

A Lefschetz Trace Formula for p^n -torsion Étale Cohomology

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Abstract. A Lefschetz trace formula is proved for p^n -torsion étale cohomology of open algebraic varieties defined over a finite field of characteristic p . These are analogues of Deligne's conjecture on Lefschetz trace formula for ℓ -adic cohomology, which was proved by Fujiwara.

1. Introduction

The aim of this paper is to prove a certain Lefschetz trace formula for p^n -torsion étale cohomology of open algebraic varieties defined over a finite field of characteristic p . A version of this formula for ℓ -adic étale cohomology ($\ell \neq p$) was originally conjectured by Deligne, and proved by Fujiwara [7] in full generality. It is an important tool in arithmetic geometry; for example, it is used in the proof of the local Langlands correspondence for GL_n over a p -adic field by Harris-Taylor [13], and the Langlands correspondence for GL_n over a function field by Lafforgue [17].

We first recall Deligne's conjecture. We fix prime numbers p and ℓ such that $\ell \neq p$. Let \mathbb{F}_q be a finite field of characteristic p with q elements and k an algebraic closure of \mathbb{F}_q . Let X_0 and Y_0 be separated \mathbb{F}_q -schemes of finite type and $a_0: Y_0 \rightarrow X_0 \times_{\mathrm{Spec} \mathbb{F}_q} X_0$ a morphism of \mathbb{F}_q -schemes. For an \mathbb{F}_q -scheme S_0 , we denote by S the base change $S_0 \times_{\mathrm{Spec} \mathbb{F}_q} \mathrm{Spec} k$. Similar notation will be used for morphisms of \mathbb{F}_q -schemes. We put $a_1 = \mathrm{pr}_1 \circ a_0$ and $a_2 = \mathrm{pr}_2 \circ a_0$, where pr_1 (resp. pr_2) is the first (resp. second) projection of $X \times_{\mathrm{Spec} k} X$. We denote by Fr_X the q -th power Frobenius endomorphism on X . We put $a_1^{(m)} = \mathrm{Fr}_X^m \circ a_1$. We write $a^{(m)}$ for the correspondence such that $(a^{(m)})_1 = a_1^{(m)}$ and $(a^{(m)})_2 = a_2$. We denote by $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ the derived category of bounded complexes of $\overline{\mathbb{Q}}_\ell$ -sheaves with constructible cohomology. (For more detail of this category, see for example [5].)

THEOREM 1.1 (Deligne's conjecture [7, Corollary 5.4.5]). *We assume that a_1 is proper and a_2 is quasi-finite. Then there exists an integer N depending only on X, Y and*

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the correspondence a such that, for any integer $m \geq N$, any object $K \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$ and any morphism $u \in \text{Hom}(a_1^{(m)*} K, a_2^! K)$, we have

$$\text{Tr}(u_! \mid R\Gamma_c(X, K)) = \sum_{z \in \text{Fix } a^{(m)}} \text{naive-loc}_z(u).$$

In the above equality, we define $u_!$ by the composite

$$R\Gamma_c(X, K) \xrightarrow{\text{adj}} R\Gamma_c(Y, a_1^{(m)*} K) \xrightarrow{u} R\Gamma_c(Y, a_2^! K) \xrightarrow{\text{adj}} R\Gamma_c(X, K)$$

and $\text{naive-loc}_z(u)$ by the trace of the composite

$$K_w = (a_1^{(m)*} K)_z \xrightarrow{u_z} (a_2^! K)_z \hookrightarrow \bigoplus_{z' \in a_2^{-1}(w)} (a_2^! K)_{z'} = (a_{21} a_2^! K)_w \xrightarrow{\text{adj}} K_w,$$

where we put $w = a_1^{(m)}(z) = a_2(z)$.

Now it is natural to ask if such a trace formula also holds for étale cohomology with \mathbb{Z}_p -coefficients. For this question, we can find the following negative answer. We put $X_0 = Y_0 = \mathbb{A}_{\mathbb{F}_p}^1$ and $a_0 = \Delta_{\mathbb{A}_{\mathbb{F}_p}^1/\mathbb{F}_p}$, where $\mathbb{A}_{\mathbb{F}_p}^1$ is the affine line over \mathbb{F}_p and $\Delta_{\mathbb{A}_{\mathbb{F}_p}^1/\mathbb{F}_p} : \mathbb{A}_{\mathbb{F}_p}^1 \rightarrow \mathbb{A}_{\mathbb{F}_p}^1 \times_{\text{Spec } \mathbb{F}_p} \mathbb{A}_{\mathbb{F}_p}^1$ is the diagonal. We define $H_c^i(\mathbb{A}_k^1, \mathbb{Z}_p) = \varprojlim_n H_c^i(\mathbb{A}_k^1, \mathbb{Z}/p^n)$. If the formula held in this situation, we should have

$$\sum_i (-1)^i \text{Tr}(\text{id} \mid H_c^i(\mathbb{A}_k^1, \mathbb{Z}_p)) = \#\{z \in \mathbb{A}_k^1 \mid z^{p^m} = z\}$$

for sufficiently large m . The left-hand side is equal to zero since $H_c^i(\mathbb{A}_k^1, \mathbb{Z}/p^n)$ is zero for any i and n . However the right-hand side is equal to p^m , which is a contradiction. Nevertheless, we have

$$\sum_i (-1)^i \text{Tr}(\text{id} \mid H_c^i(\mathbb{A}_k^1, \mathbb{Z}/p^n)) = \#\{z \in \mathbb{A}_k^1 \mid z^{p^m} = z\} \pmod{p^n}$$

for any integer $m \geq n$. In addition, for étale cohomology with p -torsion coefficients, a special version of the Lefschetz trace formula was already proved in [2, Fonction L modulo ℓ^n et modulo p]. Thus the following question is still meaningful:

QUESTION 1.2. Does an analogous statement of Deligne’s conjecture hold for étale cohomology with p^n -torsion coefficients? Namely, can we find a sufficiently large integer N such that for any integer $m \geq N$, any bounded complex K of flat constructible \mathbb{Z}/p^n -modules on X , and any $u \in \text{Hom}(a_1^{(m)*} K, a_2^! K)$, we have

$$\text{Tr}(u_! \mid R\Gamma_c(X, K)) = \sum_{z \in \text{Fix } a^{(m)}} \text{naive-loc}_z(u)?$$

The results of this article give some affirmative answers to the question. We have two main results. One of them requires almost no assumption on the schemes, but it can only be applied to the graph of an automorphism of finite order and to p -torsion coefficients. The precise statement is as follows:

THEOREM 1.3. *Let X_0 be a separated \mathbb{F}_q -scheme of finite type and $a_{01}: X_0 \rightarrow X_0$ an \mathbb{F}_q -automorphism of order r on X_0 . Then for any constructible \mathbb{F}_p -module G_0 on X_0 , any integer $m \geq 1$ and any morphism $u_0 \in \text{Hom}(a_{01}^{(m)*} G_0, G_0)$ of order r , we have*

$$\text{Tr}(u_! \mid R\Gamma_c(X, G)) = \sum_{x \in \text{Fix}(a^{(m)})} \text{Tr}(u_x \mid G_x),$$

where G (resp. u) is the pull-back of G_0 (resp. u_0) to X .

The other result is a statement for more general correspondences and for p^n -torsion coefficients. However it requires the smoothness of X and Y , lifts of X_0, Y_0, a_0 , and the Frobenius endomorphism of X_0 over the ring of truncated Witt vectors of length n , smooth compactifications of the lifts, and freeness of some cohomology groups. To state it precisely, we define some notation. For an \mathbb{F}_q -scheme S_0 , we denote by Φ_{S_0} the p -th power map on the structure sheaf of S_0 . For any field κ of characteristic p , we write $W_n(\kappa)$ for the ring of Witt vectors of length n over κ . For any $W_n(\mathbb{F}_q)$ -scheme \mathcal{S}_0 , we denote by \mathcal{S} the base change $\mathcal{S}_0 \times_{\text{Spec } W_n(\mathbb{F}_q)} \text{Spec } W_n(k)$. The pull-back of a sheaf G_0 on \mathcal{S}_0 to \mathcal{S} is denoted by G . Similar notation is used for morphisms of $W_n(\mathbb{F}_q)$ -schemes and those of sheaves on $W_n(\mathbb{F}_q)$ -schemes. Since the canonical morphism $\mathcal{S}_0 \times_{\text{Spec } W_n(\mathbb{F}_q)} \text{Spec } \mathbb{F}_q \rightarrow \mathcal{S}_0$ induces an equivalence between the category of étale sheaves on \mathcal{S}_0 and that on $\mathcal{S}_0 \times_{\text{Spec } W_n(\mathbb{F}_q)} \text{Spec } \mathbb{F}_q$, we identify étale sheaves on $\mathcal{S}_0 \times_{\text{Spec } W_n(\mathbb{F}_q)} \text{Spec } \mathbb{F}_q$ with their push-forwards by the canonical morphism.

THEOREM 1.4. *Let X_0 and Y_0 be smooth \mathbb{F}_q -schemes and $a_0: Y_0 \rightarrow X_0 \times_{\text{Spec } \mathbb{F}_q} X_0$ an \mathbb{F}_q -morphism. Suppose that there exist smooth $W_n(\mathbb{F}_q)$ -schemes \mathcal{X}_0 and \mathcal{Y}_0 and a $W_n(\mathbb{F}_q)$ -morphism $\alpha_0: \mathcal{Y}_0 \rightarrow \mathcal{X}_0 \times_{\text{Spec } W_n(\mathbb{F}_q)} \mathcal{X}_0$ such that the triple $(\mathcal{X}_0, \mathcal{Y}_0, \alpha_0)$ is a lift of (X_0, Y_0, a_0) over $W_n(\mathbb{F}_q)$ and there exist proper smooth $W_n(\mathbb{F}_q)$ -schemes $\overline{\mathcal{X}}_0$ and $\overline{\mathcal{Y}}_0$, a $W_n(\mathbb{F}_q)$ -morphism $\overline{\alpha}_0: \overline{\mathcal{Y}}_0 \rightarrow \overline{\mathcal{X}}_0 \times_{\text{Spec } W_n(\mathbb{F}_q)} \overline{\mathcal{X}}_0$, and a σ_0 -semi-linear ring homomorphism $\Phi_{\mathcal{O}_{\overline{\mathcal{X}}_0}}: \mathcal{O}_{\overline{\mathcal{X}}_0} \rightarrow \mathcal{O}_{\overline{\mathcal{X}}_0}$ such that*

- (a) \mathcal{X}_0 (resp. \mathcal{Y}_0) is an open $W_n(\mathbb{F}_q)$ -subscheme of $\overline{\mathcal{X}}_0$ (resp. $\overline{\mathcal{Y}}_0$) and the diagram

$$\begin{array}{ccc} \mathcal{Y}_0 & \xrightarrow{\alpha_0} & \mathcal{X}_0 \times_{\text{Spec } W_n(\mathbb{F}_q)} \mathcal{X}_0 \\ \downarrow & & \downarrow \\ \overline{\mathcal{Y}}_0 & \xrightarrow{\overline{\alpha}_0} & \overline{\mathcal{X}}_0 \times_{\text{Spec } W_n(\mathbb{F}_q)} \overline{\mathcal{X}}_0 \end{array}$$

is cartesian,

- (b) α_1 is proper, $\bar{\alpha}_2$ is étale, and $\bar{\alpha}$ is a closed immersion,
- (c) $\mathcal{D}_0 := \overline{\mathcal{X}}_0 \setminus \mathcal{X}_0$ is a Cartier divisor which is flat over $W_n(\mathbb{F}_q)$,
- (d) $\Phi_{\mathcal{O}_{\overline{\mathcal{X}}_0}}$ is a lift of the p -th power map of $\mathcal{O}_{\overline{\mathcal{X}}_0} \otimes_{W_n(\mathbb{F}_q)} \mathbb{F}_q$,
- (e) the defining ideal \mathcal{I}_0 of \mathcal{D}_0 is $\Phi_{\mathcal{O}_{\overline{\mathcal{X}}_0}}$ -stable.

Let \overline{G}_0 be a locally free constructible \mathbb{Z}/p^n -module on \overline{X}_0 . We put $G_0 = \overline{G}_0|_{X_0}$. Suppose that

- (f) $H_c^i(X, G)$ (resp. $H^i(\overline{\mathcal{X}}, \overline{G} \otimes_{\mathbb{Z}/p^n} \mathcal{I})$) is free over \mathbb{Z}/p^n (resp. $W_n(k)$) for any i .

Then there exists an integer N such that, for any integer $m \geq N$ and any homomorphism $u_0 \in \text{Hom}(a_{01}^{(m)*} G_0, a_{02}^* G_0)$, we have the following equality

$$\sum_i (-1)^i \text{Tr}(u_! | H_c^i(X, G)) = \sum_{z \in \text{Fix}(a^{(m)})} \text{Tr}(u_z | G_z).$$

If $n = 1$, then we can take $N = 1$.

- REMARK 1.5. (i) If X_0 is a curve, the condition (e) follows from the conditions (a), (b), (c), and (d) (see [15, 1.1.2]).
- (ii) The existence of the lifts can be proved if X_0 is an ordinary abelian variety, due to Serre-Tate’s theory (see [16]). However, if X_0 is a curve and the genus of X_0 is at least two, then Fr_{X_0} cannot be lifted ([18, Lemma I.5.4]). In the case of general dimensions, such non-existence for proper smooth \mathbb{F}_q -schemes with positive Kodaira dimensions is proved by Dupuy [4].
- (iii) The condition (f) holds if X_0 is an open subscheme of a projective curve or an ordinary elliptic curve, and G_0 is the constant sheaf \mathbb{Z}/p^n . The author expects that the assumption (f) can be removed if we can calculate the traces in the derived category of perfect \mathbb{Z}/p^n -complexes instead of the category of \mathbb{Z}/p^n -modules.

If, in addition to the assumptions of Theorem 1.4, the schemes are proper, then we can verify such a trace formula not only for sheaves but for perfect complexes:

THEOREM 1.6. We assume that X_0 and Y_0 are proper and smooth over \mathbb{F}_q . Let K_0 be a complex in $D_{\text{perf}}(X_0, \mathbb{Z}/p^n)$. We assume the following conditions:

- There exists a lift $(\mathcal{X}_0, \mathcal{Y}_0, \alpha_0)$ of the triple (X_0, Y_0, a_0) to $W_n(\mathbb{F}_q)$ such that \mathcal{X} and \mathcal{Y} are smooth over $W_n(k)$, α is closed immersion, and α_2 is étale.
- There exists a lift $\Phi_{\mathcal{X}_0} : \mathcal{O}_{\mathcal{X}_0} \rightarrow \mathcal{O}_{\mathcal{X}_0}$ of the p -th power map Φ_{X_0} on \mathcal{O}_{X_0} .
- $H^i(X, K)$ (resp. $H^i(\mathcal{X}, K \otimes_{\mathbb{Z}/p^n} \mathcal{O}_{\mathcal{X}})$) is free over \mathbb{Z}/p^n (resp. $W_n(k)$) for each i .

Then there exists an integer N such that, for any $m \geq N$ and any $u_0 \in \text{Hom}(a_{01}^{(m)*} K_0, a_{02}^* K_0)$, we have

$$\text{Tr}(u_* | R\Gamma(X, K)) = \sum_{z \in \text{Fix } a^{(m)}} \text{Tr}(u_z | K_z).$$

Since the proof is similar to that of Theorem 1.4, we omit it.

Although the assumptions of Theorem 1.4 are slightly complicated, the conditions (d), (e), and (f) become unconditional if $n = 1$. Moreover, we can then take $N = 1$. Thus we obtain the following:

COROLLARY 1.7. *Suppose that there exist proper smooth \mathbb{F}_q -schemes \overline{X}_0 and \overline{Y}_0 and an \mathbb{F}_q -morphism $\overline{a}_0: \overline{Y}_0 \rightarrow \overline{X}_0 \times_{\text{Spec } \mathbb{F}_q} \overline{X}_0$ satisfying the conditions (a), (b), and (c) in Theorem 1.4. Then, for any integer $m \geq 1$, any smooth constructible \mathbb{F}_p -module \overline{G}_0 on X_0 , and any $u_0 \in \text{Hom}(a_{01}^{(m)*} G_0, a_{02}^* G_0)$, we have*

$$\sum_i (-1)^i \text{Tr}(u_0 | H_c^i(X, G)) = \sum_{z \in \text{Fix } a^{(m)}} \text{Tr}(u_z | G_z),$$

where we put $G_0 = \overline{G}_0|_{X_0}$.

As an application of Corollary 1.7, we obtain a relation between p -torsion étale cohomology and ℓ -adic cohomology. Under the notation and the assumptions of Corollary 1.7, we put $Y_0 = X_0, a_{02} = \text{id}, G_0 = \mathbb{Z}/p$, and $u_0 = \text{id}$. Then we have

$$\sum_i (-1)^i \text{Tr}((\text{Fr}_X^m \circ a_1)^* | H_c^i(X, \mathbb{Z}/p)) = \# \text{Fix}(\text{Fr}_X^m \circ a_1) \pmod{p}$$

for any $m \geq 1$. By the Lefschetz trace formula in [7], there exists an integer $M \geq 1$ such that, for any integer $m \geq M$, we have

$$\sum_i (-1)^i \text{Tr}((\text{Fr}_X^m \circ a_1)^* | H_c^i(X, \mathbb{Q}_\ell)) = \# \text{Fix}(\text{Fr}_X^m \circ a_1). \quad \mathbb{1}$$

Therefore for any $m \geq M$ we have

$$\begin{aligned} & \sum_i (-1)^i \text{Tr}((\text{Fr}_X^m \circ a_1)^* | H_c^i(X, \mathbb{Q}_\ell)) \pmod{p} \\ &= \sum_i (-1)^i \text{Tr}((\text{Fr}_X^m \circ a_1)^* | H_c^i(X, \mathbb{Z}/p)). \quad \mathbb{1} \end{aligned}$$

Thus, in some sense, we have the ℓ -independence (mod p) including $\ell = p$ of the alternating sum $\sum_i (-1)^i \text{Tr}((\text{Fr}_X^m \circ a_1)^*)$.

The article is organized as follows: In Section 2 we recall some cohomological operations. In Section 3 we give a proof of Theorem 1.3. The assertion for $a_1 = \text{id}$ is already known ([2, Fonction L modulo ℓ^n et modulo p , Théorème 4.1]). By using a method in [3], we extend the result to the case where a_1 is an automorphism of finite order. In Sections 4 and 5, we prove some facts needed in Section 6. In Section 4, we present a generalization of the so-called ‘‘Woods Hole formula’’ ([10, Exp. III, Corollaire 6.12]), which is a trace formula for coherent cohomology. In Section 5, we recall some facts on semi-linear algebra. A proof of Theorem 1.4 is given in Section 6, which goes as follows: First, by using an exact sequence

of Artin-Schreier type (Lemma 6.1), we rewrite the left-hand side of the formula in terms of coherent cohomology. Then we apply to it the trace formula proved in Section 4 and we obtain the result.

Notation

- Every ring appearing in this paper is assumed to be commutative with 1.
- In this paper, we fix a prime number p . We denote by \mathbb{F}_q the finite field of q elements, where q is a power of p . We fix an algebraic closure k of \mathbb{F}_q . For any field κ of characteristic p , we write $W_n(\kappa)$ for the ring of Witt vectors of length n over κ . We write $\sigma_0: W_n(\mathbb{F}_q) \rightarrow W_n(\mathbb{F}_q)$ (resp. $\sigma: W_n(k) \rightarrow W_n(k)$) for the lift of the p -th power map on \mathbb{F}_q (resp. k).
- For any $W_n(\mathbb{F}_q)$ -scheme \mathcal{S}_0 , we denote by \mathcal{S} the base change $\mathcal{S}_0 \times_{\text{Spec } W_n(\mathbb{F}_q)} \text{Spec } W_n(k)$. The pull-back of a sheaf G_0 on \mathcal{S}_0 to \mathcal{S} is denoted by G . Similar notation is used for morphisms of $W_n(\mathbb{F}_q)$ -schemes and those of sheaves on $W_n(\mathbb{F}_q)$ -schemes.
- Let \mathcal{X}_0 and \mathcal{Y}_0 be $W_n(\mathbb{F}_q)$ -schemes. We assume that a lift $\text{Fr}_{\mathcal{X}_0}$ of the Frobenius endomorphism on $\mathcal{X}_0 \times_{\text{Spec } W_n(\mathbb{F}_q)} \text{Spec } \mathbb{F}_q$ exists. For morphisms of $W_n(\mathbb{F}_q)$ -schemes $f_0: \mathcal{Y}_0 \rightarrow \mathcal{X}_0$ and $\alpha_0: \mathcal{Y}_0 \rightarrow \mathcal{X}_0 \times_{\text{Spec } W_n(\mathbb{F}_q)} \mathcal{X}_0$, we put $f_0^{(m)} = f_0 \circ \text{Fr}_{\mathcal{X}_0}^m$ and $\alpha_0^{(m)} = (\alpha_{01}^{(m)}, \alpha_{02}^{(m)})$, where we write α_{01} (resp. α_{02}) for the composite of α_0 and the first (resp. second) projection of $\mathcal{X}_0 \times_{W_n(\mathbb{F}_q)} \mathcal{X}_0$. We also define $f^{(m)}$ and $\alpha^{(m)}$ in the same way.
- Let A be a ring. We denote by $D(A)$ the derived category of the category of A -modules. We denote by $D_{\text{perf}}(A)$ the full subcategory of $D(A)$ which consists of perfect complexes, that is, complexes which are quasi-isomorphic to bounded complexes of projective A -modules of finite type.
- In this article, sheaves and cohomology are always considered on the étale site of a scheme.
- Let X be a scheme and \mathcal{A} a sheaf of rings on X . We denote by $D(X, \mathcal{A})$ the derived category of the category of \mathcal{A} -modules on X and by $D^-(X, \mathcal{A})$ the full subcategory of $D(X, \mathcal{A})$ consisting of complexes K^\bullet with $H^i(K^\bullet) = 0$ for all sufficiently large i .
- We denote by $\otimes_{\mathcal{A}}^L: D^-(X, \mathcal{A}) \times D^-(X, \mathcal{A}) \rightarrow D^-(X, \mathcal{A})$ the derived tensor product functor. If \mathcal{F} is a flat \mathcal{A} -module, we write $\otimes_{\mathcal{A}} \mathcal{F}$ instead of $\otimes_{\mathcal{A}}^L \mathcal{F}$.
- We denote by $D_{\text{perf}}(X, \mathcal{A})$ the full subcategory of $D(X, \mathcal{A})$ which consists of perfect complexes, that is, complexes which are locally quasi-isomorphic to bounded complexes of locally free \mathcal{A} -modules of finite type.
- Let Λ be a constant sheaf whose value is a Noetherian torsion ring. We denote by $D_{\text{ctf}}^b(X, \Lambda)$ the category of complexes of Λ -modules on X which are of finite tor-dimension and with constructible cohomology.
- If $\mathcal{A} = \mathcal{O}_X$, we simply write $D(\mathcal{O}_X)$ (resp. $D^-(\mathcal{O}_X)$, $D_{\text{perf}}(\mathcal{O}_X)$) for $D(X, \mathcal{O}_X)$

(resp. $D^-(X, \mathcal{O}_X)$, $D_{\text{perf}}(X, \mathcal{O}_X)$).

2. Cohomological operations

In the section, we recall some cohomological operations.

2.1. Cohomological operations. Let X and Y be quasi-compact and quasi-separated schemes and $f: Y \rightarrow X$ a separated morphism of finite type of schemes.

2.1.1. Sheaves of Λ -modules. Let Λ be a Noetherian torsion ring. We denote by $\mathcal{C}(X, \Lambda)$ (resp. $\mathcal{C}(Y, \Lambda)$) the category of étale sheaves of Λ -modules on X (resp. Y). We denote by $f_*: \mathcal{C}(Y, \Lambda) \rightarrow \mathcal{C}(X, \Lambda)$ (resp. $f^*: \mathcal{C}(X, \Lambda) \rightarrow \mathcal{C}(Y, \Lambda)$) the direct (resp. inverse) image functor associated to f . Since f_* is a left exact functor and f is finite-dimensional, it induces a right derived functor $Rf_*: D(Y, \Lambda) \rightarrow D(X, \Lambda)$. We often denote Rf_* by f_* for simplicity. Since f^* is an exact functor, it induces a functor from $D(X, \Lambda)$ to $D(Y, \Lambda)$ which we also denote by f^* . For any complex $K \in D(Y, \Lambda)$, we write $\text{adj}: K \rightarrow f_* f^* K$ for the adjunction map.

Let $j: V \hookrightarrow U$ be an open immersion of schemes. We write $j_!: \mathcal{C}(V, \Lambda) \rightarrow \mathcal{C}(U, \Lambda)$ for the extension-by-zero functor. Since $j_!$ is exact, it induces a functor from $D(V, \Lambda)$ to $D(U, \Lambda)$ which we also denote by $j_!$. By Nagata's compactification theorem [1], the morphism $f: Y \rightarrow X$ factors as $f = \overline{f} \circ j_Y$, where $\overline{f}: \overline{Y} \rightarrow X$ is a proper morphism of schemes and $j_Y: Y \hookrightarrow \overline{Y}$ is an open immersion. We define the direct image functor with proper support $f_!: D(Y, \Lambda) \rightarrow D(X, \Lambda)$ by the composite of $j_{Y!}$ and \overline{f}_* . We note that, if X is the spectrum of an Artinian local ring with separably closed residue field, then the functor f_* (resp. $f_!$) on $\mathcal{C}(Y, \Lambda)$ is equal to the global section functor $\Gamma(Y, \bullet)$ (resp. the global section with proper support functor $\Gamma_c(Y, \bullet)$).

The functor f^* sends $D_{\text{ctf}}^b(X, \Lambda)$ to $D_{\text{ctf}}^b(Y, \Lambda)$. In addition, by [9, Exp. XIV, Théorème 1.1] and [9, Exp. XVII, Théorème 5.2.10], the functor $f_!$ sends $D_{\text{ctf}}^b(Y, \Lambda)$ to $D_{\text{ctf}}^b(X, \Lambda)$. In particular, if X is the spectrum of an Artinian local ring with separably closed residue field, then $R\Gamma_c(Y, K)$ belongs to $D_{\text{perf}}(\Lambda)$ for any $K \in D_{\text{ctf}}^b(Y, \Lambda)$.

If f is étale, then the functors $f_!$ and f^* are exact, and $f_!$ is the left adjoint functor of f^* (see for example [6, Chapter I, Lemma 8.2]). Hence, for any $K \in D(X, \Lambda)$, we can define a natural homomorphism $\text{Tr}_{f,K}: f_! f^* K \rightarrow K$, called the trace map.

2.1.2. \mathcal{O}_X -modules. We denote by $\mathcal{C}(\mathcal{O}_X)$ (resp. $\mathcal{C}(\mathcal{O}_Y)$) the category of \mathcal{O}_X -modules (resp. \mathcal{O}_Y -modules). We can naturally define the direct image functor $f_*: \mathcal{C}(\mathcal{O}_Y) \rightarrow \mathcal{C}(\mathcal{O}_X)$ and its right derived functor $D(\mathcal{O}_Y) \rightarrow D(\mathcal{O}_X)$, which we also denote by f_* . If f is proper and smooth, then by [11, Exp. III, Example 4.1.1 and Corollaire 4.8.1], the functor f_* sends $D_{\text{perf}}(\mathcal{O}_Y)$ to $D_{\text{perf}}(\mathcal{O}_X)$.

We write f^{-1} for the inverse image functor of étale sheaves. We define the functor f^*

as follows:

$$f^*: \mathcal{C}(\mathcal{O}_X) \rightarrow \mathcal{C}(\mathcal{O}_Y): \mathcal{G} \mapsto f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y.$$

Since f^* is right exact, we can define its left derived functor $D^-(\mathcal{O}_X) \rightarrow D^-(\mathcal{O}_Y)$, which we also denote by f^* . If a complex \mathcal{K} is in $D_{\text{perf}}(\mathcal{O}_X)$, then $f^*\mathcal{K}$ is in $D_{\text{perf}}(\mathcal{O}_Y)$.

For any complex $\mathcal{K} \in D^-(\mathcal{O}_X)$, we write $\text{adj}: \mathcal{K} \rightarrow f_*f^*\mathcal{K}$ for the adjunction map which coincides with the composite of the adjunction map $\mathcal{K} \rightarrow f_*f^{-1}\mathcal{K}$ defined in (2.1.1) and the canonical morphism $f_*f^{-1}\mathcal{K} \rightarrow f_*f^*\mathcal{K}$.

If f is finite and étale, then we have the trace map

$$\text{Tr}_{f,\mathcal{K}}: f_*f^*\mathcal{K} \rightarrow \mathcal{K}$$

for any $\mathcal{K} \in D(\mathcal{O}_X)$. This morphism is functorial with respect to \mathcal{K} .

2.2. Cohomological correspondences. Let S be the spectrum of an Artinian local ring. Let X and Y be separated schemes of finite type over S .

2.2.1. Correspondences. Let $a: Y \rightarrow X \times_S X$ be an S -morphism. Such a morphism is called a correspondence. We denote by $\text{Fix } a$ the fiber product $Y \times_{X \times_S X} X$ of the correspondence a and the diagonal $\Delta_{X/S}: X \rightarrow X \times_S X$:

$$\begin{array}{ccc} \text{Fix } a & \longrightarrow & X \\ \downarrow & & \downarrow \Delta_{X/S} \\ Y & \xrightarrow{a} & X \times_S X. \end{array}$$

We write a_1 (resp. a_2) for the composite of a and the first (resp. second) projection of $X \times_S X$. We assume that a_1 is proper and a_2 is étale.

Let K be a complex in $D(X, \Lambda)$ and $u \in \text{Hom}(a_1^*K, a_2^*K)$. We define u_1 by the composition

$$R\Gamma_c(X, K) \xrightarrow{R\Gamma_c(\text{adj})} R\Gamma_c(Y, a_1^*K) \xrightarrow{R\Gamma_c(u)} R\Gamma_c(Y, a_2^*K) \xrightarrow{R\Gamma_c(\text{Tr})} R\Gamma_c(X, K).$$

For any geometric point z of $\text{Fix } a$, we put $K_z = (a_1^*K)_z = (a_2^*K)_z$. Then the stalk

$$u_z: K_z = (a_1^*K)_z \rightarrow (a_2^*K)_z = K_z$$

is an endomorphism on K_z .

Furthermore we assume that X and Y are proper over S . Then a_2 is finite. For any complex \mathcal{K} in $D^-(\mathcal{O}_X)$ and $u \in \text{Hom}(a_1^*\mathcal{K}, a_2^*\mathcal{K})$, we also define u_* by the composition

$$R\Gamma(X, \mathcal{K}) \xrightarrow{R\Gamma(\text{adj})} R\Gamma(Y, a_1^*\mathcal{K}) \xrightarrow{R\Gamma(u)} R\Gamma(Y, a_2^*\mathcal{K}) \xrightarrow{R\Gamma(\text{Tr})} R\Gamma(X, \mathcal{K}).$$

Let \mathcal{K} be a complex in $D^-(\mathcal{O}_X)$ and $u \in \text{Hom}(a_1^*\mathcal{K}, a_2^*\mathcal{K})$. For any connected component β of $\text{Fix } a$, we put $\mathcal{K}_\beta = i_\beta^*a_1^*\mathcal{K} = i_\beta^*a_2^*\mathcal{K}$ and

$$u_\beta = i_\beta^*u: \mathcal{K}_\beta = i_\beta^*a_1^*\mathcal{K} \rightarrow i_\beta^*a_2^*\mathcal{K} = \mathcal{K}_\beta,$$

where we write i_β for the closed immersion $\beta \hookrightarrow Y$.

2.2.2. Push-forward for correspondences. We consider open immersions $j_X: X \hookrightarrow \overline{X}$ and $j_Y: Y \hookrightarrow \overline{Y}$, and correspondences $a: Y \rightarrow X \times_S X$ and $\overline{a}: \overline{Y} \rightarrow \overline{X} \times_S \overline{X}$ such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{a} & X \times_S X \\ j_Y \downarrow & & \downarrow j_X \times j_X \\ \overline{Y} & \xrightarrow{\overline{a}} & \overline{X} \times_S \overline{X} \end{array}$$

is commutative. We assume that a_1 and \overline{a}_1 are proper. Let K be a complex in $D(X, \Lambda)$. Then we define

$$BC_2: j_{Y!}a_2^*K \rightarrow \overline{a}_2^*j_{X!}K$$

by the composition

$$j_{Y!}a_2^*K \xrightarrow{j_{Y!}a_2^*(\text{adj})} j_{Y!}a_2^*j_X^*j_{X!}K = j_{Y!}j_Y^*\overline{a}_2^*j_{X!}K \xrightarrow{\text{adj}} \overline{a}_2^*j_{X!}K.$$

By the properness of a_1 and \overline{a}_1 , we also have the morphism

$$j_{X!}K \xrightarrow{j_{X!}(\text{adj})} j_{X!}a_{1*}a_1^*K = j_{X!}a_{1!}a_1^*K = \overline{a}_{1!}j_{Y!}a_1^*K = \overline{a}_{1*}j_{Y!}a_1^*K.$$

Thus, by adjointness, we can define

$$BC_1: \overline{a}_1^*j_{X!}K \rightarrow j_{Y!}a_1^*K.$$

For any $u \in \text{Hom}(a_1^*K, a_2^*K)$, we define $j_{!}u \in \text{Hom}(\overline{a}_1^*j_{X!}K, \overline{a}_2^*j_{X!}K)$ by the composition

$$\overline{a}_1^*j_{X!}K \xrightarrow{BC_1} j_{Y!}a_1^*K \xrightarrow{j_{Y!}u} j_{Y!}a_2^*K \xrightarrow{BC_2} \overline{a}_2^*j_{X!}K.$$

2.2.3. Frobenius correspondences. Let X_0 be an \mathbb{F}_q -scheme and G_0 an abelian sheaf on X_0 . Since the morphism $\text{fr}_{X_0} = (\text{id}_{X_0}, \Phi_{\mathcal{O}_{X_0}})$ induces the identity on the étale topoi of X_0 , we have $\text{fr}_{X_0}^*G_0 = G_0$. Hence we have $\text{Fr}_{X_0}^*G_0 = G_0$. Let $f: X \rightarrow X_0$ be the base change morphism by $\mathbb{F}_q \rightarrow k$. Then we have the following canonical isomorphisms

$$\text{Fr}_X^*G = \text{Fr}_X^*f^*G_0 \xrightarrow{\sim} (f \circ \text{Fr}_X)^*G_0 = (\text{Fr}_{X_0} \circ f)^*G_0 \xrightarrow{\sim} f^*\text{Fr}_{X_0}^*G_0 = f^*G_0 = G.$$

We call the composite of the above morphisms the Frobenius correspondence of G and denote it by Fr_G . The Frobenius correspondences commute with base change, that is, for any morphism $g_0: Y_0 \rightarrow X_0$ of \mathbb{F}_q -schemes, the pull-back $g^*\text{Fr}_G$ is the Frobenius correspondence of

g^*G . For any integer $m \geq 1$, we put

$$\mathrm{Fr}_G^m = \mathrm{Fr}_G \circ \mathrm{Fr}_X^* \mathrm{Fr}_G \circ \cdots \circ \mathrm{Fr}_X^{(m-1)*} \mathrm{Fr}_G.$$

In view of this isomorphism, we may work with the morphism $u_0 \in \mathrm{Hom}(a_{01}^* G_0, a_{02}^* G_0)$ instead of $u_0 \in \mathrm{Hom}(a_{01}^{(m)*} G_0, a_{02}^* G_0)$ as in Question 1.2.

Let \mathcal{X}_0 be a $W_n(\mathbb{F}_q)$ -scheme. Since the étale topology of \mathcal{X} and that of $\mathcal{X} \times_{\mathrm{Spec} W_n(k)} \mathrm{Spec} k$ are naturally equivalent, we can define the Frobenius correspondence for any sheaf on \mathcal{X}_0 .

We often denote by Fr the Frobenius correspondence for a sheaf on X_0 or on \mathcal{X}_0 for short.

3. The case of an automorphism of finite order

First, we recall the Lefschetz trace formula for the Frobenius correspondence which is proved in [2, Fonction L modulo ℓ^n et modulo p]. Then we deduce Theorem 1.3 by using the method in the proof of [3, Proposition 3.3].

THEOREM 3.1 ([2, Fonction L modulo ℓ^n et modulo p , Théorème 4.1]). *Let X_0 be a separated scheme of finite type over \mathbb{F}_q , and A a Noetherian reduced ring of characteristic p . Then for any object K_0 in $D_{\mathrm{ctf}}^b(X_0, A)$ and any integer $m \geq 1$, we have*

$$\mathrm{Tr}((\mathrm{Fr}_K^m)_! \mid R\Gamma_c(X, K)) = \sum_{x \in \mathrm{Fix}(\mathrm{Fr}_X^m \times \mathrm{id})} \mathrm{Tr}((\mathrm{Fr}_K^m)_x \mid K_x).$$

PROOF OF THEOREM 1.3. By replacing the base field \mathbb{F}_q with \mathbb{F}_{q^m} , we may assume that $m = 1$.

We put $a_0 := (a_{01}, \mathrm{id}): X_0 \rightarrow X_0 \times_{\mathrm{Spec} \mathbb{F}_q} X_0$. Since X_0 is separated of finite type over \mathbb{F}_q and a_{01} is of finite order, we can construct a finite partition X_{0i} of X_0 such that X_{0i} is locally closed in X_0 , affine, a_{01} -stable for each i , and G_0 is smooth on X_{0i} . Then we have

$$\mathrm{Tr}((u \circ \mathrm{Fr})_! \mid R\Gamma_c(X, G)) = \sum_i \mathrm{Tr}((u \circ \mathrm{Fr})_! \mid R\Gamma_c(X_i, G_i))$$

and

$$\sum_{x \in \mathrm{Fix}(a^{(1)})} \mathrm{Tr}((u \circ \mathrm{Fr})_x \mid G_x) = \sum_i \sum_{x \in \mathrm{Fix}(a_i^{(1)})} \mathrm{Tr}((u \circ \mathrm{Fr})_x \mid G_x),$$

where a_i (resp. G_i) is the restriction of a (resp. G) to X_i . Hence we may assume that X_0 is affine and G_0 is smooth.

We put $X_1 = X_0 \times_{\mathrm{Spec} \mathbb{F}_q} \mathrm{Spec} \mathbb{F}_{q^r}$, $G_1 = G_0|_{X_1}$ and $u_1 = u_0|_{X_1}$. We denote by σ the isomorphism on the scheme $\mathrm{Spec} \mathbb{F}_{q^r}$ associated to the q -th power map on \mathbb{F}_{q^r} . Since X_1 is affine, by using $a_{01} \times_{\mathrm{Spec} \mathbb{F}_q} \sigma^{-1}$ as a descent datum, we can construct a scheme X'_0 over \mathbb{F}_q such that $X'_0 \times_{\mathrm{Spec} \mathbb{F}_q} \mathrm{Spec} \mathbb{F}_{q^r} \simeq X_1$ and $a_1^{(1)}$ is the relative Frobenius endomorphism of X

with respect to X'_0 . Since G_1 is smooth, we can also construct a sheaf G'_0 on X'_0 such that $G'_0|_{X_1} \simeq G_1$ and $u \circ \text{Fr}$ is equal to the Frobenius correspondence on G with respect to G'_0 . Therefore by Theorem 3.1 we obtain

$$\text{Tr}((u \circ \text{Fr})_! \mid R\Gamma_c(X, G)) = \sum_{x \in \text{Fix}(a^{(1)})} \text{Tr}((u \circ \text{Fr})_x \mid G_x).$$

□

4. Woods Hole formula

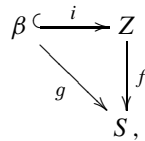
In this section we prove a trace formula for coherent sheaves, which is a generalization of the Woods Hole formula ([10, Exp. III, Corollaire 6.12]). We use this theorem in Section 6.

THEOREM 4.1. *Let S be the spectrum of an Artinian local ring. Let X and Y be proper smooth schemes over S . Let $a: Y \hookrightarrow X \times_S X$ be a closed immersion over S . Assume that a_2 is étale and the homomorphism $da_1: a_1^* \Omega_{X/S} \rightarrow \Omega_{Y/S}$ is zero. Then for any $\mathcal{K} \in D_{\text{perf}}(\mathcal{O}_X)$ and $u \in \text{Hom}(a_1^* \mathcal{K}, a_2^* \mathcal{K})$, we have*

$$\text{Tr}(u_* \mid R\Gamma(X, \mathcal{K})) = \sum_{\beta \in \pi_0(\text{Fix } a)} \text{Tr}_{\beta/S}(\text{Tr}(u_\beta \mid \mathcal{K}_\beta)).$$

REMARK 4.2. Since X and Y are smooth over S and $da_1 = 0$, the scheme Y meets transversally the diagonal of X , that is, $\text{Fix } a$ is étale over S [8, Corollaire 17.13.6]. Hence $\text{Fix } a$ is a finite direct sum of the spectra of local Artinian rings which are étale over S .

Before starting the proof, we recall residue symbols in [14, Chapter III, §9]. We consider the following commutative diagram of schemes



where i is a closed immersion, f is smooth of relative dimension d , and g is étale and finite. We denote by \mathcal{I}_β the defining ideal of β . Since g is étale, the natural morphism

$$\mathcal{I}_\beta / \mathcal{I}_\beta^2 \rightarrow \Omega_{Z/S}^1 \otimes_{\mathcal{O}_Z} \mathcal{O}_Z / \mathcal{I}_\beta$$

is an isomorphism. Taking \wedge^d of the both sides of the above isomorphism, we have

$$\bigwedge^d \mathcal{I}_\beta / \mathcal{I}_\beta^2 \xrightarrow{\sim} \Omega_{Z/S}^d \otimes_{\mathcal{O}_Z} \mathcal{O}_Z / \mathcal{I}_\beta.$$

We note that $\bigwedge^d \mathcal{I}_\beta / \mathcal{I}_\beta^2$ is an invertible sheaf on β . \square

Now we take an \mathcal{O}_Z -sequence s_1, \dots, s_d generating \mathcal{I}_β and a global section ω of $\Omega_{Z/S}^d$. We denote by $\bar{\omega}$ the global section of $\Omega_{Z/S}^d \otimes_{\mathcal{O}_Z} \mathcal{O}_Z / \mathcal{I}_\beta$ obtained from ω and by $\bar{s}_1, \dots, \bar{s}_d$ the global sections of $\mathcal{I}_\beta / \mathcal{I}_\beta^2$ obtained from s_1, \dots, s_d . By the above isomorphism, we can regard $\bar{\omega}$ as a global section of $\bigwedge^d \mathcal{I}_\beta / \mathcal{I}_\beta^2$. For any $\tau \in \Gamma(\beta, \mathcal{O}_\beta)$, we define

$$\text{Res}_{Z/S} \left(\begin{array}{ccc} \tau & \omega & \\ s_1 & \dots & s_d \end{array} \right) := \text{Tr}_{\beta/S} \left(\tau \cdot \bar{\omega} \otimes (d\bar{s}_1 \wedge \dots \wedge d\bar{s}_d)^{-1} \right),$$

which is an element of $\Gamma(S, \mathcal{O}_S)$.

PROOF OF THEOREM 4.1. We put $m = \dim X$. We denote by d_1 (resp. d_2) the composite of $d : \mathcal{O}_{X \times_S X} \rightarrow \Omega_{X \times_S X/S}$ and the first (resp. second) projection of $\Omega_{X \times_S X/S} \xrightarrow{\sim} \text{pr}_1^* \Omega_{X/S} \oplus \text{pr}_2^* \Omega_{X/S}$.

By the corollary of the Lefschetz-Verdier trace formula [10, Exp. III, Théorème 6.10 and Remarques 6.11], we obtain

$$\begin{aligned} & \text{Tr}(u_* \mid R\Gamma(X, \mathcal{K})) \\ &= \sum_{\beta \in \pi_0(\text{Fix } a)} \text{Res}_{X \times_S X/S} \left(\text{Tr}(u_\beta \mid \mathcal{K}_\beta) \begin{array}{cccc} d_2 t_1 \wedge \dots \wedge d_2 t_m \wedge d_1 s_1 \wedge \dots \wedge d_1 s_m & & & \\ t_1 & \dots & t_m & s_1 \dots s_m \end{array} \right), \end{aligned}$$

where s_1, \dots, s_m is an $\mathcal{O}_{X \times_S X}$ -sequence generating the defining ideal of Y nearby β , and t_1, \dots, t_m is an $\mathcal{O}_{X \times_S X}$ -sequence defined as follows. Since $a_2 \circ i_\beta$ is a closed immersion and β is étale over S , there are an open neighborhood X' of β in X and an étale morphism $X' \rightarrow S[T_1, \dots, T_m]$ such that the following diagram is commutative:

$$\begin{array}{ccc} \beta & \xrightarrow{\quad} & X' \\ \downarrow & & \downarrow \\ S & \xrightarrow{\quad} & S[T_1, \dots, T_m], \end{array}$$

where the bottom row is the zero section. We denote by x_1, \dots, x_m the sections of $\mathcal{O}_{X'}$ corresponding to T_1, \dots, T_m . By the above diagram, we obtain $a_2(x_i) \in \mathcal{I}_\beta$ for each i . In addition, $1 \otimes x_1 - x_1 \otimes 1, \dots, 1 \otimes x_m - x_m \otimes 1$ generate the defining ideal of the diagonal $\Delta_X : X \hookrightarrow X \times_S X$ nearby β since X' is étale over $S[T_1, \dots, T_m]$. We put $t_i = 1 \otimes x_i - x_i \otimes 1$.

Then we obtain

$$\begin{aligned} & \text{Res} \left(\text{Tr}(u_\beta \mid \mathcal{K}_\beta) \begin{array}{cccc} d_2 t_1 \wedge \dots \wedge d_2 t_m \wedge d_1 s_1 \wedge \dots \wedge d_1 s_m & & & \\ t_1 & \dots & t_m & s_1 \dots s_m \end{array} \right) \\ &= \text{Res} \left(\text{Tr}(u_\beta \mid \mathcal{K}_\beta) \begin{array}{cccc} d(1 \otimes x_1) \wedge \dots \wedge d(1 \otimes x_m) \wedge d_1 s_1 \wedge \dots \wedge d_1 s_m & & & \\ t_1 & \dots & t_m & s_1 \dots s_m \end{array} \right) \end{aligned}$$

$$\begin{aligned}
 &= \text{Res} \left(\text{Tr}(u_\beta \mid \mathcal{K}_\beta) \begin{array}{ccccccc} d(1 \otimes x_1) \wedge \cdots \wedge d(1 \otimes x_m) \wedge ds_1 \wedge \cdots \wedge ds_m \\ t_1 & \dots & t_m & s_1 & \dots & s_m \end{array} \right) \\
 &= \text{Res}_{Y/S} \left(\text{Tr}(u_\beta \mid \mathcal{K}_\beta) \begin{array}{ccccccc} d(a_2(x_1)) \wedge \cdots \wedge d(a_2(x_m)) \\ a_2(x_1) - a_1(x_1) & \dots & a_2(x_m) - a_1(x_m) \end{array} \right),
 \end{aligned}$$

where we write Res instead of $\text{Res}_{X \times_S X/S}$ for simplicity. At the last equality we use the property of [14, Chapter III, §9, (R3)].

By the definition of x_1, \dots, x_m , for each i , we can write

$$a_2(x_i) = \sum_{j=1}^m c_{ij}(a_2(x_j) - a_1(x_j))$$

nearby β , where each c_{ij} is a section of \mathcal{O}_Y . Since $a_1 \circ i_\beta = a_2 \circ i_\beta$ and $da_1 = 0$ by assumption, we have

$$da_2(x_i) = \sum_{j=1}^m c_{ij} da_2(x_j)$$

as sections of $i_\beta^* \Omega_{Y/S}$. Thus we obtain $c_{ij} = \delta_{ij}$ in \mathcal{O}_β , where δ_{ij} is Kronecker's delta. Therefore, by [14, Chapter III, §9, (R6)], we have

$$\begin{aligned}
 &\text{Res}_{Y/S} \left(\text{Tr}(u_\beta \mid \mathcal{K}_\beta) \begin{array}{ccccccc} d(a_2(x_1)) \wedge \cdots \wedge d(a_2(x_m)) \\ a_2(x_1) - a_1(x_1) & \dots & a_2(x_m) - a_1(x_m) \end{array} \right) \\
 &= \text{Res}_{Y/S} \left(\text{Tr}(u_\beta \mid \mathcal{K}_\beta) \begin{array}{ccccccc} \det(c_{ij}) \cdot d(a_2(x_1) - a_1(x_1)) \wedge \cdots \wedge d(a_2(x_m) - a_1(x_m)) \\ a_2(x_1) - a_1(x_1) & \dots & a_2(x_m) - a_1(x_m) \end{array} \right) \\
 &= \text{Tr}_{\beta/S}(\text{Tr}(u_\beta \mid \mathcal{K}_\beta) \det(c_{ij})|_\beta) \\
 &= \text{Tr}_{\beta/S}(\text{Tr}(u_\beta \mid \mathcal{K}_\beta)).
 \end{aligned}$$

□

5. Some semi-linear algebra

In this section, we recall some semi-linear algebra, which we will use in Section 6. We put $d = [\mathbb{F}_q : \mathbb{F}_p]$.

LEMMA 5.1. *Let M_0 be a $W_n(\mathbb{F}_q)$ -module of finite type and $\Phi_0: M_0 \rightarrow M_0$ a σ_0 -semi-linear endomorphism. We put $F_0 = \Phi_0^d$ and $M = M_0 \otimes_{W_n(\mathbb{F}_q)} W_n(k)$. We write $\Phi: M \rightarrow M$ (resp. $F: M \rightarrow M$) for the σ -semi-linear (resp. $W_n(k)$ -linear) extension of Φ_0 (resp. F_0) to M . Then the following assertions hold:*

- (1) *The morphism $1 - \Phi: M \rightarrow M$ is surjective.*
- (2) *We have $M^\Phi / p(M^\Phi) = (M/pM)^\Phi$.*

(3) *The canonical morphism $M^\Phi \otimes_{\mathbb{Z}/p^n} W_n(k) \rightarrow M$ is injective. Moreover, Φ and F are nilpotent on the cokernel of this morphism.*

PROOF. (1) We prove the assertion by induction on n . If $n = 1$, this assertion follows from [12, Exp. XXII, Proposition 1.2]. We assume that the assertion holds for $n - 1$. We have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & pM & \longrightarrow & M & \longrightarrow & M/pM & \longrightarrow & 0 \\ & & \downarrow 1-\Phi & & \downarrow 1-\Phi & & \downarrow 1-\Phi & & \\ 0 & \longrightarrow & pM & \longrightarrow & M & \longrightarrow & M/pM & \longrightarrow & 0 \end{array}$$

where the horizontal sequences are exact. We note that pM (resp. M/pM) has a natural $W_{n-1}(k)$ -module (resp. k -vector space) structure of finite type. By the induction hypothesis (resp. the case $n = 1$), the endomorphism $1 - \Phi$ on pM (resp. M/pM) is surjective. Hence, by the above exact sequence, the endomorphism $1 - \Phi$ on M is surjective.

(2) By the snake lemma, we have the following exact sequence

$$0 \rightarrow (pM)^\Phi \rightarrow M^\Phi \rightarrow (M/pM)^\Phi \rightarrow 0.$$

Hence we have $M^\Phi / (pM)^\Phi = (M/pM)^\Phi$. In addition, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M[p] & \longrightarrow & M & \xrightarrow{\times p} & pM & \longrightarrow & 0 \\ & & \downarrow 1-\Phi & & \downarrow 1-\Phi & & \downarrow 1-\Phi & & \\ 0 & \longrightarrow & M[p] & \longrightarrow & M & \xrightarrow{\times p} & pM & \longrightarrow & 0, \end{array}$$

where we put $M[p] = \text{Ker}(M \xrightarrow{\times p} M)$. As we have shown above, the endomorphism $1 - \Phi$ on $M[p]$ is surjective. Applying the snake lemma to this diagram, we have $p(M^\Phi) = (pM)^\Phi$. Therefore we obtain $M^\Phi / p(M^\Phi) = (M/pM)^\Phi$.

(3) We prove the assertion by induction on n . If $n = 1$, it follows from [12, Exp. XXII, Corollaire 1.1.10]. We assume that it holds for $n - 1$.

First, we will prove the former statement. By the proved assertion (2), we have the following exact sequence

$$0 \longrightarrow (pM)^\Phi \longrightarrow M^\Phi \longrightarrow (M/pM)^\Phi \longrightarrow 0.$$

Tensoring this diagram with $W_n(k)$, we obtain the following exact sequence

$$0 \rightarrow (pM)^\Phi \otimes_{\mathbb{Z}/p^{n-1}} W_{n-1}(k) \rightarrow M^\Phi \otimes_{\mathbb{Z}/p^n} W_n(k) \rightarrow (M/pM)^\Phi \otimes_{\mathbb{F}_p} k \rightarrow 0.$$

Thus we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
 0 \longrightarrow & (pM)^\Phi \otimes_{\mathbb{Z}/p^{n-1}} W_{n-1}(k) & \longrightarrow & M^\Phi \otimes_{\mathbb{Z}/p^n} W_n(k) & \longrightarrow & (M/pM)^\Phi \otimes_{\mathbb{F}_q} k & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \longrightarrow & pM & \longrightarrow & M & \longrightarrow & M/pM & \longrightarrow 0.
 \end{array}$$

By the induction hypothesis (resp. the case $n = 1$), the left (resp. right) vertical arrow of the above diagram is injective. Hence the middle vertical morphism is also injective, that is, the former assertion for n also holds.

We prove the latter statement. We put

$$M' = \text{Coker}(M^\Phi \otimes_{\mathbb{Z}/p^n} W_n(k) \rightarrow M).$$

By the snake lemma, we have

$$M'/pM' \simeq \text{Coker} \left((M/pM)^\Phi \otimes_{\mathbb{F}_q} k \rightarrow M/pM \right).$$

Now we need the following lemma:

LEMMA 5.2. *Let M' be a $W_n(k)$ -module and $\Phi: M' \rightarrow M'$ be a \mathbb{Z}/p^n -linear endomorphism. We assume that Φ is nilpotent on M'/pM' . Then Φ is nilpotent on M' .*

PROOF. By assumption, there exists an integer $e \geq 1$ such that $\Phi^e = 0$ on M'/pM' . Let x_0 be an element of M' . For each integer $i \geq 0$, we can inductively find $x_{i+1} \in M'$ such that $\Phi^e(x_i) = px_{i+1}$. Thus we have $\Phi^{ne}(x) = p^n x_n = 0$. Therefore Φ is nilpotent. \square

We return to the proof of Lemma 5.1(3). By the case $n = 1$, the endomorphism Φ on M'/pM' is nilpotent. Therefore, by Lemma 5.2, Φ is also nilpotent on M' .

We will show that F is nilpotent on M' . We fix an integer e such that $\Phi^{de} = 0$ on M' . Since F coincides with Φ^d on M_0 , we have $F^e = 0$ on the image of M_0 in M' . Hence $F^e = 0$ on M' , that is, the morphism F is nilpotent on M' . \square

LEMMA 5.3. *Under the notation of Lemma 5.1, we assume that M is free over $W_n(k)$ and M^Φ is free over \mathbb{Z}/p^n . Then there exists an integer $N \geq 1$ such that the following assertion holds: Let φ be a $W_n(k)$ -linear endomorphism on M which stabilizes M^Φ . Then, for any $m \geq N$, we have*

$$\text{Tr}(\varphi \circ F^m \mid M^\Phi) = \text{Tr}(\varphi \circ F^m \mid M).$$

If $n = 1$ and $\varphi \circ F = F \circ \varphi$, then we can take $N = 1$.

PROOF. By Lemma 5.1(3), we have the exact sequence

$$0 \rightarrow M^\Phi \otimes_{\mathbb{Z}/p^n} W_n(k) \rightarrow M \rightarrow M' \rightarrow 0,$$

where we put $M' = \text{Coker}(M^\Phi \otimes_{\mathbb{Z}/p^n} W_n(k) \rightarrow M)$. By assumption, $M^\Phi \otimes W_n(k)$ is a free $W_n(k)$ -module of finite rank. Since $W_n(k)$ is a Gorenstein ring, the $W_n(k)$ -module $M^\Phi \otimes W_n(k)$ is injective. Hence the above sequence splits. Thus M' is a finite projective $W_n(k)$ -module, and we have

$$\text{Tr}(\varphi \circ F^m | M) = \text{Tr}(\varphi \circ F^m | M^\Phi) + \text{Tr}(\varphi \circ F^m | M').$$

By Lemma 5.1(3), we can take an integer N such that the equality $\Phi^{dN} = 0$ on M' holds. Then, for any $m \geq N$, we have $\text{Tr}(\varphi \circ F^m | M') = 0$ and the equality in Lemma 5.3 holds.

We assume that $n = 1$ and $\varphi \circ F = F \circ \varphi$. Then $\varphi \circ F$ is nilpotent on M' . Hence all of the eigenvalues of $\varphi \circ F$ are zero. Therefore, for any $m \geq 1$, we have $\text{Tr}(\varphi \circ F^m | M') = 0$. \square

6. Proof of Theorem 1.4

In this section, we shall prove Theorem 1.4. Before starting the proof, we recall a short exact sequence of Artin-Schreier type.

LEMMA 6.1. *Let k be a perfect field of characteristic p and $\overline{\mathcal{X}}$ a flat $W_n(k)$ -scheme. We denote by σ the Frobenius endomorphism of $W_n(k)$. Suppose that there exists a σ -semi-linear endomorphism $\Phi: \mathcal{O}_{\overline{\mathcal{X}}} \rightarrow \mathcal{O}_{\overline{\mathcal{X}}}$ which is a lift of p -th power endomorphism on the structure sheaf of $\overline{\mathcal{X}} \times_{\text{Spec } W_n(k)} \text{Spec } k$. Let \mathcal{D} be a closed flat $W_n(k)$ -subscheme of $\overline{\mathcal{X}}$ whose defining ideal \mathcal{I} is Φ -stable. We put $\mathcal{X} = \overline{\mathcal{X}} \setminus \mathcal{D}$. We denote by j the open immersion $\mathcal{X} \hookrightarrow \overline{\mathcal{X}}$. Let \overline{G} be a locally free constructible \mathbb{Z}/p^n -module on $\overline{\mathcal{X}}$. We put $G = \overline{G}|_{\mathcal{X}}$. Then we have the following short exact sequence*

$$0 \longrightarrow j_!G \longrightarrow \overline{G} \otimes \mathcal{I} \xrightarrow{1-\Phi} \overline{G} \otimes \mathcal{I} \longrightarrow 0.$$

PROOF. For any integer $1 \leq m \leq n$, we put $\overline{\mathcal{X}}_m := \overline{\mathcal{X}} \times_{\text{Spec } W_n(k)} \text{Spec } W_m(k)$. We define \mathcal{D}_m and Φ_m similarly. First, we prove the exactness of the sequence

$$0 \longrightarrow \mathbb{Z}/p^m \longrightarrow \mathcal{O}_{\overline{\mathcal{X}}_m} \xrightarrow{1-\Phi} \mathcal{O}_{\overline{\mathcal{X}}_m} \longrightarrow 0$$

by induction on m . The case of $m = 1$ is the usual Artin-Schreier sequence. We assume the assertion holds for $m - 1$. Since $\mathcal{O}_{\overline{\mathcal{X}}}$ is flat over $W_n(k)$, the sequence

$$0 \longrightarrow \mathcal{O}_{\overline{\mathcal{X}}_{m-1}} \longrightarrow \mathcal{O}_{\overline{\mathcal{X}}_m} \longrightarrow \mathcal{O}_{\overline{\mathcal{X}}_1} \longrightarrow 0.$$

is exact. Furthermore, we have the inclusion $\mathbb{Z}/p^m \subset \text{Ker}(1 - \Phi_m)$ since Φ is σ -semi-linear. Hence, by the induction hypothesis, the assertion also holds for m . We can also obtain such a sequence for \mathcal{D} by the assumptions that \mathcal{D} is flat over $W_n(k)$ and $\Phi\mathcal{I} \subset \mathcal{I}$.

We denote by i the closed immersion $\mathcal{D} \hookrightarrow \overline{\mathcal{X}}$. By tensoring the obtained sequences with \overline{G} and $i^{-1}\overline{G}$, we have the following short exact sequences

$$0 \longrightarrow \overline{G} \longrightarrow \overline{G} \otimes \mathcal{O}_{\overline{\mathcal{X}}} \xrightarrow{1-\Phi} \overline{G} \otimes \mathcal{O}_{\overline{\mathcal{X}}} \longrightarrow 0$$

and

$$0 \longrightarrow i^{-1}\overline{G} \longrightarrow (i^{-1}\overline{G}) \otimes \mathcal{O}_{\mathcal{D}} \xrightarrow{1-\Phi} (i^{-1}\overline{G}) \otimes \mathcal{O}_{\mathcal{D}} \longrightarrow 0.$$

From these sequences, we obtain the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \overline{G} & \longrightarrow & \overline{G} \otimes \mathcal{O}_{\overline{\mathcal{X}}} & \xrightarrow{1-\Phi} & \overline{G} \otimes \mathcal{O}_{\overline{\mathcal{X}}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & i_*i^{-1}\overline{G} & \longrightarrow & i_*((i^{-1}\overline{G}) \otimes \mathcal{O}_{\mathcal{D}}) & \xrightarrow{1-\Phi} & i_*((i^{-1}\overline{G}) \otimes \mathcal{O}_{\mathcal{D}}) \longrightarrow 0, \end{array}$$

where the horizontal sequences are exact. Here, the left vertical arrow is the morphism which appears in the short exact sequence

$$0 \longrightarrow j_*i^{-1}\overline{G} \longrightarrow \overline{G} \longrightarrow i_*i^{-1}\overline{G} \longrightarrow 0$$

and the middle and right vertical arrows are the composite of the canonical morphisms

$$\overline{G} \otimes \mathcal{O}_{\overline{\mathcal{X}}} \rightarrow \overline{G} \otimes (\mathcal{O}_{\overline{\mathcal{X}}}/\mathcal{I}) \xrightarrow{\sim} i_*((i^{-1}\overline{G}) \otimes \mathcal{O}_{\mathcal{D}}).$$

Hence we obtain the required sequence by using the snake lemma. □

PROOF OF THEOREM 1.4. We denote by i (resp. j) the closed (resp. open) immersion $\mathcal{D}_0 \hookrightarrow \overline{\mathcal{X}}_0$ (resp. $\mathcal{X}_0 \hookrightarrow \overline{\mathcal{X}}_0$). We use the same notation for the closed (resp. open) immersion $\mathcal{D} \hookrightarrow \overline{\mathcal{X}}$ (resp. $\mathcal{X} \hookrightarrow \overline{\mathcal{X}}$).

We denote by $f_{\overline{\mathcal{X}}_0}$ the endomorphism $(\text{id}_{\overline{\mathcal{X}}_0}, \Phi_{\mathcal{O}_{\overline{\mathcal{X}}_0}})$ on $\overline{\mathcal{X}}_0$ and by $\text{Fr}_{\overline{\mathcal{X}}_0}$ the $W_n(\mathbb{F}_q)$ -endomorphism $(\text{id}_{\overline{\mathcal{X}}_0}, \Phi_{\mathcal{O}_{\overline{\mathcal{X}}_0}}^d)$ on $\overline{\mathcal{X}}_0$, where we put $d = [\mathbb{F}_q : \mathbb{F}_p]$. We write $\text{Fr}_{\overline{\mathcal{X}}}$ for the base change of $\text{Fr}_{\overline{\mathcal{X}}_0}$ by $W_n(\mathbb{F}_q) \rightarrow W_n(k)$. We put $\overline{\mathcal{G}}_0 = \overline{G}_0 \otimes \mathcal{O}_{\overline{\mathcal{X}}_0}$ and $\mathcal{G}_0 = \overline{\mathcal{G}}_0|_{\mathcal{X}_0} = G_0 \otimes \mathcal{O}_{\mathcal{X}_0}$. Recall that \mathcal{I}_0 is the defining ideal of the complement $\mathcal{D}_0 = \overline{\mathcal{X}}_0 \setminus \mathcal{X}_0$. We put $\mathcal{G}'_0 = \overline{\mathcal{G}}_0 \otimes \mathcal{I}_0$, which is a locally free $\mathcal{O}_{\overline{\mathcal{X}}_0}$ -module since \mathcal{I}_0 is locally generated by a non-zero divisor of $\mathcal{O}_{\overline{\mathcal{X}}_0}$. We put $\Phi_{\overline{\mathcal{G}}_0} = \text{id}_{\overline{\mathcal{G}}_0} \otimes \Phi_{\mathcal{O}_{\overline{\mathcal{X}}_0}}$. We also define $\Phi_{\overline{\mathcal{G}}}$ as well. By the assumption (e), \mathcal{G}'_0 is $\Phi_{\overline{\mathcal{G}}_0}$ -stable and we define $\Phi_{\mathcal{G}'_0}$ by the restriction of $\Phi_{\overline{\mathcal{G}}_0}$ to \mathcal{G}'_0 . We define $\Phi_{\mathcal{G}'}$ in the same way. We sometimes denote by Φ these homomorphisms for short. For a σ_0 -semi-linear morphism of $\mathcal{O}_{\overline{\mathcal{X}}_0}$ -modules $\Psi: \mathcal{G}_1 \rightarrow \mathcal{G}_2$, we denote by Ψ' the $\mathcal{O}_{\overline{\mathcal{X}}_0}$ -linear

homomorphism $f_{\mathcal{X}_0}^* \mathcal{G}_1 \rightarrow \mathcal{G}_2$ associated to Ψ . We put

$$\text{Fr}_{\overline{\mathcal{G}}_0} = \text{id}_{\overline{\mathcal{G}}_0} \otimes \Phi'^d : \text{Fr}_{\mathcal{X}_0}^* \overline{\mathcal{G}}_0 \rightarrow \overline{\mathcal{G}}_0.$$

Then, by the assumption (e), the morphism $\text{Fr}_{\overline{\mathcal{G}}_0} : \text{Fr}_{\mathcal{X}_0}^* \overline{\mathcal{G}}_0 \rightarrow \overline{\mathcal{G}}_0$ is factorized as

$$\begin{array}{ccc} \text{Fr}_{\mathcal{X}_0}^* \mathcal{G}'_0 & \dashrightarrow & \mathcal{G}'_0 \\ \downarrow & & \downarrow \\ \text{Fr}_{\mathcal{X}_0}^* \overline{\mathcal{G}}_0 & \xrightarrow{\text{Fr}_{\overline{\mathcal{G}}_0}} & \overline{\mathcal{G}}_0. \end{array} \tag{1}$$

We denote by $\text{Fr}_{\mathcal{G}'_0}$ the top horizontal morphism in the diagram (1). Changing the base of the above diagram by $W_n(\mathbb{F}_q) \rightarrow W_n(k)$, we obtain the diagram

$$\begin{array}{ccc} \text{Fr}_{\mathcal{X}}^* \mathcal{G}' & \longrightarrow & \mathcal{G}' \\ \downarrow & & \downarrow \\ \text{Fr}_{\mathcal{X}}^* \overline{\mathcal{G}} & \longrightarrow & \overline{\mathcal{G}}. \end{array} \tag{2}$$

We write $\text{Fr}_{\overline{\mathcal{G}}}$ (resp. $\text{Fr}_{\mathcal{G}'}$) for the bottom (resp. top) horizontal morphism in the diagram (2). For simplicity, we sometimes write Fr for these morphisms and their pull-backs. We remark that the composite

$$\mathcal{G}'_0 \longrightarrow \text{Fr}_{\overline{\mathcal{G}}_0}^* \text{Fr}_{\mathcal{X}_0}^* \mathcal{G}'_0 = \text{Fr}_{\mathcal{X}_0}^* \mathcal{G}'_0 \xrightarrow{\text{Fr}_{\mathcal{G}'_0}} \mathcal{G}'_0$$

coincides with $\Phi_{\mathcal{G}'_0}^d$ and $\Phi_{\mathcal{G}'}$ is the σ -semi-linear extension of $\Phi_{\mathcal{G}'_0}$. Since α_{01} is proper, the following diagram is cartesian as that of topological spaces:

$$\begin{array}{ccc} \mathcal{Y}_0 & \xrightarrow{\alpha_{01}} & \mathcal{X}_0 \\ j_{\mathcal{Y}_0} \downarrow & & \downarrow j_{\mathcal{X}_0} \\ \overline{\mathcal{Y}}_0 & \xrightarrow{\overline{\alpha}_{01}} & \overline{\mathcal{X}}_0. \end{array}$$

Thus we have $\overline{\alpha}_{02}^{-1}(\overline{\mathcal{X}}_0 \setminus \mathcal{X}_0) \subset \overline{\alpha}_{01}^{-1}(\overline{\mathcal{X}}_0 \setminus \mathcal{X}_0)$ as topological spaces, and we have

$$(\overline{\alpha}_{01}^{-1} \mathcal{I}_0 \cdot \mathcal{O}_{\overline{\mathcal{Y}}_0})_{\text{red}} \subset (\overline{\alpha}_{02}^{-1} \mathcal{I}_0 \cdot \mathcal{O}_{\overline{\mathcal{Y}}_0})_{\text{red}}.$$

Since $\overline{\alpha}_{02}$ is étale, we obtain

$$\overline{\alpha}_{01}^{-1} \mathcal{I}_0 \cdot \mathcal{O}_{\overline{\mathcal{Y}}_0} \subset (\overline{\alpha}_{01}^{-1} \mathcal{I}_0 \cdot \mathcal{O}_{\overline{\mathcal{Y}}_0})_{\text{red}} \subset (\overline{\alpha}_{02}^{-1} \mathcal{I}_0 \cdot \mathcal{O}_{\overline{\mathcal{Y}}_0})_{\text{red}} = \overline{\alpha}_{02}^{-1} \mathcal{I}_0 \cdot \mathcal{O}_{\overline{\mathcal{Y}}_0}. \tag{3}$$

Since \overline{G}_0 is smooth and \mathcal{X}_0 is dense in $\overline{\mathcal{X}}_0$, there exists a unique morphism $\overline{u}_0: \overline{\alpha}_{01}^* \overline{G}_0 \rightarrow \overline{\alpha}_{02}^* \overline{G}_0$ such that $\overline{u}_0|_{\mathcal{Y}_0} = u_0$ (see for example [9, Exp. XVI, Proposition 3.2]). By the inclusion (3) and the flatness of $\overline{\alpha}_{02}$, the morphism $\overline{u}_0 \otimes \text{id}: \overline{\alpha}_{01}^* \overline{\mathcal{G}} \rightarrow \overline{\alpha}_{02}^* \overline{\mathcal{G}}$ is factorized as

$$\begin{array}{ccc} \overline{\alpha}_{01}^* \mathcal{G}'_0 & \xrightarrow{u'_0} & \overline{\alpha}_{02}^* \mathcal{G}'_0 \\ \downarrow & & \downarrow \\ \overline{\alpha}_{01}^* \overline{\mathcal{G}}_0 & \xrightarrow{\overline{u}_0 \otimes \text{id}} & \overline{\alpha}_{02}^* \overline{\mathcal{G}}_0. \end{array} \tag{4}$$

Therefore we have the following diagrams

$$\begin{array}{ccc} \overline{\alpha}_{01}^* j! G_0 & \xrightarrow{j! u_0} & \overline{\alpha}_{02}^* j! G_0 \\ \downarrow & & \downarrow \\ \overline{\alpha}_{01}^* \mathcal{G}'_0 & \xrightarrow{u'_0} & \overline{\alpha}_{02}^* \mathcal{G}'_0 \end{array} \tag{5}$$

and

$$\begin{array}{ccccccc} R\Gamma(\overline{\mathcal{X}}_0, \mathcal{G}'_0) & \xrightarrow{R\Gamma(\text{adj})} & R\Gamma(\overline{\mathcal{Y}}_0, \overline{\alpha}_{01}^* \mathcal{G}') & \xrightarrow{R\Gamma(u'_0)} & R\Gamma(\overline{\mathcal{Y}}_0, \overline{\alpha}_{02}^* \mathcal{G}') & \xrightarrow{R\Gamma(\text{Tr})} & R\Gamma(\overline{\mathcal{X}}_0, \mathcal{G}'_0) \\ R\Gamma(\Phi) \downarrow & & & & & & R\Gamma(\Phi) \downarrow \\ R\Gamma(\overline{\mathcal{X}}_0, \mathcal{G}'_0) & \xrightarrow{R\Gamma(\text{adj})} & R\Gamma(\overline{\mathcal{Y}}_0, \overline{\alpha}_{01}^* \mathcal{G}') & \xrightarrow{R\Gamma(u'_0)} & R\Gamma(\overline{\mathcal{Y}}_0, \overline{\alpha}_{02}^* \mathcal{G}') & \xrightarrow{R\Gamma(\text{Tr})} & R\Gamma(\overline{\mathcal{X}}_0, \mathcal{G}'_0). \end{array} \tag{6}$$

We note that the composite

$$\begin{aligned} R\Gamma_c(\mathcal{X}_0, G_0) & \xrightarrow{R\Gamma(\text{adj})} R\Gamma(\overline{\mathcal{Y}}_0, \overline{\alpha}_{01}^* j! G_0) \\ & \xrightarrow{R\Gamma(j! u_0)} R\Gamma(\overline{\mathcal{Y}}_0, \overline{\alpha}_{02}^* j! G_0) \\ & \xrightarrow{R\Gamma(\text{Tr})} R\Gamma_c(\mathcal{X}_0, G_0) \end{aligned}$$

coincides with u_{01} .

LEMMA 6.2. *The diagram (5) is commutative. Moreover, if $n = 1$, then the diagram (6) is also commutative.*

PROOF. The commutativity of the square (5) follows from the fact that, in the following diagram

$$\begin{array}{ccccc}
 & & \bar{\alpha}_{01}^* j! G_0 & \xrightarrow{\quad} & \bar{\alpha}_{02}^* j! G_0 \\
 & \swarrow & \downarrow & & \swarrow \\
 \bar{\alpha}_{01}^* \bar{G}_0 & \xrightarrow{\quad} & \bar{\alpha}_{02}^* \bar{G}_0 & & \\
 \downarrow & & \downarrow & & \downarrow \\
 & & \bar{\alpha}_{01}^* \mathcal{G}'_0 & \xrightarrow{\quad} & \bar{\alpha}_{02}^* \mathcal{G}'_0 \\
 & \swarrow & \downarrow & & \swarrow \\
 \bar{\alpha}_{01}^* \bar{\mathcal{G}}_0 & \xrightarrow{\quad} & \bar{\alpha}_{02}^* \bar{\mathcal{G}}_0 & & \\
 & & & & (*)
 \end{array}$$

all the squares except (5) are commutative and the morphism (*) is injective.

Now we assume $n = 1$. We write $\Phi_{\bar{\mathcal{Y}}_0}$ for the p -th power map on $\mathcal{O}_{\bar{\mathcal{Y}}_0}$. We put $f_{\bar{\mathcal{Y}}_0} = (\text{id}_{\bar{\mathcal{Y}}_0}, \Phi_{\bar{\mathcal{Y}}_0})$. As before, $\Phi_{\bar{\mathcal{Y}}_0}$ induces σ_0 -semi-linear endomorphisms on $\bar{\alpha}_{01}^* \bar{G}_0$, $\bar{\alpha}_{02}^* \bar{G}_0$, $\bar{\alpha}_{01}^* \mathcal{G}'_0$, and $\bar{\alpha}_{02}^* \mathcal{G}'_0$. We also denote these by $\Phi_{\bar{\mathcal{Y}}_0}$. For any σ_0 -semi-linear endomorphisms Ψ , we write Ψ' for the corresponding $\mathcal{O}_{\bar{\mathcal{Y}}_0}$ -homomorphism.

For the commutativity of (6), we shall show that of the following diagram

$$\begin{array}{ccccccc}
 R\Gamma(\bar{\mathcal{X}}_0, \mathcal{G}'_0) & \xrightarrow{R\Gamma(\text{adj})} & R\Gamma(\bar{\mathcal{Y}}_0, \bar{\alpha}_{01}^* \mathcal{G}'_0) & \xrightarrow{R\Gamma(u'_0)} & R\Gamma(\bar{\mathcal{Y}}_0, \bar{\alpha}_{02}^* \mathcal{G}'_0) & \xrightarrow{R\Gamma(\text{Tr})} & R\Gamma(\bar{\mathcal{X}}_0, \mathcal{G}'_0) \\
 R\Gamma(\Phi) \downarrow & & R\Gamma(\bar{\alpha}_{01}^* \Phi_{\bar{\mathcal{Y}}_0}) \downarrow & & R\Gamma(\bar{\alpha}_{02}^* \Phi_{\bar{\mathcal{Y}}_0}) \downarrow & & R\Gamma(\Phi) \downarrow \\
 R\Gamma(\bar{\mathcal{X}}_0, \mathcal{G}'_0) & \xrightarrow{R\Gamma(\text{adj})} & R\Gamma(\bar{\mathcal{Y}}_0, \bar{\alpha}_{01}^* \mathcal{G}'_0) & \xrightarrow{R\Gamma(u'_0)} & R\Gamma(\bar{\mathcal{Y}}_0, \bar{\alpha}_{02}^* \mathcal{G}'_0) & \xrightarrow{R\Gamma(\text{Tr})} & R\Gamma(\bar{\mathcal{X}}_0, \mathcal{G}'_0).
 \end{array} \tag{7}$$

The commutativity of the right and left squares of the diagram (7) follows from the commutative diagrams

$$\begin{array}{ccc}
 f_{\bar{\mathcal{X}}_0}^* \mathcal{G}'_0 & \xrightarrow{\text{adj}} & \bar{\alpha}_{01*} \bar{\alpha}_{01}^* f_{\bar{\mathcal{X}}_0}^* \mathcal{G}'_0 = \bar{\alpha}_{01*} f_{\bar{\mathcal{Y}}_0}^* \bar{\alpha}_{01}^* \mathcal{G}'_0 \\
 \Phi' \downarrow & & \bar{\alpha}_{01*} \bar{\alpha}_{01}^* \Phi' \downarrow \swarrow \bar{\alpha}_{01*} \Phi'_{\bar{\mathcal{Y}}_0} \\
 \mathcal{G}'_0 & \xrightarrow{\text{adj}} & \bar{\alpha}_{01*} \bar{\alpha}_{01}^* \mathcal{G}'_0
 \end{array}$$

and

$$\begin{array}{ccc}
 \bar{\alpha}_{02*} f_{\bar{\mathcal{Y}}_0}^* \bar{\alpha}_{02}^* \mathcal{G}'_0 & = & f_{\bar{\mathcal{X}}_0}^* \bar{\alpha}_{02*} \bar{\alpha}_{02}^* \mathcal{G}'_0 \xrightarrow{\text{Tr}} f_{\bar{\mathcal{X}}_0}^* \mathcal{G}'_0 \\
 \bar{\alpha}_{02*} \Phi'_{\bar{\mathcal{Y}}_0} \searrow & & \downarrow \bar{\alpha}_{02*} \bar{\alpha}_{02}^* \Phi' \quad \downarrow \Phi' \\
 & & \bar{\alpha}_{02*} \bar{\alpha}_{02}^* \mathcal{G}'_0 \xrightarrow{\text{Tr}} \mathcal{G}'_0.
 \end{array}$$

The commutativity of the middle of the diagram (7) follows from the commutative square

$$\begin{array}{ccc}
 \bar{\alpha}_{01}^* \bar{\mathcal{G}}_0 & \xrightarrow{\bar{u}_0 \otimes \text{id}} & \bar{\alpha}_{02}^* \bar{\mathcal{G}}_0 \\
 \bar{\alpha}_{01}^* \Phi_{\bar{\mathcal{Y}}_0} \downarrow & & \downarrow \bar{\alpha}_{02}^* \Phi_{\bar{\mathcal{Y}}_0} \\
 \bar{\alpha}_{01}^* \bar{\mathcal{G}}_0 & \xrightarrow{\bar{u}_0 \otimes \text{id}} & \bar{\alpha}_{02}^* \bar{\mathcal{G}}_0
 \end{array}$$

and the commutative diagram (4). □

By Lemma 6.1, we obtain the exact sequence

$$0 \longrightarrow j_! G \longrightarrow \mathcal{G}' \xrightarrow{1-\Phi} \mathcal{G}' \longrightarrow 0. \tag{8}$$

This induces the following long exact sequence

$$\dots \longrightarrow H_c^i(\mathcal{X}, G) \longrightarrow H^i(\bar{\mathcal{X}}, \mathcal{G}') \xrightarrow{1-H^i(\Phi)} H^i(\bar{\mathcal{X}}, \mathcal{G}') \longrightarrow \dots$$

Since $\bar{\mathcal{X}}_0$ is proper over $W_n(\mathbb{F}_q)$ and \mathcal{G}'_0 is a coherent $\mathcal{O}_{\bar{\mathcal{X}}_0}$ -module, the $W_n(\mathbb{F}_q)$ -module $H^i(\bar{\mathcal{X}}_0, \mathcal{G}'_0)$ is finitely generated for each i , the canonical homomorphism $H^i(\bar{\mathcal{X}}_0, \mathcal{G}'_0) \otimes_{W_n(\mathbb{F}_q)} W_n(k) \rightarrow H^i(\bar{\mathcal{X}}, \mathcal{G}')$ is an isomorphism, and we have

$$H^i(\text{Fr}_{\mathcal{G}'_0}) \otimes_{W_n(\mathbb{F}_q)} W_n(k) = H^i(\text{Fr}_{\mathcal{G}'}) .$$

By Lemma 5.1(1), the σ -semi-linear map $1 - H^i(\Phi_{\mathcal{G}'})$ is surjective. Thus, for each index i , we have the following exact sequence

$$0 \longrightarrow H_c^i(\mathcal{X}, G) \longrightarrow H^i(\bar{\mathcal{X}}, \mathcal{G}') \xrightarrow{1-H^i(\Phi)} H^i(\bar{\mathcal{X}}, \mathcal{G}') \longrightarrow 0.$$

By Lemma 6.2, we have the following commutative square

$$\begin{array}{ccc}
 H_c^i(\mathcal{X}, G) & \longrightarrow & H^i(\bar{\mathcal{X}}, \mathcal{G}') \\
 \downarrow u_! & & \downarrow u' \\
 H_c^i(\mathcal{X}, G) & \longrightarrow & H^i(\bar{\mathcal{X}}, \mathcal{G}') .
 \end{array}$$

Therefore, by Lemma 5.3 and the assumption (f), we have

$$\begin{aligned}
 \text{Tr}((u \circ \text{Fr}^m)_! | H_c^i(\mathcal{X}, G)) &= \text{Tr}(u_! \circ (\text{Fr}_G)_!^m | H_c^i(\mathcal{X}, G)) \\
 &= \text{Tr}(u'_* \circ (\text{Fr}_{\mathcal{G}'})_*^m | H^i(\bar{\mathcal{X}}, \mathcal{G}')) \\
 &= \text{Tr}((u' \circ \text{Fr}^m)_* | H^i(\bar{\mathcal{X}}, \mathcal{G}'))
 \end{aligned} \tag{9}$$

for any integer $m \geq N$, where we take an integer N large enough so that $N \geq n$ and

$$\mathrm{Tr}(u'_! \circ (\mathrm{Fr}_G)_!^m \mid H_c^i(\mathcal{X}, G)) = \mathrm{Tr}(u'_* \circ (\mathrm{Fr}_{\mathcal{G}'})_*^m \mid H^i(\overline{\mathcal{X}}, \mathcal{G}'))$$

holds for any $m \geq N$ and any index i . We note once again that if $n = 1$ we can take $N = 1$ by Lemma 5.3 and Lemma 6.2.

Since $d\mathrm{Fr}_{\overline{\mathcal{X}}}^m = 0$, $\overline{\alpha}_2$ is étale, and \mathcal{G}' is locally free of finite rank, we can apply Theorem 4.1 to $\mathrm{Tr}((u' \circ \mathrm{Fr}^m)_* \mid R\Gamma(\overline{\mathcal{X}}, \mathcal{G}'))$:

$$\mathrm{Tr}((u' \circ \mathrm{Fr}^m)_* \mid R\Gamma(\overline{\mathcal{X}}, \mathcal{G}')) = \sum_{\beta \in \pi_0(\mathrm{Fix} \overline{\alpha}^{(m)})} \mathrm{Tr}((u' \circ \mathrm{Fr})_\beta \mid \mathcal{G}'_\beta). \quad (10)$$

We remark that the morphism $\beta \rightarrow S$ is an isomorphism since $W_n(k)$ is a strict Henselian ring.

If $\beta \in \pi_0(\mathrm{Fix} \alpha^{(m)})$, then we have $\mathrm{Tr}((u' \circ \mathrm{Fr})_\beta \mid \mathcal{G}'_\beta) = \mathrm{Tr}((u \circ \mathrm{Fr})_\beta \mid G_\beta)$ by the definition of u' . If β does not belong to $\pi_0(\mathrm{Fix} \alpha^{(m)})$, then we have $\mathcal{I}_\beta = 0$. Thus $\mathcal{G}'_\beta = 0$ and we obtain $\mathrm{Tr}((u' \circ \mathrm{Fr})_\beta \mid \mathcal{G}'_\beta) = 0$. Therefore we obtain Theorem 1.4 by the equalities (9) and (10). \square

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