(s)|e^{\sigma\gamma s}(\tanh s)^{\gamma-1}\,ds\,.$$

Hence for $\varepsilon,\delta>0$,

$$\int_0^\infty |I_{21}(x)|(\tanh x)^{\varepsilon}e^{\delta x}dx \le c\int_0^\infty |f(x)|(\tanh x)^{\varepsilon+\gamma}e^{(\sigma\gamma+\delta)x}\,dx\,. \tag{20}$$

As for I_{22}, we note that $|\frac{d}{dt}B(t,s)| \le ce^{\sigma ns}(\tanh s)^{n-2}$. Hence I_{22} is dominated by

$$\begin{aligned}
&\int_x^\infty |W_\mu^\sigma(f)(s)|\Big(\int_x^s \frac{d}{dt}B(t,s)(t-x)^{-\mu}\,dt\Big)\,ds \\
&\le \int_x^\infty |W_\mu^\sigma(f)(s)|e^{\sigma ns}(\tanh s)^{n-\mu-1}\,ds \\
&\le \int_x^\infty |f(t)|\Big(\int_x^t e^{\sigma ns}(\tanh s)^{n-\mu-1}(\cosh\sigma t-\cosh\sigma s)^{\mu-1}\,ds\Big)\,d\cosh\sigma t \\
&\le \int_x^\infty |f(t)|e^{\sigma\gamma t}(\tanh t)^{\gamma-1}\Big(\int_x^t s^{-\mu}(t-s)^{\mu-1}ds\Big)\,dt \\
&\le \int_x^\infty |f(t)|e^{\sigma\gamma t}(\tanh t)^{\gamma-1}\,dt\,.
\end{aligned}$$

Therefore, I_{22} also satisfies (20). As for (ii), we note that

$$\begin{aligned}
&W_{-\gamma}^\sigma \circ W_\gamma^{\mathbf{R}}(f)(x) \\
&= W_{-\mu}^\sigma \circ W_{-n}^\sigma \circ W_n^{\mathbf{R}} \circ W_\mu^{\mathbf{R}}(f)(x) \\
&= W_{-\mu}^\sigma \circ W_{-n}^\sigma\Big(\int_x^\infty W_\mu^{\mathbf{R}}(f)(s)(s-x)^{n-1}\,ds\Big) \\
&= W_{-\mu}^\sigma\Big(cW_\mu^{\mathbf{R}}(f)(x)(\sinh\sigma x)^{-n} \\
&\quad + \sum_{k=1}^{n-1}\sum_{l=0}^{[\frac{n-k}{2}]+1} c_{kl}(\cosh\sigma x)^{n-k-2l}(\sinh\sigma x)^{-2n+k+2l}W_{\gamma-k}^{\mathbf{R}}(f)(x)\Big)\,,
\end{aligned} \tag{21}$$

where $[\cdot]$ is the Gauss symbol. The first term in (21) can be written as

$$\begin{aligned}
&W_{-\mu}^\sigma(W_\mu^{\mathbf{R}}(f)(x)(\sinh\sigma x)^{-n}) \\
&= c\int_x^\infty \frac{d}{d\cosh\sigma s}(W_\mu^{\mathbf{R}}(f)(s)(\sinh\sigma s)^{-n})(\cosh\sigma s-\cosh\sigma x)^{-\mu}\,d\cosh\sigma s \\
&= vc\int_x^\infty \Big(\frac{d}{ds}W_\mu^{\mathbf{R}}(f)(s)(\sinh\sigma x)^{-n} \\
&\qquad + \frac{d}{ds}(W_\mu^{\mathbf{R}}(f)(s)(\sinh\sigma s)^{-n}-(\sinh\sigma x)^{-n})\Big)(\cosh\sigma s-\cosh\sigma x)^{-\mu}\,ds \\
&= cW_{-\mu}^\sigma \circ W_\mu^{\mathbf{R}}(f)(x)(\sinh\sigma x)^{-n} \\
&\qquad + c\int_x^\infty W_\mu^{\mathbf{R}}(f)(s)((\sinh\sigma s)^{-n}-(\sinh\sigma x)^{-n}) \\
&\qquad\qquad \times(\cosh\sigma s-\cosh\sigma x)^{-\mu-1}\sinh\sigma s\,ds \\
&= J_{11}(x)+J_{12}(x)\,.
\end{aligned}$$

Clearly, it follows from the previous result for $0 < \gamma < 1$ that for $\varepsilon > \gamma$, $\delta > \sigma\gamma$,

$$\int_0^\infty |J_{11}(x)|(\tanh x)^\varepsilon e^{\delta x} dx \le c \int_0^\infty |f(x)|(\tanh x)^{\varepsilon-\gamma} e^{(-\sigma\gamma+\delta)x} dx .$$

As for J_{12}, we note that J_{12} is estimated as

$$\begin{aligned}
&\int_x^\infty |f(t)|\Big(\int_x^t (t-s)^{\mu-1} \frac{(\sinh\sigma s)^{-n} - (\sinh\sigma x)^{-n}}{s-x} \\
&\qquad\qquad \times \frac{s-x}{\cosh\sigma s - \cosh\sigma x}(\cosh\sigma s - \cosh\sigma x)^{-\mu} \sinh\sigma s \, ds\Big)\, dt \\
&\le e^{-\sigma\gamma x}(\tanh x)^{-\gamma-1} \int_x^\infty |f(t)|\Big(\int_x^t (t-s)^{\mu-1}(s-x)^{-\mu}\, ds\Big)\, dt \\
&\le e^{-\sigma\gamma x}(\tanh x)^{-\gamma-1} \int_x^\infty |f(t)|\, dt .
\end{aligned}$$

Here we used the fact that $\cosh\sigma s - \cosh\sigma x = 2\sinh\frac{\sigma(s+x)}{2}\sinh\frac{\sigma(s-x)}{2} \ge ce^{\sigma s}$ $(\tanh s)$ $\tanh(s-x)$. Since $-\gamma - 1 + \varepsilon > -\gamma - 1 + 2\gamma - 1 - \mu = n - 2 \ge -1$ for $n \ge 1$ and $-\sigma\gamma + \delta > 0$, it follows that

$$\int_0^\infty |I_{12}(x)|(\tanh x)^\varepsilon e^{\delta x} dx \le c \int_0^\infty |f(x)|(\tanh x)^{\varepsilon-\gamma} e^{(-\sigma\gamma+\delta)x}\, dx .$$

To estimate the second term in (21), we note that

$$\begin{aligned}
&W^\sigma_{-\mu}\big((\cosh\sigma x)^{n-k-2l}(\sinh\sigma x)^{-2n+k+2l} W^{\mathbf{R}}_{\gamma-k}(f)\big)(x) \\
&= c\int_x^\infty \frac{d}{d\sinh\sigma s}\big((\cosh\sigma x)^{n-k-2l}(\sinh\sigma x)^{-2n+k+2l} W^{\mathbf{R}}_{\gamma-k}(f)\big)(s) \\
&\qquad\qquad \times (\cosh\sigma s - \cosh\sigma x)^{-\mu}\, d\sinh\sigma s \\
&= c\int_x^\infty \Big(\frac{d}{ds}\big((\cosh\sigma x)^{n-k-2l}(\sinh\sigma x)^{-2n+k+2l} W^{\mathbf{R}}_{\gamma-k}(f)\big)(s) \\
&\qquad + (\cosh\sigma x)^{n-k-2l}(\sinh\sigma x)^{-2n+k+2l} W^{\mathbf{R}}_{\gamma-k-1}(f)(s)\Big) \\
&\qquad\qquad \times (\cosh\sigma s - \cosh\sigma x)^{-\mu} ds \\
&= J_{21}(x) + J_{22}(x) .
\end{aligned}$$

J_{21} is dominated as

$$\begin{aligned}
&\int_x^\infty e^{-ns}(\tanh s)^{-2n+k-1}|W^{\mathbf{R}}_{\gamma-k}(f)(s)|(\cosh\sigma s - \cosh\sigma x)^{-\mu}\, ds \\
&\le \int_x^\infty |W^{\mathbf{R}}_{n-k}(f)(t)| \\
&\qquad \times \Big(\int_x^t (t-s)^{\mu-1} e^{-ns}(\tanh s)^{-2n+k-1}(\cosh\sigma s - \cosh\sigma x)^{-\mu} ds\Big)\, dt
\end{aligned}$$

$$\leq ce^{-\gamma x}(\tanh x)^{-2n+k-1-\mu}\int_x^\infty |W^{\mathbf{R}}_{n-k}(f)(t)|\Big(\int_x^t (t-s)^{\mu-1}(s-x)^{-\mu}ds\Big)\,dt$$
$$\leq ce^{-\gamma x}(\tanh x)^{-2n+k-1-\mu}\int_x^\infty |W^{\mathbf{R}}_{n-k}(f)(t)|\,dt\,.$$

Since $-2n+k-1-\mu+\varepsilon \geq -2n-\mu+\varepsilon > -1$ and $-\sigma\gamma+\delta > 0$, by repeating integration by parts, it follows that

$$\int_0^\infty |I_{21}(x)|(\tanh x)^\varepsilon e^{\delta x}\,dx \leq c\int_0^\infty |W^{\mathbf{R}}_{n-k}(f)(x)|(\tanh x)^{\varepsilon-2n+k-\mu}e^{(-\sigma\gamma+\delta)x}\,dx$$
$$\leq c\int_0^\infty |f(x)|(\tanh x)^{\varepsilon-\gamma}e^{(-\sigma\gamma+\delta)x}\,dx\,.$$

As for J_{22}, by changing k to $k+1$, we can deduce the same estimate. This completes the proof of Lemma 1. □

PROPOSITION 4. *Let $F = W_+(f)$ for $f \in L^1(\Delta)$. Then for $0 \leq \gamma \leq \alpha+\frac{1}{2}$ and $\delta > 0$,*

$$\|W^{\mathbf{R}}_{-\gamma}(F)\|_{L^1((\tanh x)^\gamma e^{\delta x})} \leq c\|f\|_{L^1((\tanh x)^{2\alpha+1}e^{(\rho+\delta)x})}\,.$$

Moreover, if $1 \leq \gamma < \alpha+\frac{1}{2}$ is an integer or $\gamma = 1$ if $\alpha = \frac{1}{2}$, then the above inequality holds for $\delta = 0$.

PROOF. We first consider the case that $\delta > 0$ and $\gamma = \alpha+\frac{1}{2}$. We recall that

$$W^{\mathbf{R}}_{-(\alpha+\frac{1}{2})}(F) = W^{\mathbf{R}}_{-(\alpha+\frac{1}{2})}\circ W_+(f) = W^{\mathbf{R}}_{-(\alpha+\frac{1}{2})}\circ W^1_{\alpha-\beta}\circ W^2_{\beta+\frac{1}{2}}(f)$$
$$= \Big(W^{\mathbf{R}}_{-(\alpha+\frac{1}{2})}W^1_{\alpha+\frac{1}{2}}\Big)\circ\Big(W^1_{-(\beta+\frac{1}{2})}W^{\mathbf{R}}_{\beta+\frac{1}{2}}\Big)\circ\Big(W^{\mathbf{R}}_{-(\beta+\frac{1}{2})}W^2_{\beta+\frac{1}{2}}\Big)(f)$$
$$= A\circ B\circ C(f)\,.$$

Then it follows from Lemma 1 that

$$\big\|W^{\mathbf{R}}_{-(\alpha+\frac{1}{2})}(F)\big\|_{L^1((\tanh x)^{\alpha+\frac{1}{2}}e^{\delta x})} = \|A\circ B\circ C(f)\|_{L^1((\tanh x)^{\alpha+\frac{1}{2}}e^{\delta x})}$$
$$\leq c\|B\circ C(f)\|_{L^1((\tanh x)^{2\alpha+1}e^{(\alpha+\frac{1}{2}+\delta)x})}$$
$$\leq c\|C(f)\|_{L^1((\tanh x)^{2\alpha+1-(\beta+\frac{1}{2})}e^{(\alpha-\beta+\delta)x})}$$
$$\leq c\|f\|_{L^1((\tanh x)^{2\alpha+1}e^{\rho x})}\,.$$

We suppose that $0 < \gamma < \alpha+\frac{1}{2}$ and put $\gamma = \alpha+\frac{1}{2}-\mu$, $\mu > 0$. Then it follows from the previous case that

$$\big\|W^{\mathbf{R}}_{-\gamma}(F)\big\|_{L^1((\tanh x)^\gamma e^{\delta x})} = \big\|W^{\mathbf{R}}_{\mu-(\alpha+\frac{1}{2})}(F)\big\|_{L^1((\tanh x)^{\alpha+\frac{1}{2}-\mu}e^{\delta x})}$$
$$\leq \big\|W^{\mathbf{R}}_{-(\alpha+\frac{1}{2})}(F)\big\|_{L^1((\tanh x)^{\alpha+\frac{1}{2}}e^{\delta x})}$$

$$\le c\|f\|_{L^1((\tanh x)^{2\alpha+1}e^{(\rho+\delta)x})} .$$

Hence we obtain the desired result for $\delta > 0$. Last we consider the case that γ is an integer. First we suppose that $1 \le \gamma = n < \alpha + \frac{1}{2}$. Since

$$\Big|\Big(\frac{d}{dx}\Big)^n A(x, y)\Big| \le ce^{(\rho-1)y}e^x(\tanh y)^{\alpha+\frac{1}{2}}(\tanh(y-x))^{\alpha-\frac{1}{2}-n} ,$$

it follows that

$$\begin{aligned}
&\int_0^\infty |W_{-n}^{\mathbf{R}}F(x)|(\tanh x)^n\,dx \\
&= \int_0^\infty \Big|\int_x^\infty f(y)\Big(\frac{d}{dx}\Big)^n A(x, y)\,dy\Big|(\tanh x)^n\,dx \\
&\le \int_0^\infty |f(y)|\Big(\int_0^y \Big|\Big(\frac{d}{dx}\Big)^n A(x, y)\Big|(\tanh x)^n dx\Big)\,dy \\
&\le c\int_0^\infty |f(y)|e^{(\rho-1)y}(\tanh y)^{\alpha+\frac{1}{2}}\Big(\int_0^y e^x(\tanh(y-x))^{\alpha-\frac{1}{2}-n}(\tanh x)^n dx\Big)\,dy \\
&\le c\int_0^\infty |f(y)|e^{\rho y}(\tanh y)^{2\alpha+1}\,dy .
\end{aligned}$$

When $\gamma = 1$ and $\alpha = \frac{1}{2}$, we see from the explicit form of $A(x, y)$ that

$$\Big|\Big(\frac{d}{dx}\Big)A(x, y)\Big| \le ce^{(\rho-1)y}e^x\tanh y .$$

Hence the desired inequality similarly holds. This completes the proof of Proposition 4. □

5. Weak-L^1 functions

Let w be a positive measurable function on $\mathbf{R}_+$. We say that a function $f(x)$ on $\mathbf{R}_+$ satisfies the weak-$L^1(w)$ estimate provided that there exists a constant c such that

$$\lambda\int_{\{x\in\mathbf{R}_+|f(x)>\lambda\}} w(x)dx \le c .$$

We call such a function a weak-$L^1(w)$ function. Here we recall that some maximal functions of $f \in L^1(\mathbf{R})$ are weak-L^1 functions. For example, the classical Hardy-Littlewood maximal operator $M_{\mathrm{HL}}^{\mathbf{R}}$ on $\mathbf{R}$, which is defined by

$$M_{\mathrm{HL}}^{\mathbf{R}}F(x) = \sup_{0<t<\infty}\frac{1}{2t}\int_{x-t}^{x+t}|F(y)|\,dy ,$$

satisfies the so-called maximal theorem, that is, $M_{\mathrm{HL}}^{\mathbf{R}}$ is bounded from $L^p(\mathbf{R})$ to $L^p(\mathbf{R})$ for $1 < p \le \infty$ and $M_{\mathrm{HL}}^{\mathbf{R}}f$ is a weak-L^1 function for $f \in L^1(\mathbf{R})$. More generally, for a function

ϕ on $\mathbf{R}$, the radial maximal function $M_\phi^{\mathbf{R}} F$ is defined by

$$M_\phi^{\mathbf{R}} F(x) = \sup_{0<t<\infty} |F \otimes \phi_t(x)|,$$

where ϕ_t is the dilation of ϕ: $\phi_t(x) = \frac{1}{t}\phi(\frac{x}{t})$ and $\otimes$ is the Euclidean convolution. Since

$$M_{\mathrm{HL}}^{\mathbf{R}} F = \frac{1}{2} M_{\chi_{[-1,1]}}^{\mathbf{R}} |F|,$$

where χ_S is the characteristic function of $S \subset \mathbf{R}$, $M_{\chi_{[-1,1]}}^{\mathbf{R}}$ satisfies the maximal theorem. Furthermore, if $\phi \in \mathcal{S}(\mathbf{R})$, then $M_\phi^{\mathbf{R}}$ also satisfies the maximal theorem. We denote by $M_{\mathrm{HL}}^{\mathbf{R},0}$ and $M_\phi^{\mathbf{R},0}$ the local maximal operators, which are defined by replacing $\sup\limits_{0<t<\infty}$ in the above definitions by $\sup\limits_{0<t<1}$.

EXAMPLE 2. We shall give some examples of weak-$L^1(\Delta)$ functions on $\mathbf{R}_+$.

(1) Let $B(x) = \displaystyle\int_0^x \Delta(x)dx$. Then it is obvious that

$$f(x) = \frac{1}{B(x)}$$

is a weak-$L^1(\Delta)$ function on $\mathbf{R}_+$.

(2) Let $F \in L^1(\mathbf{R})$ and suppose that supp $F \subset [2, \infty)$. Then

$$f(x) = \frac{1}{\Delta(x)} M_{\mathrm{HL}}^{\mathbf{R},0} F(x)$$

is a weak-$L^1(\Delta)$ function on $\mathbf{R}_+$. Actually, we divide $\mathbf{R}_+$ as

$$\mathbf{R}_+ = \bigcup_{k=0}^{\infty} I_k,$$

where $I_k = [k, k+1]$. We let $f_k = f\chi_{I_k}$ and $F_k = F\chi_{I_k}$, where χ_{I_k} is the characteristic function of I_k. Since $M_{\mathrm{HL}}^{\mathbf{R},0}$ is local and F is supported on $[2, \infty)$, it follows that f is supported on $[1, \infty)$ and moreover,

$$\left\{ j \mid \mathrm{supp}(M_{\mathrm{HL}}^{\mathbf{R},0} F_j) \cap I_k \neq \emptyset \right\} = \{k-1, k, k+1\}$$

for $k \geq 1$. Hence, by noting that $\Delta(x)$ is increasing on $\mathbf{R}_+$ and $\Delta(x) \sim e^{2\rho x}$ for $x \geq 1$, we see that

$$\lambda \int_{\{x \in \mathbf{R}_+ | f(x) > \lambda\}} \Delta(x)\, dx = \lambda \sum_{k=1}^{\infty} \int_{\{x \in I_k | f_k(x) > \lambda\}} \Delta(x)\, dx$$

$$
\begin{aligned}
&\le 3\lambda \sum_{k=2}^{\infty} \Delta(k+1) \int_{\{x\in I_k \mid M_{\mathrm{HL}}^{\mathbf{R},0} F_k(x) > \frac{1}{3}\Delta(k)\lambda\}} dx \\
&\le 3^2 \sum_{k=2}^{\infty} \frac{\Delta(k+1)\lambda}{\Delta(k)\lambda} \|F_k\|_{L^1(\mathbf{R})} \\
&\le c\|F\|_{L^1(\mathbf{R})} .
\end{aligned}
$$

Therefore, $f(x)$ is a weak-$L^1(\Delta)$ function on $\mathbf{R}_+$.

(3) Let ϕ be a smooth function on $\mathbf{R}$ supported on $[-1, 1]$. Let $F \in L^1(\mathbf{R})$. We suppose that supp $F \subset [0, 2]$ and $W_{-\gamma}^{\mathbf{R}}(F) \in L^1((\tanh x)^\gamma)$ for $\gamma \ge 0$. Then

$$
f(x) = \frac{(\tanh x)^\gamma}{\Delta(x)} M_\phi^{\mathbf{R},0} W_{-\gamma}^{\mathbf{R}}(F)(x)
$$

is a weak-$L^1(\Delta)$ function on $\mathbf{R}_+$. To obtain this result, first we note that $f(x)$ is supported on $[0, 3]$ and for $0 \le x \le 3$,

$$
\begin{aligned}
f(x) &\le \frac{(\tanh x)^\gamma}{\Delta(x)} \sup_{\frac{x}{2}<t<1} W_{-\gamma}^{\mathbf{R}}(F) \otimes \phi_t(x) \\
&\quad + \frac{(\tanh x)^\gamma}{\Delta(x)} \sup_{0<t<\frac{x}{2}} \frac{\|\phi\|_\infty}{t} \int_{x-t}^{x+t} \left| W_{-\gamma}^{\mathbf{R}}(F)(y) \right| dy \\
&= I_1(x) + I_2(x) .
\end{aligned}
$$

As for I_1, we note that

$$
W_{-\gamma}^{\mathbf{R}}(F) \otimes \phi_t = F \otimes W_{-\gamma}^{\mathbf{R}}(\phi_t) = t^{-\gamma} F \otimes W_{-\gamma}^{\mathbf{R}}(\phi)_t .
$$

Since $\frac{x}{2} < t$, it follows that

$$
\begin{aligned}
|I_1(x)| &\le c \frac{(\tanh x)^\gamma}{\Delta(x)} \frac{\left\| W_{-\gamma}^{\mathbf{R}}(\phi) \right\|_{L^\infty(\mathbf{R})}}{t^{1+\gamma}} \|F\|_{L^1(\mathbf{R})} \\
&\le c \frac{\|F\|_{L^1(\mathbf{R})}}{|B(x)|} .
\end{aligned}
$$

By (1), $I_1(x)$ is a weak-$L^1(\Delta)$ function on $\mathbf{R}_+$. As for I_2, since $t \le 1$, we can apply a covering argument used in the proof of the weak-L^1 estimate for $M_{\mathrm{HL}}^{\mathbf{R}}$. When x belongs to $S_\lambda = \{z \mid \mathbf{R}_+ \mid I_2(z) > \lambda\}$, there exists t such that

$$
\int_{x-t}^{x+t} \left| W_{-\gamma}^{\mathbf{R}}(F)(y) \right| dy > ct \frac{\Delta(x)}{(\tanh x)^\gamma} \lambda .
$$

We note that, since $0 < x < 3$ and $t < \frac{x}{2}$, $\Delta(x) \sim (\tanh x)^{2\alpha+1}$ and $x + 2t \le 2x$. Then it follows that

$$\begin{aligned}|B(x,2t)| = &\int_{x-2t}^{x+2t} \Delta(y)dy \\ \leq &\Delta(x+2t)t \leq c\Delta(x)t \\ \leq &c\frac{1}{\lambda}(\tanh x)^{\gamma}\int_{x-t}^{x+t}|F(y)|dy\,.\end{aligned}$$

Since $y \geq x - t \geq x - \frac{x}{2} = \frac{x}{2}$, it follows that

$$\lambda|B(x,2t)| \leq c\int_{x-t}^{x+t}\left|W_{-\gamma}^{\mathbf{R}}(F)(y)\right|(\tanh y)^{\gamma}dy\,.$$

Then the covering argument yields that

$$\lambda|S_\lambda| \leq c\|W_{-\gamma}^{\mathbf{R}}(F)\|_{L^1((\tanh x)^{\gamma})}\,.$$

Finally, we obtain that

$$\lambda\int_{\{x\in\mathbf{R}_+ | f(x)>\lambda\}}\Delta(x)dx \leq c\left\{\|F\|_{L^1(\mathbf{R}_+)} + \|W_{-\gamma}^{\mathbf{R}}(F)\|_{L^1((\tanh x)^{\gamma})}\right\}.$$

Combining Example 2 (2) and (3), we can deduce the following.

PROPOSITION 5. *Let $\gamma \geq 0$ and Φ be the even extension of $W_{\gamma}^{\mathbf{R}}(\chi_{[0,1]})$. Let $F \in L^1(\mathbf{R})$ and suppose that $W_{-\gamma}^{\mathbf{R}}(F) \in L^1((\tanh x)^{\gamma})$. Then*

$$f(x) = \frac{(\tanh x)^{\gamma}}{\Delta(x)}M_{\Phi}^{\mathbf{R},0}W_{-\gamma}^{\mathbf{R}}(F)(x)$$

satisfies the weak-$L^1(\Delta)$ estimate:

$$\lambda\int_{\{x\in\mathbf{R}_+ | f(x)>\lambda\}}\Delta(x)dx \leq c\left(\|F\|_{L^1(\mathbf{R}_+)} + \|W_{-\gamma}^{\mathbf{R}}(F)\|_{L^1((\tanh x)^{\gamma})}\right).$$

PROOF. Let ψ be an even smooth function on $\mathbf{R}$ such that $\psi(x) = 1$ if $|x| \leq 2$ and $\psi(x) = 0$ if $|x| > 3$. Since

$$\begin{aligned}f(x) \leq &\frac{(\tanh x)^{\gamma}}{\Delta(x)}M_{\Phi}^{\mathbf{R},0}\big((1-\psi(x))W_{-\gamma}^{\mathbf{R}}(F)\big)(x) \\ &+ \frac{(\tanh x)^{\gamma}}{\Delta(x)}M_{\Phi}^{\mathbf{R},0}\big(\psi(x)W_{-\gamma}^{\mathbf{R}}(F)\big)(x) \\ = &f_1(x) + f_0(x)\,,\end{aligned}$$

it is enough to obtain the desired estimate for f_0 and f_1. As for f_1, we note that $(1-\psi(x))W_{-\gamma}^{\mathbf{R}}(F)(x)$ is supported on $[2,\infty)$. Then it follows from Example 2 (2) that

$\lambda \int_{\{x\in\mathbf{R}_+ | f_1(x)>\lambda\}} \Delta(x)\, dx$ is dominated by

$$\left\|(1-\psi(x))W^{\mathbf{R}}_{-\gamma}(F)\right\|_{L^1(\mathbf{R}_+)} \le c\left\|W^{\mathbf{R}}_{-\gamma}(F)\right\|_{L^1((\tanh x)^\gamma)}.$$

As for f_0, we note that $\psi(x)W^{\mathbf{R}}_{-\gamma}(F)(x)$ is supported on [0, 3] and

$$\left|W^{\mathbf{R}}_{-\gamma}(\Phi_t(\cdot)\psi(\cdot+x))(y)\right| \le c\frac{1}{t^{1+\gamma}}.$$

Then, applying the same argument used in Example 2 (3), we can deduce that $\lambda \int_{\{x\in\mathbf{R}_+ | f_1(x)>\lambda\}} \Delta(x)dx$ is dominated by $\|F\|_{L^1(\mathbf{R}_+)} + \|W^{\mathbf{R}}_{-\gamma}(F)\|_{L^1((\tanh x)^\gamma)}$. This completes the proof of Proposition 5. □

6. Shape of maximal functions

Similarly as in the Euclidean case, for the Jacobi hypergroup, we can introduce the Hardy-Littlewood maximal operator M_{HL}, the Poisson maximal operator M_{P} and the Littlewood-Paley g-function $g(f)$ as in (1), where the normalized characteristic function $\widetilde{\chi}_t$ is defined by

$$\widetilde{\chi}_t(x) = \frac{1}{|B(t)|}\chi_{B(t)}(x),$$

where $\chi_{B(t)}(x)$ denotes the characteristic function of $B(t) = [0, t]$ and $|B(t)|$ the volume of $B(t)$ with respect to $\Delta(x)\,dx$, and the Poisson kernel p_t is defined as the function whose Jacobi transform is given by

$$\hat{p}_t(\lambda) = e^{-t\sqrt{\lambda^2+\rho^2}}.$$

Then it follows from [2], Theorem 4.3.1 that

$$p_t(x) \sim e^{-\rho x}e^{-\rho\sqrt{t^2+x^2}}\begin{cases}\dfrac{t}{(t+x)^2}, & t+x\le 1,\\[2mm] \dfrac{t}{(t+x)^{\frac{3}{2}}}, & t+x>1\end{cases} \tag{22}$$

and

$$\frac{d^\ell}{dt^\ell}\frac{d^k}{dx^k}p_t(x) \sim \min\left\{t, 1+\frac{x}{t}\right\}^{-1}\min\{1, t+x\}^{-k}p_t(x).$$

The aim of this section is to give an alternative proof of the weak-$L^1(\Delta)$ estimates for the operators $M = M_{\mathrm{HL}}, M_{\mathrm{P}}, g$ (see Theorem 1). Actually, we show that Mf, $f\in L^1(\Delta)$, has a standard shape of weak-$L^1(\Delta)$ functions obtained in §5.

6.1. Basic idea. The process to obtain a standard shape of $M(f)$ for $M = M_{\mathrm{HL}}, M_{\mathrm{P}}, g$ is based on (13) in §2 and key estimates obtained in §4. Actually, we use the fact that the convolution $*$ in the above definitions of M is replaced by the Euclidean convolution $\otimes$ through the Abel transform (see (9)) and thus, letting $F = W_+(f)$ and $K_t = W_+(k_t)$, where $k_t = \widetilde{\chi}_t$, p_t and $t\frac{dp_t}{dt}$ respectively, we see that

$$f * k_t(x) \sim \frac{e^{\rho x}}{\Delta(x)}\Big(\sum_{\gamma\in\Gamma_0}(\tanh x)^{\gamma} W^{\mathbf{R}}_{-\gamma}(F)\otimes K_t(x) + \sum_{\gamma\in\Gamma_1}(\tanh x)^{\gamma}\int_x^{\infty} W^{\mathbf{R}}_{-\gamma}(F)\otimes K_t(s)A_{\gamma}(x,s)\,ds\Big). \tag{23}$$

Therefore, if $K_t = W_+(k_t)$ is related to a Euclidean dilation, then the maximal operator M is reduced to a Euclidean one. We use the following facts.

LEMMA 2. (1) *Let Φ be the even extension of $W^{\mathbf{R}}_{\alpha+\frac{1}{2}}(\chi_{[0,1]})$ and $\widetilde{\chi}_t$ the normalized characteristic function on $B(t)$. Then for* $0 < t < 1$,

$$e^{\rho x} W_+(\widetilde{\chi}_t)(x) \sim \Phi_t(x)\,.$$

(2) *Let $\Psi(x)$ be the function on $\mathbf{R}$ defined by $\Psi(x) = \frac{1}{1+x^2}$ if $|x| \le 1$ and $\Psi(x) = \frac{1}{(1+|x|)^{\frac{3}{2}}}$ if $|x| > 1$. Then for* $0 < t < 1$,

$$e^{\rho x} W_+(p_t)(x) \sim u_t(x)\Psi_t(x)\,,$$

where $u_t(x) = 1$ if $|x| \le 1$ and $u_t(x) = \sqrt{t}$ if $|x| > 1$. Moreover, for $0 < t < 1$,

$$e^{\rho x} W_+\Big(t\frac{dp_t}{dt}\Big)(x) \sim u_t(x)\Psi_t(x)\,.$$

PROOF. (1) Clearly, $\Phi(x) = c(1-|x|)^{\alpha+\frac{1}{2}}\chi_{[-1,1]}(x)$ and $\Phi_t(x) = \frac{c}{t}\big(1-\frac{|x|}{t}\big)^{\alpha+\frac{1}{2}}\chi_{[-t,t]}(x)$. Both $W_+(\widetilde{\chi}_t)(x)$ and $\Phi_t(x)$ are even and supported on $[-t,t]$. For $0 \le x \le t < 1$, it follows from (18) that

$$\begin{aligned} e^{\rho x} W_+(\chi_t)(x) &= e^{\rho x}\int_x^{\infty}\widetilde{\chi}_t(s)A(x,s)ds \\ &\sim \frac{1}{t^{2\alpha+2}}\int_x^t s(s^2-x^2)^{\alpha-\frac{1}{2}}\,ds \\ &= \frac{c}{t}\Big(1-\frac{x}{t}\Big)^{\alpha+\frac{1}{2}}\Big(1+\frac{x}{t}\Big)^{\alpha+\frac{1}{2}} \\ &\sim \Phi_t(x)\,. \end{aligned}$$

(2) It follows from [1], p. 289 that

$$W_+(p_t)(x) = ct\big(t^2+x^2\big)^{-\frac{1}{2}}K_1\big(\rho\sqrt{t^2+x^2}\big)\,,$$

where K_1 is the modified Bessel function of the second kind. Since $K_1(z) = O(z^{-\frac{1}{2}}e^{-z})$ if $x \to \infty$, $O(z^{-1})$ if $x \to 0$, and $e^{-\rho\sqrt{t^2+x^2}} \sim e^{-\rho x}$ for $0 < t < 1$, it follows that $e^{\rho x}W_+(p_t)(x) \sim u_t(x)\Psi_t(x)$. Since $\frac{dp_t}{dt} \sim t^{-1}p_t$ (see (22)), $e^{\rho x}W_+\big(t\frac{dp_t}{dt}\big)(x) \sim u_t(x)\Psi_t(x)$. □

6.2. The case of M_{HL}. We shall give a proof of Theorem 1. We define

$$M^0_{\mathrm{HL}}f(x) = \sup_{0<t<1} |f| * \widetilde{\chi}_t(x) \quad \text{and} \quad M^1_{\mathrm{HL}}f(x) = \sup_{t\geq 1} |f| * \widetilde{\chi}_t(x)\,. \tag{24}$$

Then it is easy to see that Theorem 1 is true if each $M^0_{\mathrm{HL}}f$ and $M^1_{\mathrm{HL}}f$ satisfies the desired inequality. Usually, the weak $L^1(\Delta)$-estimate for M^0_{HL} is proved by applying the covering argument used in the Euclidean case (see [13]). Our alternative proof is based on the following new estimate of $M^0_{\mathrm{HL}}f$ in (26).

THEOREM 4. *Let $f \in L^1(\Delta)$ and $F = W_+(f)$. Then*

$$M^1_{\mathrm{HL}}f(x) \leq c\frac{\|f\|_{L^1(\Delta)}}{\Delta(x)}\,, \tag{25}$$

$$M^0_{\mathrm{HL}}f(x) \leq c\frac{\|f\|_{L^1(\Delta)}}{B(x)} + c\sum_{\gamma\in\Gamma_0}\frac{(\tanh x)^\gamma}{\Delta(x)}M^{\mathbf{R},0}_{\Phi}(e^{\rho x}W_{-\gamma}(F))(x)\,. \tag{26}$$

Especially, M_{HL} *satisfies the weak-$L^1(\Delta)$ estimate*:

$$\lambda\int_{\{x\in\mathbf{R}_+ | M_{\mathrm{HL}}f(x)>\lambda\}}\Delta(x)\,dx \leq c\|f\|_{L^1(\Delta)}\,. \tag{27}$$

PROOF. In the following, we denote $|f|$ by f for simplicity. The weak-$L^1(\Delta)$ estimate (27) follows from (25) and (26), because of Example 2 (1) and Proposition 5 with the fact that

$$\|e^{\rho x}W_{-\gamma}(F)\|_{L^1((\tanh x)^\gamma)} \leq c\|f\|_{L^1(\Delta)}$$

(see Proposition 4). First we shall consider M^1_{HL}. We divide f into $f = f_0 + f_1$, where $f_0 = f\cdot\chi_{B(1)}$ and $f_1 = f - f_0$. Since

$$M^1_{\mathrm{HL}}f \leq M^1_{\mathrm{HL}}f_0 + M^1_{\mathrm{HL}}f_1\,,$$

it is enough to show that both $M^1_{\mathrm{HL}}f_0$ and $M^1_{\mathrm{HL}}f_1$ satisfy (25). As for $M^1_{\mathrm{HL}}f_1$, we use (23) to handle $f * \widetilde{\chi}_t$ in the definition of H^1_{HL} and note that

$$e^{\rho x}W^{\mathbf{R}}_{-\gamma}(F)\otimes K_t = (e^{\rho x}W^{\mathbf{R}}_{-\gamma}(F))\otimes(e^{\rho x}K_t) = (e^{\rho x}F)\otimes(e^{\rho x}W^{\mathbf{R}}_{-\gamma}(K_t))\,,$$

where $F = W_+(f_1)$ and $K_t = W_+(\widetilde{\chi}_t)$. Then it follows from Proposition 4 with $\gamma = 0$ and $\delta = \rho$ that

$$\|e^{\rho x}F\|_{L^1(\mathbf{R}_+)} \leq c\|f_1\|_{L^1(\Delta)}\,.$$

Moreover, since $t \geq 1$, it follows from the proof of Proposition 3 that for $0 \leq \gamma \leq \alpha + \frac{1}{2}$ and $0 < x < t$,

$$e^{\rho x}|W^{\mathbf{R}}_{-\gamma}(K_t)(x)| \leq c e^{\rho x} W^{\mathbf{R}}_{-\gamma}\Big(\frac{1}{|B(t)|}\int_x^t \frac{d}{dy}R(x, y)\cdot(\cosh y - \cosh x)^{\alpha+\frac{1}{2}}\,dy\Big)$$
$$\leq c\frac{e^{\rho x}}{e^{2\rho t}}\int_x^t e^{\rho y}\,dy \leq c\,.$$

Hence, for $\gamma \in \Gamma_0 \cup \Gamma_1$, we see that

$$\big\|e^{\rho x} W^{\mathbf{R}}_{-\gamma}(F \otimes K_t)\big\|_{L^\infty(\mathbf{R}_+)} \leq \|e^{\rho x} F\|_{L^1(\mathbf{R}_+)}\big\|e^{\rho x} W^{\mathbf{R}}_{-\gamma}(K_t)\big\|_{L^\infty(\mathbf{R}_+)}$$
$$\leq c\|f_1\|_{L^1(\Delta)}\,.$$

Therefore, by noting (14), it follows from (23) that

$$|f_1 * \widetilde{\chi}_t(x)| \leq c\frac{\|f_1\|_{L^1(\Delta)}}{\Delta(x)}\,.$$

As for $M^1_{\mathrm{HL}} f_0$, we note that $f_0 * \widetilde{\chi}_t$ is supported on $[0, 1+t)$ and $t \geq 1$. Since $\|\widetilde{\chi}_t\|_{L^\infty(\Delta)} = \frac{1}{|B(t)|} \sim \frac{1}{\Delta(t)}$ and $\frac{1}{\Delta(1+t)} \leq c\frac{1}{\Delta(x)}$ for $0 \leq x \leq 1+t$, it follows that

$$|f_0 * \chi_t(x)| \leq \|f_0\|_{L^1(\Delta)}\|\widetilde{\chi}_t\|_{L^\infty(\Delta)}$$
$$\leq c\frac{\|f_0\|_{L^1(\Delta)}}{|B(t)|}$$
$$\leq c\frac{\|f_0\|_{L^1(\Delta)}}{\Delta(x)}\,.$$

Thus we have obtained (25).

Next we shall consider M^0_{HL}. By noting (23), to prove (26) it is enough to prove that the following (28) and (29) are estimated as (26):

$$\frac{e^{\rho x}}{\Delta(x)}(\tanh x)^\gamma \sup_{0<t<1} W^{\mathbf{R}}_{-\gamma}(F) \otimes K_t(x)\,, \tag{28}$$

$$\frac{e^{\rho x}}{\Delta(x)}(\tanh x)^\gamma \sup_{0<t<1}\int_x^\infty W^{\mathbf{R}}_{-\gamma}(F) \otimes K_t(s) A_\gamma(x, s)\,ds\,. \tag{29}$$

As shown in Lemma 2, $e^{\rho x} K_t \sim \Phi_t$ and therefore, (28) is dominated by

$$\frac{(\tanh x)^\gamma}{\Delta(x)} M^{\mathbf{R},0}_{\Phi}\big(e^{\rho x} W^{\mathbf{R}}_{-\gamma}(F)\big)(x)\,.$$

As for (29), when $0 \leq \frac{x}{2} < t$, we see that

$$e^{\rho x}(\tanh x)^\gamma \int_x^\infty W^{\mathbf{R}}_{-\gamma}(F \otimes K_t)(s) A_\gamma(x, s)\,ds$$

$$\leq c(\tanh x)^{\gamma}\|e^{\rho x}F\|_{L^1(\mathbf{R}_+)}\big\|e^{\rho x}W^{\mathbf{R}}_{-\gamma}(K_t)\big\|_{L^\infty(\mathbf{R}_+)}\int_x^\infty A_\gamma(x,s)\,ds$$
$$\leq c(\tanh x)^{\gamma}\|f\|_{L^1(\Delta)}\frac{1}{t^{\gamma+1}}$$
$$\leq c\frac{\|f\|_{L^1(\Delta)}}{\tanh x}\,.$$

On the other hand, when $\frac{x}{2} > t$, we use Proposition 1. Then it follows that

$$e^{\rho x}(\tanh x)^{\gamma}\int_x^\infty W^{\mathbf{R}}_{-\gamma}(F)\otimes K_t(s)A_\gamma(x,s)\,ds$$
$$= e^{\rho x}(\tanh x)^{\gamma}\int_x^\infty W^{\mathbf{R}}_{-(\gamma+\xi_\gamma)}(F)\otimes K_t(s)\widetilde{A}_\gamma(x,s)\,ds$$
$$\leq c(\tanh x)^{\gamma+\xi_\gamma-1}\int_{-\infty}^\infty\Big(\int_x^\infty\big(e^{\rho x}W^{\mathbf{R}}_{-(\gamma+\xi_\gamma)}(F)\big)(s-y)ds\Big)(e^{\rho x}K_t)(y)\,dy\,.$$

Since K_t is supported on $[-t,t]$, we may suppose that $|y|\leq t$. Then it follows that $s-y\geq x-y\geq x-t\geq\frac{x}{2}$ and thus,

$$(\tanh x)^{\gamma+\xi_\gamma}\leq(\tanh(s-y))^{\gamma+\xi_\gamma}\,.$$

Therefore, by Proposition 4, the last integral is dominated by

$$(\tanh x)^{-1}\big\|e^{\rho x}W^{\mathbf{R}}_{-(\gamma+\xi_\gamma)}(F)\big\|_{L^1((\tanh x)^{\gamma+\xi_\gamma})}\|e^{\rho x}K_t\|_{L^1(\mathbf{R}_+)}$$
$$\leq c(\tanh x)^{-1}\|f\|_{L^1(\Delta)}\|\widetilde{\chi}_t\|_{L^1(\Delta)}$$
$$\leq c\frac{\|f\|_{L^1(\Delta)}}{\tanh x}\,.$$

Here we used tha fact that $\gamma+\xi_\gamma\leq\alpha+\frac{1}{2}$ for $\gamma\in\Gamma_1$. Finally, we have proved that (29) is dominated by

$$\frac{1}{\Delta(x)}\frac{\|f\|_{L^1(\Delta)}}{\tanh x}\leq c\frac{\|f\|_{L^1(\Delta)}}{|B(x)|}\,.$$

This completes the proofs of Theorem 4 and Theorem 1 for M_{HL}. □

6.3. The case of M_{P}. Let M^0_{P} and M^1_{P} be the operators defined by

$$M^0_{\mathrm{P}}f(x)=\sup_{0<t<1}|f*p_t(x)|\quad\text{and}\quad M^1_{\mathrm{P}}f(x)=\sup_{t\geq1}|f*p_t(x)|\,.$$

Similarly as Theorem 4, we shall prove the following.

THEOREM 5. *M^0_{P} and M^1_{P} satisfy the same inequalities in Theorem* 4 *replaced $M^{\mathbf{R},0}_{\Phi}$ with $M^{\mathbf{R},0}_{\Psi}$. Especially, M_{P} satisfies the weak-$L^1(\Delta)$ estimate.*

PROOF. We shall prove that M_{P}^1 and M_{P}^0 also satisfy (25) and (26) respectively. We put $F = W_+(f)$ and $K_t = W_+(p_t)$. As for M_{P}^1, we let $f = f_0 + f_1$ as in 6.2. Since $e^{\rho x} K_t(x) \sim u_t(x)\Psi_t(x)$ (see Lemma 2 (2)), it follows that for $t \geq 1$,

$$\left\| e^{\rho x} W_{-\gamma}^{\mathbf{R}}(K_t) \right\|_{L^\infty(\mathbf{R}_+)} \leq c\,.$$

Therefore, (25) holds for f_1 similarly as in 6.2. On the other hand, we note that f_0 is supported on $B(1)$ and for $t \geq 1$ and $p_t(x) \leq c e^{-2\rho x}$ (see (22)). Hence, it follows that

$$|f_0 * p_t(x)| \leq c e^{-2\rho x} \|f_0\|_{L^1(\Delta)}\,.$$

Therefore (25) also holds for f_0. We have obtained that M_{P}^1 satisfies (25).

As for M_{P}^0, we shall estimate (28) and (29) for $K_t = W_+(p_t)$. First we note that

$$e^{\rho x} K_t(x) \sim u_t(x)\Psi_t(x) \leq \Psi_t(x)$$

(see Lemma 2 (2)). Hence (28) is dominated by

$$\frac{(\tanh x)^\gamma}{\Delta(x)} M_\Psi^{\mathbf{R},0}\big(e^{\rho x} W_{-\gamma}^{\mathbf{R}}(F)\big)(x)\,.$$

On the other hand, for (29), let ϕ be an even smooth function supported on $\pm[\frac{3}{4}, 3]$ and 1 on $\pm[\frac{5}{4}, \frac{1}{2}]$. Furthermore, we suppose that

$$\sum_{k \in \mathbf{Z}} \phi\Big(\frac{x}{2^k}\Big) \sim 1\,.$$

We note that

$$\begin{aligned}
&e^{\rho x} W_{-\gamma}^{\mathbf{R}}(F \otimes K_t)(x) \\
&\sim \int_{-\infty}^{\infty} \big(e^{\rho x} W_{-\gamma}^{\mathbf{R}}(F)\big)(x-y) u_t(y)\Psi_t(y)\,dy \\
&= \sum_{k \in \mathbf{Z}}^{\infty} \int_{-\infty}^{\infty} \big(e^{\rho x} W_{-\gamma}^{\mathbf{R}}(F)\big)(x-y) u_t(y)\Psi_t(y)\phi\Big(\frac{y}{2^k t}\Big)\,dy \\
&\sim \sum_{k \in \mathbf{Z}} \frac{a_{k,t}}{t} \Psi(2^k) \int_{-\infty}^{\infty} \big(e^{\rho x} W_{-\gamma}^{\mathbf{R}}(F)\big)(x-y)\phi\Big(\frac{y}{2^k t}\Big)\,dy\,,
\end{aligned}$$

where $a_{k,t} = 1$ if $k \leq n_t$ and $a_{k,t} = \sqrt{t}$ if $k > n_t$ with $n_t = -\frac{\log t}{\log 2}$. Therefore, since $0 < t < 1$, it follows that

$$\begin{aligned}
&e^{\rho x}(\tanh x)^\gamma \left| \int_x^\infty W_{-\gamma}^{\mathbf{R}}(F) \otimes K_t(s) A_\gamma(x,s)\,ds \right| \\
&\leq c \sum_{k \in \mathbf{Z}} \Psi(2^k) 2^{k+1} (\tanh x)^\gamma \left| \int_x^\infty e^{\rho s} W_{-\gamma}^{\mathbf{R}}(F) \otimes \phi_{2^{k+1}t}(s) A_\gamma(x,s)\,ds \right|
\end{aligned}$$

$$= \sum_{k\in\mathbf{Z}} \Psi(2^k)2^{k+1} I_k(t,x)\,.$$

Since ϕ is compactly supported, we can apply the argument used in the proof of Theorem 4 replaced t with $2^{k+1}t$. Hence it follows that

$$I_k(t,x) \le c(\tanh x)^{-1}\|f\|_{L^1(\Delta)}\,.$$

Therefore, since

$$\sum_{k\in\mathbf{Z}} \Psi(2^k)2^{k+1} \le \|\Psi\|_{L^1(\mathbf{R})}\,,$$

(29) is dominated by $B(x)^{-1}\|f\|_{L^1(\Delta)}$. This completes the proof of Theorem 5. □

6.4. The case of g. We define

$$g^0(f)(x) = \Big(\int_0^1 \Big|f * t\frac{dp_t}{dt}(x)\Big|^2 \frac{dt}{t}\Big)^{\frac12}\,,$$
$$g^1(f)(x) = \Big(\int_1^\infty \Big|f * t\frac{dp_t}{dt}(x)\Big|^2 \frac{dt}{t}\Big)^{\frac12}$$

and

$$g_\Psi^{\mathbf{R},0}(F)(x) = \Big(\int_0^1 \Big|F \otimes t\frac{d\Psi_t}{dt}(x)\Big|^2 \frac{dt}{t}\Big)^{\frac12}\,,$$

where Ψ is given by Lemma 2 (2). We note that $g_\Psi^{\mathbf{R},0}$ satisfies the weak-L^1 estimate on $\mathbf{R}$. We shall prove the following.

THEOREM 6. *Let $f \in L^1(\Delta)$ and $F = W_+(f)$. Then*

$$g^1(f)(x) \le c\frac{\|f\|_{L^1(\Delta)}}{\Delta(x)}\,, \tag{30}$$

$$g^0(f)(x) \le c\frac{\|f\|_{L^1(\Delta)}}{B(x)} + c\sum_{\gamma\in\Gamma_0}\frac{(\tanh x)^\gamma}{\Delta(x)} g_\Psi^{\mathbf{R},0}\big(W_{-\gamma}^{\mathbf{R}}(F)\big)(x)\,. \tag{31}$$

Especially, g satisfies the weak-$L^1(\Delta)$ estimate.

PROOF. We shall prove (30). We divide $f = f_0 + f_1$ as before. As for $g^1 f_1$, we use (23) to handle $f_1 * t\frac{dp_t}{dt}$ and thus, we estimate the following terms:

$$\frac{e^{\rho x}}{\Delta(x)}\Big(\sum_{\gamma\in\Gamma_0}(\tanh x)^\gamma\Big(\int_1^\infty \big|W_{-\gamma}^{\mathbf{R}}(F)\otimes K_t(x)\big|^2\frac{dt}{t}\Big)^{\frac12}$$
$$+\sum_{\gamma\in\Gamma_1}\Big(\int_1^\infty\Big|(\tanh x)^\gamma\int_x^\infty W_{-\gamma}^{\mathbf{R}}(F)\otimes K_t(s)A_\gamma(x,s)ds\Big|^2\frac{dt}{t}\Big)^{\frac12}\Big)\,, \tag{32}$$

where $F = W_+(f_1)$ and $K_t(x) = W_+(t\frac{dp_t}{dt})(x)$. First we shall deduce that

$$\int_1^\infty \left|e^{\rho x} W^{\mathbf{R}}_{-\gamma}(K_t)(x)\right|^2 \frac{dt}{t} \le c\,. \tag{33}$$

By using (22) with $t > 1$, we see that

$$\begin{aligned}
&\left|e^{\rho x} W^{\mathbf{R}}_{-\gamma}(K_t)(x)\right| \\
&\le c e^{\rho x} W^{\mathbf{R}}_{-\gamma}\Big(\int_x^\infty \frac{d}{ds}\Big(t\frac{dp_t}{dt}(s)R(x,s)\Big)\cdot(\cosh y - \cosh s)^{\alpha+\frac{1}{2}} ds\Big) \\
&\le c e^{\rho x} \int_x^\infty \frac{t^2}{(t+s)^{\frac{3}{2}}} e^{-\rho s} e^{-\rho\sqrt{t^2+s^2}} e^{\rho s}\, ds\,.
\end{aligned}$$

When $x > \frac{t}{2}$, since $t < 2x \le 2s$, it follows that

$$\begin{aligned}
\left|e^{\rho x} W^{\mathbf{R}}_{-\gamma}(K_t)(x)\right| &\le c e^{\rho x} \frac{t}{(t+x)^{\frac{1}{2}}} \int_x^\infty \frac{s}{\sqrt{t^2+s^2}} e^{-\rho\sqrt{t^2+s^2}}\, ds \\
&\le c e^{\rho x} \frac{t}{(t+x)^{\frac{1}{2}}} e^{-\rho\sqrt{t^2+x^2}}\,.
\end{aligned}$$

Hence it follows that

$$\int_1^\infty \left|e^{\rho x} W^{\mathbf{R}}_{-\gamma}(K_t)(x)\right|^2 \frac{dt}{t} \le c e^{2\rho x} \int_1^\infty \frac{t}{\sqrt{t^2+x^2}} e^{-2\rho\sqrt{t^2+x^2}}\, dt \le c\,.$$

When $x \le \frac{t}{2}$, we see that

$$\left|e^{\rho x} W^{\mathbf{R}}_{-\gamma}(K_t)(x)\right| \le c e^{\frac{\rho t}{2}} e^{-\rho t} \int_x^\infty \frac{t^2}{(t+s)^{\frac{3}{2}}}\, ds \le c e^{-\frac{\rho t}{2}} t^{\frac{3}{2}}\,.$$

Hence we have deduced (33).

By using (33), we see that

$$\begin{aligned}
&\Big(\int_1^\infty \left|e^{\rho x} (\tanh x)^\gamma W^{\mathbf{R}}_{-\gamma}(F) \otimes K_t(x)\right|^2 \frac{dt}{t}\Big)^{\frac{1}{2}} \\
&\le c \int_{-\infty}^\infty |(e^{\rho x}F)(x-y)| \Big(\int_1^\infty \left|e^{\rho x} W^{\mathbf{R}}_{-\gamma}(K_t)(y)\right|^2 \frac{dt}{t}\Big)^{\frac{1}{2}} dy \\
&\le c\|e^{\rho x}F\|_{L^1(\mathbf{R}_+)} \le c\|f_1\|_{L^1(\Delta)}
\end{aligned}$$

and

$$\begin{aligned}
&\Big(\int_1^\infty \Big|e^{\rho x} (\tanh x)^\gamma \int_x^\infty W^{\mathbf{R}}_{-\gamma}(F) \otimes K_t(s) A_\gamma(x,s)\, ds dx\Big|^2 \frac{dt}{t}\Big)^{\frac{1}{2}} \\
&\le (\tanh x)^\gamma \int_x^\infty |e^{\rho s}F| \otimes \Big(\int_1^\infty e^{\rho s} W^{\mathbf{R}}_{-\gamma}(K_t)(s)|^2 \frac{dt}{t}\Big)^{\frac{1}{2}} |A_\gamma(x,s)|\, ds dx
\end{aligned}$$

$$\le c\|e^{\rho x}F\|_{L^1(\mathbf{R}_+)}\int_x^\infty |A_\gamma(x,s)|\,ds \le c\|f_1\|_{L^1(\Delta)}\,.$$

Therefore, (30) follows from (23).

As for $g^1 f_0$, if $t > 1$ and $t < 1+\frac{x}{t}$, then $t\frac{dp_t}{dt} \sim p_t$ (see (22)). Hence (33) holds and thus, similarly as above, (30) follows. We suppose that $t > 1$ and $t \ge 1+\frac{x}{t}$. Then $t\frac{dp_t}{dt} \sim \frac{t^2}{t+x}p_t \sim \frac{t^3}{(t+x)^{\frac{5}{2}}}e^{-\rho x}e^{-\rho\sqrt{t^2+x^2}}$ (see (22)). Hence

$$\begin{aligned}\int_1^\infty \Big|t\frac{dp_t}{dt}(x)\Big|^2\frac{dt}{t} &\le ce^{-2\rho x}\int_1^\infty \frac{t^5}{(t+x)^5}e^{-2\rho\sqrt{t^2+x^2}}\,dt\\ &\le ce^{-2\rho x}\int_1^\infty \frac{t}{(t^2+x^2)^{\frac{1}{2}}}e^{-2\rho\sqrt{t^2+x^2}}\,dt\\ &\le ce^{-4\rho x}\,.\end{aligned}$$

Since f_0 is supported on [0, 1], it is easy to see that

$$\begin{aligned}\big|g^1(f_0)(x)\big| &\le c|f_0| * \Big(\int_1^\infty \Big|t\frac{dp_t}{dt}(x)\Big|^2\frac{dt}{t}\Big)^{\frac{1}{2}}\\ &\le c\|f_0\|_{L^1(\Delta)}e^{-2\rho x}\,.\end{aligned}$$

Therefore, $g^1(f_0)$ also satisfies (30). We have obtained that g^1 satisfies (30).

Next we shall consider g^0. We use (23) to handle $f * t\frac{dp_t}{dt}$ and estimate the following terms.

$$\begin{aligned}&\frac{e^{\rho x}}{\Delta(x)}\Big(\sum_{\gamma\in\Gamma_0}(\tanh x)^\gamma\Big(\int_0^1\big|W^{\mathbf{R}}_{-\gamma}(F)\otimes K_t(x)\big|^2\frac{dt}{t}\Big)^{\frac{1}{2}}\\ &\qquad+\sum_{\gamma\in\Gamma_1}\Big(\int_0^1\Big|(\tanh x)^\gamma\int_x^\infty W^{\mathbf{R}}_{-\gamma}(F)\otimes K_t(s)A_\gamma(x,s)ds\Big|^2\frac{dt}{t}\Big)^{\frac{1}{2}}\Big),\end{aligned}\qquad(34)$$

where $F = W_+(f)$ and $K_t(x) = W_+(t\frac{dp_t}{dt})(x)$. Since $e^{\rho x}K_t \sim u_t(x)\Psi_t(x)$ for $t < 1$ (see Lemma 2 (2)), the first term is dominated by

$$\sum_{\gamma\in\Gamma_0}\frac{(\tanh x)^\gamma}{\Delta(x)}g_\Psi^{\mathbf{R},0}\big(e^{\rho x}W^{\mathbf{R}}_{-\gamma}(F)\big)(x)\,.$$

We shall consider the second term of (34). When $0 \le \frac{x}{2} < t < 1$, we note that $e^{\rho x}W^{\mathbf{R}}_{-\gamma}(K_t)$ $\sim W^{\mathbf{R}}_{-\gamma}(\Psi_t)$ and

$$\int_{\frac{x}{2}}^1 \Big|W^{\mathbf{R}}_{-\gamma}(\Psi_t)(x)\Big|^2\frac{dt}{t} \le \frac{c}{x^{2+2\gamma}}\,.$$

Hence, it follows from (14) that

$$e^{\rho x}(\tanh x)^{\gamma}\Big(\int_0^1\Big|\int_x^{\infty}W_{-\gamma}^{\mathbf{R}}(F)\otimes K_t(s)A_{\gamma}(x,s)ds\Big|^2\frac{dt}{t}\Big)^{\frac{1}{2}}$$
$$\le(\tanh x)^{\gamma}\int_x^{\infty}(e^{\rho x}F)\otimes\Big(\int_0^1\big|e^{\rho x}W_{-\gamma}^{\mathbf{R}}(K_t)(s)\big|^2dt\Big)^{\frac{1}{2}}A_{\gamma}(x,s)ds$$
$$\le c(\tanh x)^{\gamma}\|e^{\rho x}F\|_{L^1(\mathbf{R}_+)}\frac{c}{x^{1+\gamma}}\int_x^{\infty}A_{\gamma}(x,s)ds$$
$$\le c\frac{\|f\|_{L^1(\Delta)}}{\tanh x}.$$

When $\frac{x}{2}>t$, similarly as in the proof of Theorem 5, we may replace K_t by ϕ_t where ϕ is an even smooth compactly supported function on $\mathbf{R}$. We divide $\phi_t=\chi_{\frac{x}{2}}\phi_t+\chi_{\frac{x}{2}}^c\phi_t$, where $\chi_{\frac{x}{2}}$ and $\chi_{\frac{x}{2}}^c$ denote the characteristic functions of $[0,\frac{x}{2}]$ and $[\frac{x}{2},\infty)$ respectively. Moreover, we rewrite the action of $W_{-\gamma}^{\mathbf{R}}$ as

$$W_{-\gamma}^{\mathbf{R}}(F)\otimes\chi_{\frac{x}{2}}^c\phi_t=F\otimes W_{-\gamma}^{\mathbf{R}}\big(\chi_{\frac{x}{2}}^c\phi_t\big),$$

$$W_{-\gamma}^{\mathbf{R}}(F)\otimes\chi_{\frac{x}{2}}\phi_t=W_{-(1-\xi_{\gamma})}^{\mathbf{R}}\big(W_{-(\gamma+\xi_{\gamma})}^{\mathbf{R}}(F)\otimes W_1^{\mathbf{R}}\big(\chi_{\frac{x}{2}}\phi_t\big)\big).$$

Changing the order of integration and using the duality of operators, we see that

$$e^{\rho x}(\tanh x)^{\gamma}\int_x^{\infty}W_{-\gamma}^{\mathbf{R}}(F)\otimes\phi_t(s)A_{\gamma}(x,s)ds$$
$$\le c(\tanh x)^{\gamma}\int\Big(\int_x^{\infty}(e^{\rho x}F)(s-y)A_{\gamma}(x,s)ds\Big)\big(e^{\rho x}W_{-\gamma}^{\mathbf{R}}\big(\chi_{\frac{x}{2}}^c\phi_t\big)\big)(y)dy$$
$$+c(\tanh x)^{\gamma}\int\Big(\int(e^{\rho x}W_{-(\gamma+\xi_{\gamma})}^{\mathbf{R}}(F))(s-y)\widetilde{W}_{-(1-\xi_{\gamma})}^{\mathbf{R}}(\chi_x^cA_{\gamma}(x,\cdot))(s)ds\Big)$$
$$\times\big(e^{\rho x}W_1^{\mathbf{R}}\big(\chi_{\frac{x}{2}}\phi_t\big)\big)(y)dy$$
$$=I_{11}(x)+I_{12}(x).$$

As for I_{11}, since $y>\frac{x}{2}>t$ and ϕ is a smooth compactly supported function, it follows that

$$\Big(\int_0^{\frac{x}{2}}\Big|W_{-\gamma}^{\mathbf{R}}(\phi_t)(y)\Big|^2\frac{dt}{t}\Big)^{\frac{1}{2}}\le\frac{c}{y^{1+\gamma}}\Big(\int_{\frac{2y}{x}}^{\infty}s^{2+2\gamma}\Big|s\frac{d\phi}{ds}(s)\Big|^2\frac{ds}{s}\Big)^{\frac{1}{2}}$$
$$\le\frac{c}{x^{1+\gamma}}.$$

Therefore, since $\|e^{\rho x}F\|_{L^1(\mathbf{R}_+)}\le c\|f\|_{L^1(\Delta)}$, similarly as before, we see that

$$\Big(\int_0^1|I_{11}(x)|^2\frac{dt}{t}\Big)^{\frac{1}{2}}\le c\frac{\|f\|_{L^1(\Delta)}}{\tanh x}.$$

As for I_{12}, we note that $(e^{\rho x} W_1^{\mathbf{R}}(\chi_{\frac{x}{2}}\phi_t))(y)$ is supported on $[0, \frac{x}{2}]$. Then for $y \le \frac{x}{2}$, we see that $s - y \ge x - y \ge x - \frac{x}{2} = \frac{x}{2}$ and thus,

$$\tanh x \le \tanh(s-y)\,.$$

Hence I_{12} can be written as

$$c(\tanh x)^{-1} \int \Big(\int (\tanh(s-y)^{\gamma+\xi_\gamma}) e^{\rho x} W^{\mathbf{R}}_{-(\gamma+\xi_\gamma)}(F))(s-y) \\ \times (\tanh x)^{1-\xi_\gamma} \widetilde{W}^{\mathbf{R}}_{-(1-\xi_\gamma)}(\chi_x^c A_\gamma(x,\cdot))(s)\, ds\Big)\big(e^{\rho x} W_1^{\mathbf{R}}\big(\chi_{\frac{x}{2}} K_t\big)\big)(y)\, dy\,.$$

Here we note that

$$\Big(\int_0^1 \big|e^{\rho y} W_1^{\mathbf{R}}(\chi_x \phi_t)(y)\big|^2 \frac{dt}{t}\Big)^{\frac{1}{2}} \le c\,.$$

Actually, since $\phi_t(y) = c\frac{d}{dy}(W_1^{\mathbf{R}}(\phi)(\frac{y}{t}))$, it follows that

$$\int_0^1 \big|e^{\rho y} W_1^{\mathbf{R}}(\chi_x \phi_t)(y)\big|^2 \frac{dt}{t} \le c\int_y^\infty \big|W_1^{\mathbf{R}}(\phi)(s)\big|^2 \frac{ds}{s} \\ \le c\,.$$

Moreover, since $\gamma + \xi_\gamma \le \alpha + \frac{1}{2}$ for $\gamma \in \Gamma_1$, it follows from Proposition 1 that

$$(\tanh x)^{1-\xi_\gamma} \int \widetilde{W}^{\mathbf{R}}_{-(1-\xi_\gamma)}(\chi_x^c A_\gamma(x,\cdot))(s)\, ds \\ = (\tanh x)^{1-\xi_\gamma} \widetilde{W}^{\mathbf{R}}_{\xi_\gamma}(\chi_x^c A_\gamma(x,\cdot))(s)\, ds = (\tanh x)^{1-\xi_\gamma} \widetilde{A}(x,\cdot)(s) \\ \le c\,.$$

Hence by Proposition 4, we see that

$$\Big(\int_0^1 |I_{12}(x)|^2 \frac{dt}{t}\Big)^{\frac{1}{2}} \le c(\tanh x)^{-1} \big\|e^{\rho x} W^{\mathbf{R}}_{-(\gamma+\xi_\gamma)}(F)\big\|_{L^1((\tanh x)^{\gamma+\xi_\gamma})} \\ \le c\frac{\|f\|_{L^1(\Delta)}}{\tanh x}\,.$$

Therefore, the second term of (34) is dominated by $\frac{\|f\|_{L^1(\Delta)}}{|B(x)|}$. This completes the proof of Theorem 6. □

7. The endpoint of the KS phenomenon

The Kunze-Stein phenomenon for the Jacobi analysis (see Theorem 2) is originally not difficult. The following proof was found in [7], Theorem 5.5: We suppose that $f \in L^p(\Delta)$

and $1 \le p < 2$. Since q, the conjugate of p, is greater than 2, it follows (3) and (2) that

$$\begin{aligned}\|\hat{f}\|_\infty &\le \int_0^\infty |f(x)||\phi_\lambda(x)|\Delta(x)\,dx \\ &\le \|f\|_{L^p(\Delta)}\Big(\int_0^\infty (1+x)^q e^{-q\rho x} e^{2\rho x} dx\Big)^{\frac{1}{q}} \\ &= c_p\|f\|_{L^p(\Delta)}\,.\end{aligned}$$

Let $g \in L^2(\Delta)$. Since $\widehat{f * g} = \hat{f} \cdot \hat{g}$, it follows from (4) that

$$\begin{aligned}\|f * g\|^2_{L^2(\Delta)} &= \int_0^\infty |\hat{f}(\lambda)|^2|\hat{g}(\lambda)|^2|C(\lambda)|^{-2}\,d\lambda \\ &\le c_p^2\|f\|^2_{L^p(\Delta)}\int_0^\infty |\hat{g}(\lambda)|^2|C(\lambda)|^{-2}\,d\lambda \\ &= c_p^2\|f\|^2_{L^p(\Delta)}\|g\|^2_{L^2(\Delta)}\,.\end{aligned}$$

This completes the proof of Theorem 2.

Now we shall give an alternative proof of the endpoint estimate (see Theorem 3) based on (13) and the key estimates in §4. In order to show that $f * g \in L^{2,\infty}(\Delta)$ for $f, g \in L^{2.1}(\Delta)$, by noting the fact that the dual space of $L^{2,1}(\Delta)$ is given by $L^{2,\infty}(\Delta)$, it suffices to prove that for all $h \in L^{2,1}(\Delta)$,

$$\Big|\int_0^\infty f * g(x)h(x)\Delta(x)\,dx\Big| \le c\|f\|_{L^{2,1}(\Delta)}\|g\|_{L^{2,1}(\Delta)}\|h\|_{L^{2,1}(\Delta)}\,. \tag{35}$$

First we may suppose that f, g, h are supported on $[1,\infty)$. Actually, we note that the integral of the left hand is written as $f * g * h(0)$ and, if one of f, g, h were supported on $[0, 1)$, say f, then we see that

$$\begin{aligned}|f * g * h(0)| &\le \|f * g\|_{L^2(\Delta)}\|h\|_{L^2(\Delta)} \\ &\le \|f\|_{L^1(\Delta)}\|g\|_{L^2(\Delta)}\|f\|_{L^2(\Delta)} \\ &\le \Big(\int_0^1 \Delta(x)\,dx\Big)^{\frac{1}{2}}\|f\|_{L^2(\Delta)}\|g\|_{L^2(\Delta)}\|f\|_{L^2(\Delta)}\,.\end{aligned}$$

Since $L^{2,1}(\Delta) \subset L^2(\Delta)$, (35) is clear. We suppose that f, g, h are all supported on $[1,\infty)$ and use (13) again. Then it follows that

$$\begin{aligned}&\int_0^\infty f * g(x)h(x)\Delta(x)\,dx \\ &\sim \sum_{\gamma\in\Gamma_0}\int_0^\infty (\tanh x)^\gamma W^{\mathbf{R}}_{-\gamma}(F\otimes G)(x)H(x)\,dx \\ &\quad + \sum_{\gamma\in\Gamma_1}\int_0^\infty (\tanh x)^\gamma\Big(\int_x^\infty W^{\mathbf{R}}_{-\gamma}(F\otimes G)(s)A_\gamma(x,s)ds\Big)H(x)\,dx\,,\end{aligned}$$

$$= \sum_{\gamma\in\Gamma_0} I^1_\gamma + \sum_{\gamma\in\Gamma_1} I^2_\gamma\,,$$

where $F = W_+(f)$, $G = W_+(g)$ and $H(x) = h(x)e^{\rho x}$.

According to the cases in Proposition 2, we shall estimate I^1_γ and I^2_γ.

(i) Let $\alpha > \frac{1}{2}$ or $(\alpha, \beta) = \left(\frac{1}{2}, \pm\frac{1}{2}\right)$. We may suppose that $\gamma \geq 1$, because $H(x)$ is supported on $[1, \infty)$. Then, it follows from Propositions 3 and 4 that

$$\begin{aligned}\left\|W^{\mathbf{R}}_{-\gamma}(F)\otimes G\right\|_{L^\infty(\mathbf{R}_+)} &= \left\|W^{\mathbf{R}}_{-1}(F)\otimes W^{\mathbf{R}}_{-(\gamma-1)}(G)\right\|_{L^\infty(\mathbf{R}_+)}\\ &\leq \left\|W^{\mathbf{R}}_{-1}(F)\right\|_{L^1(\mathbf{R}_+)}\left\|W^{\mathbf{R}}_{-(\gamma-1)}(G)\right\|_{L^\infty(\mathbf{R}_+)}\\ &\leq c\|f\|_{L^1(e^{\rho x})}\|g\|_{L^1(e^{\rho x})}\,.\end{aligned}$$

Hence, I^1_γ and, by (14), I^2_γ are dominated by $\|f\|_{L^1(e^{\rho x})}\|g\|_{L^1(e^{\rho x})}\|h\|_{L^1(e^{\rho x})}$. If a function a on $\mathbf{R}_+$ is supported on $[1, \infty)$, then $\|a\|_{L^1(e^{\rho x})} \leq c\|a\|_{L^{2,1}(e^{2\rho x})} \leq c\|a\|_{L^{2,1}(\Delta)}$ (see [9], Lemma 3). Therefore (35) follows.

(ii) Let $\alpha = \frac{1}{2}$ and $-\frac{1}{2} < \beta < \frac{1}{2}$. Then, I^1_1 is dominated similarly as in (i). It follows from (15) that I^2_γ can be rewritten as

$$\int_0^\infty \left(\int_x^\infty F'\otimes G'(s)B(x,s)ds\right)H(x)\,dx\,, \tag{36}$$

where $|B(x,s)| \leq c$ for all $x \geq 1$. Since

$$\begin{aligned}\|F'\otimes G'\|_{L^1(\mathbf{R}_+)} &\leq \|F'\|_{L^1(\mathbf{R}_+)}\|G'\|_{L^1(\mathbf{R}_+)}\\ &\leq c\|f\|_{L^1(e^{\rho x})}\|g\|_{L^1(e^{\rho x})}\,,\end{aligned}$$

(36) is dominated by $\|f\|_{L^1(e^{\rho x})}\|g\|_{L^1(e^{\rho x})}\|h\|_{L^1(e^{\rho x})}$. The desired result follows as in (i).

(iii) When $-\frac{1}{2} < \alpha < \frac{1}{2}$, it follows from (16) that I^1_γ can be rewritten as

$$\int_0^\infty W^{\mathbf{R}}_{-(\alpha+\frac{1}{2})}(F\otimes G)(x)H(x)\,dx\,. \tag{37}$$

Then, by denoting $\alpha + \frac{1}{2}$ by γ_α, we see that

$$\begin{aligned}&W^{\mathbf{R}}_{-\gamma_\alpha}(F\otimes G)(x) \qquad (38)\\ &= \int_0^\infty\left(\int_{|x-y|}^\infty f(s)W^{\mathbf{R}}_{-\frac{1}{2}\gamma_\alpha}A(|x-y|,s)\,ds\right)\left(\int_{|y|}^\infty g(t)W^{\mathbf{R}}_{-\frac{1}{2}\gamma_\alpha}A(|y|,t)\,dt\right)dy\\ &= \int_1^\infty\int_1^\infty f(s)g(t)\left(\int_0^\infty W^{\mathbf{R}}_{-\frac{1}{2}\gamma_\alpha}A(|x-y|,s)\chi^c_{|x-y|}(s)W^{\mathbf{R}}_{-\frac{1}{2}\gamma_\alpha}A(|y|,t)\chi^c_{|y|}(t)\,dy\right)\\ &\qquad\times dsdt\,.\end{aligned}$$

We shall consider tha case that $x > y$ and $y > 0$. The other cases can be treated similarly. Since $t, s \geq 1$, the integral in the inside parentheses is dominated by

$$e^s e^t \int_{x-s}^{t} (s - x + y)^{\frac{1}{2}\alpha - \frac{3}{4}} (t - y)^{\frac{1}{2}\alpha - \frac{3}{4}} dy = e^s e^t (t - x + s)^{\alpha - \frac{1}{2}} .$$

If $(t - x + s) \geq 1$, then it is dominated by $e^s e^t$ and thus,

$$\left\| W^{\mathbf{R}}_{-\gamma_\alpha}(F \otimes G) \right\|_{L^\infty(e^{\rho x})} \leq c \|f\|_{L^1(e^{\rho x})} \|g\|_{L^1(e^{\rho x})} .$$

Hence the desired result follows as in (i). If $(t - x + s) < 1$, then $x < t + s < x + 1$ and thus, (37) is dominated by

$$\begin{aligned}
&\iiint_{t,s\geq 1, x<t+s<x+1} |f(s)g(t)| e^s e^t (t - x + s)^{\alpha - \frac{1}{2}} ds dt |H(x)|\, dx \\
&= \int_1^\infty \int_x^{x+1} \Big(\int |f(s)| e^s |g(t - s)| e^{t-s}\, ds \Big) (t - x)^{\alpha - \frac{1}{2}} |H(x)|\, dt dx \\
&\leq c \|f\|_{L^2(e^{2\rho x})} \|g\|_{L^2(e^{2\rho x})} \int_1^\infty \Big(\int_x^{x+1} (t - x)^{\alpha - \frac{1}{2}} dt \Big) |H(x)|\, dx \\
&\leq c \|f\|_{L^2(\Delta)} \|g\|_{L^2(\Delta)} \|h\|_{L^1(e^{\rho x})} .
\end{aligned}$$

Therefore, (35) follows. Last we note that I^2_γ can be rewritten as

$$\int_1^\infty \Big(\int_x^\infty W^{\mathbf{R}}_{-2\alpha}(F \otimes G)(s) C(x, s) ds \Big) H(x)\, dx ,$$

where

$$\int_x^\infty |C(x, s)| ds \leq c$$

for all $x > 0$ (see (16)). Replacing $W^{\mathbf{R}}_{-\gamma_\alpha}$ by $W^{\mathbf{R}}_{-2\alpha}$ in (38), we see that the integral in the inside parentheses in (38) is bounded by $e^s e^t$ and thus,

$$\left\| W^{\mathbf{R}}_{-2\alpha}(F \otimes G) \right\|_{L^\infty(e^{\rho x})} \leq c \|f\|_{L^1(e^{\rho x})} \|g\|_{L^1(e^{\rho x})} .$$

Therefore, (35) follows as in (i).

This completes the proof of Theorem 3. □

REMARK 1. In his proof Liu used the kernel form of the convolution

$$f * g(z) = \int_0^\infty \int_0^\infty f(x) g(y) K(x, y, z) \Delta(x) \Delta(y)\, dx dy$$

and the fact that $K(x, y, z)\Delta(x)\Delta(y)\Delta(z) \leq c e^{\rho(x+y+z)}$ if $x, y, z \geq 1$. He quoted this estimate from the explicit form of $K(x, y, z)$ obtained in [12]. Here we note that

$$W_+(K(x, y, \cdot))(z) = \frac{1}{\Delta(x)\Delta(y)} A(\cdot, x) \otimes A(\cdot, y)(z)$$

and thus,

$$K(x, y, z)\Delta(x)\Delta(y)\Delta(z) = W_-(A(\cdot, x) \otimes A(\cdot, y))(z)\Delta(z)\,.$$

The above estimate of $K(x, y, z)$ easily follows from this relation. Roughly speaking, our argument used in the proof of Theorem 3 is a transfer of Liu's one by using this relation.

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