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Norm Conditions on Maps between Certain Subspaces of Continuous Functions

Razieh Sadat GHODRAT, Fereshteh SADY and Arya JAMSHIDI

Tarbiat Modares University

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Abstract. For a locally compact Hausdorff space X, let $C_0(X)$ be the Banach space of continuous complex-valued functions on X vanishing at infinity endowed with the supremum norm $\|\cdot\|_X$. We show that for locally compact Hausdorff spaces X and Y and certain (not necessarily closed) subspaces A and B of $C_0(X)$ and $C_0(Y)$, respectively, if $T : A \longrightarrow B$ is a surjective map satisfying one of the norm conditions

i) $\|(Tf)^s(Tg)^t\|_Y = \|f^s g^t\|_X$,

or

ii) $\|\,|Tf|^s + |Tg|^t\,\|_Y = \|\,|f|^s + |g|^t\,\|_X$,

for some $s, t \in \mathbb{N}$ and all $f, g \in A$, then there exists a homeomorphism $\varphi : \mathrm{ch}(B) \longrightarrow \mathrm{ch}(A)$ between the Choquet boundaries of A and B such that $|Tf(y)| = |f(\varphi(y))|$ for all $f \in A$ and $y \in \mathrm{ch}(B)$. We also give a result for the case where A is closed (or, in general, satisfies a special property called Bishop's property) and $T : A \longrightarrow B$ is a surjective map satisfying the inclusion $R_\pi((Tf)^s(Tg)^t) \subseteq R_\pi(f^s g^t)$ of peripheral ranges. As an application, we characterize such maps between subspaces of the form $A_1 f_1 + A_2 f_2 + \cdots + A_n f_n$, where for each $1 \le i \le n$, A_i is a uniform algebra on a compact Hausdorff space X and f_i is a strictly positive continuous function on X. Our results in case (ii) improve similar results in [30], for subspaces rather than uniform algebras, without the additional assumption that T is $\mathbb{R}^+$-homogeneous.

1. Introduction

The study of maps between various Banach algebras preserving the spectrum or the norm of algebra elements is an active research area in modern Banach algebra theory. It is known that under certain natural conditions, such maps are forced to be linear or multiplicative. By the well-known Gleason-Kahane-Żelazko theorem, a linear surjective spectrum-preserving map $T : A \longrightarrow B$ between commutative semisimple Banach algebras is an algebra isomorphism. On the other hand, by a result of Kowalski and Słodkowski [17], an arbitrary map $T : A \longrightarrow B$ satisfying $T(0) = 0$ and $\sigma(T(a) - T(b)) \subseteq \sigma(a - b)$, for $a, b \in A$, is linear and multiplicative. Here $\sigma(\cdot)$ denotes the spectrum of algebra elements. Another major concern is to characterize such maps as weighted composition operators or multiples of

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algebra homomorphisms. Nagasawa [25] and de Leeuw, Rudin and Wermer [19] extended the Banach-Stone theorem to uniform algebras and show that a surjective linear isometry $T : A \longrightarrow B$ between uniform algebras A and B on compact Hausdorff spaces X and Y, respectively, is of the form $Tf = w\Phi(f)$, $f \in A$, where w is an invertible element in B and $\Phi : A \longrightarrow B$ is an algebra isomorphism.

In [24], Molnár described surjective maps T on $C(X)$, where X is a first countable compact Hausdorff space, satisfying $T(1) = 1$ and $\sigma(TfTg) = \sigma(fg)$ for $f, g \in C(X)$, as weighted composition operators of the form $Tf = f \circ \varphi$ for some homeomorphism $\varphi : X \longrightarrow X$ and so that T is an algebra isomorphism. This result has been extended in various directions. For instances, generalizations of this result have been given in [7, 26, 27] for uniform algebras, in [22] for maps between uniform algebras satisfying peripherally multiplicative condition (rather than multiplicative condition on the spectrum), in [9, 21] and [30] for maps T between uniform algebras satisfying multiplicative norm condition $\|TfTg\|_Y = \|fg\|_X$ and those satisfying norm condition $\||Tf| + |Tg|\|_Y = \||f| + |g|\|_X$, respectively. For more results see also [3, 6, 10, 11, 28].

Most recently, in [23] a large number of previous results are obtained for a pair of maps, not necessarily linear, between multiplicative subsets of function algebras satisfying various spectral conditions. Related results for a pair of maps, satisfying certain norm conditions, whose ranges are absolutely multiplicative subsets of uniform algebras, were also given in [9].

In this paper we first study surjective maps $T : A \longrightarrow B$ between certain subspaces A and B (not necessarily closed) of $C_0(X)$ and $C_0(Y)$, for locally compact Hausdorff spaces X and Y, respectively, satisfying one of the norm conditions

(i) $\|(Tf)^s(Tg)^t\|_Y = \|f^s g^t\|_X$,

or

(ii) $\|\,|Tf|^s + |Tg|^t\,\|_Y = \|\,|f|^s + |g|^t\,\|_X$,

for some $s, t \in \mathbb{N}$ and all $f, g \in A$. We show that there exists a homeomorphism $\varphi : \mathrm{ch}(B) \longrightarrow \mathrm{ch}(A)$ such that $|Tf| = |f \circ \varphi|$ on $\mathrm{ch}(B)$, for all $f \in A$. Since A and B are not assumed to be neither multiplicative nor closed (or complete under some norms), our results in case (i) generalize the previous results on such maps between uniform algebras or (Banach) function algebras. We should note that the case (ii) has been considered in [30] for uniform algebras on compact Hausdorff spaces when $s = t = 1$ and it was shown that, under the additional assumption that T is $\mathbb{R}^+$-homogenous, there exists a continuous map $\tau : \mathrm{ch}(A) \longrightarrow \mathrm{ch}(B)$ such that the equality $|Tf(\tau(x))| = |f(x)|$ holds for all $f \in A$ and $x \in \mathrm{ch}(A)$. However, we get the same result (with a homeomorphism τ) for maps between certain subspaces of $C_0(X)$ and $C_0(Y)$, not assumed to be closed, rather than uniform algebras, without assuming that T is $\mathbb{R}^+$-homogenous. Next we show that if A is closed (or has a certain property called Bishop's property) and $T : A \longrightarrow B$ satisfies the stronger condition $R_\pi((Tf)^s(Tg)^t) \subseteq R_\pi(f^s g^t)$ on peripheral ranges for all $f, g \in A$, then $(Tf)^d(y) = \gamma(y) f^d(\varphi(y))$ for all $f \in A$ and $y \in \mathrm{ch}(B)$, where d is the greatest common divisor of s and t, φ is as above and γ is a unimodular continuous function on $\mathrm{ch}(B)$. This is similar to the results stated in [8, 9, 23] for such maps between certain (absolutely) multiplicative subsets of uniform algebras and function algebras and in [3] for pointed Lipschitz algebras.

As an application, we consider subspaces of $C(X)$ of the form $A_1 f_1 + A_2 f_2 + \cdots + A_n f_n$ where for each $i = 1, ..., n$, A_i is a uniform algebra on a compact Hausdorff space X and $f_i \in C(X)$ is strictly positive. We should note that a discussion on surjective linear isometries between subspaces of the form Af, where A is a uniform algebra on a compact Hausdorff space X and $f \in C(X)$ is strictly positive, was given in [1]. Moreover, the results of [16] and [12] provide the form of surjective isometries, respectively into isometries, not assumed to be linear, on such spaces. However, the known results on maps satisfying norm conditions (i) and (ii), can not be applied for such subspaces, since they are not (absolutely) multiplicative.

2. Preliminaries

For an arbitrary topological space X, let $C(X)$ be the space of all continuous complex-valued functions on X. When X is a locally compact Hausdorff space, $C_0(X)$ denotes the Banach space of all continuous complex-valued functions on X vanishing at infinity, with the supremum norm $\|.\|_X$. In this case X_∞ is the one-point compactification of X.

For a locally compact Hausdorff space X and a function $f \in C_0(X)$, by $M(f)$ we mean the maximum set of f, i.e. $M(f) = \{x \in X : |f(x)| = \|f\|_X\}$ which is clearly a compact subset of X whenever f is nonzero. The *peripheral range* of $f \in C_0(X)$ is defined by $R_\pi(f) = \{f(x) : x \in M(f)\}$. For a subspace A of $C_0(X)$, we say that $f \in A$ is a *peaking function* if $R_\pi(f) = \{1\}$ and in this case we say that the set $\{x \in X : f(x) = 1\}$ is a *peak set* for A. For a subspace A of $C_0(X)$, we denote the dual space of A (with respect to the supremum norm) by A^* and the Choquet boundary of A by $\mathrm{ch}(A)$. We recall that each extreme point of the unit ball of A^* is of the form $\alpha\varphi_x$ for some scalar α with $|\alpha| = 1$ and $x \in X$, where φ_x is the evaluation functional at x. Moreover, $\mathrm{ch}(A)$ consists of all points $x \in X$ such that φ_x is an extreme point of the unit ball of A^*. For each subspace A of $C_0(X)$, $\mathrm{ch}(A)$ is a boundary of A, that is for each $f \in A$, $M(f) \cap \mathrm{ch}(A) \neq \emptyset$ (see [29, Page 184]) . Clearly in the case where X is compact and A contains the constant function 1, a point $x \in X$ is in $\mathrm{ch}(A)$ if and only if φ_x is an extreme point of $T_A = \{L \in A^* : L(1) = 1 = \|L\|\}$.

For a compact Hausdorff space X, a *uniform algebra* on X is a closed subalgebra A of $C(X)$ containing the constants and separating the points of X. In general for a locally compact Hausdorff space X, a uniform algebra on X is a closed subalgebra A of $C_0(X)$, such that for distinct points $x, y \in X$, there exists $f \in A$ with $f(x) \neq f(y)$ and for each $x \in X$ there exists $f \in A$ with $f(x) \neq 0$. It is well known that for a uniform algebra A on a locally compact Hausdorff space X, $\mathrm{ch}(A)$ consists of all points $x \in X$ such that for each neighborhood U of x there exists $f \in A$ such that $f(x) = 1 = \|f\|_X$ and $|f| < 1$ on $X \backslash U$ (see [20, Theorem 4.7.22] for compact case and [27, Theorem 2.1] for general case).

3. Main Results

We begin this section by introducing some notations and a definition. Let X be a locally compact Hausdorff space and A be a subspace of $C_0(X)$. For a point $x \in X$ we fix the following notations:

$$V_x(A) = \{f \in A : f(x) = 1 = \|f\|_X\} \quad , \quad F_x(A) = \{f \in A : |f(x)| = 1 = \|f\|_X\}$$

DEFINITION 3.1. Let X be a locally compact Hausdorff space and A be a subspace of $C_0(X)$. We call a closed subset E of X a *weak peak set* for A if for each neighborhood U of E and $0 < \epsilon \leq 1$ there exists a function $f \in A$ such that $f|_E = 1 = \|f\|_X$ and $|f| < \epsilon$ on $X \backslash U$. A *weak peak point* for A is a point $x \in X$ such that $\{x\}$ is a weak peak set for A.

Clearly every weak peak set for A is necessarily compact. We denote the set of all weak peak points for A by $\Theta(A)$.

We emphasis on the distinction between the notion of weak peak points defined above and the one given in some literatures as the intersections of peak sets. Indeed, the above definition is compatible with the one given in [13, Definition 1]. However, the two definitions are the same in uniform algebra case. We should also note that some authors call weak peak points as strong boundary points (see [16]) while the others use the notation of strong boundary points for the points $x \in X$ satisfying the above condition for just $\epsilon = 1$. Clearly for subalgebras of $C_0(X)$ these definitions are the same.

We note that for any subspace A of $C_0(X)$, $\Theta(A) \subseteq \mathrm{ch}(A)$ (see [2, Theorem 2.2.1] for compact case and [12] for locally compact case) and, as we noted before, for each uniform algebra A on X, $\Theta(A) = \mathrm{ch}(A)$.

For a subspace A of $C_0(X)$, it is easy to see that weak peak points for A can be described as points $x \in X$ such that $\mu(\{x\}) = 0$ for all $\mu \in A^{\perp}$ (see for example [13] for compact case).

For a subset E of a locally compact Hausdorff space X, we say that a bounded continuous function f on X *belongs locally* to a subspace A of $C_0(X)$ at E if there exists a neighborhood U of E and $g \in A$ with $\|g\|_X = \|f\|_X$ and $f|_U = g|_U$. For example, for a locally compact Hausdorff space Ω, the constant function 1 on Ω belongs locally to $C_0(\Omega)$ at each compact subset of Ω.

PROPOSITION 3.2. *Let X be a locally compact Hausdorff space and A be a subspace of $C_0(X)$. If E is a weak peak set for A such that the constant function* 1 *belongs locally to A at E, then E is an intersection of peak sets for A. In particular, if X is compact and A contains the constant function* 1, *then each weak peak set for A is an intersection of peak sets for A.*

PROOF. We first note that for each neighborhood W of E and $\epsilon > 0$, the function $g \in A$ with the property that $g|_E = 1 = \|g\|_X$ and $|g| < \epsilon$ on $X \backslash W$, can be chosen to be a peaking function. Indeed, by hypothesis, there exists a neighborhood U of E and $f_0 \in A$ with norm 1 such that $f_0|_U = 1$. Fixing $0 < \epsilon \leq \frac{1}{2}$, for each neighborhood W of E there exists an $f \in A$ such that $f|_E = 1 = \|f\|_X$ and $|f| < \epsilon$ on $X \backslash (W \cap U)$. Setting $g_W = \epsilon f_0 + (1-\epsilon) f$ we see that $g_W \in A$ is a peaking function with $g_W|_E = 1 = \|g_W\|_X$ and $|g_W| < 2\epsilon$ on $X \backslash W$. Therefore, E is the intersection $\cap_W M(g_W)$ of peak sets, where the intersection is taken over all neighborhoods W of E. □

We recall that by [5, Theorem 2.12.7] for a uniform algebra A on a compact Hausdorff space X, a closed subset E of X is a weak peak set for A if and only if $\mu \in A^{\perp}$ implies $\mu_E \in A^{\perp}$, where μ_E is the restriction of a regular Borel measure μ on X to E. However, the following important theorem in [5] shows, in particular, that for a closed subspace A of $C(X)$, every closed subset E of X satisfying this implication is a weak peak set for A.

THEOREM 3.3 ([5, Theorem 2.12.5]). *Let A be a closed subspace of $C(X)$, for a compact Hausdorff space X, and E be a closed subset of X such that $\mu_E \in A^{\perp}$ for all measures $\mu \in A^{\perp}$. Let $f \in A|_E$ and p be a strictly positive continuous function on X such that $|f(y)| \leq p(y)$ for all $y \in E$. Then there is $g \in A$ such that $g|_E = f$ and $|g(x)| \leq p(x)$ for all $x \in X$.*

The above theorem yields the following corollary:

COROLLARY 3.4. *Let $A_1, ..., A_n$ be uniform algebras on a compact Hausdorff space X, $f_1, ..., f_n \in C(X)$ be strictly positive and $A = A_1 f_1 + \cdots + A_n f_n$. If E is a weak peak set for A_i, for some $i = 1, ..., n$ such that $1 \in A_i f_i|_E$, then E is a weak peak set for A. In particular, for each $i = 1, ..., n$, $\mathrm{ch}(A_i) = \Theta(A_i) \subseteq \Theta(A)$.*

PROOF. Since for each uniform algebra B on X and strictly positive $f \in C(X)$, Bf is a closed subspace of $C(X)$ and $\mu \in Bf^{\perp}$ implies that $f\mu \in B^{\perp}$, using the above theorem we conclude that each weak peak set E for some A_i with $1 \in A_i f_i|_E$, is a weak peak set for $A_i f_i$ and consequently a weak peak set for A. In particular $\cup_{i=1}^n \Theta(A_i) \subseteq \Theta(A)$. □

To state our results, we define two maps $\Phi_+, \Phi_\times : C(X) \times C(X) \longrightarrow C(X)$ by $\Phi_+(f, g) = |f| + |g|$ and $\Phi_\times(f, g) = fg$ for all $f, g \in C(X)$, where X is a compact Hausdorff space. As we mentioned before, for the case where $\Phi = \Phi_\times$, the following result is well known whenever A and B are uniform algebras or multiplicative subsets of uniform algebras. The case where $\Phi = \Phi_+$, $s = t = 1$ and A and B are uniform algebras has also been investigated in [30] under the additional assumption that T is $\mathbb{R}^+$-homogeneous.

We first state our result for compact case and then obtain the locally compact case as a corollary (Corollary 3.14). In the next theorem for a compact Hausdoeff space X, by a *good subspace* of $C(X)$ we mean a subspace A of $C(X)$ satisfying one of the following conditions:

(i) A contains constants,

or

(ii) A is a subalgebra of $C(X)$,

or

(iii) $\mathrm{ch}(A)$ is a closed subset of X.

THEOREM 3.5. *Let X, Y be compact Hausdorff spaces, A, B be good subspaces of $C(X)$ and $C(Y)$, not assume to be closed, with $\mathrm{ch}(A) = \Theta(A)$ and $\mathrm{ch}(B) = \Theta(B)$. Then for every surjective map $T : A \longrightarrow B$ satisfying the norm condition*

$$\|\Phi((Tf)^s, (Tg)^t)\|_Y = \|\Phi(f^s, g^t)\|_X \qquad (f, g \in A)$$

where $\Phi = \Phi_+$ or $\Phi = \Phi_\times$ and $s, t \in \mathbb{N}$, there exists a homeomorphism $\varphi : \mathrm{ch}(B) \longrightarrow \mathrm{ch}(A)$ such that $|Tf(\varphi(y_0))| = |f(\varphi(y_0))|$ for all $f \in A$ and $y_0 \in \mathrm{ch}(B)$.

The proof of the above theorem will be given through the following lemmas. In the next lemmas, we assume that X, Y are compact Hausdorff spaces, A and B are subspaces of $C(X)$, not necessarily closed, with $\mathrm{ch}(A) = \Theta(A)$ and $\mathrm{ch}(B) = \Theta(B)$. We also assume that $T : A \longrightarrow B$ is a surjective map satisfying $\|\Phi((Tf)^s, (Tg)^t)\|_Y = \|\Phi(f^s, g^t)\|_X$ for all $f, g \in A$ where $\Phi = \Phi_+$ or $\Phi = \Phi_\times$.

LEMMA 3.6. *Let $\mathcal{A}$ be a subspace of $C(X)$ and $s, t \in \mathbb{N}$. Then given $x_0 \in \Theta(\mathcal{A})$, $\Phi(|f(x_0)|^s, 1) = \inf\{\|\Phi(f^s, h^t)\|_X : h \in V_{x_0}(\mathcal{A})\}$ for the case where $\Phi = \Phi_\times$ and $f \in \mathcal{A}$ or the case where $\Phi = \Phi_+$ and $f \in \mathcal{A}$ with $\|f\|_X \le 1$.*

PROOF. Let $\Phi = \Phi_\times$ or $\Phi = \Phi_+$ and let $f \in \mathcal{A}$ be given. Clearly $\Phi(|f(x_0)|^s, 1) \le \|\Phi(f^s, h^t)\|_X$ for all $h \in V_{x_0}(\mathcal{A})$. Given $0 < \epsilon \le 1$ we set $U = \{x \in X : |f^s(x) - f^s(x_0)| < \epsilon\}$. Then since $x_0 \in \Theta(\mathcal{A})$, there exists $h \in V_{x_0}(\mathcal{A})$ such that $|h| < \min(\frac{\epsilon}{\|f\|_X^s}, \epsilon)$ on $X \backslash U$. Since $|h^t| \le |h|$, it is easy to see that $\|f^s h^t\|_X \le |f(x_0)|^s + \epsilon$ and, moreover, $|f^s(x)| + |h^t(x)| \le |f^s(x_0)| + \epsilon + 1$ for all $x \in U$. Now if $\|f\|_X \le 1$, then for each $x \in X \backslash U$, $|f(x)|^s + |h(x)|^t \le 1 + \epsilon$ and hence $\|\, |f|^s + |h|^t \,\|_X \le |f(x_0)|^s + \epsilon + 1$ which proves the desired equalities. □

LEMMA 3.7. *Let $y_0 \in \mathrm{ch}(B)$. Then for each $r > 0$, the intersection $\cap_{Tf \in rV_{y_0}(B)} M(f)$ is nonempty.*

PROOF. Let $y_0 \in \mathrm{ch}(B)$ and $r > 0$ be given. It suffices to show that the family $\{M(f) : Tf \in rV_{y_0}(B)\}$ of compact subsets of X has finite intersection properties. Let $f_1, ..., f_n \in A$ such that $Tf_i \in rV_{y_0}(B)$ for $i = 1, ..., n$. Since T is surjective there exists $h \in A$ such that $Th = \frac{1}{n}\Sigma_{i=1}^n Tf_i$. We note that the norm condition on T in both cases implies that $\|Tf\|_Y = \|f\|_X$ for $f \in A$. Indeed, in the case where $\Phi = \Phi_\times$ we have $\|(Tf)^{s+t}\|_Y = \|f^{s+t}\|_X$ and hence $\|Tf\|_Y = \|f\|_X$ and in the case where $\Phi = \Phi_+$ since $\|\, |T0|^s + |T0|^t \,\|_Y = 0$ it follows that $T0 = 0$ and consequently $\|(Tf)^s\|_Y = \|f^s\|_X$, i.e. $\|Tf\|_Y = \|f\|_X$. Hence in both cases T is norm preserving. Therefore, $\|h\|_X = \|Th\|_Y = r = Th(y_0)$. Since $\Theta(A) = \mathrm{ch}(A)$ and $\mathrm{ch}(A)$ is a boundary for A, there exists $x_0 \in \Theta(A)$ such that $|h(x_0)| = r = \|h\|_X$. We claim that $x_0 \in M(f_i)$ for $i = 1, ..., n$. Assume on the contrary that $|f_i(x_0)| < r$ for some $1 \le i \le n$. Then there exists a neighborhood U of x_0 such that $|f_i| < r$ on U and since $x_0 \in \Theta(A)$ we can find $h' \in V_{x_0}(A)$ such that $|h'| < 1$ on $X \backslash U$. Then clearly $\|f_i^s h'^t\|_X < r^s$ and $\|\, |f_i|^s + |h'|^t \,\|_X < r^s + 1$.

Assume first that $\Phi = \Phi_\times$, that is $\Phi(f, g) = fg$ for all $f, g \in C(X)$ $(f, g \in C(Y))$. In this case

$$\|(Tf_i)^s(Th')^t\|_Y = \|\Phi((Tf_i)^s, (Th')^t)\|_Y = \|\Phi(f_i^s, h'^t)\|_X = \|f_i^s h'^t\|_X < r^s .$$

We may assume that $s \geq t$. Then since $\|Th'\|_Y = 1$ and consequently $\|Tf_i Th'\|_Y^s = \|(Tf_i)^s(Th')^s\|_Y \leq \|(Tf_i)^s(Th')^t\|_Y < r^s$ it follows that $\|Tf_i Th'\|_Y < r$. Thus for each $y \in Y$

$$\begin{aligned}
|(Th')^s(y)(Th)^t(y)| \leq |(Th')^t(y)(Th)^t(y)| &= \Big|Th'(y) \cdot \Big(\frac{1}{n}Tf_i(y) + \frac{1}{n}\Sigma_{j\neq i}Tf_j(y)\Big)\Big|^t \\
&\leq \Big(\frac{1}{n}|Tf_i(y) \cdot Th'(y)| + \frac{(n-1)r}{n}\Big)^t \\
&\leq \Big(\frac{1}{n}\|Tf_i Th'\|_Y + \frac{(n-1)r}{n}\Big)^t \\
&< \Big(\frac{r}{n} + \frac{(n-1)r}{n}\Big)^t = r^t .
\end{aligned}$$

Hence $\|h'^s h^t\|_X = \|(Th')^s(Th)^t\|_Y < r^t$ while $|h'^s h^t(x_0)| = r^t$, a contradiction.

Assume now that $\Phi = \Phi_+$, that is $\Phi(f, g) = |f|+|g|$ for all $f, g \in C(X)$ $(f, g \in C(Y))$. Since $\|\,|Tf_i|^s + |Th'|^t\,\|_Y = \|\,|f_i|^s + |h'|^t\,\|_X < r^s + 1$ and $\|Tf_i\|_Y = r$ and $\|Th'\|_Y = 1$ this inequality easily implies that $\|\,|Tf_i| + |Th'|\,\|_Y < r + 1$. Hence for each $y \in Y$,

$$\begin{aligned}
|Th(y)| + |Th'(y)| &\leq \frac{1}{n}|Tf_i(y)| + \frac{1}{n}\Sigma_{j\neq i}|Tf_j(y)| + |Th'(y)| \\
&\leq \frac{1}{n}(|Tf_i(y)| + |Th'(y)|) + \frac{1}{n}(\Sigma_{j\neq i}|Tf_j(y)| + (n-1)|Th'(y)|) \\
&\leq \frac{1}{n}\|\,|Tf_i| + |Th'|\,\|_Y + \frac{1}{n}((n-1)r + n - 1) \\
&< \frac{1}{n}(r+1) + \frac{(n-1)r}{n} + \frac{n-1}{n} = r + 1 ,
\end{aligned}$$

that is $|Th(y)|+|Th'(y)| < r+1$. Since $\|Th\|_Y = r$ and $\|Th'\|_Y = 1$ this inequality implies that $\|\,|Th|^s + |Th'|^t\,\|_Y < r^s + 1$. Therefore, $\||h|^s + |h'|^t\|_X < r^s + 1$ while $|h(x_0)| = r$ and $h'(x_0) = 1$, a contradiction.

This argument shows that $x_0 \in \cap_{i=1}^n M(f_i)$, as desired. □

LEMMA 3.8. *Let $\mathcal{A}$ be a subspace of $C(X)$. Then for each nonempty intersection $E = \cap E_\alpha$ of peak sets E_α for $\mathcal{A}$ we have $E \cap \mathrm{ch}(\mathcal{A}) \neq \emptyset$.*

PROOF. Since each E_α is a peak set for $\mathcal{A}$, there exists a function $h_\alpha \in \mathcal{A}$ with $h_\alpha = 1$ on E_α and $|h_\alpha| < 1$ on $X\backslash E_\alpha$. For each α, set $F_\alpha = \{L \in \mathcal{A}_1^* : L(h_\alpha) = 1\}$ where $\mathcal{A}_1^*$ is the closed unit ball of $\mathcal{A}^*$. Then clearly each F_α is a convex weak-star compact subset of $\mathcal{A}_1^*$. Setting $F = \cap_\alpha F_\alpha$ we see that the weak-star compact set F which contains E, is an extreme subset of $\mathcal{A}_1^*$. Hence, by the Krein-Milman Theorem, F contains an extreme point of $\mathcal{A}_1^*$ and consequently there exist $\lambda \in \mathbb{T}$ and $x \in \mathrm{ch}(\mathcal{A})$ such that $\lambda\varphi_x \in F_\alpha$ for all α. Therefore, $\lambda h_\alpha(x) = 1$ and consequently $|h_\alpha(x)| = 1$ for all α. Since h_α is a peaking function at E_α, it follows that $x \in E_\alpha$ for all α. Therefore, $E \cap \mathrm{ch}(\mathcal{A}) \neq \emptyset$. □

LEMMA 3.9. *If A is, in addition, a good subspace, then for each $y_0 \in \mathrm{ch}(B)$ and $r > 0$, $\cap_{Tf\in rV_{y_0}} M(f) \cap \mathrm{ch}(A) \neq \emptyset$.*

PROOF. Given $y_0 \in \mathrm{ch}(B)$ and $r > 0$, by Lemma 3.7, $\cap_{Tf\in rV_{y_0}(B)} M(f) \neq \emptyset$. Let x_0^r be an arbitrary element in this intersection. Assume that $1 \in A$ and for each $f \in A$ with $Tf \in rV_{y_0}(B)$, let $f_r^* \in A$ be defined by $f_r^*(x) = \frac{r^2+\overline{f(x_0^r)}f}{2r^2}$, $x \in X$. Then it is easy to see that f_r^* is a peaking function for A with $M(f_r^*) \subseteq M(f)$ and, moreover, $x_0^r \in M(f_r^*)$ for all $f \in A$ with $Tf \in rV_{y_0}(B)$. Therefore, $\cap_{Tf\in rV_{y_0}(B)} M(f_r^*)$ is a nonempty intersection of peak sets and so, by the above lemma, it contains a point of $\mathrm{ch}(A)$. In particular, $\cap_{Tf\in rV_{y_0}(B)} M(f) \cap \mathrm{ch}(A) \neq \emptyset$.

In the case that A is a subalgebra of $C(X)$ it suffices to apply the same argument for the peaking function $f_r^* = \frac{r^2\overline{f(x_0^r)}f+\overline{f(x_0^r)^2}f^2}{2r^4}$ for A.

Finally in the case where $\mathrm{ch}(A)$ is closed, it suffices to note that the family $\{M(f) \cap \mathrm{ch}(A) : Tf \in rV_{y_0}(B)\}$ of compact subsets of X has finite intersection property. Indeed, for $f_1, ..., f_n \in A$ with $Tf_i \in rV_{y_0}(B)$ there exists $h \in A$ with $Th = \frac{1}{n}\Sigma_{i=1}^n Tf_i$ and since $\|h\|_X = \|Th\|_Y = r$ there exists $x_0 \in \mathrm{ch}(A)$ with $|h(x_0)| = \|h\|_X$. Then the argument given in Lemma 3.7 shows that $x_0 \in M(f_i)$ for $i = 1, ..., n$, as desired. □

LEMMA 3.10. *Let A and B satisfy the hypotheses of Theorem 3.5. Then for each $y_0 \in \mathrm{ch}(B)$ and $r > 0$ there exists a point $x_0^r \in \mathrm{ch}(A)$ such that $T^{-1}(rV_{y_0}(B)) \subseteq rF_{x_0^r}(A)$ and $T(rV_{x_0^r}(A)) \subseteq rF_{y_0}(B)$.*

PROOF. Given $y_0 \in \mathrm{ch}(B)$ and $r > 0$, let x_0^r be an element in the intersection $\cap_{Tf\in rV_{y_0}(B)} M(f) \cap \mathrm{ch}(A)$. Then clearly $T^{-1}(rV_{y_0}(B)) \subseteq rF_{x_0^r}(A)$. By a similar argument $\cap_{f\in rV_{x_0^r}(A)} M(Tf) \cap \mathrm{ch}(B) \neq \emptyset$ and consequently there exists a point $z_0^r \in \mathrm{ch}(B)$ such that $T(rV_{x_0^r}(A)) \subseteq rF_{z_0^r}(B)$. Hence it suffices to show that $y_0 = z_0^r$. Assume on the contrary that $y_0 \neq z_0^r$. Then considering disjoint neighborhoods U and W of y_0 and z_0^r, respectively, we can find elements $g \in V_{y_0}(B)$ and $h \in V_{z_0^r}(B)$ such that $|g| < 1$ on $Y\backslash U$ and $|h| < 1$ on $Y\backslash W$. Clearly $\|(rg)^s h^t\|_Y < r^s$ and $\|\,|rg|^s + |h|^t\,\|_Y < r^s + 1$. Let $f, f' \in A$ with $Tf = rg$ and $Tf' = h$. Then $f \in T^{-1}(rV_{y_0}(B)) \subseteq rF_{x_0^r}(A)$ and consequently $|f(x_0^r)| = r = \|f\|_X$. In particular, $\alpha f \in rV_{x_0^r}(A)$ for some $\alpha \in \mathbb{T}$. Therefore, $T(\alpha f) \in T(rV_{x_0^r}(A)) \subseteq rF_{z_0^r}(B)$, that is $\|T(\alpha f)\|_Y = |(T(\alpha f))(z_0^r)| = r$. This implies that $\|(T(\alpha f))^s h^t\|_Y = r^s = |(T(\alpha f))^s(z_0^r)h^t(z_0^r)|$ and $\|\,|T(\alpha f)|^s + |h|^t\,\|_Y = r^s + 1 = |T(\alpha f)(z_0^r)|^s + |h(z_0^r)|^t$. Therefore, if $\Phi = \Phi_\times$, then

$$\|(rg)^s h^t\|_Y = \|(Tf)^s\,(Tf')^t\|_X = \|f^s f'^t\|_X = \|(\alpha f)^s f'^t\|_X = \|T(\alpha f)^s h^t\|_Y = r^s$$

which is a contradiction. If $\Phi = \Phi_+$, then

$$\begin{aligned}\| \, |rg|^s + |h|^t \, \|_Y &= \| \, |Tf|^s + |Tf'|^t \, \|_Y = \| \, |f|^s + |f'|^t \, \|_X \\ &= \| \, |\alpha f|^s + |f'|^t \, \|_X = \| \, |T(\alpha f)|^s + |h|^t \, \|_X = r^s + 1\end{aligned}$$

which is again, a contradiction. □

LEMMA 3.11. *Under the hypotheses of Theorem* 3.5, *for each* $y_0 \in \mathrm{ch}(B)$, *there exists a point* $x_0 \in \mathrm{ch}(A)$ *such that for all* $r > 0$, $\cap_{Tf \in rV_{y_0}(B)} M(f) \cap \mathrm{ch}(A) = \{x_0\}$.

PROOF. Let $y_0 \in \mathrm{ch}(B)$ and let x_0 be an arbitrary element in $\cap_{Tf \in V_{y_0}(B)} M(f) \cap \mathrm{ch}(A)$. Then it suffices to show that for each $r > 0$ and each $x_0^r \in \cap_{Tf \in rV_{y_0}(B)} M(f) \cap \mathrm{ch}(A)$ we have $x_0^r = x_0$. Assume on the contrary that $x_0^r \neq x_0$ for some $r > 0$. We note that, by the proof of the above lemma, $T^{-1}(rV_{y_0}(B)) \subseteq rF_{x_0^r}(A)$ and $T(rV_{x_0^r}(A)) \subseteq rF_{y_0}(B)$ and similarly $T^{-1}(V_{y_0}(B)) \subseteq F_{x_0}(A)$ and $T(V_{x_0}(A)) \subseteq F_{y_0}(B)$.

Choosing disjoint neighborhoods of x_0^r and x_0 in X we can find easily functions $f \in V_{x_0}(A)$ and $g \in V_{x_0^r}(A)$ such that $\|(rg)^s f^t\|_X < r^s$ and $\| \, |rg|^s + |f|^t \, \|_X < r^s + 1$. In particular, $T(rg) \in rF_{y_0}(B)$ and $Tf \in F_{y_0}(B)$. According to the case where $\Phi = \Phi_\times$ or $\Phi = \Phi_+$ we have either

$$\|T(rg)^s Tf^t\|_Y = \|(rg)^s f^t\|_X < r^s$$

or

$$\| \, |T(rg)|^s + |Tf|^t \, \|_Y = \| \, |rg|^s + |f|^t \, \|_X < r^s + 1$$

which both are impossible, since $|T(rg)(y_0)| = r$ and $|Tf(y_0)| = 1$. This contradiction completes the proof. □

Using the above lemmas we can define a function $\varphi : \mathrm{ch}(B) \longrightarrow \mathrm{ch}(A)$ which associates to each $y_0 \in \mathrm{ch}(B)$, the unique point $x_0 \in \cap_{Tf \in V_{y_0}(B)} M(f) \cap \mathrm{ch}(B)$. Therefore, by the above lemma, for each $r > 0$, $\cap_{Tf \in rV_{y_0}(B)} M(f) \cap \mathrm{ch}(B) = \{x_0\}$ and consequently, as we noted before, $T(rV_{\varphi(y_0)}(A)) \subseteq rF_{y_0}(B)$.

LEMMA 3.12. *Let A and B be as in Theorem* 3.5. *Then* $|Tf(y_0)| = |f(\varphi(y_0))|$ *holds for all* $f \in A$ *and* $y_0 \in \mathrm{ch}(B)$.

PROOF. Let $f \in A$ (with $\|f\|_X \leq 1$ for the case where $\Phi = \Phi_+$) and let $y_0 \in \Theta(B)$ be such that $|f(\varphi(y_0))| < |Tf(y_0)|$. Then $\Phi(|f(\varphi(y_0))|^s, 1) < \Phi(|Tf(y_0)|^s, 1)$ and since, by Lemma 3.6, $|\Phi(|f(\varphi(y_0))|^s, 1)| = \inf\{\|\Phi(f^s, h^t)\|_X : h \in V_{\varphi(y_0)}(A)\}$ it follows that there exists $h \in V_{\varphi(y_0)}(A)$ such that $\|\Phi((Tf)^s, (Th)^t)\|_Y = \|\Phi(f^s, h^t)\|_X < \Phi(|Tf(y_0)|^s, 1)$. Since $Th \in T(V_{\varphi(y_0)}(A)) \subseteq F_{y_0}(B)$ it follows that

$$\Phi(|Tf(y_0)|^s, 1) > \|\Phi((Tf)^s, (Th)^t)\|_X \geq \Phi(|Tf(y_0)|^s, |Th(y_0)|^t) = \Phi(|Tf(y_0)|^s, 1)\,,$$

which is a contradiction. Hence $|Tf(y_0)| \leq |f(\varphi(y_0))|$ for all $f \in A$ and $y_0 \in \Theta(B)$. The other inequality is proven in a similar manner, that is $|Tf(y_0)| = |f(\varphi(y_0))|$. Hence it suffices to show that this equality holds for arbitrary $f \in A$ when $\Phi = \Phi_+$. So assume that T satisfies $\| \, |Tf|^s + |Tg|^t \, \|_Y = \| \, |f|^s + |g|^t \, \|_X$ for all $f, g \in A$. Given an arbitrary nonzero function $f \in A$, using Lemma 3.6 for $\frac{f}{\|f\|_X}$ we conclude that $|f(\varphi(y_0))|^s + \|f\|_X^s = \inf\{\| \, |f|^s + |h|^t \, \|_X : h \in \|f\|_X^{\frac{s}{t}} V_{\varphi(y_0)}(A)\}$. If $|f(\varphi(y_0))| < |Tf(y_0)|$, then $|f(\varphi(y_0))|^s + \|f\|_X^s < |Tf(y_0)|^s + \|f\|_X^s$ and consequently there exists $h \in \|f\|_X^{\frac{s}{t}} V_{\varphi(y_0)}(A)$ such that

$$\| \, |Tf|^s + |Th|^t \, \|_Y = \| \, |f|^s + |h|^t \, \|_X < |Tf(y_0)|^s + \|f\|_X^s .$$

On the other hand, using Lemma 3.10 for $r = \|f\|_X^{\frac{s}{t}}$ we have $Th \in rF_{y_0}(B)$, that is $|Th(y_0)| = r = \|Th\|_Y$ and so $|Tf(y_0)|^s + |Th(y_0)|^t = |Tf(y_0)|^s + r^t = |Tf(y_0)|^s + \|f\|_X^s$, a contradiction. □

PROOF OF THEOREM 3.5. By the above lemmas we need only to show that the function $\varphi : \mathrm{ch}(B) \longrightarrow \mathrm{ch}(A)$ is a homeomorphism. We first note that φ is surjective. Indeed, given $x_0 \in \mathrm{ch}(A)$, using the same argument as in Lemma 3.10, there exists a point $y_0 \in \mathrm{ch}(B)$ such that $\cap_{f \in V_{x_0}(A)} M(Tf) \cap \mathrm{ch}(B) = \{y_0\}$. Since, by the definition of φ, $\cap_{Tf \in V_{y_0}(B)} M(f) \cap \mathrm{ch}(A) = \{\varphi(y_0)\}$, the argument given in Lemma 3.10 implies that $\varphi(y_0) = x_0$, i.e. φ is surjective.

Similar argument shows that φ is injective.

To prove the continuity of φ, let $y_0 \in \mathrm{ch}(B)$ and U be an open neighborhood of $\varphi(y_0)$ in X. Then choosing $h \in V_{\varphi(y_0)}(A)$ with $|h| < \frac{1}{2}$ on $X \backslash U$, the equality $|Th| = |h \circ \varphi|$ on $\mathrm{ch}(B)$ implies that for the open subset $W = \{y \in \mathrm{ch}(B) : |Th(y)| > \frac{1}{2}\}$ of $\mathrm{ch}(B)$ we have $\varphi(W) \subseteq U \cap \mathrm{ch}(A)$. Hence φ is continuous. Similarly φ^{-1} is also continuous. □

As we noted before, for a uniform algebra A on a compact Hausdorff spaces X and strictly positive function $f \in C(X)$ we have $\mathrm{ch}(A) \subseteq \Theta(Af) \subseteq \mathrm{ch}(Af)$. Hence Theorem 3.5 can be applied for subspaces of the form Af whenever $\mathrm{ch}(A) = X$. As an example of uniform algebras A on X with this property we can refer to Dirichlet algebras or, in general, logmodular algebras (see [20, Theorem 1.6.3]). However we have the following more general results.

COROLLARY 3.13. *Let $A_1, ..., A_n$ be uniform algebras on a compact Hausdorff space X with $\cup_{i=1}^n \mathrm{ch}(A_i) = X$. Let $s, t \in \mathbb{N}$, $f_1, ..., f_n$ be strictly positive functions in $C(X)$ and $B = A_1 f_1 + \cdots + A_n f_n$. Then for each surjective map $T : B \longrightarrow B$ satisfying $\|\Phi((Tf)^s, (Tg)^t)\|_X = \|\Phi(f^s, g^t)\|_X$ for all $f, g \in B$, where $\Phi = \Phi_+$ or $\Phi_\times$, there exists a homeomorphism $\varphi : X \longrightarrow X$ such that $|Tf(y)| = |f(\varphi(y))|$ for all $f \in B$ and $y \in X$.*

Clearly for each family $\{A_\alpha\}_{\alpha \in I}$ of uniform algebras on a compact Hausdorff space X with $\cup_\alpha \mathrm{ch}(A_\alpha) = X$ and a family $\{f_\alpha\}_{\alpha \in I}$ of strictly positive functions in $C(X)$, the above corollary can also be stated for the subspace B consisting of all finite sums $g_{\alpha_1} f_{\alpha_1} + \cdots + g_{\alpha_n} f_{\alpha_n}$ where $g_{\alpha_i} \in A_{\alpha_i}$, $i = 1, ..., n$.

We now give our result for locally compact case. For a locally compact Hausdorff space Ω, since any subspace A of $C_0(\Omega)$ can be considered as a subspace of $C(\Omega_\infty)$ with the same Choquet boundary and the same weak peak points, we get the following corollary.

COROLLARY 3.14. *Let Ω_1, Ω_2 be locally compact Hausdorff spaces and A, B be subspaces of $C_0(\Omega_1)$ and $C_0(\Omega_2)$, respectively such that* $\mathrm{ch}(A) = \Theta(A)$ *and* $\mathrm{ch}(B) = \Theta(B)$. *Assume further that each of A and B satisfies one of the following conditions*

(i) *it is a subalgebra, or*

(ii) *its Choquet boundary is compact.*

Then for every surjective map $T : A \longrightarrow B$ satisfying $\|\Phi((Tf)^s, (Tg)^t)\|_Y = \|\Phi(f^s, g^t)\|_X$ for all $f, g \in A$, where $\Phi = \Phi_+$ or $\Phi_\times$ and $s, t \in \mathbb{N}$, there exists a homeomorphism φ : $\mathrm{ch}(B) \longrightarrow \mathrm{ch}(A)$ *such that $|Tf(y)| = |f(\varphi(y))|$ for all $f \in A$ and $y \in \mathrm{ch}(B)$.*

We note that the above corollary is an improvement of the known results for uniform algebras on a locally compact Hausdorff space Ω or Banach subalgebras of $C_0(\Omega)$, without assuming that the subalgebras under consideration are uniformly closed or complete under some norms.

Next corollary gives [30, Proposition 1] without the additional assumption that T is $\mathbb{R}^+$-homogeneous.

COROLLARY 3.15. *If A and B are uniform algebras on locally compact Hausdorff spaces X and Y, respectively, and $T : A \longrightarrow B$ is a surjective map such that $\| |Tf| + |Tg| \|_Y = \| |f| + |g| \|_X$ for all $f, g \in A$, then there exists a homeomorphism φ :* $\mathrm{ch}(B) \longrightarrow \mathrm{ch}(A)$ *such that $|Tf(y)| = |f(\varphi(y))|$ for all $f \in A$ and $y \in \mathrm{ch}(B)$.*

In the rest of the paper we provide conditions for surjective maps $T : A \longrightarrow B$ between subspaces A and B of $C_0(X)$ and $C_0(Y)$, respectively, to be a weighted composition operator.

Following [4], for a locally compact Hausdorff space X we call a subspace A of $C_0(X)$, *extremely regular* if given $0 < \epsilon < 1$, for each $x \in X$ and a neighborhood U of x there exists a function $f \in A$ with $f(x) = \|f\|_X = 1$ and $|f| < \epsilon$ on $X \backslash U$, and *extremely regular of type zero* if for each $x \in X$ and a neighborhood U of x there exists a function $f \in A$ with $f(x) = \|f\|_X = 1$ and $f = 0$ on $X \backslash U$. Clearly for an extremely regular subspace A of $C_0(X)$, $\Theta(A) = \mathrm{ch}(A) = X$ and so A satisfies the hypotheses of Corollary 3.14(ii), whenever X is compact.

Here are some examples of extremely regular subspaces (of type zero):

EXAMPLE 3.16. (i) For a compact Hausdorff space X, by [4], the kernel of each continuous measure $\mu \in M(X)$ (that is its atomic part is zero), is a (maximal) extremely regular subspace of $C(X)$ of type zero.

(ii) For a compact metric space (X, d) and $0 < \alpha \leq 1$, let $\mathrm{Lip}(X, \alpha)$ be the Banach space of Lipschitz functions of order α on X, endowed with the norm $\|f\| = \|f\|_X + p_\alpha(f)$ where for $f \in \mathrm{Lip}(X,\alpha)$

$$p_\alpha(f) = \sup_{x\neq y} \frac{|f(x)-f(y)|}{d^\alpha(x,y)}\,.$$

For each $x \in X$ and open neighborhood U of x, let $f_{x,U} \in \mathrm{Lip}(X,\alpha)$ be defined by $f_{x,U}(y) = \min\big(\frac{d^\alpha(y,X\backslash U)}{d^\alpha(x,X\backslash U)}, 1\big)$. Then clearly any subspace of $\mathrm{Lip}(X,\alpha)$ containing all functions of the form $f_{x,U}$ is an extremely regular subspace of $C(X)$ of type zero.

(ii) For a locally compact group G and $1 \le p < \infty$, the Figa-Talamanca-Herz algebra $A_p(G)$ and every subspace of $A_p(G)$ containing functions of the form $\chi_U * \chi_{Ux_0}$ where U is a compact symmetric neighborhood of the identity of G and $x_0 \in G$, is an extremely regular subspace of $C_0(G)$ of type zero (see [11, Page 319].

Next theorem, in particular, proves the additive and multiplicative version of Bishop's lemma simultaneously (see [30]).

THEOREM 3.17. *Let X be a locally compact Hausdorff space, B be a closed subspace of $C_0(X)$ and let $\Phi = \Phi_\times$ or $\Phi = \Phi_+$ and $s, t \in \mathbb{N}$. Then for a compact subset E of X the following statements are equivalent*:

(i) *E is a weak peak set for B.*

(ii) *For each neighborhood U of E and a bounded continuous function f on X (with $\|f\|_X \le 1$ for the case where $\Phi = \Phi_+$) if p is a continuous function on X with $\inf_{x\in X} p(x) > 0$ and $|f|\big|_E = p\big|_E$, then there exists a function $h \in B$ with $h|_E = 1 = \|h\|_X$, $|\Phi(f^s, h^t)| \le \Phi(p^s, 1)$ on X and $M\big(\frac{\Phi(f^s,h^t)}{\Phi(p^s,1)}\big) = M(h) \subseteq U$.*

PROOF. (i) $\Rightarrow$ (ii): The proof is a minor modification of [30, Lemma 1]. Assume that f and p are as in (ii) and set $m_0 = \inf_{x\in X} p(x)$. Then by hypotheses $m_0 > 0$ and f is necessarily nonzero. For each $n \in \mathbb{N}$ we set

$$U_n = \left\{x \in U : \big||f^s(x)| - p^s(x)\big| < \frac{m_0^s}{2^{n+1}}\right\}.$$

Then clearly $U_n \supseteq U_{n+1}$ for each $n \in \mathbb{N}$ and $E \subset \cap_{n=1}^\infty U_n = \{x \in U : |f^s(x)| = p^s(x)\}$. Since E is a weak peak set, it follows that for each $n \in \mathbb{N}$, there exists a function $h_n \in B$ with $h_n|_E = 1 = \|h_n\|_X$ and $|h_n| < \min\big(\frac{m_0^s}{2^n\|f\|_X^s}, \frac{m_0^s}{2^n}, \frac{1}{2^n}\big)$ on $X\backslash U_n$. Clearly the function $h = \Sigma_{n=1}^\infty \frac{h_n}{2^n}$ is an element of B with $h|_E = 1 = \|h\|_X$. Moreover, $M(h) \subseteq \cap_{n=1}^\infty M(h_n) \subseteq \cap_{n=1}^\infty U_n$.

Now let $x \in X$ and $\Phi = \Phi_\times$. If $x \in M(h)$, then since $x \in \cap_{n=1}^\infty U_n$ we have $|f^s(x)| = p^s(x)$ and so $|f^s(x)h^t(x)| = p^s(x) = \Phi(p^s(x), 1)$.

If $x \in \cap_{n=1}^\infty U_n \backslash M(h)$, then clearly $x \in U$, $|f^s(x)| = p^s(x)$ and $|h(x)| < 1$ which conclude that $|f^s(x)h^t(x)| < p^s(x) = \Phi(p^s(x), 1)$.

If $x \in X\backslash \cap_{n=1}^\infty U_n$, then either $x \notin U_n$ for all $n \in \mathbb{N}$ or there exists $n \ge 2$ such that $x \in U_{n-1} \backslash U_n$. In the first case $|h_n(x)| < \min\big(\frac{m_0^s}{2^n \|f\|_X^s}, \frac{m_0^s}{2^n}, \frac{1}{2^n}\big)$ for all $n \in \mathbb{N}$, which implies that $|f^s(x)h^t(x)| \leq |f^s(x)h(x)| < \Sigma_{n=1}^{\infty} \frac{m_0^s}{4^n} < m_0^s \leq p^s(x)$. In the second case $x \in U_1, ..., U_{n-1}$ and $x \notin U_i$ for all $i \geq n$. Since $\big| |f^s(x)| - p^s(x) \big| < \frac{m_0^s}{2^n}$ we have

$$
\begin{aligned}
|f^s(x)h^t(x)| &\leq \Big(p^s(x) + \frac{m_0^s}{2^n}\Big)|h(x)| \leq p^s(x)\Big(1 + \frac{1}{2^n}\Big)\Big(\Sigma_{i=1}^{n-1} \frac{1}{2^i} + \Sigma_{i=n}^{\infty} \frac{1}{4^i}\Big) \\
&= p^s(x)\Big(1 + \frac{1}{2^n}\Big)\Big(1 - \frac{1}{2^{n-1}} + \frac{1}{3 \cdot 4^{n-1}}\Big) < p^s(x) .
\end{aligned}
$$

The above argument shows that for $\Phi = \Phi_\times$, $|\Phi(f^s, h^t)| = |f^s h^t| \leq p^s = \Phi(p^s, 1)$ on X and $M\big(\frac{\Phi(f^s, h^t)}{\Phi(p^s, 1)}\big) = M\big(\frac{f^s h^t}{p^s}\big) = M(h) \subseteq U$.

A similar argument can be applied for the case where $\Phi = \Phi_+$ (with $\|f\|_X \leq 1$).

(ii) $\Rightarrow$ (i) It is clear. □

We should note that in part (ii) of the above theorem, if either B contains the constant function 1 locally at E or is a subalgebra of $C_0(X)$, then since each h_n can be chosen to be a peaking function, it follows that in these cases the function $h \in B$ is also a peaking function.

COROLLARY 3.18. *Let X be a locally compact Hausdorff space and B be a closed subspace of $C_0(X)$. Let $x_0 \in \Theta(B)$, U be an open neighborhood of x_0 and $f \in C_0(X)$ with $f(x_0) \neq 0$. Then for each $s, t \in \mathbb{N}$ there exists a function $h \in V_{x_0}(B)$ such that $M(f^s h^t) = M(h) \subseteq U$. If, furthermore, either* 1 *belongs locally at x_0 to B or B is a subalgebra of $C_0(X)$ then the function h can be chosen to be a peaking function with $R_\pi(f^s h^t) = \{f^s(x_0)\}$.*

PROOF. Assume without loss of generality that $f(x_0) = 1$. Then it suffices to apply the proof of the above theorem for the constant function $p = 1$ and sets $U_n = \big\{x \in U : |f^s(x) - 1| < \frac{1}{2^{n+1}}\big\}$. □

REMARK. Let X be a locally compact Hausdorff space and B be a, not necessarily closed, subspace of $C_0(X)$ such that there exists a complete norm $\|\cdot\|$ on B with $\|\cdot\| \geq \|.\|_X$. Assume that $\Theta(B, \|.\|)$ is the set of points $x_0 \in X$ such that for each $0 < \epsilon \leq 1$ and neighborhood U of x_0, there exists a function $f \in B$ such that $f(x_0) = 1 = \|f\|$ and $|f| < \epsilon$ on $X \backslash U$. Then the proof of the preceding theorem shows that part (ii) of this theorem also holds for each $x_0 \in \Theta(B, \|\cdot\|)$. Hence in the case that $\Theta(B, \|.\|) = \mathrm{ch}(B)$ which clearly implies $\Theta(B) = \mathrm{ch}(B)$, the above corollary holds for each $x_0 \in \mathrm{ch}(B)$. Motivated by this, we consider the following definition.

DEFINITION 3.19. Let X be a locally compact Hausdorff space. We say that a (not necessarily closed) subspace A of $C_0(X)$ has *Bishop's property* (for a pair $(s, t) \in \mathbb{N} \times \mathbb{N}$) if for each $x_0 \in \mathrm{ch}(A)$ and $f \in A$ with $f(x_0) \neq 0$ there exists a peaking function $h \in A$ such that $h(x_0) = 1 = \|h\|_X$, $R_\pi(f^s h^t) = \{f^s(x_0)\}$.

Here are some examples of subspaces having Bishop's property:

EXAMPLE 3.20. (i) By Corollary 3.18, every closed subspace B of $C_0(X)$ with $\Theta(B)=\mathrm{ch}(B)$ which either contains the constant function 1 locally at each point of $\mathrm{ch}(B)$ or is a subalgebra of $C_0(X)$ has Bishop's property for all $(s,t)\in\mathbb{N}\times\mathbb{N}$. In particular, uniform algebras on a locally compact Hausdorff space X and closed extremely regular subspaces of $C(Y)$, for a compact Hausdorff space Y, containing constants have Bishop's property.

(ii) Let X be a compact metric space with a distinguished base point e_X and let $\mathrm{Lip}_0(\mathrm{X})$ be the algebra of all complex Lipschitz functions f on X such that $f(e_X)=0$. Then by Lemma 2.1 in [15] for any $x\in X$ and $f\in \mathrm{Lip}_0(\mathrm{X})$ with $f(x)\neq 0$ there exists $h\in\mathrm{Lip}_0(\mathrm{X})$ with $h(x)=1=\|h\|_X$ and $R_\pi(fh)=\{f(x)\}$. Since the function h in this lemma is positive it follows that $\mathrm{Lip}_0(\mathrm{X})$ has Bishop's property for $(s,1)$, $s\in\mathbb{N}$.

(iii) For a locally compact group G with a left Haar measure λ, since for each compact symmetric neighborhood U of the identity of G and $x_0\in G$, the element $f_{U,x_0}=\lambda(U)^{-1}\chi_U*\chi_{Ux_0}$ of $A_p(G)$ satisfies $\|f_{U,x_0}\|=1$, $f_{U,x_0}(x_0)=1$ and $f_{U,x_0}=0$ on $G\backslash Ux_0$, it follows from the above remark that $A_p(G)$ and its closed subalgebras containing all f_{U,x_0} have Bishop's property for all $(s,t)\in\mathbb{N}\times\mathbb{N}$.

Next theorem generalizes previous results proven for function algebras, see for example [14]. As we noted before similar results have been also proven in [23] for multiplicative subsets of function algebras. As before we first state our result for compact case and conclude the locally compact case as a corollary.

THEOREM 3.21. *Let X,Y be compact Hausdorff spaces and A,B be subspaces of $C(X)$ and $C(Y)$ satisfying the hypotheses of Theorem 3.5. Assume, further, that A has Bishop's property for some $(s,t)\in\mathbb{N}\times\mathbb{N}$. Then for each surjective map $T:A\longrightarrow B$ satisfying*

$$R_\pi((Tf)^s(Tg)^t)\subseteq R_\pi(f^sg^t)\quad (f,g\in A)$$

there exists a homeomorphism $\varphi:\mathrm{ch}(B)\longrightarrow\mathrm{ch}(A)$ and a continuous function $w:\mathrm{ch}(B)\longrightarrow\mathbb{T}$ such that $(Tf)^s(y)=w(y)f^s(\varphi(y))$. Moreover, $(Tf)^t(y)=\frac{1}{w(y)}f^t(\varphi(y))$ also holds for all $f\in A$ and $y\in\mathrm{ch}(B)$, whenever A has Bishop's property for (t,s). In particular, in this case, $Tf^d(y)=\gamma(y)f^d(\varphi(y))$ for all $f\in A$ and $y\in\mathrm{ch}(B)$ where d is the greatest common divisor of s and t, and $\gamma:\mathrm{ch}(B)\longrightarrow\mathbb{T}$ is a continuous function.

PROOF. The proof is a modification of [14, Theorem 2.1]. Clearly T satisfies the norm condition $\|(Tf)^s(Tg)^t\|_Y=\|f^sg^t\|_X$ for all $f,g\in A$. Hence by Theorem 3.5, there exists a homeomorphism $\varphi:\mathrm{ch}(B)\longrightarrow\mathrm{ch}(A)$ such that for each $f\in A$, $|Tf|=|f\circ\varphi|$ on $\mathrm{ch}(B)$. Assume now that $y_0\in\mathrm{ch}(B)$ and $f\in A$ are given and set $x_0=\varphi(y_0)$. Choosing an arbitrary peaking function $h\in V_{x_0}(A)$, by hypotheses, we have $R_\pi((Th)^{s+t})=R_\pi((Th)^s(Th)^t)\subseteq R_\pi(h^{s+t})=\{1\}$ and consequently $R_\pi((Th)^{s+t})=\{1\}$. Since $|Th(y_0)|=|h(x_0)|=1$ it follows that $(Th)^{s+t}(y_0)=1$. Setting $w(y_0)=\frac{1}{(Th)^t(y_0)}$, we have clearly $|w(y_0)|=1$. We note that the definition of $w(y_0)$ is independent of the choice of peaking function $h\in V_{x_0}(A)$.

Indeed, if $h_1, h_2 \in V_{x_0}(A)$ are peaking functions, then since

$$R_\pi((Th_2)^s(Th_1)^t) \subseteq R_\pi(h_2^s h_1^t) = \{1\}$$

and $R_\pi((Th_2)^{s+t}) = \{1\}$ we have $R_\pi((Th_2)^s(Th_1)^t) = \{1\} = R_\pi((Th_2)^{s+t})$ and consequently $((Th_2)^s(Th_1)^t)(y_0) = 1 = (Th_2)^{s+t}(y_0)$. Therefore, $(Th_1)^t(y_0) = (Th_2)^t(y_0)$, as desired. Hence the function w is well defined.

We now show that for each $f \in A$ and $y_0 \in \mathrm{ch}(B)$, $(Tf)^s(y_0) = w(y_0)f^s(x_0)$ where $x_0 = \varphi(y_0)$. Clearly the equality holds if $f(x_0) = 0$. So we assume that $f(x_0) \neq 0$. Then since A has Bishop's property, there exists a peaking function $h \in V_{x_0}(A)$ with $R_\pi(f^s h^t) = \{f^s(x_0)\}$. Since $R_\pi((Tf)^s(Th)^t) \subseteq R_\pi(f^s h^t) = \{f^s(x_0)\}$ and moreover, $|(Tf)^s(Th)^t(y_0)| = |f^s(x_0)h^t(x_0)| = |f^s(x_0)|$ it follows that $\frac{1}{w(y_0)}(Tf)^s(y_0) = (Tf)^s(y_0)(Th)^t(y_0) = f^s(x_0)$ and consequently $(Tf)^s(y_0) = w(y_0)f^s(x_0)$.

If A also has Bishop's property for (t, s), then a similar argument shows that there exists a function $u : \mathrm{ch}(B) \longrightarrow \mathbb{T}$ such that $(Tf)^t(y_0) = u(y_0)f^t(\varphi(y_0))$ for all $f \in A$ and $y_0 \in \mathrm{ch}(B)$. Since for each $x_0 \in \mathrm{ch}(A)$ and peaking function $h \in V_{x_0}(A)$, $R_\pi((Th)^{s+t}) \subseteq R_\pi(h^{s+t}) = 1$ and $|Th(y_0)| = |h(x_0)| = 1$ we have $(Th)^{s+t}(y_0) = 1$ which shows that $u(y_0)w(y_0) = 1$. Hence $(Tf)^t(y_0) = \frac{1}{w(y_0)} f^t(y_0)$, as desired. Since for the greatest common divisor d of s and t there are integers k_1 and k_2 such that $d = k_1 s + k_2 t$ it follows that $(Tf)^d(y_0) = \gamma(y_0)f^d(y_0)$ where $\gamma(y_0) = w(y_0)^{k_1 - k_2}$.

Now it suffices to show that w is continuous. For this, since for each $y_0 \in \mathrm{ch}(B)$ and $f \in A$, $(Tf)^s(y_0) = w(y_0)f^s(\varphi(y_0))$, choosing a function $f \in A$ with $f(\varphi(y_0)) \neq 0$ we have $w = \frac{(Tf)^s}{f^s \circ \varphi}$ on a neighborhood of y_0 which implies that w is continuous at y_0. □

COROLLARY 3.22. *Let Ω_1, Ω_2 be locally compact Hausdorff spaces, A and B be subspaces of $C_0(\Omega_1)$ and $C_0(\Omega_2)$ satisfying the hypotheses of Corollary* 3.14. *If A has Bishop's property for $(s, t) \in \mathbb{N} \times \mathbb{N}$, then any surjective map $T : A \longrightarrow B$ satisfying*

$$R_\pi((Tf)^s(Tg)^t) \subseteq R_\pi(f^s g^t) \quad (f, g \in A)$$

has the same description as in the above theorem.

As we noted before, all uniform algebras on locally compact Hausdorff spaces as well as $\mathrm{Lip}_0(X)$ for a compact metric space X and $A_p(G)$, $1 \leq p < \infty$, for a locally compact group G satisfy the hypotheses of the above theorem and corollary (for appropriate $s, t \in \mathbb{N}$). For another example, we can refer to closed extremely regular subspaces of $C(X)$ containing constants, where X is a compact Hausdorff space, in particular subspaces of the form $\overline{A_1 f_1 + \cdots + A_n f_n} + \mathbb{C}1$ where each A_i is a uniform algebra on X and moreover, $\cup_{i=1}^n \mathrm{ch}(A_i) = X$.

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Present Addresses:
RAZIEH SADAT GHODRAT
DEPARTMENT OF PURE MATHEMATICS, FACULTY OF MATHEMATICAL SCIENCES,
TARBIAT MODARES UNIVERSITY,
TEHRAN, 14115–134, IRAN.
e-mail: rs.ghodrat@modares.ac.ir

FERESHTEH SADY
DEPARTMENT OF PURE MATHEMATICS, FACULTY OF MATHEMATICAL SCIENCES,
TARBIAT MODARES UNIVERSITY,
TEHRAN, 14115–134, IRAN.
e-mail: sady@modares.ac.ir

ARYA JAMSHIDI
DEPARTMENT OF PURE MATHEMATICS, FACULTY OF MATHEMATICAL SCIENCES,
TARBIAT MODARES UNIVERSITY,
TEHRAN, 14115–134, IRAN.
e-mail: arya.jamshidi@modares.ac.ir