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On a Class of Epstein Zeta Functions

Sami OMAR

American University of the Middle East

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Abstract. X.-J. Li gave in [4] a criterion for the Riemann hypothesis in terms of the positivity of a set of coefficients. In this paper, we investigate exactly how the Li criterion for the Riemann hypothesis fails for a class of Epstein zeta functions. This enables to derive some interesting consequences regarding $c_K = \frac{h_K \log d_K}{\sqrt{d_K}}$ of a quadratic imaginary field K of absolute discriminant d_K and class number h_K. Similar results are stated for the period ratios of elliptic curves with complex multiplication.

1. A class of Epstein zeta functions

Let Q be a 2×2 real symmetric positive-definite matrix of determinant 1. Let $Q[x]$ denote the quadratic form defined by $x \mapsto x^t Qx$, where $x \in \mathbb{R}^2$. We define the Epstein zeta function as follows

$$Z(Q,s) = \frac{1}{2} \sum_{x \in \mathbb{Z}^2 - \{0\}} Q[x]^{-s}, \quad \mathrm{Re}(s) > 1 .$$

The series converges absolutely and uniformly on compacts in the half-plane $\mathrm{Re}(s) > 1$. Moreover, $Z(Q,s)$ can be continued over the whole complex plane, defining a meromorphic function whose only pole is at $s = 1$. Let $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ denote the open upper-half plane. The following map is a bijection between H and the set of such matrices Q

(1) $$z = x + iy \mapsto Q_z = \begin{pmatrix} \frac{1}{y} & -\frac{x}{y} \\ -\frac{x}{y} & \frac{x^2+y^2}{y} \end{pmatrix} .$$

This enables us to define for all $z \in H$ the functions

(2) $$F_z(s) = 2s(s-1)\pi^{-s}\Gamma(s)Z(Q_z,s) .$$

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Another way of stating the above is in terms of Eisenstein series, defined for $z \in H$ by the sum

$$E_s(z) = \frac{1}{2} y^s \sum_{m,n} |mz + n|^{-2s} \quad \text{for } \mathrm{Re}(s) > 1\,,$$

where the sum ranges over all pairs of relatively prime integers $(m, n) \neq (0, 0)$. It is easily shown that

$$\zeta(2s) E_s(z) = Z(Q_z, s) \quad \text{for all } z \in H,\ s \in \mathbb{C}\,,$$

and so we get

$$F_z(s) = 2s(s-1)\pi^{-s}\Gamma(s)\zeta(2s)E_s(z) \quad \text{for all } z \in H,\ s \in \mathbb{C}\,.$$

Let

$$D = \left\{ z \in H \mid -\frac{1}{2} < \mathrm{Re}(z) \le \frac{1}{2}, |z| \ge 1 \text{ and if } |z| = 1, \text{ then } \mathrm{Re}(z) \ge 0 \right\} \tag{3}$$

denote the usual fundamental domain for the action of the modular group $\Gamma = SL_2(\mathbb{Z})$ on H. The following theorem reviews some properties of $F_z(s)$ [10, Chapter III] .

THEOREM 1. *For all $z \in H$, we have*
i) *The function $F_z(s)$ is an entire function satisfying $F_z(0) = 1$.*
ii) *For all $s \in \mathbb{C}$, $F_z(s) = F_z(1-s)$.*
iii) *For all $\gamma \in \Gamma$, $F_{\gamma z}(s) = F_z(s)$.*
iv) *The function $F_z(s)$ has the following Fourier expansion*

$$F_z(s) = 2s(s-1)\left(y^s \Lambda(s) + y^{1-s}\Lambda(1-s) + \sum_{n \neq 0} k_n y^{\frac{1}{2}} K_{s-1/2}(2\pi|n|y) e^{2\pi i n x} \right),$$

where $\Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s)$, $k_n = 2|n|^{s-1/2}\sum_{0<d|n} d^{1-2s}$, and $K_s(z)$ is the K-Bessel function defined for $w \in \mathbb{R}$, $w > 0$, and $s \in \mathbb{C}$ by the equation

$$K_s(w) = \frac{1}{2}\int_0^\infty \exp\left[-\frac{w}{2}\left(t + \frac{1}{t} \right) \right] t^{s-1} dt\,.$$

For instance, when $s = 1/2$, we have the formula $K_{1/2}(w) = \left(\frac{\pi}{2w}\right)^{1/2} e^{-w}$. For $z \in H$, we write the Taylor expansion for F_z at $s = 1$ as follows

$$F_z(s) = \sum_{k=0}^{\infty} a_k(z)(s-1)^k\,. \tag{4}$$

It is well-known that Epstein zeta functions in general do not satisfy the Riemann hypothesis, though we shall prove this result in Section 3.2 using the Li criterion.

2. Generalizations of Kronecker's limit formula

In general, Kronecker's limit formula gives the value of the constant term of the Laurent expansion at $s = 1$ of zeta functions. In the present paper, we will focus not only on $a_1(z)$ but also on the second coefficient $a_2(z)$. For this purpose, we will need to understand the asymptotic properties of the functions $a_1(z)$ and $a_2(z)$ in the next section.

Approximations to the functions $a_1(z)$ and $a_2(z)$. The functions $a_k(z)$ cannot be written in terms of known functions, but the zero-th coefficient in the Fourier expansion of $F_z(s)$ in Theorem 1 enables us to obtain good approximations $\tilde{a}_k(z)$ for $a_k(z)$. For $z \in H$, $y = \mathrm{Im}(z)$, we define functions $\tilde{a}_k(z)$ by the formula

$$\sum_{k=0}^{\infty} \tilde{a}_k(z)(s-1)^k = 2s(s-1)\big(y^s \Lambda(s) + y^{1-s}\Lambda(1-s)\big)\,, \tag{5}$$

where we recall the definition $\Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s)$ from Theorem 1. By using formula (5), we can compute the first coefficients of the Taylor expansion.

PROPOSITION 1. *We have*

$$\tilde{a}_0(z) = 1\,,$$

$$\tilde{a}_1(z) = \frac{\pi}{3}y - \log y + \gamma - \log(4\pi) + 1\,,$$

$$\begin{aligned}\tilde{a}_2(z) = {} & \frac{\pi}{3}y\log y + \Big(2\beta_1 + \frac{\pi}{3}\Big)y + \frac{1}{2}\log^2 y - (1+\gamma-\log(4\pi))\log y \\ & + (\gamma - \log(4\pi) + 2\lambda_2)\,,\end{aligned}$$

$$\begin{aligned}\tilde{a}_3(z) = {} & \frac{\pi}{6}y\log^2 y + \Big(2\beta_1 + \frac{\pi}{3}\Big)y\log y + (2\beta_1+\beta_2)y - \frac{1}{6}\log^3 y \\ & + \frac{1}{2}(1+\gamma-\log(4\pi))\log^2 y - (\gamma - \log(4\pi) + 2\lambda_2)\log y \\ & + 2(\lambda_2+\lambda_3)\,,\end{aligned}$$

where $\beta_1 = \Lambda'(1) \approx -1.49$, $\beta_2 = \Lambda''(1) \approx 7.00$, $\lambda_2 \approx 2.00$ *is the coefficient of* $(s-1)$ *in the Laurent expansion of* $\Lambda(1-s)$ *at* $s=1$ *and* $\lambda_3 \approx -3.99$ *is the coefficient of* $(s-1)^2$ *in the Laurent expansion of* $\Lambda(1-s)$ *at* $s=1$.

PROOF. Let $f(s) = 2s(s-1)y^s\Lambda(s)$ and $g(s) = 2s(s-1)y^{1-s}\Lambda(1-s)$. We can easily check that

$$\begin{aligned}f(s) = {} & \frac{\pi}{3}y(s-1) + \Big(\frac{\pi}{3}y\log y + \Big(2\Lambda'(1) + \frac{\pi}{3}\Big)y\Big)(s-1)^2 + \Big(\frac{\pi}{6}y\log^2 y \\ & + \Big(2\Lambda'(1) + \frac{\pi}{3}\Big)y\log y + \big(2\Lambda'(1) + \Lambda''(1)\big)y\Big)(s-1)^3 + \cdots\end{aligned}$$

and

$$g(s) = 1 + (1 + 2\lambda_1 - \log y)(s-1) + \Big(\frac{1}{2}\log^2 y - (1+2\lambda_1)\log y + 2\lambda_1 + 2\lambda_2\Big)(s-1)^2 + \Big(-\frac{1}{6}\log^3 y + \frac{1}{2}(1+2\lambda_1)\log^2 y - (2\lambda_1 + 2\lambda_2)\log y + 2\lambda_2 + 2\lambda_3\Big)(s-1)^3 + \cdots$$

By computing and replacing

$$\lambda_1 = \zeta(0)\Gamma'(1) + 2\zeta'(0) - \zeta(0)\log\pi = \frac{\gamma}{2} - \log 2\pi + \frac{1}{2}\log\pi$$

in the expressions of $f(s)$ and $g(s)$, we deduce immediately the above formulas for $\tilde{a}_0(z)$, $\tilde{a}_1(z)$, $\tilde{a}_2(z)$ and $\tilde{a}_3(z)$.

As we notice, $\tilde{a}_n(z)$ is a polynomial in y and $\log y$, and in particular, is independent of x. These functions are in fact remarkably good approximations to the functions $a_n(z)$. Actually, the following theorem shows that for $1 \le n \le 2$, the error $|a_n(z) - \tilde{a}_n(z)|$ tends to zero exponentially quickly in both n and $y = \mathrm{Im}(z)$.

THEOREM 2. *For all $z \in D$, we have*

$$|a_1(z) - \tilde{a}_1(z)| \le 0.34\, e^{-\pi y}$$

and

$$|a_2(z) - \tilde{a}_2(z)| \le 2.06\, e^{-\pi y}\,.$$

REMARK 1. We observe that $\tilde{a}_1(z)$ is precisely the same approximation of Kronecker's limit formula. Actually, the Kronecker limit formula states that $a_1(z) = \gamma + 1 - \log(4\pi) - \log(y|\eta(z)|^4)$, where η denotes the Dedekind eta function $\eta(z) = e^{\frac{2\pi i z}{24}} \prod_{n=1}^{\infty}(1 - e^{2\pi i n z})$, for $z \in H$. This is easily proved using the Fourier expansion of Theorem 1 above and the fact that $K_{1/2}(2\pi y) = \frac{1}{2} y^{-\frac{1}{2}} e^{-2\pi y}$. Therefore,

$$-\log\Big(y|\eta(x+iy)|^4\Big) = \frac{\pi}{3}y - \log y - 4\sum_{n=1}^{\infty}\log\big|1 - e^{2\pi i n z}\big|$$

and $a_1(z) = \tilde{a}_1(z) + \epsilon(z)$, where $\epsilon(z) = -4\sum_{n=1}^{\infty}\log|1 - e^{2\pi i n z}|$. The error term $\epsilon(z)$ is very small since by Theorem 2, $|\epsilon(z)| \le 0.34\, e^{-\pi y}$.

PROOF OF THEOREM 2. By Formula (5) and Theorem 1, $a_m(z) - \tilde{a}_m(z)$ is the coefficient of $(s-1)^m$ in the Taylor expansion at $s = 1$ of

$$2s(s-1)\sum_{n\neq 0} k_n \sqrt{y} K_{s-\frac{1}{2}}(2\pi|n|y) e^{2\pi i n x}\,.$$

Let S_m be the Taylor coefficient of $(s-1)^m$ in $\sum_{n\neq 0} k_n\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)$. Therefore, $|a_1(z)-\tilde{a}_1(z)| \leq 2|S_0|$ and $|a_2(z)-\tilde{a}_2(z)| \leq 2|S_0| + 2|S_1|$. Let J_p (resp. I_p) denote the coefficient of $(s-1)^p$ in the expansion of $k_n = 2\sum_{0<d|n} \frac{\sqrt{|n|}}{d}(|n|d^{-2})^{s-1}$ (resp. $K_{s-\frac{1}{2}}$). Next, we shall find bounds for I_0, I_1, J_0 and J_1. By using the integral formula for $K_{s-\frac{1}{2}}$ in Theorem 1, we have

$$I_1 = \frac{1}{2}\int_0^\infty \exp\left(-\frac{w}{2}\left(t+\frac{1}{t}\right)\right)\frac{\log t}{\sqrt{t}}dt\,,$$

where $w = 2\pi|n|y$. First, we observe by the change of variables $t \mapsto \frac{1}{t}$ that

$$\int_0^1 \exp\left(-\frac{w}{2}\left(t+\frac{1}{t}\right)\right)\frac{\log t}{\sqrt{t}}dt = \int_1^\infty \exp\left(-\frac{w}{2}\left(t+\frac{1}{t}\right)\right)\frac{(-\log t)}{\sqrt{t}}\frac{dt}{t}\,.$$

Since for all $t > 0$, $t + \frac{1}{t} \geq \frac{1}{2}(t+\frac{1}{t}) + 1$, and $\frac{\log t}{t} \leq e^{-1}$, for $t \geq 1$, we obtain

$$\begin{aligned}|I_1| &\leq \int_1^\infty \exp\left(-\frac{w}{4}\left(t+\frac{1}{t}\right)\right)\exp\left(-\frac{w}{2}\right)\frac{\log t}{\sqrt{t}}dt\\ &\leq e^{-\frac{w}{2}}\int_1^\infty e^{-\frac{1}{4}wt}\frac{\log t}{\sqrt{t}}dt\\ &\leq e^{-\frac{w}{2}}e^{-1}\int_1^\infty e^{-\frac{1}{4}wt}t\,dt\,.\end{aligned}$$

Therefore

$$|I_1| \leq e^{-\frac{w}{2}}e^{-1}\frac{16}{w^2} \leq e^{-\pi|n|y}\frac{4e^{-1}}{(\pi n y)^2}\,.$$

We recall that $I_0 = K_{\frac{1}{2}}(2\pi|n|y) = \frac{1}{2\sqrt{|n|y}}e^{-2\pi|n|y}$. By considering $s = \pi|n|y$, $\alpha = 2$ in the classical inequality $e^{-s} \leq \alpha^\alpha e^{-\alpha}s^{-\alpha}$, $\forall s > 0$, we obtain

$$I_0 \leq \frac{1}{2\sqrt{|n|y}}e^{-\pi|n|y}\,2^2e^{-2}(\pi|n|y)^{-2}\,.$$

Now, regarding the bounds for J_0 and J_1, we have

$$|J_0| = \left|2\sum_{0<d|n}\frac{\sqrt{|n|}}{d}\right| \leq 2|n|^{\frac{3}{2}}$$

and

$$|J_1| = \left|2\sum_{0<d|n}\frac{\sqrt{|n|}}{d}\log|n|\right| \leq 2\sqrt{n}\log|n|\,|n| \leq 2e^{-1}|n|^{\frac{5}{2}}\,.$$

Since $y \geq \frac{\sqrt{3}}{2}$, it follows that

$$2\sqrt{y}|I_0||J_0| \leq 8e^{-2}(\pi y)^{-2}e^{-\pi|n|y} \leq 0.15\, e^{-\pi|n|y}$$

and

$$|S_0| \leq 2\sqrt{y}\sum_{n=1}^{\infty}|I_0||J_0| \leq 0.15 \sum_{n=1}^{\infty}e^{-\pi ny} = 0.15\, e^{-\pi y}\frac{e^{\pi y}}{e^{\pi y}-1} \leq 0.17\, e^{-\pi y}\,.$$

Therefore, $|a_1(z) - \tilde{a}_1(z)| \leq 0.34\, e^{-\pi y}$. Similarly, we have

$$2\sqrt{y}|I_0||J_1| \leq 8e^{-3}(\pi y)^{-2}e^{-\pi|n|y}$$

and

$$2\sqrt{y}|I_1||J_0| \leq 16e^{-1}\sqrt{y}(\pi y)^{-2}e^{-\pi|n|y}\,.$$

Then

$$\begin{aligned}
|S_1| &\leq 2\sqrt{y}\sum_{n=1}^{\infty}(|I_0||J_1| + |I_1||J_0|)\\
&\leq \sum_{n=1}^{\infty}e^{-\pi ny}\left(8e^{-3}(\pi y)^{-2} + 16e^{-1}\sqrt{y}(\pi y)^{-2}\right)\\
&\leq 0.8 \sum_{n=1}^{\infty}e^{-\pi ny} \leq 0.8\, e^{-\pi y}\frac{e^{\pi y}}{e^{\pi y}-1} \leq 0.86\, e^{-\pi y}\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
|a_2(z) - \tilde{a}_2(z)| &\leq 2|S_0| + 2|S_1|\\
&\leq 0.34\, e^{-\pi y} + 1.72\, e^{-\pi y}\\
&\leq 2.06\, e^{-\pi y}\,.
\end{aligned}$$

3. The Li criterion

Let ξ be an entire function of order < 2, such that $\xi(0) \neq 0$ and satisfying the following functional equation

$$\xi(1-s) = w\xi^*(s)\,, \tag{6}$$

where $\xi^*(s) = \xi(\bar{s})$ and w is constant (necessarily $|w| = 1$). We write the Taylor expansion of $\log\xi(\frac{t}{t-1})$ at the origin as follows

$$\log\xi\left(\frac{t}{t-1}\right) = \log\xi(0) + \sum_{n=1}^{\infty}b_n\frac{t^n}{n}\,. \tag{7}$$

The numbers b_n do not depend on the choice of logarithm and are called Li's coefficients of ξ. For $n \geq 1$, let us denote by $\mathcal{P}_n(\xi)$ the inequality

$$\mathrm{Re}(b_n) \geq 0\,. \tag{8}$$

Bombieri and Lagarias stated the following theorem which is the generalization of the Li criterion for the Riemann hypothesis. For more details, see [1].

THEOREM 3. *The zeros of* ξ *lie on the line* $\mathrm{Re}(z) = \frac{1}{2}$ *if and only if* $\mathcal{P}_n(\xi)$ *is satisfied for all* $n \geq 1$.

Li proved first the above theorem for the function $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, where ζ is the Riemann zeta function [4], and extended this to the case of the Dirichlet and Hecke L-functions [5]. More recently, Omar and Mazhouda generalized the Li criterion to the frame of the Selberg class and established an explicit and asymptotic formula for the Li coefficients b_n [6, 7]. If we write the Taylor expansion of $\xi(s)$ at $s = 1$ as follows

$$\xi(s) = \sum_{n=0}^{\infty} a_n (s-1)^n\,, \tag{9}$$

then it is not difficult to see by (7) that we can write $\overline{b_n} = a_0^{-n} P_n(a_0, \ldots, a_n)$ where P_n is a homogeneous polynomial of degree n with rational coefficients. In particular, we have

$$\overline{b_1} = \frac{a_1}{a_0} \qquad \text{and} \qquad \overline{b_2} = \frac{2(a_0 a_2 + a_0 a_1) - a_1^2}{a_0^2}\,. \tag{10}$$

COROLLARY 4. *The set* $S = \{F_z(s) \mid z \in D\}$ *consists of entire functions in* s *to which we can apply the Li criterion.*

Next, we will study for what $n \in \mathbb{N}$ does $\mathcal{P}_n(F_z)$ fail in general?

3.1. The inequality $\mathcal{P}_1(F_z)$. It is well known that if the zeros of ξ belong to the critical strip $\{z \in \mathbb{C} \mid 0 \leq \mathrm{Re}(z) \leq 1\}$, then $\mathcal{P}_1(\xi)$ is satisfied. However, the zeros of an Epstein zeta function $Z(Q, s)$ are not confined to lie in the critical strip. Therefore, we have no reason a priori to expect $\mathcal{P}_1(F_z)$ to hold. Nonetheless, we prove the following theorem.

THEOREM 5. *For all* $z \in D$, $\mathcal{P}_1(F_z)$ *is satisfied.*

PROOF. We have seen that $a_0(z) = 1$, so by equation (10) the inequality $\mathcal{P}_1(F_z)$ is equivalent to $a_1(z) \geq 0$ for all $z \in D$. Since $y \geq \sqrt{3}/2$, we have $|a_1(z) - \tilde{a}_1(z)| \leq 0.34e^{-\pi y} \leq 0.03$. Therefore,

$$a_1(x+iy) = \gamma + 1 - \log 4\pi + \frac{\pi}{3}y - \log y + \epsilon, \quad \text{where } |\epsilon| \leq 0.03\,.$$

The function $\frac{\pi}{3}y - \log y$ is minimized on $y \geq \sqrt{3}/2$ at $y = 3/\pi$, so

$$a_1(z) \geq \gamma + 2 - \log 12 - 0.03 > 0.06 > 0\,.$$

3.2. The inequality $\mathcal{P}_2(F_z)$. By relying on Li's criterion, the next theorem shows that the Riemann hypothesis does not hold for F_z.

THEOREM 6. *For $z \in D$ and $y = \mathrm{Im}(z)$ is sufficiently large, $\mathcal{P}_2(F_z)$ is not satisfied.*

PROOF. As we saw previously, the Li coefficients b_n related to F_z can be expressed as a homogeneous polynomial of degree n in the functions $a_i(z)$, for $0 \le i \le n$. We are now in a position to understand the asymptotic behaviour of the inequality $\mathcal{P}_2(F_z)$, as $y = \mathrm{Im}(z) \to \infty$. By (10), we have the equivalence

$$\mathcal{P}_2(F_z) \Longleftrightarrow 2(a_2 + a_1) - a_1^2 \ge 0\,,$$

where we have used the fact that $a_0(z) = 1$ and that the functions $a_i(z)$ are real valued. Using the formula for $\tilde{a}_1$ and $\tilde{a}_2$ from Section 2, we have

$$\begin{aligned}2(a_2(z) + a_1(z)) - a_1^2(z) &\sim 2(\tilde{a}_2(z) + \tilde{a}_1(z)) - \tilde{a}_1^2(z)\\ &\sim \frac{4\pi}{3} y \log y + \lambda y - \left(\frac{\pi}{3} y\right)^2 \longrightarrow -\infty \quad \text{as } y \to \infty\,,\end{aligned}$$

where $\lambda = 4\beta_1 + \frac{2\pi}{3}(1 + \log(4\pi) - \gamma)$.

4. Hecke's theorem and Gauss' problem for quadratic imaginary number fields

The previous result has interesting consequences for the Dedekind zeta function ζ_K of quadratic imaginary number fields K. Let d_K be the absolute discriminant of K and set

$$\xi_K(s) = 2s(s-1)d_K^{\frac{s}{2}}(2\pi)^{-s}\Gamma(s)\zeta_K(s)\,. \tag{11}$$

The function ξ_K is entire of order 1 and non-vanishing at $s = 0$. Moreover, it satisfies the functional equation $\xi_K(1-s) = \xi_K(s)$. Therefore, we may apply Theorem 3 to ξ_K. The following theorem is due to Hecke [9].

THEOREM 7. *Let h_K denote the class number of K and w the number of roots of unity in K. Then, there exist h_K numbers $z_1, \dots, z_{h_K} \in D$ such that*

$$\zeta_K(s) = \frac{2^s d_K^{-\frac{s}{2}}}{w} \sum_{i=1}^{h_K} 2Z(Q_{z_i}, s) \quad \textit{i.e.} \quad \xi_K(s) = \frac{2}{w} \sum_{i=1}^{h_K} F_{z_i}(s)\,. \tag{12}$$

Now, we can express the inequality $\mathcal{P}_2(\xi_K)$ in terms of the functions a_0, a_1, a_2 and the points $z_1, \dots, z_{h_K}$. By the homogeneity of the equations (10), we have

$$\mathcal{P}_2(\xi_K) \quad \Longleftrightarrow \quad 2(a_1' + a_2') \ge (a_1')^2\,, \tag{13}$$

where

$$a_1' = \frac{1}{h_K} \sum_{k=1}^{h_K} a_1(z_k) = \big(\gamma + 1 - \log(4\pi)\big) - \frac{1}{h_K} \sum_{k=1}^{h_K} \log\big(y_k |\eta(x_k + i y_k)|^4\big)\,, \tag{14}$$

(15) $$a_2' = \frac{1}{h_K} \sum_{k=1}^{h_K} a_2(z_k)\,.$$

We have just seen in the previous paragraph that the expression

$$2(\tilde{a}_2(z) + \tilde{a}_1(z)) - \tilde{a}_1(z)^2 \sim -\left(\frac{\pi}{3}y\right)^2$$

as $y = \mathrm{Im}(z) \to \infty$, so the inequality $\mathcal{P}_2(\xi_K)$ imposes strong constraints on the distribution of the points z_i as we shall see shortly.

REMARK 2. It can be shown that $\mathcal{P}_2(\xi_K)$ implies ζ_K has no zeros in a small region close to 1, for K an arbitrary number field. Conversely, a (larger) zero-free region near 1 implies that $\mathcal{P}_2(\xi_K)$ must hold. We refer to [8] for the exact statements.

4.1. The distribution of the points $z_1, \dots, z_{h_k}$**.** Let K be a quadratic imaginary number field, h_K its class number, and $z_1, \dots, z_{h_K}$ the points defined in Theorem 7. Let $y_i = \mathrm{Im}(z_i)$ for $i = 1, \dots, h_K$, and define $\log^+ t = \sup\{1, \log t\}$ for $t \in \mathbb{R}$. We will write $y = \mathrm{Im}(z)$.

COROLLARY 8. *There exists an absolute explicit constant* $C_1 > 0$ (*i.e. not depending on* K), *such that if* $\mathcal{P}_2(\xi_K)$ *is satisfied, then*

$$\frac{1}{h_K} \sum_{i=1}^{h_K} y_i \le C_1 \log^+ h_K\,.$$

PROOF. By the proof of Theorem 5, the function $z \mapsto a_1(z)$ is bounded below by a strictly positive number on D. Now, $a_1(z) \sim \tilde{a}_1(z) \sim \frac{\pi}{3}y$ as $y = \mathrm{Im}(z) \to \infty$, so this implies that there exists a constant $c_1 > 0$ such that

(16) $$a_1(z) \ge c_1 y \quad \text{for all } z \in D\,.$$

This constant is of course calculable and does not depend on K. We will refrain from repeating this in the sequel. Now by Theorem 2 and the formula $\tilde{a}_1(z)$ and $\tilde{a}_2(z)$ of Section 2, we have

$$\frac{a_1(z) + a_2(z)}{y \log^+ y} \sim \frac{\tilde{a}_1(z) + \tilde{a}_2(z)}{y \log^+ y} \to \frac{\pi}{3} \quad \text{as } y = \mathrm{Im}(z) \to \infty\,.$$

Since $y \log^+ y$ is bounded below by some strictly positive number on D, there exists a constant $c_2 > 0$ such that

(17) $$a_1(z) + a_2(z) \le c_2 y \log^+ y \quad \text{for all } z \in D\,.$$

Using equations (13), (14) and (15), we obtain for some positive constant $c_3 > 0$,

(18) $$\frac{1}{h_K} \sum_{i=1}^{h_K} y_i \log^+ y_i \ge c_3 \left(\frac{1}{h_K} \sum_{i=1}^{h_K} y_i\right)^2\,.$$

The function $y \mapsto \log^+ y$ is increasing, so the left hand side above is bounded by $h_K^{-1} T \log^+ T$, where $T = \sum_{i=1}^{h_K} y_i$. It follows that

$$\frac{T}{\log^+ T} \leq c_3^{-1} h_K \, .$$

Therefore, it is easy to deduce that there exists a strictly positive constant C_1 such that $T \leq C_1 h_K \log^+ h_K$.

COROLLARY 9. *Let $E_1, \dots, E_{h_K}$ denote the set of elliptic curves with complex multiplication by the full ring of integers $\mathbb{O}_K$ of K. Then, the numbers z_i, for $i = 1, \dots, h_K$ are the period ratios of these curves. If $\mathcal{P}_2(\xi_K)$ is satisfied, then these period ratios have small average imaginary part, and similarly gives $\frac{1}{h_K} \sum_{i=1}^{h_K} \log^+ j(E_i) = O\left(\log^+ h_K\right)$, where j denotes the elliptic modular function (since we know from the q-expansion that $j(z) = O(e^{2\pi y})$).*

The above result can be derived from Corollary 8 since for a CM elliptic curve E, the invariant $j(E)$ corresponds to the imaginary part of the period ratio of E.

4.2. Binary quadratic forms of negative discriminant and the Gauss problem. We can interpret the points z_i of Theorem 7 in terms of binary quadratic forms. If d_K is the absolute discriminant of K, then

$$z_i = \frac{-b_i + \sqrt{-d_K}}{2a_i} \quad \text{for } i = 1, \dots, h_K \, ,$$

where $\left\{(x, y) \mapsto a_i x^2 + b_i xy + c_i y^2 \mid 1 \leq i \leq h_K\right\}$ denotes the set of all reduced binary quadratic forms of discriminant $-d_K$. We recall this is just the set of relatively prime triplets $a, b, c \in \mathbb{Z}$ satisfying $-d_K = b^2 - 4ac$, with $a > 0$, $-a < b \leq a \leq c$, and $c > a$ if $b < 0$. It is well known that if ζ_K has no Siegel zeros (i.e. eventual real zeros of ζ_K in the disk with center 1 and radius $O(\frac{1}{\log d_K})$), then there is an effective lower bound on h_K of the following form

$$h_K \geq c_4 \frac{\sqrt{d_K}}{\log d_K} \, ,$$

for some absolute constant $c_4 > 0$ [2]. The classical proof of the above inequality uses estimates on the zeros. However, as an application of our functions $a_k(z)$, we show how to prove the same result directly from the inequality $\mathcal{P}_2(\xi_K)$ without resorting to estimates on zeros of ζ_K.

COROLLARY 10. *Let $n \geq 1$, and $a_1, \dots, a_n$ be the leading terms of distinct reduced binary quadratic forms of absolute discriminant d_K, as above. There exist absolute constants*

$c_5 > 0$ *and* $d_0 > 0$ (*which depends only on* c_5) *such that*, *if* $d_K \geq d_0$ *and* $\mathcal{P}_2(\xi_K)$ *holds*, *then*

$$h_K \geq c_5 \left(\sum_{i=1}^{n} \frac{1}{a_i} \right) \frac{\sqrt{d_K}}{\log d_K} .$$

PROOF. The inequality $\mathcal{P}_2(\xi_K)$ implies that

$$2h_K \sum_{i=1}^{h_K} (a_1(z_i) + a_2(z_i)) \geq \left(\sum_{i=1}^{h_K} a_1(z_i) \right)^2 .$$

Using the inequalities (16) and (17) derived from the approximations $\tilde{a}_i(z)$ for $a_i(z)$, we have

$$2c_2 h_K \sum_{i=1}^{h_K} y_i \log^+ y_i \geq c_1^2 \left(\sum_{i=1}^{h_K} y_i \right)^2 .$$

Set $T = \sum_{i=1}^{h_K} y_i$. It is clear that $\sum_{i=1}^{h_K} y_i \log^+ y_i \leq T \log^+ T$, whence

$$h_K \geq \frac{c_1^2}{2c_2} \frac{T}{\log^+ T} = \frac{c_1^2}{2c_2} \left(\sum_{i=1}^{h_K} \frac{1}{2a_i} \right) \frac{\sqrt{d_K}}{\log^+ T} .$$

Now, we know that there is a point z_i corresponding to the trivial class $(1) \in \mathbb{O}_K$, whose imaginary part $y_i = O(\sqrt{d_K})$. It follows that $T \to \infty$ as $d_K \to \infty$. Now, we know by Minkowski's theorem that $h_K \leq \frac{1}{\pi}\sqrt{d_K} \log d_K$, and thus by Corollary 8 we deduce that $\log T = O(\log d_K)$. Therefore, there exists $c_5 > 0$ such that for large d_K

$$\log^+ T = \log T \leq \frac{c_1^2}{4c_2 c_5} \log d_K .$$

If $d_K \equiv 1 \bmod 24$, then d_K is a perfect square modulo $4a$ for $a = 1, 2, 3, 6$. Therefore, in the above corollary $\sum_{i=1}^{n} \frac{1}{a_i} > 3$. We also notice experimentally that this term can clearly be arbitrarily high using different values of h_K. In the below table, we first compute the values $c_K = \frac{h_K \log d_K}{\sqrt{d_K}}$ and $\tilde{c}_K = \frac{h_K \log d_K}{\sqrt{d_K} \sum_{i=1}^{n} a_i^{-1}}$ for d_K around 10^6 using PARI-GP, and we notice that c_K can fluctuate enormously, whereas $\tilde{c}_K$ much less so for large absolute discriminants d_K.

d_K	h_K	c_K	$\tilde{c}_K$	d_K	h_K	c_K	$\tilde{c}_K$
1000003	105	1.45	1.14	1000052	306	4.22	2.71
1000007	630	8.70	2.61	1000055	828	11.43	3.04
1000011	368	5.08	2.83	1000056	364	5.02	2.86
1000015	430	5.94	2.44	1000059	240	3.31	1.87
1000019	342	4.72	2.75	1000063	394	5.40	3.21
1000020	320	4.42	2.68	1000067	318	4.39	2.73
1000023	706	9.75	3.43	1000068	372	5.13	2.48
1000024	274	3.78	2.56	1000072	264	3.62	2.61
1000027	168	2.32	1.87	1000079	974	13.45	3.57
1000031	928	12.82	3.26	1000083	184	2.53	2.04
1000036	192	2.65	1.91	1000084	105	1.45	1.26
1000039	877	12.11	2.97	1000087	300	4.14	2.30
1000040	688	9.50	2.62	1000088	372	5.13	2.39
1000043	192	2.65	2.13	1000091	342	4.72	2.82
1000047	508	7.01	2.63	1000095	720	9.94	3.18
1000051	276	3.81	2.52	1000099	187	2.58	1.87

Finally, we believe that we can extend the results of this paper to other classes of Epstein zeta functions. For instance, it is interesting to write down the approximation formulas for $a_k(z)$ in the case of real quadratic fields by relying on the work of Zagier [12] and Yamamoto [11]. Furthermore, it might be feasible to deal with a general number field case by relying on the ideas of Katayama [3]. These problems will be considered in a sequel to this article.

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Present Address:
DEPARTMENT OF MATHEMATICS AND STATISTICS,
AMERICAN UNIVERSITY OF THE MIDDLE EAST,
EGAILA, KUWAIT
e-mail: sami.omar@fst.rnu.tn