On Filter-regular Sequences of Multi-graded Modules

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Abstract. Let S be an $\mathbf{N}^d$-graded algebra over a noetherian ring and a finitely generated $\mathbf{N}^d$-graded S-module M. This paper will study the relationship of filter-regular sequences of M to joint reductions and homogeneous parameter systems. As an application, we show that any maximal filter-regular sequence is a joint reduction of $(S_1, \ldots, S_d)$ with respect to M, and any maximal strong-filter-regular sequence is a reduction of S_+ with respect to M. And we characterize the existence of parts of homogeneous parameter systems for M consisting of elements of total degree 1 via strong-filter-regular sequences.

1. Introduction

Throughout this paper, let $(A, \mathfrak{m})$ be a noetherian local ring with maximal ideal $\mathfrak{m}$, infinite residue field $k = A/\mathfrak{m}$. Let $S = \bigoplus_{n_1,\ldots,n_d \geq 0} S_{(n_1,\ldots,n_d)}$ be a finitely generated standard $\mathbf{N}^d$-graded algebra over A, i.e., S is generated over A by elements of total degree 1. Let $M = \bigoplus_{n_1,\ldots,n_d \geq 0} M_{(n_1,\ldots,n_d)}$ be a finitely generated $\mathbf{N}^d$-graded S-module that satisfies $M_{(n_1,\ldots,n_d)} = S_{(n_1,\ldots,n_d)} M_{(0,\ldots,0)}$ for all $n_1, \ldots, n_d \geq 0$. Set

$$S_+ = \bigoplus_{n_1+\cdots+n_d>0} S_{(n_1,\ldots,n_d)}; \quad S_{++} = \bigoplus_{n_1,\ldots,n_d>0} S_{(n_1,\ldots,n_d)}; \quad S_i = S_{(0,\underbrace{\ldots,1,\ldots,0}_{i})}$$

for all $i = 1, \ldots, d$.

In this paper, we explore the relation of filter-regular sequences in multi-graded modules with superficial sequences of ideals in local rings; joint reductions; and parts of homogeneous parameter systems for M.

Filter-regular sequences were introduced by Stuckrad and Vogel in [12]. The theory of these sequences became as an important tool to study some classes of singular rings and has been continually developed (see e.g. [1, 5, 14, 22, 25]).

DEFINITION 1.1 (Definition 2.1). Let M be a finitely generated $\mathbf{N}^d$-graded S-module.

(i) Let $S_{(1,1,\dots,1)}$ be not contained in $\sqrt{\mathrm{Ann}M}$. A homogeneous element $x \in S$ is called an S_{++}-*filter-regular element with respect to M* if $(0_M : x)_{(n_1,\dots,n_d)} = 0$ for all large $n_1, \dots, n_d$.

(ii) Let S_i be not contained in $\sqrt{\mathrm{Ann}M}$. A homogeneous element $x \in S$ is called an S_i-*filter-regular element with respect to M* if for $n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_d \geq 0$ there exists $N \in \mathbf{N}$ such that $(0_M : x)_{(n_1,\dots,n_i,\dots,n_d)} = 0$ for all $n_i \geq N$.

We first study the relation of filter-regular sequences with joint reductions.

The concept of joint reductions of $\mathfrak{m}$-primary ideals was given by Rees in 1984 [10]. This important concept was extended to the set of arbitrary ideals by [2, 7, 9, 17, 19, 21]. In the case of graded modules, we have the following notion.

DEFINITION 1.2 (Definition 3.1). Let $\mathfrak{R}$ be a subset of $\bigcup_{i=1}^d S_i$. Set $\mathfrak{R}_i = \mathfrak{R} \bigcap S_i$ for $1 \leqslant i \leqslant d$. Then $\mathfrak{R}$ is called a *joint reduction of $(S_1, \dots, S_d)$ with respect to M* if for all large integers $n_1, n_2, \dots, n_d$, $M_{(n_1+1,\dots,n_d+1)} = \sum_{i=1}^d (\mathfrak{R}_i) M_{(n_1+1,\dots,n_i,\dots,n_d+1)}$.

Then we obtain the following interesting result.

THEOREM 1.3 (Theorem 3.2). *Let $\mathfrak{R}$ be a subset of $\bigcup_{i=1}^d S_i$. Then the following statements hold.*

(i) *If $\mathfrak{R}$ is a maximal S_{++}-filter-regular sequence with respect to M, then $\mathfrak{R}$ is a joint reduction of $(S_1, \dots, S_d)$ with respect to M.*

(ii) *If $\mathfrak{R}$ is a maximal strong-filter-regular sequence with respect to M, then $(\mathfrak{R})$ is a reduction of S_+ with respect to M.*

As applications of Theorem 1.3 for joint reductions of ideals, we show that any maximal weak-(FC)-sequence is a joint reduction (Corollary 3.5, Section 3); and in the case of $\mathfrak{m}$-primary ideals, each maximal superficial sequence is also a joint reduction (Corollary 3.6, Section 3).

It is well known that one of obstructions in studying graded modules is that these modules often do not have homogeneous parameter systems. Using strong-filter-regular sequences, we characterize the existence of parts of homogeneous parameter systems for M consisting of elements of total degree 1 by the following theorem.

THEOREM 1.4 (Theorem 4.1). *Set $\mathfrak{S}_{\hat{i}} = \left\{ S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_d \right\}$ for $1 \leq i \leq d$. Let $k_1, k_2, \dots, k_d$ be non-negative integers with $k_1 + k_2 + \cdots + k_d = s \leq \dim M$. Then the following statements are equivalent.*

(i) *For any $1 \leq i \leq d$, $\dim_S \frac{M}{(\mathfrak{U}, S_i)M} \leq \dim_S \frac{M}{(\mathfrak{U})M} - k_i$ for all $\mathfrak{U} \subset \mathfrak{S}_{\hat{i}}$.*

(ii) *There is a strong-filter-regular sequence $x_1, \dots, x_s$ with respect to M consisting of k_1 elements of $S_1, \dots, k_d$ elements of S_d such that if $y_1, \dots, y_{k_i} \in S_i$ is a subsequence of $x_1, \dots, x_s$, then $\dim_S \frac{M}{(\mathfrak{U}, y_1, \dots, y_{k_i})M} = \dim_S \frac{M}{(\mathfrak{U})M} - k_i$ for all $\mathfrak{U} \subset \mathfrak{S}_{\hat{i}}$ and $1 \leq i \leq d$. And $x_1, \dots, x_s$ is a part of a parameter system for M.*

(iii) *There is a part of a parameter system $x_1, \ldots, x_s$ for M consisting of k_1 elements of $S_1, \ldots, k_d$ elements of S_d such that if $y_1, \ldots, y_{k_i} \in S_i$ is a subsequence of $x_1, \ldots, x_s$, then $\dim_S \frac{M}{(\mathfrak{U}, y_1, \ldots, y_{k_i})M} = \dim_S \frac{M}{(\mathfrak{U})M} - k_i$ for all $\mathfrak{U} \subset \mathfrak{S}_{\hat{i}}$ and $1 \leq i \leq d$.*

Theorem 1.4 yields interesting consequences such as the existence of homogeneous parameter ideals generated by elements of total degree 1 for $\mathbf{N}^d$-graded modules over artinian rings (Corollary 4.7; Corollary 4.8, Section 4) and different consequences and examples (Corollary 4.2; Corollary 4.5; Example 4.4; Example 4.6).

This paper is divided into four sections. Section 2 is devoted to the discussion of filter-regular sequences and strong-filter-regular sequences such as the universal existence of strong-filter-regular sequences; the relationship of filter-regular sequences of multi-graded modules with superficial sequences and weak-(FC)-sequences of ideals; and other properties that will be used in this paper. Section 3 explores the relationship between maximal filter-regular sequences and joint reductions and gives some applications for joint reductions of ideals. In Section 4 we characterize the existence of parts of homogeneous parameter systems for M. As an application we get some results in the case of artinian rings.

2. Filter-Regular Sequences

In this section, we will define the notion of filter-regular sequences and strong-filter-regular sequences in multi-graded modules, and study some important properties of these sequences.

DEFINITION 2.1. Let $S = \bigoplus_{n_1, \ldots, n_d \geq 0} S_{(n_1, \ldots, n_d)}$ be a finitely generated standard $\mathbf{N}^d$-graded algebra over a noetherian local ring A and let $M = \bigoplus_{n_1, \ldots, n_d \geq 0} M_{(n_1, \ldots, n_d)}$ be a finitely generated $\mathbf{N}^d$-graded S-module. Then we have the following notions:

(i) Let $S_{(1,1,\ldots,1)}$ be not contained in $\sqrt{\operatorname{Ann} M}$. A homogeneous element $x \in S$ is called an S_{++}-*filter-regular element with respect to M* if $(0_M : x)_{(n_1, \ldots, n_d)} = 0$ for all large $n_1, \ldots, n_d$. Let $x_1, \ldots, x_t$ be homogeneous elements in S. We call that $x_1, \ldots, x_t$ is an S_{++}-filter-regular sequence with respect to M if x_{i+1} is an S_{++}-filter-regular element with respect to $M_i = \frac{M}{(x_1, \ldots, x_i)M}$ for all $i = 0, \ldots, t-1$.
An S_{++}-filter-regular sequence $x_1, \ldots, x_t$ with respect to M is called a *maximal S_{++}-filter-regular sequence* if $S_{(1,1,\ldots,1)} \subseteq \sqrt{\operatorname{Ann} M_t}$.

(ii) Let S_i be not contained in $\sqrt{\operatorname{Ann} M}$. A homogeneous element $x \in S$ is called an S_i-*filter-regular element with respect to M* if for $n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_d \geq 0$ there exists $N \in \mathbf{N}$ such that $(0_M : x)_{(n_1, \ldots, n_i, \ldots, n_d)} = 0$ for all $n_i \geq N$. Let $x_1, \ldots, x_t$ be homogeneous elements in S. We call that $x_1, \ldots, x_t$ is a *strong-filter-regular sequence* with respect to M if x_{i+1} is an S_j-filter-regular element for some j with respect to $M_i = \frac{M}{(x_1, \ldots, x_i)M}$ for all $i = 0, \ldots, t-1$.

A strong-filter-regular sequence $x_1, \ldots, x_t$ with respect to M is called a *maximal strong-filter-regular sequence* if $S_i \subseteq \sqrt{\mathrm{Ann} M_t}$ for all $i = 1, \ldots, d$.

REMARK 2.2.   We have some following comments for filter-regular sequences.

(i) $S_{(1,1,\ldots,1)} \subseteq \sqrt{\mathrm{Ann} M}$ if and only if there exists $u > 0$ such that $S_{(u,\ldots,u)} M = 0$. Since $S_{(u,\ldots,u)} M_{(m_1,\ldots,m_d)} = M_{(m_1+u,\ldots,m_d+u)}$, it follows that $S_{(1,1,\ldots,1)} \subseteq \sqrt{\mathrm{Ann} M}$ if and only if $M_{(n_1,\ldots,n_d)} = 0$ for all $n_1, \ldots, n_d \gg 0$. Then in this case, we get $(0_M : x)_{(n_1,\ldots,n_d)} \subseteq M_{(n_1,\ldots,n_d)} = 0$ for all large $n_1, \ldots, n_d$. Hence any homogeneous element of S has the property as an S_{++}-filter-regular element.

If $S_i \subseteq \sqrt{\mathrm{Ann} M}$ then $S_{(0,\ldots,a_i,\ldots,0)} M = 0$ for some a_i. Hence

$$M_{(n_1,\ldots,n_i+a_i,\ldots,n_d)} = S_{(0,\ldots,a_i,\ldots,0)} M_{(n_1,\ldots,n_d)} = 0$$

for all $n_1, \ldots, n_d$. Thus $(0_M : x)_{(n_1,\ldots,n_d)} \subseteq M_{(n_1,\ldots,n_d)} = 0$ for all $n_i \gg 0$ and for all $n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_d \geq 0$. In this case, any homogeneous element of S carries the property of an S_i-filter-regular element.

The above facts only obstruct and do not carry usefulness. That is why in Definition 2.1, one has to exclude the cases that $S_{(1,1,\ldots,1)} \subseteq \sqrt{\mathrm{Ann} M}$ and $S_i \subseteq \sqrt{\mathrm{Ann} M}$ in defining S_{++}-filter-regular elements and S_i-filter-regular elements, respectively.

(ii) By [22], a homogeneous element $x \in S$ is an S_{++}-filter-regular element with respect to M if and only if $0_M : x \subseteq 0_M : S_{++}^\infty$.

(iii) One showed that if $S_{(1,1,\ldots,1)}$ is not contained in $\sqrt{\mathrm{Ann} M}$ then for each $i = 1, \ldots, d$, there exists an S_{++}-filter-regular element $x \in S_i$ with respect to M (see [22, Proposition 2.2]).

Next, we show that the existence of strong-filter-regular sequences is universal by the following proposition.

PROPOSITION 2.3.   *Assume that S_i is not contained in $\sqrt{\mathrm{Ann} M}$. Then there exists an S_i-filter-regular element $y \in S_i$ with respect to M. And a homogeneous element x is an S_i-filter-regular element with respect to M if and only if $0_M : x \subseteq 0_M : S_i^\infty$.*

PROOF.   Note that since M is noetherian, there exists a positive integer n such that $0_M : S_i^\infty = 0_M : S_i^n$. For any $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d \geq 0$, set

$$M' = [0_M : S_i^n] \bigcap \left[\bigoplus_{n_i \geq 0} M_{(a_1,\ldots,a_{i-1},n_i,a_{i+1},\ldots,a_d)} \right].$$

Then M' is a noetherian graded $[\bigoplus_{n_i \geq 0} S_{(0,\ldots,n_i,\ldots,0)}]$-submodule of

$$\bigoplus_{n_i \geq 0} M_{(a_1,\ldots,a_{i-1},n_i,a_{i+1},\ldots,a_d)} \, .$$

Hence by [22, Lemma 2.3], there exists a positive integer u_i such that

$$M'_{(a_1,\ldots,a_{i-1},n_i,a_{i+1},\ldots,a_d)} = S_{(0,\ldots,n_i-u_i,\ldots,0)} M'_{(a_1,\ldots,a_{i-1},u_i,a_{i+1},\ldots,a_d)}$$

for all $n_i \geq u_i$. Consequently

$$[0_M : S_i^n]_{(a_1,\ldots,a_{i-1},n_i,a_{i+1},\ldots,a_d)} = S_{(0,\ldots,n_i-u_i,\ldots,0)}[0_M : S_i^n]_{(a_1,\ldots,a_{i-1},u_i,a_{i+1},\ldots,a_d)}$$

for all $n_i \geq u_i$. From this fact follows that

$$[0_M : S_i^n]_{(a_1,\ldots,a_{i-1},n_i,a_{i+1},\ldots,a_d)} = 0$$

for all $n_i \geq n + u_i$. So for any $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d \geq 0$ we have

$$[0_M : S_i^\infty]_{(a_1,\ldots,a_{i-1},n_i,a_{i+1},\ldots,a_d)} = 0$$

for all $n_i \gg 0$. Now note that

$$\mathrm{Ass}_S\big(M/0_M : S_i^\infty\big) = \big\{P \in \mathrm{Ass}_S M \mid S_i \nsubseteq P\big\}.$$

Since k is an infinite field and $\coprod = \mathrm{Ass}_S\big(M/0_M : S_i^\infty\big)$ is finite, there exists $y \in S_i \backslash \bigcup_{P \in \coprod} P$. Then $0_M : y \subseteq 0_M : S_i^\infty$. Therefore, for $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d \geq 0$ we obtain

$$[0_M : y]_{(a_1,\ldots,a_{i-1},n_i,a_{i+1},\ldots,a_d)} \subseteq [0_M : S_i^\infty]_{(a_1,\ldots,a_{i-1},n_i,a_{i+1},\ldots,a_d)} = 0$$

for all $n_i \gg 0$. Thus y is an S_i-filter-regular element with respect to M.

Next, it has already been shown that if a homogeneous element x satisfies

$$0_M : x \subseteq 0_M : S_i^\infty$$

then x is an S_i-filter-regular element with respect to M. Now assume that x is an S_i-filter-regular element with respect to M. We need to show that

$$0_M : x \subseteq 0_M : S_i^\infty.$$

It is enough to prove that $(0_M : x)_{(v_1,\ldots,v_d)} \subseteq 0_M : S_i^\infty$ for all $v_1, \ldots, v_d$. Indeed, for any $z \in (0_M : x)_{(v_1,\ldots,v_d)}$ then $x S_{(n_1,\ldots,n_d)} z = 0$ for every $n_1, \ldots, n_d$. Hence

$$S_i^n z \subseteq (0_M : x)_{(v_1,\ldots,v_{i-1},n+v_i,v_{i+1},\ldots,v_d)} = 0$$

for all large n. It implies that $z \in 0_M : S_i^\infty$. Hence $0_M : x \subseteq 0_M : S_i^\infty$. So x is an S_i-filter-regular element with respect to M if and only if

$$0_M : x \subseteq 0_M : S_i^\infty. \qquad \blacksquare$$

REMARK 2.4. From Proposition 2.3, it follows that there is not any strong-filter-regular sequence in $\bigcup_{i=1}^d S_i$ if and only if $S_+ \subseteq \sqrt{\mathrm{Ann}M}$. In this case, $M_{(n_1,\ldots,n_d)} = 0$ for $n_1 + \cdots + n_d \gg 0$. Consequently, there is not any strong-filter-regular sequence in $\bigcup_{i=1}^d S_i$ if and only if $M_{(n_1,\ldots,n_d)} = 0$ for all $n_1 + \cdots + n_d \gg 0$.

In the case that A is artinian, $\ell_A[M_{(0,\ldots,0,n_i,0,\ldots,0)}]$ is a polynomial for all $n_i \gg 0$. Denote by $P_i(n_i)$ the polynomial of $\ell_A[M_{(0,\ldots,0,n_i,0,\ldots,0)}]$.

Now, if we assign the degree -1 to the zero polynomial then we have the following proposition.

PROPOSITION 2.5.   *Let A be an artinian ring. Then for each $1 \le i \le d$, the length of every maximal strong-filter-regular sequence in S_i with respect to M is equal to* $\deg P_i(n_i)+1$.

PROOF.   Note that if $x \in S_i$ is an S_i-filter-regular element with respect to M, then

$$\ell_A[(M/xM)_{(0,\ldots,0,n_i,0,\ldots,0)}] = \ell_A[M_{(0,\ldots,0,n_i,0,\ldots,0)}] - \ell_A[M_{(0,\ldots,0,n_i-1,0,\ldots,0)}]$$

for all $n_i \gg 0$ by [22, Remark 2.6]. On the other hand, it easily seen that $P_i(n_i) = 0$ if and only if $S_i \subseteq \sqrt{\mathrm{Ann}M}$. Hence using the same argument as in the proof of [22, Proposition 3.3(iii)], we get the proof of this proposition.   ∎

REMARK 2.6.   Proposition 2.5 shows that if A is an artinian ring, then the set of lengths of maximal strong-filter-regular sequences with respect to M is bounded above by $\sum_{i=1}^{d} \deg P_i(n_i) + d$.

Now, we study the relationship between filter-regular sequences and weak-(FC)-sequences of ideals. This is a kind of variant of superficial sequences.

The author of [15] in 2000 characterized mixed multiplicities as the Hilbert-Samuel multiplicity by (FC)-sequences (see e.g. [2, 3, 7, 16, 17, 18, 19, 20, 24]).

DEFINITION 2.7 ([15]).   Let N be a finitely generated A-module. Let $I_1, \ldots, I_d$ be ideals such that $I = I_1 \ldots I_d$ is not contained in $\sqrt{\mathrm{Ann}N}$. An element $x \in A$ is called a *weak-(FC)- element* of N with respect to $(I_1, \ldots, I_d)$ if there exists $i \in \{1, \ldots, d\}$ such that $x \in I_i$ and the following conditions are satisfied:

(i) x is an I-filter-regular element with respect to N, i.e., $0_N : x \subseteq 0_N : I^\infty$.
(ii) $xN \bigcap I_1^{n_1} \ldots I_i^{n_i+1} \ldots I_d^{n_d} N = x I_1^{n_1} \ldots I_i^{n_i} \ldots I_d^{n_d} N$ for all $n_1, \ldots, n_d \gg 0$.

Let $x_1, \ldots, x_t$ be a sequence in A. For any $0 \le i < t$, set $N_i = \frac{N}{(x_1,\ldots,x_i)N}$. Then $x_1, \ldots, x_t$ is called a *weak-(FC)-sequence of N with respect to $(I_1, \ldots, I_d)$* if x_{i+1} is a weak-(FC)-element of N_i with respect to $(I_1, \ldots, I_d)$ for all $i = 0, \ldots, t - 1$.

A weak-(FC)-sequence $x_1, \ldots, x_t$ of N with respect to $(I_1, \ldots, I_d)$ is called a *maximal weak-(FC)-sequence* if $I \subseteq \sqrt{\mathrm{Ann}N_t}$.

Set

$$\mathcal{S} = F_{I_1}(I_1, \ldots, I_d; A) = \bigoplus_{n_1,\ldots,n_d \ge 0} \frac{I_1^{n_1} \ldots I_d^{n_d}}{I_1^{n_1+1} I_2^{n_2} \cdots I_d^{n_d}}$$

and

$$\mathcal{M} = F_{I_1}(I_1, \ldots, I_d; N) = \bigoplus_{n_1, \ldots, n_d \geq 0} \frac{I_1^{n_1} \cdots I_d^{n_d} N}{I_1^{n_1+1} I_2^{n_2} \cdots I_d^{n_d} N}.$$

Then $\mathcal{S}$ is a finitely generated standard multi-graded algebra over A/I_1 and $\mathcal{M}$ is a finitely generated multi-graded $\mathcal{S}$-module.

Assume that $x \in I_i$ is a weak-(FC)-element of N with respect to $(I_1, \ldots, I_d)$ and x^* is the image of x in $\mathcal{S}_i = (F_{I_1}(I_1, \ldots, I_d; A))_i$. Then by [25, Proposition 4.5(i)], x^* is an $\mathcal{S}_{++} = F_{I_1}(I_1, \ldots, I_d; A)_{++}$-filter-regular element of $\mathcal{M}$. Note that $I = I_1 \ldots I_d \subseteq \sqrt{\mathrm{Ann}_A N}$ if and only if

$$\mathcal{S}_{(1,1,\ldots,1)} \subseteq \sqrt{\mathrm{Ann}_{\mathcal{S}} \mathcal{M}}.$$

We immediately get the following result.

PROPOSITION 2.8. *Let $x_1, \ldots, x_t$ be a sequence in $\bigcup_{i=1}^d I_i$ and $x_1^*, \ldots, x_t^*$ the images of $x_1, \ldots, x_t$ in $\bigcup_{i=1}^d \mathcal{S}_i$, respectively. Then if $x_1, \ldots, x_t$ is a weak-(FC)-sequence of N with respect to $(I_1, \ldots, I_d)$, $x_1^*, \ldots, x_t^*$ is an $\mathcal{S}_{++}$-filter-regular sequence with respect to $\mathcal{M}$. In this case, $x_1, \ldots, x_t$ is a maximal weak-(FC)-sequence of N with respect to $(I_1, \ldots, I_d)$ if and only if $x_1^*, \ldots, x_t^*$ is a maximal $\mathcal{S}_{++}$-filter-regular sequence with respect to $\mathcal{M}$.*

The notion of superficial elements goes back to P. Samuel [26]. The classical theory of superficial elements is an important tool in local algebra and has been continually developed (see e.g. [4, 6, 11, 13]).

DEFINITION 2.9. Let N be a finitely generated A-module. Let $I_1, \ldots, I_d$ be ideals such that $I = I_1 \ldots I_d$ is not contained in $\sqrt{\mathrm{Ann}_A N}$. An element x is called an I_i-*superficial element of N with respect to $(I_1, \ldots, I_d)$* if $x \in I_i$ and there exists a non-negative integer c such that

$$(I_1^{n_1} \ldots I_i^{n_i+1} \ldots I_d^{n_d} N : x) \bigcap I_1^{n_1} \ldots I_i^c \ldots I_d^{n_d} N = I_1^{n_1} \ldots I_d^{n_d} N$$

for all $n_i \geq c$ and for all non-negative integers $n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_d$.
Let $x_1, \ldots, x_t$ be a sequence in A. For any $0 \leq i < t$, set $N_i = \frac{N}{(x_1, \ldots, x_i)N}$. Then $x_1, \ldots, x_t$ is called a *superficial sequence of N with respect to $(I_1, \ldots, I_d)$* if x_{i+1} is a superficial element of N_i with respect to $(I_1, \ldots, I_d)$ for all $i = 0, \ldots, t - 1$. A superficial sequence $x_1, \ldots, x_t$ of N with respect to $(I_1, \ldots, I_d)$ is called a *maximal superficial sequence* if $I \subseteq \sqrt{\mathrm{Ann}_A N_t}$ (see [11]).

Now we need to prove that if $I_1, \ldots, I_d$ are $\mathfrak{m}$-primary and x is an I_i-superficial element of N with respect to $(I_1, \ldots, I_d)$, then x is a weak-(FC)-element of N with respect to $(I_1, \ldots, I_d)$. Indeed, by [4, Lemma 17.2.4],

$$I_1^{n_1} \ldots I_d^{n_d} N : x = (0_N : x) + I_1^{n_1} \ldots I_i^{n_i-1} \ldots I_d^{n_d} N$$

for all $n_1, \ldots, n_s \gg 0$. Multiplying the last equation by x yields

$$I_1^{n_1} \cdots I_d^{n_d} N \bigcap xN = x I_1^{n_1} \cdots I_i^{n_i-1} \cdots I_d^{n_d} N$$

for $n_1, \ldots, n_d \gg 0$. Hence x satisfies the condition (ii) of a weak-(FC)- element of N with respect to $(I_1, \ldots, I_d)$. By [4, Lemma 17.2.4], we have

$$(0_N : x) \bigcap I_1^{n_1} \cdots I_d^{n_d} N = 0$$

for $n_1, \ldots, n_d \gg 0$. Hence $0_N : x \subseteq 0_N : I^\infty$. So x is an I-filter-regular element with respect to N, i.e., x satisfies the condition (i) of a weak-(FC)- element of N with respect to $(I_1, \ldots, I_d)$. Thus x is a weak-(FC)- element of N with respect to $(I_1, \ldots, I_d)$. Note that a superficial sequence $x_1, \ldots, x_m$ of N with respect to $(I_1, \ldots, I_d)$ is a maximal superficial sequence if and only if $I \subseteq \sqrt{\mathrm{Ann} N_m}$. Therefore $x_1, \ldots, x_m$ is a maximal superficial sequence of N with respect to $(I_1, \ldots, I_d)$ if and only if $x_1, \ldots, x_m$ is also a maximal weak-(FC) sequence of N with respect to $(I_1, \ldots, I_d)$.

This fact yields the following.

PROPOSITION 2.10. *Let $I_1, \ldots, I_d$ be $\mathfrak{m}$-primary ideals. Assume that $x_1, \ldots, x_t$ is a superficial sequence of N with respect to $(I_1, \ldots, I_d)$. Then $x_1, \ldots, x_t$ is a weak-(FC)-sequence of N with respect to $(I_1, \ldots, I_d)$. In this case, $x_1, \ldots, x_t$ is a maximal superficial sequence of N with respect to $(I_1, \ldots, I_d)$ if and only if $x_1, \ldots, x_t$ is a maximal weak-(FC)-sequence of N with respect to $(I_1, \ldots, I_d)$.*

Then as an immediate consequence of Proposition 2.8 and Proposition 2.10, we have the following result.

PROPOSITION 2.11. *Let $I_1, \ldots, I_d$ be $\mathfrak{m}$-primary ideals. Assume that $x_1, \ldots, x_t$ is a sequence in $\bigcup_{i=1}^{d} I_i$ and $x_1^*, \ldots, x_t^*$ the images of $x_1, \ldots, x_t$ in $\bigcup_{i=1}^{d} \mathcal{S}_i$, respectively. Then if $x_1, \ldots, x_t$ is a superficial sequence of N with respect to $(I_1, \ldots, I_d)$, $x_1^*, \ldots, x_t^*$ is an $\mathcal{S}_{++}$-filter-regular sequence with respect to $\mathcal{M}$. In this case, $x_1, \ldots, x_t$ is a maximal superficial sequence of N with respect to $(I_1, \ldots, I_d)$ if and only if $x_1^*, \ldots, x_t^*$ is a maximal $\mathcal{S}_{++}$-filter-regular sequence with respect to $\mathcal{M}$.*

REMARK 2.12. Let $I_1, \ldots, I_d$ be $\mathfrak{m}$-primary ideals. Assume that x is an I_i-superficial element of N with respect to $(I_1, \ldots, I_d)$ and x^* the image of x in $\mathcal{S}_i$. Then x^* is an $\mathcal{S}_{++}$-filter-regular element with respect to $\mathcal{M}$ by Proposition 2.11. Hence

$$(I_1^{n_1+1} I_2^{n_2} \cdots I_i^{n_i+1} \cdots I_d^{n_d} N : x) \bigcap I_1^{n_1} \cdots I_d^{n_d} N = I_1^{n_1+1} I_2^{n_2} \cdots I_d^{n_d} N$$

for all $n_1, \ldots, n_d \gg 0$. And by [22, Proposition 2.5], this is equivalent to

$$x^* \in \mathcal{S}_i \setminus \bigcup_{P \in \mathrm{Ass}_{\mathcal{S}} \mathcal{M}/(0_{\mathcal{M}} : \mathcal{S}_{++}^\infty)} P.$$

3. Filter-Regular Sequences and Joint Reductions

In this section, we explore the relationship between maximal filter-regular sequences and reductions.

The concept of joint reductions of $\mathfrak{m}$-primary ideals was given by Rees in 1984 [10]. And he proved that mixed multiplicities of $\mathfrak{m}$-primary ideals are multiplicities of ideals generated by joint reductions. This concept was extended to the set of arbitrary ideals by [7, 9, 17, 19]. In the case of graded modules, we have the following.

DEFINITION 3.1. Let $S = \bigoplus_{n_1,\ldots,n_d \geq 0} S_{(n_1,\ldots,n_d)}$ be a finitely generated standard $\mathbf{N}^d$-graded algebra over a noetherian local ring A and let $M = \bigoplus_{n_1,\ldots,n_d \geq 0} M_{(n_1,\ldots,n_d)}$ be a finitely generated $\mathbf{N}^d$-graded S-module. Let $\mathfrak{R}$ be a subset of $\bigcup_{i=1}^{d} S_i$. Set $\mathfrak{R}_i = \mathfrak{R} \bigcap S_i$ for all $i = 1, \ldots, d$ and $(\emptyset) = 0_S$. Then $\mathfrak{R}$ is called a *joint reduction of $(S_1, \ldots, S_d)$ with respect to M* if

$$M_{(n_1+1,\ldots,n_d+1)} = \sum_{i=1}^{d} (\mathfrak{R}_i) M_{(n_1+1,\ldots,n_i,\ldots,n_d+1)} \text{ for all large integers } n_1, n_2, \ldots, n_d .$$

Recall that an ideal $\mathfrak{J}$ is called *a reduction of an ideal $\mathcal{I}$ with respect to A-module N* if $\mathfrak{J} \subseteq \mathcal{I}$ and $\mathcal{I}^{n+1} N = \mathfrak{J}\mathcal{I}^n N$ for all large n by [8].

Then the relationship between maximal filter-regular sequences and reductions is stated by the following theorem.

THEOREM 3.2. *Let $\mathfrak{R}$ be a subset of $\bigcup_{i=1}^{d} S_i$. Then the following statements hold.*

(i) *If $\mathfrak{R}$ is a maximal S_{++}-filter-regular sequence with respect to M, then $\mathfrak{R}$ is a joint reduction of $(S_1, \ldots, S_d)$ with respect to M.*

(ii) *If $\mathfrak{R}$ is a maximal strong-filter-regular sequence with respect to M, then $(\mathfrak{R})$ is a reduction of S_+ with respect to M.*

PROOF. Set $\mathfrak{R}_i = \mathfrak{R} \bigcap S_i$ for all $i = 1, \ldots, d$. Assume that $\mathfrak{R}$ is a maximal S_{++}-filter-regular sequence with respect to M. Then we have

$$S_{(1,1,\ldots,1)} \subseteq \sqrt{\text{Ann}[M/(\mathfrak{R})M]} .$$

Hence by Remark 2.2(ii), $[M/(\mathfrak{R})M]_{(n_1,\ldots,n_d)} = 0$ for all large $n_1, \ldots, n_d$. From this it follows that

$$M_{(n_1+1,\ldots,n_d+1)} = \sum_{i=1}^{d} (\mathfrak{R}_i) M_{(n_1+1,\ldots,n_i,\ldots,n_d+1)}$$

for all large integers $n_1, n_2, \ldots, n_d$. Thus, $\mathfrak{R}$ is a joint reduction of $(S_1, \ldots, S_d)$ with respect to M. We have (i). Now, if $\mathfrak{R}$ is a maximal strong-filter-regular sequence with respect to M, then it follows that there is not any strong-filter-regular sequence in $\bigcup_{i=1}^{d} S_i$ with respect to

$M/(\mathfrak{R})M$. Hence by Remark 2.4, $[M/(\mathfrak{R})M]_{(n_1,\ldots,n_d)} = 0$ for all $n_1 + \cdots + n_d \gg 0$. This is equivalent to

$$\sum_{i=1}^{d}(\mathfrak{R}_i)M_{(n_1,\ldots,n_i-1,\ldots,n_d)} = M_{(n_1,\ldots,n_d)}$$

for all $n_1 + \cdots + n_d \gg 0$. Here we assign that $M_{(n_1,\ldots,-1,\ldots,n_d)} = 0$. Direct computation shows that

$$S_+^n M = \sum_{n_1+\cdots+n_d \geq n} M_{(n_1,\ldots,n_d)}$$

and

$$\left[(\mathfrak{R}) \sum_{n_1+\cdots+n_d \geq n} M_{(n_1,\ldots,n_d)} \right]_{(m_1,\ldots,m_d)} = \sum_{i=1}^{d}(\mathfrak{R}_i)M_{(m_1,\ldots,m_i-1,\ldots,m_d)}$$

for all $m_1 + \cdots + m_d \geq n + 1$. Hence we have

$$(\mathfrak{R})S_+^n M = (\mathfrak{R}) \sum_{n_1+\cdots+n_d \geq n} M_{(n_1,\ldots,n_d)}$$

$$= \sum_{m_1+\cdots+m_d \geq n+1} \left[\sum_{i=1}^{d}(\mathfrak{R}_i)M_{(m_1,\ldots,m_i-1,\ldots,m_d)} \right]$$

$$= \sum_{m_1+\cdots+m_d \geq n+1} M_{(m_1,\ldots,m_d)} = S_+^{n+1} M$$

for all $n \gg 0$. Thus, $(\mathfrak{R})$ is a reduction of S_+ with respect to M. So we get (ii). ∎

By the proof of Theorem 3.2(ii), we immediately obtain the following corollary.

COROLLARY 3.3. $(\mathfrak{R})$ *is a reduction of S_+ with respect to M if and only if*

$$\sum_{i=1}^{d}(\mathfrak{R}_i)M_{(n_1,\ldots,n_i-1,\ldots,n_d)} = M_{(n_1,\ldots,n_d)} \quad \textit{for all } n_1 + \cdots + n_d \gg 0.$$

Now, we consider joint reductions of ideals $I_1, \ldots, I_d$ with respect to N.

DEFINITION 3.4 ([21]).   Let $I_1, \ldots, I_d$ be ideals of A and N an A-module. Let $\mathfrak{R}$ be a subset of $\bigcup_{i=1}^{d} I_i$. Set $\mathfrak{R}_i = \mathfrak{R} \bigcap I_i$ for $i = 1, \ldots, d$ and $(\emptyset) = 0_A$. Then $\mathfrak{R}$ is called a *joint reduction* of $(I_1, \ldots, I_d)$ with respect to N if

$$I_1^{n_1+1} \cdots I_d^{n_d+1} N = \sum_{i=1}^{d}(\mathfrak{R}_i)I_1^{n_1+1} \cdots I_i^{n_i} \cdots I_d^{n_d+1} N$$

for all large integers $n_1, n_2, \ldots, n_d$.

Recall that $\mathcal{S} = F_{I_1}(I_1, \ldots, I_d; A)$ and $\mathcal{M} = F_{I_1}(I_1, \ldots, I_d; N)$. Now for each $x \in I_j$, denote by x^* the image of x in $\mathcal{S}_j = (F_{I_1}(I_1, \ldots, I_d; A))_j$. Now, assume that $x_1, \ldots, x_t$ is a maximal weak-(FC)-sequence of N with respect to $(I_1, \ldots, I_d)$ then $x_1^*, \ldots, x_t^*$ is a maximal $\mathcal{S}_{++}$-filter-regular sequence with respect to $\mathcal{M}$ by Proposition 2.8. Set

$$\mathfrak{R} = \{x_1, \ldots, x_t\}; \ \mathfrak{R}_i = \mathfrak{R} \bigcap I_i$$

for all $i = 1, \ldots, d$ and

$$\mathfrak{R}^* = \{x_1^*, \ldots, x_t^*\}; \ \mathfrak{R}_i^* = \mathfrak{R}^* \bigcap \mathcal{S}_i$$

for all $i = 1, \ldots, d$. By Theorem 3.2(i), $\mathfrak{R}^*$ is a joint reduction of $(\mathcal{S}_1, \ldots, \mathcal{S}_d)$ with respect to $\mathcal{M}$. Hence

$$\mathcal{M}_{(n_1+1,\ldots,n_d+1)} = \sum_{i=1}^{d} (\mathfrak{R}_i^*) \mathcal{M}_{(n_1+1,\ldots,n_i,\ldots,n_d+1)}$$

for all large integers $n_1, n_2, \ldots, n_d$. From this it follows that

$$I_1^{n_1+1} \cdots I_d^{n_d+1} N = \sum_{i=1}^{d} (\mathfrak{R}_i) I_1^{n_1+1} \cdots I_i^{n_i} \cdots I_d^{n_d+1} N + I_1^{n_1+2} I_2^{n_2+1} \cdots I_d^{n_d+1} N$$

for all large integers $n_1, n_2, \ldots, n_d$. Therefore,

$$I_1^{n_1+1} \cdots I_d^{n_d+1} N = \sum_{i=1}^{d} (\mathfrak{R}_i) I_1^{n_1+1} \cdots I_i^{n_i} \cdots I_d^{n_d+1} N$$

for all large integers $n_1, n_2, \ldots, n_d$ by Nakayama's lemma. Thus, $\mathfrak{R}$ is a joint reduction of $(I_1, \ldots, I_d)$ with respect to N.

Then we get the following result which is a generalization of [19, Theorem 3.4] and [7, Theorem 2.9].

COROLLARY 3.5. *If $x_1, \ldots, x_t$ is a maximal weak-(FC)-sequence of N with respect to $(I_1, \ldots, I_d)$ then $\{x_1, \ldots, x_t\}$ is a joint reduction of $(I_1, \ldots, I_d)$ with respect to N.*

In the case that $I_1, \ldots, I_d$ are $\mathfrak{m}$-primary ideals then from Proposition 2.10 and Corollary 3.5, we immediately get the following result.

COROLLARY 3.6. *Let $I_1, \ldots, I_d$ be $\mathfrak{m}$-primary ideals of A. Assume that $x_1, \ldots, x_t$ is a maximal superficial sequence of N with respect to $(I_1, \ldots, I_d)$. Then $\{x_1, \ldots, x_t\}$ is a joint reduction of $(I_1, \ldots, I_d)$ with respect to N.*

4. Filter-Regular Sequences and Parameter Systems

In this section, we characterize the existence of parts of homogeneous parameter systems for M consisting of elements of total degree 1 via the relationship between strong-filter-regular

sequences and parts of homogeneous parameter systems.

The main result of this section is the following theorem.

THEOREM 4.1.   *Let* $S = \bigoplus_{n_1,\dots,n_d \geq 0} S_{(n_1,\dots,n_d)}$ *be a finitely generated standard* $\mathbf{N}^d$-*graded algebra over a noetherian local ring* $(A, \mathfrak{m})$ *and let* $M = \bigoplus_{n_1,\dots,n_d \geq 0} M_{(n_1,\dots,n_d)}$ *be a finitely generated* $\mathbf{N}^d$-*graded S-module. Let* $k_1, k_2, \dots, k_d$ *be non-negative integers with* $k_1 + k_2 + \cdots + k_d = s \leq \dim M$. *Set* $(\emptyset) = 0_S$ *and*

$$\mathfrak{S}_{\hat{i}} = \{S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_d\}$$

for all $1 \leq i \leq d$. *Then the following statements are equivalent.*

  (i) *For any* $1 \leq i \leq d$, $\dim_S \frac{M}{(\mathfrak{U}, S_i)M} \leq \dim_S \frac{M}{(\mathfrak{U})M} - k_i$ *for all* $\mathfrak{U} \subset \mathfrak{S}_{\hat{i}}$.

  (ii) *There is a strong-filter-regular sequence* $x_1, \dots, x_s$ *with respect to* M *consisting of* k_1 *elements of* $S_1, \dots, k_d$ *elements of* S_d *such that if* $y_1, \dots, y_{k_i} \in S_i$ *is a subsequence of* $x_1, \dots, x_s$, *then*

$$\dim_S \frac{M}{(\mathfrak{U}, y_1, \dots, y_{k_i})M} = \dim_S \frac{M}{(\mathfrak{U})M} - k_i$$

   for all $\mathfrak{U} \subset \mathfrak{S}_{\hat{i}}$ *and* $1 \leq i \leq d$. *And* $x_1, \dots, x_s$ *is a part of a parameter system for* M.

  (iii) *There is a part of a parameter system* $x_1, \dots, x_s$ *for* M *consisting of* k_1 *elements of* $S_1, \dots, k_d$ *elements of* S_d *such that if* $y_1, \dots, y_{k_i} \in S_i$ *is a subsequence of* $x_1, \dots, x_s$, *then*

$$\dim_S \frac{M}{(\mathfrak{U}, y_1, \dots, y_{k_i})M} = \dim_S \frac{M}{(\mathfrak{U})M} - k_i$$

   for all $\mathfrak{U} \subset \mathfrak{S}_{\hat{i}}$ *and* $1 \leq i \leq d$.

PROOF.   (i)$\Rightarrow$(ii): Set $k_1 + k_2 + \cdots + k_d = s$. The proof is by induction on s that there exists a strong-filter-regular sequence $x_1, \dots, x_s$ with respect to M consisting of k_1 elements of $S_1, \dots, k_d$ elements of S_d such that if $y_1, \dots, y_{k_i} \in S_i$ is a subsequence of $x_1, \dots, x_s$ then

$$\dim_S \frac{M}{(\mathfrak{U}, y_1, \dots, y_{k_i})M} = \dim_S \frac{M}{(\mathfrak{U})M} - k_i$$

for all $\mathfrak{U} \subset \mathfrak{S}_{\hat{i}}$ and $1 \leq i \leq d$. And $x_1, \dots, x_s$ is a part of a parameter system for M. The result is true for $s = 0$. Assume the result for $s - 1 \geq 0$. Since $s > 0$, there exists $1 \leq v \leq d$ such that $k_v > 0$. Then we have

$$\dim_S \frac{M}{(\mathfrak{U}, S_v)M} \leq \dim_S \frac{M}{(\mathfrak{U})M} - k_v < \dim_S \frac{M}{(\mathfrak{U})M}$$

for all $\mathfrak{U} \subset \mathfrak{S}_{\hat{v}}$. Set

$$U = \mathrm{Ass}_S \frac{M}{(0_M : S_v^\infty)} = \{P \in \mathrm{Ass}_S M \mid S_v \nsubseteq P\}.$$

And for any $\mathfrak{U} \subset \mathfrak{S}_{\widehat{v}}$, set

$$V_{\mathfrak{U}} = \left\{ P \in \mathrm{Ass}_S \frac{M}{(\mathfrak{U})M} \mid \dim_S \frac{M}{(\mathfrak{U})M} = \mathrm{Coht} P \right\}.$$

Since

$$\dim_S \frac{M}{(\mathfrak{U}, S_v)M} < \dim_S \frac{M}{(\mathfrak{U})M},$$

$S_v \not\subseteq P$ for any $P \in V_{\mathfrak{U}}$ and all $\mathfrak{U} \subset \mathfrak{S}_{\widehat{v}}$. Put $G = \left(\bigcup_{\mathfrak{U} \subset \mathfrak{S}_{\widehat{v}}} V_{\mathfrak{U}} \right) \bigcup U$. Then $S_v \not\subseteq P$ for any $P \in G$. Since k is an infinite field and G is finite, there exists

$$x \in S_v \setminus \bigcup_{P \in G} P.$$

Because $x \notin \bigcup_{P \in U} P$, x is an S_v-filter-regular element with respect to M. And since $x \notin \bigcup_{P \in V_{\mathfrak{U}}} P$, it follows that

$$\dim_S \frac{M}{(\mathfrak{U}, x)M} = \dim_S \frac{M}{(\mathfrak{U})M} - 1$$

for all $\mathfrak{U} \subset \mathfrak{S}_{\widehat{v}}$. In particular, $\mathfrak{U} = \emptyset$ then we get

$$\dim_S \frac{M}{xM} = \dim_S \frac{M}{(\emptyset, x)M} = \dim_S \frac{M}{(\emptyset)M} - 1 = \dim_S M - 1.$$

Hence x is a parameter element for M. Set $\overline{M} = M/xM$ and

$$h_1 = k_1, \ldots, h_{v-1} = k_{v-1}, h_v = k_v - 1, h_{v+1} = k_{v+1}, \ldots, h_d = k_d.$$

Then $h_1 + \cdots + h_d = s - 1$. Now, we need to show that

$$\dim_S \frac{\overline{M}}{(\mathfrak{U}, S_i)\overline{M}} \leq \dim_S \frac{\overline{M}}{(\mathfrak{U})\overline{M}} - h_i$$

for all $\mathfrak{U} \subset \mathfrak{S}_{\widehat{i}}$ and $1 \leq i \leq d$. Indeed, for $i \neq v$, let $\mathfrak{U} \subset \mathfrak{S}_{\widehat{i}}$. If $S_v \in \mathfrak{U}$ then we have

$$\dim_S \frac{\overline{M}}{(\mathfrak{U}, S_i)\overline{M}} = \dim_S \frac{M}{(\mathfrak{U}, S_i)M} \leq \dim_S \frac{M}{(\mathfrak{U})M} - k_i = \dim_S \frac{\overline{M}}{(\mathfrak{U})\overline{M}} - h_i.$$

If $S_v \notin \mathfrak{U}$ then $\mathfrak{U} \cup \{S_i\} \subset \mathfrak{S}_{\widehat{v}}$. Hence

$$\dim_S \frac{\overline{M}}{(\mathfrak{U}, S_i)\overline{M}} = \dim_S \frac{M}{(\mathfrak{U}, S_i, x)M} = \dim_S \frac{M}{(\mathfrak{U}, S_i)M} - 1$$

$$\leq \dim_S \frac{M}{(\mathfrak{U})M} - k_i - 1 = \dim_S \frac{M}{(\mathfrak{U}, x)M} - k_i$$

$$= \dim_S \frac{\overline{M}}{(\mathfrak{U})\overline{M}} - h_i.$$

For $i = v$,

$$\dim_S \frac{\overline{M}}{(\mathfrak{U}, S_v)\overline{M}} = \dim_S \frac{M}{(\mathfrak{U}, S_v)M}$$

$$\leq \dim_S \frac{M}{(\mathfrak{U})M} - k_v = \dim_S \frac{M}{(\mathfrak{U}, x)M} + 1 - k_v$$

$$= \dim_S \frac{\overline{M}}{(\mathfrak{U})\overline{M}} + 1 - k_v = \dim_S \frac{\overline{M}}{(\mathfrak{U})\overline{M}} - h_v$$

for all $\mathfrak{U} \subset \mathfrak{S}_{\widehat{v}}$. Consequently $\dim_S \dfrac{\overline{M}}{(\mathfrak{U}, S_i)\overline{M}} \leq \dim_S \dfrac{\overline{M}}{(\mathfrak{U})\overline{M}} - h_i$ for all $\mathfrak{U} \subset \mathfrak{S}_{\widehat{i}}$ and $1 \leq i \leq d$.

By induction, there exists a strong-filter-regular sequence $x_2, \ldots, x_s$ with respect to $\overline{M}$ consisting of h_1 elements of $S_1, \ldots, h_d$ elements of S_d and if $y_1, \ldots, y_{h_i} \in S_i$ is a subsequence of $x_2, \ldots, x_s$ then

$$\dim_S \frac{\overline{M}}{(\mathfrak{U}, y_1, \ldots, y_{h_i})\overline{M}} = \dim_S \frac{\overline{M}}{(\mathfrak{U})\overline{M}} - h_i$$

for all $\mathfrak{U} \subset \mathfrak{S}_{\widehat{i}}$ and $1 \leq i \leq d$. And $x_2, \ldots, x_s$ is a part of a parameter system for $\overline{M}$. It is a plain fact that $x = x_1, x_2, \ldots, x_s$ is a strong-filter-regular sequence consisting of $k_1 = h_1$ elements of $S_1, \ldots, k_{v-1} = h_{v-1}$ elements of S_{v-1}, $k_v = h_v + 1$ elements of S_v, $k_{v+1} = h_{v+1}$ elements of $S_{v+1}, \ldots, k_d = h_d$ elements of S_d. Next we need to prove that for any i and $z_1, \ldots, z_{k_i} \in S_i$ is a subsequence of $x_1, \ldots, x_s$, then

$$\dim_S \frac{M}{(\mathfrak{U}, z_1, \ldots, z_{k_i})M} = \dim_S \frac{M}{(\mathfrak{U})M} - k_i$$

for all $\mathfrak{U} \subset \mathfrak{S}_{\widehat{i}}$ and $1 \leq i \leq d$. Indeed, consider the following cases:

The case that $i \neq v$: Then in this case, $k_i = h_i$. Let $\mathfrak{U} \subset \mathfrak{S}_{\widehat{i}}$.
If $S_v \in \mathfrak{U}$ then since $x \in S_v$,

$$\dim_S \frac{M}{(\mathfrak{U}, z_1, \ldots, z_{k_i})M} = \dim_S \frac{\overline{M}}{(\mathfrak{U}, z_1, \ldots, z_{h_i})\overline{M}}$$

$$= \dim_S \frac{\overline{M}}{(\mathfrak{U})\overline{M}} - h_i = \dim_S \frac{M}{(\mathfrak{U})M} - k_i \ .$$

If $S_v \notin \mathfrak{U}$ then $\mathfrak{U} \subset \mathfrak{S}_{\widehat{v}}$. Hence

$$\dim_S \frac{M}{(\mathfrak{U})M} - k_i - 1 \leq \dim_S \frac{M}{(\mathfrak{U}, z_1, \ldots, z_{k_i})M} - 1$$

$$\leq \dim_S \frac{M}{(\mathfrak{U}, x, z_1, \ldots, z_{k_i})M} = \dim_S \frac{\overline{M}}{(\mathfrak{U}, z_1 \ldots, z_{k_i})\overline{M}}$$

$$= \dim_S \frac{\overline{M}}{(\mathfrak{U})\overline{M}} - h_i = \dim_S \frac{M}{(\mathfrak{U}, x)M} - k_i = \dim_S \frac{M}{(\mathfrak{U})M} - k_i - 1.$$

So $\dim_S \frac{M}{(\mathfrak{U}, z_1, \ldots, z_{k_i})M} = \dim_S \frac{M}{(\mathfrak{U})M} - k_i$.

The case that $i = v$: In this case, $x = z_1$ and $k_v = h_v + 1$. We get

$$\dim_S \frac{M}{(\mathfrak{U}, z_1, \ldots, z_{k_v})M} = \dim_S \frac{\overline{M}}{(\mathfrak{U}, z_2, \ldots, z_{k_v})\overline{M}}$$

$$= \dim_S \frac{\overline{M}}{(\mathfrak{U})\overline{M}} - h_v = \dim_S \frac{\overline{M}}{(\mathfrak{U})\overline{M}} + 1 - k_v$$

$$= \dim_S \frac{M}{(\mathfrak{U}, x)M} + 1 - k_v = \dim_S \frac{M}{(\mathfrak{U})M} - k_v,$$

for all $\mathfrak{U} \subset \mathfrak{S}_{\widehat{v}}$. Therefore

$$\dim_S \frac{M}{(\mathfrak{U}, z_1, \ldots, z_{k_i})M} = \dim_S \frac{M}{(\mathfrak{U})M} - k_i$$

for all $\mathfrak{U} \subset \mathfrak{S}_{\widehat{i}}$ and $1 \le i \le d$. Now observe that since $x = x_1$ is a parameter element for M and $x_2, \ldots, x_s$ is a part of a parameter system for M/xM, $x_1, \ldots, x_s$ is a part of a parameter system for M. The induction is complete. We get (i)$\Rightarrow$(ii). (ii)$\Rightarrow$(iii) is obvious. (iii)$\Rightarrow$(i): By (iii) we have $\dim_S \frac{M}{(\mathfrak{U}, y_1, \ldots, y_{k_i})M} = \dim_S \frac{M}{(\mathfrak{U})M} - k_i$ for all $\mathfrak{U} \subset \mathfrak{S}_{\widehat{i}}$ and $1 \le i \le d$. Hence

$$\dim_S \frac{M}{(\mathfrak{U}, S_i)M} \le \dim_S \frac{M}{(\mathfrak{U}, y_1, \ldots, y_{k_i})M} = \dim_S \frac{M}{(\mathfrak{U})M} - k_i$$

for all $\mathfrak{U} \subset \mathfrak{S}_{\widehat{i}}$ and $1 \le i \le d$. So (iii)$\Rightarrow$(i) is proved. ∎

COROLLARY 4.2. *Let S be a finitely generated standard $\mathbf{N}^d$-graded algebra over a noetherian local ring $(A, \mathfrak{m})$ and let M be a finitely generated $\mathbf{N}^d$-graded S-module. Set $S_0 = 0_S$ and $s = \dim_S M - \dim_A M_{(0,\ldots,0)}$ and*

$$k_i = \dim_S \frac{M}{(S_0, \ldots, S_{i-1})M} - \dim_S \frac{M}{(S_0, \ldots, S_i)M}$$

for any $1 \le i \le d$. Assume that

$$k_i \le \min \left\{ \dim_S \frac{M}{(\mathfrak{U})M} - \dim_S \frac{M}{(\mathfrak{U}, S_i)M} \mid \emptyset \ne \mathfrak{U} \subset \mathfrak{S}_{\widehat{i}} \right\}$$

for any $1 \le i \le d$. Then $k_1 + \cdots + k_d = s$ and the following conditions are equivalent:

  (i) $k_i \le \dim_S M - \dim_S \frac{M}{(S_i)M}$ *for all $1 \le i \le d$.*
  (ii) *There is a strong-filter-regular sequence with respect to M consisting of k_1 elements of $S_1, \ldots, k_d$ elements of S_d. And this sequence is a part of a parameter system for M.*

(iii) *There is a part of a homogeneous parameter system for M consisting of k_1 elements of $S_1, \ldots, k_d$ elements of S_d.*

PROOF. Since $k_i = \dim_S \frac{M}{(S_0,\ldots,S_{i-1})M} - \dim_S \frac{M}{(S_0,\ldots,S_i)M}$ for any $1 \le i \le d$, $k_1 + \cdots + k_d = \dim_S M - \dim_S \frac{M}{(S_0,\ldots,S_d)M} = \dim_S M - \dim_A M_{(0,\ldots,0)} = s$. (i)$\Rightarrow$(ii): By (i), for any $1 \le i \le d$, we have $\dim_S \frac{M}{(\mathfrak{U}, S_i)M} \le \dim_S \frac{M}{(\mathfrak{U})M} - k_i$ for all $\mathfrak{U} \subset \mathfrak{S}_{\widehat{i}}$. Hence (i)$\Rightarrow$(ii) is immediate from (i)$\Rightarrow$(ii) of Theorem 4.1. And (ii)$\Rightarrow$(iii) is obvious. (iii)$\Rightarrow$(i): Assume that $x_1, \ldots, x_s \in \bigcup_{i=1}^{d} S_i$ is a part of a parameter system for M consisting of k_1 elements of $S_1, \ldots, k_d$ elements of S_d and $z_1, \ldots, z_{k_i} \in S_i$ is a subsequence of $x_1, \ldots, x_s$, then $\dim_S \frac{M}{(S_i)M} \le \dim_S \frac{M}{(z_1,\ldots,z_{k_i})M} = \dim_S M - k_i$. Thus, $k_i \le \dim_S M - \dim_S \frac{M}{(S_i)M}$ for all $1 \le i \le d$. ∎

In particular, if $d = 1$ then since

$$\dim_S \frac{M}{S_0 M} - \dim_S \frac{M}{(S_0, S_1)M} = \dim_S M - \dim_S \frac{M}{(S_1)M} = \dim_S M - \dim_A M_0,$$

we have the following.

COROLLARY 4.3. *Let $S = \bigoplus_{n \ge 0} S_n$ be a finitely generated standard $\mathbf{N}$-graded algebra over a noetherian local ring $(A, \mathfrak{m})$ and let $M = \bigoplus_{n \ge 0} M_n$ be a finitely generated $\mathbf{N}$-graded S-module. Set $s = \dim_S M - \dim_A M_0$. Then there is at most s parameter elements for M contained in S_1.*

EXAMPLE 4.4. Let I be an ideal of positive height in a noetherian local ring $(A, \mathfrak{m})$. Denote by $R(I) = \bigoplus_{n \ge 0} I^n t^n$ the Rees algebra and $G(I) = \bigoplus_{n \ge 0}(I^n/I^{n+1})$ the associated graded ring of A with respect to I. Since $\dim R(I) - \dim A = 1$, there is at most one parameter element for $R(I)$ contained in It by Corollary 4.3. And since $\dim G(I) - \dim(A/I) = \dim A - \dim(A/I)$, there is at most $\dim A - \dim(A/I)$ parameter elements for $G(I)$ contained in I/I^2 by Corollary 4.3.

Now, if choose $k_i = \min\left\{ \dim_S \frac{M}{(\mathfrak{U})M} - \dim_S \frac{M}{(\mathfrak{U},S_i)M} \mid \mathfrak{U} \subset \mathfrak{S}_{\widehat{i}} \right\}$ for all $1 \le i \le d$, then as an immediate consequence of Theorem 4.1, we have the following consequence.

COROLLARY 4.5. *Let S be a finitely generated standard $\mathbf{N}^d$-graded algebra over a noetherian local ring $(A, \mathfrak{m})$ and let M be a finitely generated $\mathbf{N}^d$-graded S-module. Set $k_i = \min\left\{ \dim_S \frac{M}{(\mathfrak{U})M} - \dim_S \frac{M}{(\mathfrak{U},S_i)M} \mid \mathfrak{U} \subset \mathfrak{S}_{\widehat{i}} \right\}$ for any $1 \le i \le d$. Then there is a strong-filter-regular sequence with respect to M consisting of k_1 elements of $S_1, \ldots, k_d$ elements of S_d. And this sequence is a part of a parameter system for M.*

EXAMPLE 4.6. Assume that $\dim A > 0$; $J, I_1, \ldots, I_d$ ($d > 1$) are $\mathfrak{m}$-primary ideals. Denote by $F_J(I_1, \ldots, I_d; A) = \bigoplus_{n_1,\ldots,n_d \ge 0} \frac{I_1^{n_1}\ldots I_d^{n_d}}{J I_1^{n_1}\ldots I_d^{n_d}}$ the d-graded fiber cone of A with

respect to $J, I_1, \ldots, I_d$. Set $\mathbf{S} = F_J(I_1, \ldots, I_d; A)$; $\mathbf{S}_i = \mathbf{S}_{(0,\underbrace{\ldots,1,\ldots,0}_{i})} = I_i/JI_i$ and

$$\mathfrak{S}_{\hat{i}}^* = \{\mathbf{S}_1, \ldots, \mathbf{S}_{i-1}, \mathbf{S}_{i+1}, \ldots, \mathbf{S}_d\} \text{ for all } 1 \leq i \leq d.$$

Then $\dim \mathbf{S} = \dim A + d - 1$ (see [23]). Since $\frac{\mathbf{S}}{(\mathbf{S}_i)} \cong F_J(I_1, \ldots, I_{i-1}, I_{i+1}, \ldots, I_d; A)$ and $d > 1$, $\dim \frac{\mathbf{S}}{(\mathbf{S}_i)} = \dim A + d - 2$. So $\dim \mathbf{S} - \dim \frac{\mathbf{S}}{(\mathbf{S}_i)} = 1$ for all $1 \leq i \leq d$. Now, assume that there exists a parameter system for $\mathbf{S}$ consisting of $(\dim A + d - 1)$ elements of $\bigcup_{i=1}^d \mathbf{S}_i$. If $\dim A > 1$ then there exists i such that $\mathbf{S}_i$ contains two parameter elements x, y. In this case,

$$1 = \dim \mathbf{S} - \dim \frac{\mathbf{S}}{(\mathbf{S}_i)} \geq \dim \mathbf{S} - \dim \frac{\mathbf{S}}{(x, y)} = 2.$$

This contradiction shows that if $\dim A > 1$, then there does not exist a homogeneous parameter ideal for $\mathbf{S}$ generated by elements of total degree 1. However, since

$$1 = \min\left\{\dim_{\mathbf{S}} \frac{\mathbf{S}}{(\mathfrak{U})} - \dim_{\mathbf{S}} \frac{\mathbf{S}}{(\mathfrak{U}, \mathbf{S}_i)} \mid \mathfrak{U} \subset \mathfrak{S}_{\hat{i}}^*\right\}$$

for any $1 \leq i \leq d$, there exists a part of a parameter system for $\mathbf{S}$ consisting of 1 element of $\mathbf{S}_1, \ldots,$ 1 element of $\mathbf{S}_d$ by Corollary 4.5(ii). Thus, in this case, there is a homogeneous parameter ideal for $\mathbf{S}$ generated by homogeneous elements of total degree 1 if and only if $\dim A = 1$.

COROLLARY 4.7. *Let $(A, \mathfrak{m})$ be an artinian local ring. Let S be a finitely generated standard $\mathbf{N}^d$-graded algebra over A and let M be a finitely generated $\mathbf{N}^d$-graded S-module. Set $S_0 = 0_S$ and*

$$k_i = \dim_S \frac{M}{(S_0, \ldots, S_{i-1})M} - \dim_S \frac{M}{(S_0, \ldots, S_i)M}$$

for any $1 \leq i \leq d$. Assume that

$$k_i \leq \min\left\{\dim_S \frac{M}{(\mathfrak{U})M} - \dim_S \frac{M}{(\mathfrak{U}, S_i)M} \mid \emptyset \neq \mathfrak{U} \subset \mathfrak{S}_{\hat{i}}\right\}$$

for any $1 \leq i \leq d$. Then there is a parameter system for M consisting of k_1 elements of $S_1, \ldots, k_d$ elements of S_d if and only if $k_i \leq \dim_S M - \dim_S \frac{M}{(S_i)M}$ for all $1 \leq i \leq d$.

PROOF. Since A is an artinian ring,

$$k_1 + \cdots + k_d = \sum_{i=1}^d \left[\dim_S \frac{M}{(S_0, \ldots, S_{i-1})M} - \dim_S \frac{M}{(S_0, \ldots, S_i)M}\right] = \dim_S M.$$

Hence the proof follows from Corollary 4.2.   ∎

COROLLARY 4.8. *Let $(A, \mathfrak{m})$ be an artinian local ring. Let S be a finitely generated standard $\mathbf{N}^d$-graded algebra over A and let M be a finitely generated $\mathbf{N}^d$-graded S-module. Set $S_0 = 0_S$ and*

$$k_i = \dim_S \frac{M}{(S_0, \ldots, S_{i-1})M} - \dim_S \frac{M}{(S_0, \ldots, S_i)M}$$

for all $1 \le i \le d$. Assume that

$$k_i \le \min \left\{ \dim_S \frac{M}{(\mathfrak{U})M} - \dim_S \frac{M}{(\mathfrak{U}, S_i)M} \mid \emptyset \neq \mathfrak{U} \subset \mathfrak{S}_{\hat{i}} \right\}$$

for any $1 \le i \le d$. Then there is a maximal strong-filter-regular sequence $\mathfrak{R}$ with respect to M consisting of k_1 elements of $S_1, \ldots, k_d$ elements of S_d if and only if

$$k_i \le \dim_S M - \dim_S \frac{M}{(S_i)M}$$

for all $1 \le i \le d$. And in this case, $\mathfrak{R}$ is a parameter system for M.

PROOF. The "if" part: By Corollary 4.2, there is a strong-filter-regular sequence with respect to M consisting of k_1 elements of $S_1, \ldots, k_d$ elements of S_d. And this sequence is a part of a parameter system for M. Since $k_1 + \cdots + k_d = \dim_S M$, this sequence is a parameter system for M. Hence it is a maximal strong-filter-regular sequence. The "only if" part: Assume that $\mathfrak{R}$ is a maximal strong-filter-regular sequence with respect to M consisting of k_1 elements of $S_1, \ldots, k_d$ elements of S_d. By Theorem 3.2(ii), $(\mathfrak{R})$ is a reduction of S_+ with respect to M. Set $\dim_S M = s$. Since $\mid \mathfrak{R} \mid = s$, $\mathfrak{R}$ is a parameter system for M. Thus, $k_i \le \dim_S M - \dim_S \frac{M}{(S_i)M}$ for all $1 \le i \le d$ by Corollary 4.7. ∎

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