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Representations of Path Algebras with Applications to Subgroup Lattices and Group Characters

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Abstract. Let Q be a quiver, and w a weight function on the set of arrows of Q. In this paper, we will introduce an R-algebra $\mathsf{UD}(Q, w; R)$ over a ring R in which the information how vertices of Q are joined by its arrows with weights should be reflected well. This algebra is obtained by using $\mathbf{Z}Q$-modules where $\mathbf{Z}Q$ is the path algebra of Q over $\mathbf{Z}$. We will particularly focus on quivers and weight functions defined by the subgroup lattice of a finite group G, and defined by irreducible characters of subgroups of G. The structure of the corresponding $\mathbf{Z}$-algebras $\mathsf{UD}(Q, w; \mathbf{Z})$ and relations with the group G will be studied.

1. Introduction

Let G be a finite group, and $\mathrm{Sgp}(G)$ the totality of subgroups of G. One of the motivations of this paper is the following. For a family $\mathcal{D} \subseteq \mathrm{Sgp}(G)$ of subgroups of G, denote by $\Delta(\mathcal{D})$ the totality of chains $(H_0 < H_1 < \cdots < H_\ell)$ of subgroups in $\mathcal{D}$ with respect to the inclusion-relation $\leq$. Then a pair $(\mathcal{D}, \Delta(\mathcal{D}))$ forms a simplicial complex (ordered complex) which is called a subgroup complex of G. There are a lot of works on subgroup complexes such as their homotopy property, Lefschetz modules, classifying spaces, and so on (see [6] and various references in it for this research area). Those complexes can be thought of objects in the intersection of finite group theory, combinatorics, and algebraic topology. In order to pursue the nature of subgroup complexes further, we like to consider their representations. Since a partially ordered set $(\mathcal{D}, \leq)$ can be regarded as a quiver, representations of path algebras of quivers are studied here. Although we do not still reach a direct application to subgroup complexes, we hope that this paper will give us another approach to such complexes in the future.

The paper is organized as follows: In Section 2, we recall the concept of quivers $Q = (Q_0, Q_1, s, r)$ and path algebras RQ where R is a coefficient ring. Our standing situation is established here, and in particular a weight function w on the set of arrows of Q is considered. In Section 3, we define a right, and also a left $\mathbf{Z}Q$-modules RQ_0. Then using those actions of $\mathbf{Z}Q$, an R-algebra $\mathsf{UD}(Q, w; R)$ is introduced as an R-subalgebra of $\mathrm{End}(RQ_0)$.

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This is one of the main objects in this paper. We examine the structure of $\mathsf{UD}(Q, w; R)$, and in particular $\mathsf{UD}(Q, w; \mathbf{Z})$ over $\mathbf{Z}$ is completely determined by some integers which we call the generating constants. In Section 4, we consider a quiver Q_G and a weight function w_G associated to the subgroup lattice $(\mathrm{Sgp}(G), \leq)$ of G, and investigate the generating constants of $\mathsf{UD}(Q_G, w_G; \mathbf{Z})$. We especially show that a generating constant corresponding to subgroups $A, B \leq G$ is equal to the order of G if and only if A and B furnish a factorization of G. In Section 5, we consider a quiver Q_G^{ch} and a weight function w_G^{ch} associated to irreducible characters of subgroups of G. Some elements of $\mathsf{UD}(Q_G^{\mathrm{ch}}, w_G^{\mathrm{ch}}; \mathbf{Z})$ corresponding to Bratteli diagrams are discussed. Furthermore we see that abelian groups are characterized by a weight function w_G^{ch} which behaves in a special way. Finally we study the case where all of the generating constants are equal to 1.

2. Preliminaries

In this section, we recall the definitions of quivers and path algebras (cf. [1, Section III-1] for example), and establish our notation. One of the important things in this paper is that we equip each arrow of a quiver with a "weigh". Throughout the paper, let R be a commutative ring with the identity element.

DEFINITION 2.1. A quiver Q is a quadruple

$$Q = \big(Q_0,\ Q_1,\ (s : Q_1 \to Q_0),\ (r : Q_1 \to Q_0)\big)$$

where Q_0 ($\neq \emptyset$) and Q_1 are sets, s and r are maps from Q_1 to Q_0. Elements of Q_0 and Q_1 are called vertices and arrows of Q respectively. For an arrow $\alpha \in Q_1$, when $s(\alpha) = a$ and $r(\alpha) = b$ denote by $a \overset{\alpha}{\to} b$ or $\alpha = (a \to b)$. The start and range of α are respectively elements $s(\alpha)$ and $r(\alpha)$ in Q_0.

DEFINITION 2.2. Let $Q = (Q_0, Q_1, s, r)$ be a quiver.

(1) Q is said to be finite if Q_0 and Q_1 are both finite sets.
(2) A path Δ in Q is either a sequence $(\alpha_1\alpha_2 \cdots \alpha_k)$ $(k \geq 1)$ of arrows $\alpha_i \in Q_1$ with $r(\alpha_i) = s(\alpha_{i+1})$ for all $i = 1, \ldots, k-1$, or the symbol e_a for $a \in Q_0$ which is called the trivial path. We usually identify a vertex a with e_a. Denote by $\mathsf{P}(Q)$ the totality of paths in Q.
(3) For a non-trivial path $\Delta = (\alpha_1\alpha_2 \cdots \alpha_k)$ in Q, we define $s(\Delta) := s(\alpha_1)$ and $r(\Delta) := r(\alpha_k)$ which are called the start and range of Δ. Furthermore, define $s(e_a) := a$ and $r(e_a) := a$ for $a \in Q_0$.
(4) The path algebra RQ of Q over R is the R-free module generated by all paths in Q, and a multiplication on RQ is defined by extending bilinearly the composition

$$\Delta_1\Delta_2 := \begin{cases} (\alpha_1 \cdots \alpha_k\beta_1 \cdots \beta_m) & \text{if } r(\alpha_k) = s(\beta_1) \\ 0 & \text{otherwise} \end{cases}$$

of paths $\Delta_1 = (\alpha_1 \cdots \alpha_k)$ and $\Delta_2 = (\beta_1 \cdots \beta_m)$. Then RQ is an associative R-algebra.

Here we establish the notation which will be used in this paper. Let $Q = (Q_0, Q_1, s, r)$ be a quiver. For a non-trivial path $\Delta = (\alpha_1\alpha_2 \cdots \alpha_k) \in \mathsf{P}(Q)$ $(\alpha_i \in Q_1)$, denote by $\ell(\Delta)$ the length k of Δ. The notation $\mathsf{P}(Q)_i$ $(i \geq 1)$ stands for the totality of paths of length i, namely $\mathsf{P}(Q)_i := \{\Delta \in \mathsf{P}(Q) \mid \ell(\Delta) = i\}$ $(i \geq 1)$. Set $\mathsf{P}(Q)_0 := \{e_a \mid a \in Q_0\}$ the totality of trivial paths in Q. As mentioned in Definition 2.2, we identify a vertex a with e_a. For $a, b \in Q_0$, denote by $\mathsf{P}(Q)_{a \Rightarrow b}$ the totality of paths Δ with $s(\Delta) = a$ and $r(\Delta) = b$.

Let $w : Q_1 \longrightarrow R$ be a map, and we call it a weight function of Q. Then w can be extended on non-trivial paths by setting $w(\Delta) := \prod_{i=1}^{k} w(\alpha_i)$ for $\Delta = (\alpha_1 \cdots \alpha_k) \in \mathsf{P}(Q)$. In particular, for $\Delta_1, \Delta_2 \in \mathsf{P}(Q)$ with $\Delta_1\Delta_2 \neq 0$, we have that $w(\Delta_1\Delta_2) = w(\Delta_1)w(\Delta_2)$. It is a convention that $w(a) := 1$ for $a \in Q_0$.

Let $\mathbf{Z}$ be the ring of rational integers, and let $\mathbf{Z}Q := \bigoplus_{\Delta \in \mathsf{P}(Q)} \Delta\mathbf{Z}$ be the path algebra of Q over $\mathbf{Z}$. If Q is finite then $\mathbf{Z}Q$ possesses the identity element $\sum_{a \in Q_0} a$. Note that $\mathbf{Z}Q$ contains a $\mathbf{Z}$-subalgebra $\mathbf{Z}Q_0 := \bigoplus_{a \in Q_0} a\mathbf{Z}$ generated by all trivial paths in Q. Put $RQ_0 := R \otimes_{\mathbf{Z}} \mathbf{Z}Q_0$. In this paper, we will investigate certain $\mathbf{Z}Q$-modules RQ_0, and also an R-subalgebra of $\mathrm{End}(RQ_0)$ (see Section 3).

3. Representations of $\mathbf{Z}Q$

Keep the notation in Section 2. In this section, we define a right, and also a left $\mathbf{Z}Q$-modules RQ_0. Then using those two $\mathbf{Z}Q$-actions, we introduce an R-subalgebra $\mathsf{UD}(Q, w; R)$ of $\mathrm{End}(RQ_0)$ which is one of the main objects in this paper. Furthermore we investigate the structure of $\mathsf{UD}(Q, w; R)$, and in particular $\mathsf{UD}(Q, w; \mathbf{Z})$ over $\mathbf{Z}$ is completely determined by some integers which we call the generating constants. These constants play an important role in applications to subgroup lattices and group characters in Sections 4 and 5 respectively. Throughout this section, let $Q = (Q_0, Q_1, s, r)$ be a quiver, and w a weight function of Q.

3.1. The UD-algebra. First of all, we give the following two actions of a path $\Delta \in \mathsf{P}(Q)$ on RQ_0.

DEFINITION 3.1. For $\Delta \in \mathsf{P}(Q)$ and $a \in Q_0$, define elements Δa and $a\Delta$ in RQ_0 as follows:

$$a\Delta := w(\Delta)\big(\delta_{a,s(\Delta)}r(\Delta)\big) \in RQ_0;$$

i.e. $a\big(s(\Delta) \to \cdots \to r(\Delta)\big) := w(\Delta)r(\Delta)$ if $s(\Delta) = a$, and 0 otherwise.

$$\Delta a := w(\Delta)\big(\delta_{r(\Delta),a}s(\Delta)\big) \in RQ_0;$$

i.e. $\big(s(\Delta) \to \cdots \to r(\Delta)\big)a := w(\Delta)s(\Delta)$ if $r(\Delta) = a$, and 0 otherwise.

Then, by extending bilinearly, we have the following two maps Φ_w and Ψ_w:

$$\Phi_w : RQ_0 \times \mathbf{Z}Q \longrightarrow RQ_0 \text{ by } \Big(\sum_{a \in Q_0} d_a a, \sum_{\Delta \in \mathsf{P}(Q)} c_\Delta \Delta \Big) \mapsto \sum_{\Delta \in \mathsf{P}(Q)} \sum_{a \in Q_0} (c_\Delta d_a) a\Delta$$

$$\Psi_w : \mathbf{Z}Q \times RQ_0 \longrightarrow RQ_0 \text{ by } \Big(\sum_{\Delta \in \mathsf{P}(Q)} c_\Delta \Delta, \sum_{a \in Q_0} d_a a \Big) \mapsto \sum_{\Delta \in \mathsf{P}(Q)} \sum_{a \in Q_0} (c_\Delta d_a) \Delta a$$

PROPOSITION 3.2. *Φ_w and Ψ_w induce RQ_0 to the structures of right and left $\mathbf{Z}Q$-modules respectively.*

PROOF. For $a \in Q_0$ and $\Delta_1, \Delta_2 \in \mathsf{P}(Q)$, suppose that $r(\Delta_1) = s(\Delta_2)$. Then we have that

$$\begin{aligned}(a\Delta_1)\Delta_2 &= \big(w(\Delta_1)\delta_{a,s(\Delta_1)} r(\Delta_1)\big)\Delta_2 \\ &= w(\Delta_1)\delta_{a,s(\Delta_1)}\big(w(\Delta_2)\delta_{r(\Delta_1),s(\Delta_2)} r(\Delta_2)\big) \\ &= w(\Delta_1\Delta_2)\delta_{a,s(\Delta_1)} r(\Delta_2) = a(\Delta_1\Delta_2).\end{aligned}$$

If $r(\Delta_1) \neq s(\Delta_2)$, that is $\Delta_1\Delta_2 = 0$, then it is clear that $(a\Delta_1)\Delta_2 = 0 = a(\Delta_1\Delta_2)$. The other conditions of right $\mathbf{Z}Q$-module are straightforward. By the same way, Ψ_w makes RQ_0 a left $\mathbf{Z}Q$-module. $\square$

Using $\mathbf{Z}Q$-modules RQ_0 described in Proposition 3.2, we introduce an R-subalgebra $\mathsf{UD}(Q, w; R)$ of $\mathrm{End}(RQ_0)$.

DEFINITION 3.3. Let $Q = (Q_0, Q_1, s, r)$ be a quiver, and let $w : Q_1 \longrightarrow R$ be a weight function of Q.

(1) For each $\Delta \in \mathsf{P}(Q)$, define two endomorphisms of an R-module RQ_0 as follows:

$$\rho_w(\Delta) : RQ_0 \longrightarrow RQ_0 \quad \text{by} \quad s \mapsto \Phi_w(s, \Delta)$$

$$\lambda_w(\Delta) : RQ_0 \longrightarrow RQ_0 \quad \text{by} \quad s \mapsto \Psi_w(\Delta, s)$$

Note that, for $f, g \in \mathrm{End}(RQ_0)$, the composition map $f \circ g$ is read from left to right. So we use the notation that $a^{f \circ g} = (a^f)^g$ for $a \in RQ_0$.

(2) Let $\mathsf{UD}(Q, w; R)$ be an R-subalgebra of $\mathrm{End}(RQ_0)$ generated by $\rho_w(\Delta)$ and $\lambda_w(\Delta)$ for all paths $\Delta \in \mathsf{P}(Q)$, namely

$$\mathsf{UD}(Q, w; R) := \big\langle \rho_w(\Delta),\ \lambda_w(\Delta) \mid \Delta \in \mathsf{P}(Q) \big\rangle \leq \mathrm{End}(RQ_0)\,.$$

We call $\mathsf{UD}(Q, w; R)$ the UD-algebra (Up-Down algebra) of Q with respect to w over R.

REMARK 3.4. Let $\Delta = (a \to \cdots \to b) \in \mathsf{P}(Q)$ be a path. Then we have that

$$a^{\rho_w(\Delta)} = a\Delta = w(\Delta)b\,, \quad b^{\lambda_w(\Delta)} = \Delta b = w(\Delta)a$$

by the definition. This can be thought that $\rho_w(\varDelta)$ moves a "down" to b along $\varDelta$, and $\lambda_w(\varDelta)$ moves b "up" to a along $\varDelta$. So we regard $\mathsf{UD}(Q, w; R)$ as an R-algebra generated by "up-down operators", and the information how vertices of Q are joined by its arrows with weights should be reflected well in $\mathsf{UD}(Q, w; R)$.

The next Lemma is immediate from Proposition 3.2.

LEMMA 3.5. *For $\varDelta_1, \varDelta_2 \in \mathsf{P}(Q)$, we have that*

$$\rho_w(\varDelta_1\varDelta_2) = \rho_w(\varDelta_1) \circ \rho_w(\varDelta_2) \quad \text{and} \quad \lambda_w(\varDelta_2\varDelta_1) = \lambda_w(\varDelta_1) \circ \lambda_w(\varDelta_2)\,.$$

REMARK 3.6 (Transposed relation). Fix a sequence $B := (b_1, \dots, b_m)$ of the elements in Q_0. For $\varDelta = (b_{j_1} \to \cdots \to b_{j_k}) \in \mathsf{P}(Q)$, let $M_{\rho_w(\varDelta)}$ and $M_{\lambda_w(\varDelta)}$ be respectively representation R-matrices of $\rho_w(\varDelta)$ and $\lambda_w(\varDelta)$ with respect to B. Then we have that ${}^tM_{\rho_w(\varDelta)} = M_{\lambda_w(\varDelta)}$. Indeed since

$$b_i^{\rho_w(\varDelta)} = b_i\varDelta = w(\varDelta)\big(\delta_{b_i, s(\varDelta)} r(\varDelta)\big) = w(\varDelta)\big(\delta_{b_i, b_{j_1}} b_{j_k}\big)\,,$$

we have an (i, j)-entry $(M_{\rho_w(\varDelta)})_{i,j} = w(\varDelta)$ if $(i, j) = (j_1, j_k)$, and 0 otherwise. In other words, $M_{\rho_w(\varDelta)}$ is a matrix with the unique non-zero entry $w(\varDelta)$ in the position of $(s(\varDelta), r(\varDelta))$. Similarly since $(M_{\lambda_w(\varDelta)})_{i,j} = w(\varDelta)$ if $(i, j) = (j_k, j_1)$, and 0 otherwise, we get ${}^tM_{\rho_w(\varDelta)} = M_{\lambda_w(\varDelta)}$.

Using Lemma 3.5 and the description of $M_{\rho_w(\varDelta)}$ in Remark 3.6, we have the following.

LEMMA 3.7. *For $a, b \in Q_0$, let $e_{a,b} \in \mathrm{End}(RQ_0)$ be an endomorphism defined by $x^{e_{a,b}} := \delta_{x,a} b$ for $x \in Q_0$. Then an R-subalgebra $\langle\, \rho_w(\varDelta) \mid \varDelta \in \mathsf{P}(Q)\,\rangle$ of $\mathsf{UD}(Q, w; R)$ is described as follows*:

$$\big\langle \rho_w(\varDelta) \mid \varDelta \in \mathsf{P}(Q) \big\rangle = \sum_{a,b \in Q_0} \Bigg(\sum_{\varDelta \in \mathsf{P}(Q)_{a \Rightarrow b}} w(\varDelta) \cdot R \Bigg) e_{a,b} \quad (\textit{finite sum})$$

This is identified with a matrix algebra of the form

$$\Bigg(\sum_{\varDelta \in \mathsf{P}(Q)_{a \Rightarrow b}} w(\varDelta) \cdot R \Bigg)_{a,b \in Q_0}$$

EXAMPLE 3.8. Let Q be a quiver with $Q_0 = \{a, b\}$, $\mathsf{P}(Q) = \big\{a, b, (a \to b)\big\}$, and $k := w(a \to b) \in R$. Recall that $RQ_0 = aR \oplus bR$ is a $\mathbf{Z}Q$-module. Then endomorphisms $\rho_w(\varDelta) \in \mathrm{End}(RQ_0)$ for $\varDelta \in \mathsf{P}(Q)$ are represented as follows:

$$M_{\rho_w(a)} = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array}, \quad M_{\rho_w(b)} = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array}, \quad M_{\rho_w(a \to b)} = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \end{array}.$$

Then, by Remark 3.6, $M_{\lambda_w(\Delta)} = {}^tM_{\rho_w(\Delta)}$ for $\Delta \in \mathsf{P}(Q)$. As R-algebras, we have that

$$\langle\, \rho_w(\Delta) \mid \Delta \in \mathsf{P}(Q) \,\rangle \cong \begin{pmatrix} R & kR \\ 0 & R \end{pmatrix}.$$

3.2. Opposite paths and $\mathsf{UD}(Q, w; R)$. In order to examine the structure of $\mathsf{UD}(Q, w; R)$, we introduce the opposite path ${}^t\Delta$ of $\Delta \in \mathsf{P}(Q)$. But before doing this, we extend a quiver Q by adding arrows further.

DEFINITION 3.9. Let $Q = (Q_0, Q_1, s, r)$ be a quiver.

(1) For each arrow $\alpha = (a \to b) \in Q_1$, we define the symbol ${}^t\alpha$. Set $Q_1^{\mathrm{opp}} := \{{}^t\alpha \mid \alpha \in Q_1\}$ and $Q_1^{\mathrm{ud}} := Q_1 \cup Q_1^{\mathrm{opp}}$. Then

$$Q^{\mathrm{ud}} := \Big(Q_0,\ Q_1^{\mathrm{ud}},\ \big(s : Q_1^{\mathrm{ud}} \to Q_0\big),\ \big(r : Q_1^{\mathrm{ud}} \to Q_0\big)\Big)$$

forms a quiver where s and r are extended on Q_1^{ud} as $s({}^t\alpha) := r(\alpha) = b$ and $r({}^t\alpha) := s(\alpha) = a$ for $\alpha = (a \to b) \in Q_1$. Thus ${}^t\alpha = (b \to a)$. We call ${}^t\alpha$ the opposite arrow of α. Note that $\mathsf{P}(Q) \subseteq \mathsf{P}(Q^{\mathrm{ud}})$.

(2) For a path $\Delta = (a_0 \overset{\alpha_1}{\to} a_1 \overset{\alpha_2}{\to} \cdots \overset{\alpha_{k-1}}{\to} a_{k-1} \overset{\alpha_k}{\to} a_k) \in \mathsf{P}(Q) \subseteq \mathsf{P}(Q^{\mathrm{ud}})$ ($\alpha_i \in Q_1$), define

$${}^t\Delta := \big(a_k \overset{{}^t\alpha_k}{\longrightarrow} a_{k-1} \overset{{}^t\alpha_{k-1}}{\longrightarrow} \cdots \overset{{}^t\alpha_2}{\longrightarrow} a_1 \overset{{}^t\alpha_1}{\longrightarrow} a_0\big) \in \mathsf{P}(Q^{\mathrm{ud}})$$

which is called the opposite path of Δ. Then we have that $\ell({}^t\Delta) = \ell(\Delta) = k$.

REMARK 3.10. Let Σ be a finite connected graph with no loops. Then associating to each arrow an opposite as in Definition 3.9 is standard in the construction of the preprojective algebra $\Pi(\Sigma)$ of Σ (cf. [4, page 553]). Indeed, associate with Σ the quiver $\overline{\Sigma}$ having the same vertices as Σ and where each edge $\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$ in Σ is replaced by a pair of arrows $\bullet \rightleftarrows \bullet$. Thus for each arrow α in $\overline{\Sigma}$, there is an opposite of α. The preprojective algebra $\Pi(\Sigma)$ is a certain quotient algebra of $R\overline{\Sigma}$.

Let $\mathsf{P}(Q)^{\mathrm{opp}} := \{{}^t\Delta \mid \Delta \in \mathsf{P}(Q)\}$. Then any path $\Gamma \in \mathsf{P}(Q^{\mathrm{ud}})$ can be expressed as $\Gamma = \Gamma_1\Gamma_2\cdots\Gamma_m$ for some $\Gamma_i \in \mathsf{P}(Q) \cup \mathsf{P}(Q)^{\mathrm{opp}}$ and $m \geq 1$. Here we may assume that, for each $i = 1, \ldots, m-1$, if $\Gamma_i \in \mathsf{P}(Q)$ then $\Gamma_{i+1} \in \mathsf{P}(Q)^{\mathrm{opp}}$, or if $\Gamma_i \in \mathsf{P}(Q)^{\mathrm{opp}}$ then $\Gamma_{i+1} \in \mathsf{P}(Q)$. A weight function $w : Q_1 \longrightarrow R$ of Q can be extended on Q_1^{opp} by setting $w({}^t\alpha) := w(\alpha)$ for $\alpha \in Q_1$. For $\Gamma = \Gamma_1\cdots\Gamma_m \in \mathsf{P}(Q^{\mathrm{ud}})$, define

$$s(\Gamma) := s(\Gamma_1),\ r(\Gamma) := r(\Gamma_m),\ w(\Gamma) := \prod_{i=1}^{m} w(\Gamma_i),\ \ell(\Gamma) := \sum_{i=1}^{m} \ell(\Gamma_i)\,.$$

Now consider the path algebra RQ^{ud} of Q^{ud}, which contains an R-subalgebra $RQ_0 \subseteq RQ^{\mathrm{ud}}$. As in Section 3.1, for $\Gamma \in \mathsf{P}(Q^{\mathrm{ud}})$ and $a \in Q_0$, define

$$a\Gamma := w(\Gamma)\big(\delta_{a,s(\Gamma)} r(\Gamma)\big) \in RQ_0 \,.$$

Then we obtain an endomorphism $\rho_w(\Gamma) : RQ_0 \longrightarrow RQ_0$ defined by $a \mapsto a\Gamma$, which has the property $\rho_w(\Gamma\Gamma') = \rho_w(\Gamma) \circ \rho_w(\Gamma')$ for $\Gamma, \Gamma' \in \mathsf{P}(Q^{\mathrm{ud}})$. Note that $\rho_w({}^t\Delta)$ coincides with $\lambda_w(\Delta)$ for $\Delta \in \mathsf{P}(Q)$ (see Remark 3.6). Therefore $\mathsf{UD}(Q, w; R)$ is rewritten as follows:

$$\mathsf{UD}(Q, w; R) = \big\langle \rho_w(\Gamma) \mid \Gamma \in \mathsf{P}(Q^{\mathrm{ud}}) \big\rangle .$$

NOTATION 3.11 (Up-Down paths). For arrows $\alpha, \beta \in Q_1$ such that $r(\alpha) = r(\beta)$, we just write $\Delta = (\alpha\beta)$ for a path $\Delta = \big(\alpha\ ({}^t\beta)\big)$ in Q^{ud} where $s({}^t\beta) = r(\beta) = r(\alpha)$. Similarly, for arrows $\alpha, \beta \in Q_1$ such that $s(\alpha) = s(\beta)$, the notation $\Delta = (\alpha\beta)$ indicates a path $\Delta = \big(({}^t\alpha)\ \beta\big)$ in Q^{ud} where $r({}^t\alpha) = s(\alpha) = s(\beta)$. For example, for arrows $\alpha_1 = (a \to b)$, $\alpha_2 = (c \to b)$, $\alpha_3 = (d \to c)$, $\alpha_4 = (d \to e)$ in Q_1, the notation

$$\Delta = \big(a \overset{\alpha_1}{\to} b \overset{\alpha_2}{\leftarrow} c \overset{\alpha_3}{\leftarrow} d \overset{\alpha_4}{\to} e\big)$$

implies a path in Q^{ud} as follows:

$$\Delta = \big(a \overset{\alpha_1}{\to} b \overset{{}^t\alpha_2}{\to} c \overset{{}^t\alpha_3}{\to} d \overset{\alpha_4}{\to} e\big) .$$

So any path $\Delta \in \mathsf{P}(Q^{\mathrm{ud}})$ can be expressed as $\Delta = \big(a_0 \overset{\alpha_1}{-} a_1 \overset{\alpha_2}{-} a_2 - \cdots - a_{k-1} \overset{\alpha_k}{-} a_k\big)$ for some $\alpha_i \in Q_1$ $(i = 1, \ldots, k)$ where $-$ means $\to$ or $\leftarrow$. Throughout this paper, we frequently use this way of writing for paths without using opposite arrows.

3.3. The generating constants. We see that the algebra $\mathsf{UD}(Q, w; R)$ is realized as an R-matrix algebra in the next. From this result, the generating constants can be introduced.

PROPOSITION 3.12. *For $a, b \in Q_0$, let $e_{a,b} \in \mathrm{End}(RQ_0)$ be an endomorphism defined by $x^{e_{a,b}} := \delta_{x,a} b$ for $x \in Q_0$. Then the UD-algebra $\mathsf{UD}(Q, w; R)$ is described as follows*:

$$\mathsf{UD}(Q, w; R) = \sum_{a,b \in Q_0} \Bigg(\sum_{\Gamma \in \mathsf{P}(Q^{\mathrm{ud}})_{a \Rightarrow b}} w(\Gamma) \cdot R \Bigg) e_{a,b} \quad (\textit{finite sum})$$

This is identified with a matrix algebra of the form

$$\Bigg(\sum_{\Gamma \in \mathsf{P}(Q^{\mathrm{ud}})_{a \Rightarrow b}} w(\Gamma) \cdot R \Bigg)_{a,b \in Q_0}$$

PROOF. As in Remark 3.6, $\rho_w(\Gamma)$ for $\Gamma \in \mathsf{P}(Q^{\mathrm{ud}})_{a \Rightarrow b}$ has a representation matrix with the unique non-zero entry $w(\Gamma)$ in the position of $(s(\Gamma), r(\Gamma))$. Furthermore we had $\rho_w(\Gamma\Gamma') = \rho_w(\Gamma) \circ \rho_w(\Gamma')$ for $\Gamma, \Gamma' \in \mathsf{P}(Q^{\mathrm{ud}})$. Thus the assertion clearly holds. $\square$

COROLLARY 3.13. *Suppose that Q is finite. Suppose further that $\mathsf{P}(Q^{\mathrm{ud}})_{a\Rightarrow b} \neq \emptyset$ for any $a, b \in Q_0$, and that $w = 1$ namely $w(\alpha) = 1$ for any $\alpha \in Q_1$. Then $\mathsf{UD}(Q, w; R)$ is isomorphic to a matrix algebra $M_{|Q_0|}(R)$.*

DEFINITION 3.14 (Generating constants). Suppose that a coefficient ring is the ring $\mathbf{Z}$ of rational integers. For $a, b \in Q_0$, suppose that

$$\mathsf{P}(Q^{\mathrm{ud}})_{a\Rightarrow b} \neq \emptyset \quad \text{and} \quad \mathcal{S} := \sum_{\Gamma \in \mathsf{P}(Q^{\mathrm{ud}})_{a\Rightarrow b}} w(\Gamma) \cdot \mathbf{Z} \neq \{0\}\,.$$

Then there exists a positive integer $\mathfrak{s}_{a,b}$ such that $\mathfrak{s}_{a,b} \cdot \mathbf{Z} = \mathcal{S}$. In other words, $\mathfrak{s}_{a,b}$ is the greatest common divisor of $\{w(\Gamma) \mid \Gamma \in \mathsf{P}(Q^{\mathrm{ud}})_{a\Rightarrow b}\}$. On the other hand, if $\mathsf{P}(Q^{\mathrm{ud}})_{a\Rightarrow b} = \emptyset$ or $\mathcal{S} = \{0\}$ then we define $\mathfrak{s}_{a,b} := 0$. Then by Proposition 3.12, $\mathsf{UD}(Q, w; \mathbf{Z})$ is completely determined as

$$\mathsf{UD}(Q, w; \mathbf{Z}) = \sum_{a,b \in Q_0} (\mathfrak{s}_{a,b} \cdot \mathbf{Z}) e_{a,b} \quad \text{(finite sum)}$$

We call the integers $\mathfrak{s}_{a,b}$ ($a, b \in Q_0$) the generating constants of $\mathsf{UD}(Q, w; \mathbf{Z})$. Note that $\mathfrak{s}_{a,b} = \mathfrak{s}_{b,a}$ for any $a, b \in Q_0$. Furthermore denote by just $\mathsf{UD}(Q, w)$ the UD-algebra $\mathsf{UD}(Q, w; \mathbf{Z})$ over $\mathbf{Z}$ for short.

4. Subgroup lattices

In this section, we apply the results in Section 3 on path algebras to subgroup lattices. Suppose that a coefficient ring is the ring $\mathbf{Z}$ of rational integers. Let G be a finite group, and let $\mathrm{Sgp}(G)$ be the totality of subgroups of G including the whole group G and the trivial subgroup $\{e\}$. We first define a quiver associated to a partially ordered set (poset for short). Then the subgroup lattice $(\mathrm{Sgp}(G), \leq)$ with an ordering $\leq$ defined by the inclusion-relation gives us a quiver Q_G, and we are able to deal with the UD-algebra $\mathsf{UD}(Q_G, w_G)$ where a weight function w_G is defined by indices of subgroups. After studying some properties of the generating constants of $\mathsf{UD}(Q_G, w_G)$, we characterize those constants which are equal to the order of G. In this case, a factorization of G is related. For a subgroup $H \leq G$ and $g \in G$, we write $H^g := g^{-1}Hg$.

4.1. Quivers arising from posets. In order to apply the results in Section 3 to the lattice $(\mathrm{Sgp}(G), \leq)$, we prepare a quiver defined by a poset in general.

DEFINITION 4.1. Let $(\mathfrak{X}, \leq)$ be a poset. For elements $a, b \in \mathfrak{X}$, we define an arrow $(a \to b)$ precisely when $a > b$. Put $(Q_{\mathfrak{X}})_0 := \mathfrak{X}$ and $(Q_{\mathfrak{X}})_1 := \{(a \to b) \mid a, b \in \mathfrak{X},\ a > b\}$. Then denote by $Q_{\mathfrak{X}}$ or $Q_{(\mathfrak{X},\leq)}$ a quiver $\big((Q_{\mathfrak{X}})_0, (Q_{\mathfrak{X}})_1, s, r\big)$ where maps $s, r : (Q_{\mathfrak{X}})_1 \longrightarrow (Q_{\mathfrak{X}})_0$ are defined by $s(\alpha) := a$ and $r(\alpha) := b$ for $\alpha = (a \to b) \in (Q_{\mathfrak{X}})_1$.

DEFINITION 4.2. $\mathrm{Sgp}(G)$ can be viewed as a poset together with the inclusion-relation $\leq$. Denote by

$$Q_G := Q_{(\mathrm{Sgp}(G), \leq)}$$

a quiver associated to a poset $(\mathrm{Sgp}(G), \leq)$ (see Definition 4.1). In this case, a weight function w_G of Q_G is defined by indices of subgroups, that is, $w_G(H \to K) := |H : K| \in \mathbf{Z}$. The notation $\mathsf{UD}(G)$ or $\mathsf{UD}(G, w_G)$ stands for the UD-algebra $\mathsf{UD}(Q_G, w_G)$ over $\mathbf{Z}$. Also the notation Q_G^{ud} implies $(Q_G)^{\mathrm{ud}}$.

Before going to an application to $(\mathrm{Sgp}(G), \leq)$, we consider the structure of the UD-algebra obtained from the direct product of posets.

DEFINITION 4.3. Let $(\mathfrak{X}, \leq_{\mathfrak{X}})$ and $(\mathfrak{Y}, \leq_{\mathfrak{Y}})$ be posets. For distinct (x_1, y_1), $(x_2, y_2) \in \mathfrak{X} \times \mathfrak{Y}$, define an ordering $(x_1, y_1) \geq (x_2, y_2)$ precisely when $x_1 \geq_{\mathfrak{X}} x_2$ and $y_1 \geq_{\mathfrak{Y}} y_2$. Then $(\mathfrak{X} \times \mathfrak{Y}, \leq)$ becomes a poset. In this case, if $w_{\mathfrak{X}}$ and $w_{\mathfrak{Y}}$ are weight functions of associated quivers $Q_{\mathfrak{X}}$ and $Q_{\mathfrak{Y}}$ respectively, then we adopt a weight function w of $Q_{\mathfrak{X}\times\mathfrak{Y}}$ defined by $w\big((x_1, y_1) \to (x_2, y_2)\big) := w_{\mathfrak{X}}(x_1 \to x_2) \times w_{\mathfrak{Y}}(y_1 \to y_2)$. Note that if $x_1 = x_2$ then we set $w_{\mathfrak{X}}(x_1 \to x_2) := 1$, and similarly for $w_{\mathfrak{Y}}$.

PROPOSITION 4.4. *Under the above notation, we have a* $\mathbf{Z}$*-algebra isomorphism as follows*:

$$\mathsf{UD}(Q_{\mathfrak{X}\times\mathfrak{Y}}, w) \cong \mathsf{UD}(Q_{\mathfrak{X}}, w_{\mathfrak{X}}) \otimes_{\mathbf{Z}} \mathsf{UD}(Q_{\mathfrak{Y}}, w_{\mathfrak{Y}}).$$

PROOF. We consider the generating constants of $\mathsf{UD}(Q_{\mathfrak{X}\times\mathfrak{Y}}, w)$. For elements (x_1, y_1), $(x_2, y_2) \in \mathfrak{X} \times \mathfrak{Y}$, set $\mathsf{P} := \mathsf{P}\big(Q^{\mathrm{ud}}_{\mathfrak{X}\times\mathfrak{Y}}\big)_{(x_1,y_1)\Rightarrow(x_2,y_2)}$. Note that we have poset isomorphisms $\mathfrak{X} \cong \mathfrak{X} \times \{y_1\} \subseteq \mathfrak{X} \times \mathfrak{Y}$ and $\mathfrak{Y} \cong \{x_2\} \times \mathfrak{Y} \subseteq \mathfrak{X} \times \mathfrak{Y}$, and then we can identify $Q_{\mathfrak{X}}$ and $Q_{\mathfrak{X}\times\{y_1\}}$, and also $Q_{\mathfrak{Y}}$ and $Q_{\{x_2\}\times\mathfrak{Y}}$. Now for any path $\Gamma \in \mathsf{P}$, there exist projections Γ_1 and Γ_2 of Γ onto $\mathsf{P}(Q_{\mathfrak{X}})$ and $\mathsf{P}(Q_{\mathfrak{Y}})$ respectively such that $w(\Gamma) = w_{\mathfrak{X}}(\Gamma_1) \times w_{\mathfrak{Y}}(\Gamma_2)$. Indeed we can take certain paths

$$\Gamma_1 \in \mathsf{P}\big(Q^{\mathrm{ud}}_{\mathfrak{X}\times\{y_1\}}\big)_{(x_1,y_1)\Rightarrow(x_2,y_1)} = \mathsf{P}\big(Q^{\mathrm{ud}}_{\mathfrak{X}}\big)_{x_1\Rightarrow x_2},$$
$$\Gamma_2 \in \mathsf{P}\big(Q^{\mathrm{ud}}_{\{x_2\}\times\mathfrak{Y}}\big)_{(x_2,y_1)\Rightarrow(x_2,y_2)} = \mathsf{P}\big(Q^{\mathrm{ud}}_{\mathfrak{Y}}\big)_{y_1\Rightarrow y_2}.$$

Then by the definition of the generating constants, we have that

$$\begin{aligned} \mathfrak{s}_{(x_1,y_1),(x_2,y_2)} \cdot \mathbf{Z} &= \sum_{\Gamma\in\mathsf{P}} w(\Gamma) \cdot \mathbf{Z} \\ &= \Bigg(\sum_{\Gamma_1\in\mathsf{P}(Q^{\mathrm{ud}}_{\mathfrak{X}})_{x_1\Rightarrow x_2}} w(\Gamma_1) \cdot \mathbf{Z}\Bigg)\Bigg(\sum_{\Gamma_2\in\mathsf{P}(Q^{\mathrm{ud}}_{\mathfrak{Y}})_{y_1\Rightarrow y_2}} w(\Gamma_2) \cdot \mathbf{Z}\Bigg) \\ &= \big(\mathfrak{s}_{x_1,x_2} \cdot \mathbf{Z}\big)\big(\mathfrak{s}_{y_1,y_2} \cdot \mathbf{Z}\big) = \mathfrak{s}_{x_1,x_2}\mathfrak{s}_{y_1,y_2} \cdot \mathbf{Z}. \end{aligned}$$

Then the tensor product $\left(\mathfrak{s}_{x_1,x_2}\right)_{x_1,x_2\in\mathfrak{X}} \otimes \left(\mathfrak{s}_{y_1,y_2}\right)_{y_1,y_2\in\mathfrak{Y}}$ of two matrices of the generating constants of $\mathsf{UD}(Q_{\mathfrak{X}}, w_{\mathfrak{X}})$ and $\mathsf{UD}(Q_{\mathfrak{Y}}, w_{\mathfrak{Y}})$ gives a matrix of those of $\mathsf{UD}(Q_{\mathfrak{X}\times\mathfrak{Y}}, w)$. This leads us to the required isomorphism. $\square$

4.2. Some properties of the generating constants. In this section, we investigate some fundamental properties of the generating constants of $\mathsf{UD}(G) = \mathsf{UD}(Q_G, w_G)$ which will be used later in Section 4.3.

PROPOSITION 4.5. *For $A, B \in \mathrm{Sgp}(G)$, let $\mathfrak{s}_{A,B}$ be the generating constant of* $\mathsf{UD}(G)$.

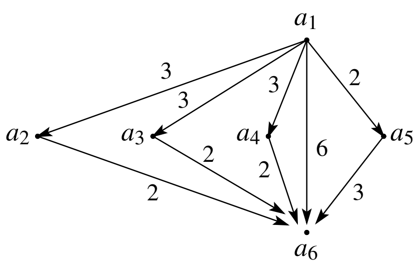
(1) $\mathfrak{s}_{A,B}$ *divides* $|A : A \cap B| \times |B : A \cap B|$.

(2) $\mathfrak{s}_{A,B}$ *divides the order of* G.

PROOF. (1) For a path $\Delta = (A \to A \cap B \leftarrow B)$ in Q_G^{ud}, $w_G(\Delta)$ is divisible by $\mathfrak{s}_{A,B}$ from Definition 3.14 where $w_G(\Delta) = |A : A \cap B| \times |B : A \cap B|$.

(2) Take paths $\Delta = (A \leftarrow G \to B)$ and $\Delta' = (A \to \{e\} \leftarrow B)$ in Q_G^{ud}. Then $w_G(\Delta) = |G : A||G : B|$ and $w_G(\Delta') = |A||B|$. By the definition of $\mathfrak{s}_{A,B}$, the greatest common divisor $(w_G(\Delta), w_G(\Delta'))$ is divisible by $\mathfrak{s}_{A,B}$. Now since $(|G|_p)^2 = |A|_p|B|_p|G : A|_p|G : B|_p$ for a prime number p, we have that $c := |A|_p|B|_p$ or $d := |G : A|_p|G : B|_p$ divides $|G|_p$. Note that, for a positive integer n, n_p is the highest power of p that divides n. It follows that $(w_G(\Delta), w_G(\Delta'))_p = \min\{c, d\}$ divides $|G|_p$. This completes the proof. $\square$

EXAMPLE 4.6 (The generating constants of $\mathsf{UD}(S_3)$). Let S_3 be the symmetric group on a set $\{1, 2, 3\}$. There are six subgroups of S_3, and we name $a_1 = S_3$, $a_2 = \langle(1, 2)\rangle$, $a_3 = \langle(1, 3)\rangle$, $a_4 = \langle(2, 3)\rangle$, $a_5 = \langle(1, 2, 3)\rangle$, and $a_6 = \{e\}$. Then a quiver Q_{S_3} with weights is drawn as follows:



Furthermore the generating constants of $\mathsf{UD}(S_3)$ can be calculated as follows:

$$\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{array}\begin{array}{c} \begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{array} \\ \begin{pmatrix} 1 & 3 & 3 & 3 & 2 & 6 \\ 3 & 1 & 1 & 1 & 6 & 2 \\ 3 & 1 & 1 & 1 & 6 & 2 \\ 3 & 1 & 1 & 1 & 6 & 2 \\ 2 & 6 & 6 & 6 & 1 & 3 \\ 6 & 2 & 2 & 2 & 3 & 1 \end{pmatrix} \end{array}$$

Note that (a_i, a_j)-entry is the generating constant $\mathfrak{s}_{a_i,a_j} \in \mathbf{Z}$.

LEMMA 4.7. *Let H be a subgroup of G. For $A, B \in \mathrm{Sgp}(H)$, let $\mathfrak{s}^G_{A,B}$ and $\mathfrak{s}^H_{A,B}$ be respectively the generating constants of $\mathsf{UD}(G)$ and $\mathsf{UD}(H)$. Then the following holds.*

(1) *$\mathfrak{s}^G_{A,B}$ divides $\mathfrak{s}^H_{A,B}$.*

(2) *If H is a normal subgroup of G, then $\mathfrak{s}^G_{A,B} = \mathfrak{s}^H_{A,B}$*

PROOF. (1) Straightforward from Definition 3.14.

(2) Let $\Delta = (A =: L_0 - L_1 - \cdots - L_k := B)$ be a path from A to B in Q^{ud}_G. Then we have a path $\Delta_{\cap H}$ from A to B in Q^{ud}_H of the form $\Delta_{\cap H} := \big((L_0 \cap H) - (L_1 \cap H) - \cdots - (L_k \cap H)\big)$ by reducing loops. Suppose that $L_i > L_{i+1}$ for some i. Then since $H \trianglelefteq G$ by our assumption, we get

$$|L_i \cap H : L_{i+1} \cap H| = \frac{|L_i : L_{i+1}|}{|L_i H : L_{i+1} H|}.$$

It follows that $w_H(\Delta_{\cap H})$ divides $w_G(\Delta)$, and thus $\mathfrak{s}^H_{A,B}$ divides $\mathfrak{s}^G_{A,B}$ as desired. □

LEMMA 4.8. *Let N be a normal subgroup of G, and set $\overline{G} := G/N$. For subgroups $N \le A, B \le G$, we have $\mathfrak{s}_{A,B} = \mathfrak{s}_{\overline{A},\overline{B}}$ where $\mathfrak{s}_{A,B}$ and $\mathfrak{s}_{\overline{A},\overline{B}}$ are the generating constants of $\mathsf{UD}(G)$ and $\mathsf{UD}(\overline{G})$ respectively.*

PROOF. Let $\Delta = (A =: L_0 - L_1 - \cdots - L_k := B)$ be a path from A to B in Q^{ud}_G. Then we obtain a path Δ_N from A to B in Q^{ud}_G of the form

$$\Delta_N := (L_0 N - L_1 N - \cdots - L_k N)$$

by reducing loops. Furthermore we obtain a path Δ_N/N from $\overline{A}$ to $\overline{B}$ in $Q^{\mathrm{ud}}_{\overline{G}}$ of the form

$$\Delta_N/N := (L_0 N/N - L_1 N/N - \cdots - L_k N/N)$$

by reducing loops. Since $w_{\overline{G}}(\Delta_N/N) = w_G(\Delta_N)$, and since $w_G(\Delta_N)$ divides $w_G(\Delta)$ by the same way as in the proof of Lemma 4.7, we see that $\mathfrak{s}_{\overline{A},\overline{B}}$ divides $\mathfrak{s}_{A,B}$.

On the other hand, since the set of paths in $Q_{\overline{G}}^{\mathrm{ud}}$ from $\overline{A}$ to $\overline{B}$ can be thought of a subset of paths in Q_G^{ud} from A to B whose vertices contain N, $\mathfrak{s}_{A,B}$ divides $\mathfrak{s}_{\overline{A},\overline{B}}$. The proof is complete. □

The following is a consequence of Proposition 4.4 on the direct product of posets.

PROPOSITION 4.9. *Let A and B be finite groups. Then* $\mathsf{UD}(A \times B, w_{A\times B}) \cong \mathsf{UD}(A, w_A) \otimes_{\mathbf{Z}} \mathsf{UD}(B, w_B)$ *if and only if* $(|A|, |B|) = 1$.

PROOF. Note that if $(|A|, |B|) = 1$ then we have a poset isomorphism $(\mathrm{Sgp}(A \times B), \leq_{A\times B}) \cong (\mathrm{Sgp}(A), \leq_A) \times (\mathrm{Sgp}(B), \leq_B)$, and that the converse is also true. Thus the assertion follows from Proposition 4.4. □

4.3. A characterization of the generating constants. In this section, we focus our attention on the generating constants $\mathfrak{s}_{A,B}$ of $\mathsf{UD}(G)$ which are equal to the order $|G|$ of G. Indeed we show that $\mathfrak{s}_{A,B} = |G|$ if and only if $G = AB$ and $A \cap B = \{e\}$.

LEMMA 4.10. *Let $\Delta = (L_0 - L_1 - \cdots - L_k)$ $(k \geq 2)$ be a path in Q_G^{ud} such that, for any $1 \leq i \leq k-1$, $L_{i-1} < L_i > L_{i+1}$ or $L_{i-1} > L_i < L_{i+1}$. Then a weight $w_G(\Delta)$ of Δ is as follows*:

(1) *If $L_0 > L_1 < L_2$ and $L_{k-2} > L_{k-1} < L_k$*

$$\begin{tikzcd}[column sep=small]
L_0 \arrow[dr] & & L_2 \arrow[dl] \arrow[dr] & & \cdots & & L_{k-2} \arrow[dl] \arrow[dr] & & L_k \arrow[dl] \\
& L_1 & & L_3 & & L_{k-3} & & L_{k-1} &
\end{tikzcd},$$

$$w_G(\Delta) = |L_0||L_k| \times \left(\frac{|L_2|\cdots|L_{k-2}|}{|L_1|\cdots|L_{k-3}||L_{k-1}|}\right)^2 = \frac{|L_0|}{|L_k|} \times \left(\frac{|L_2|\cdots|L_{k-2}||L_k|}{|L_1|\cdots|L_{k-3}||L_{k-1}|}\right)^2.$$

(2) *If $L_0 < L_1 > L_2$ and $L_{k-2} < L_{k-1} > L_k$*

$$\begin{tikzcd}[column sep=small]
& L_1 \arrow[dl] \arrow[dr] & & \cdots & & L_{k-1} \arrow[dl] \arrow[dr] & \\
L_0 & & L_2 & & L_{k-2} & & L_k
\end{tikzcd},$$

$$w_G(\Delta) = |L_0||L_k| \times \left(\frac{|L_1|\cdots|L_{k-1}|}{|L_0|\cdots|L_{k-2}||L_k|}\right)^2 = \frac{|L_0|}{|L_k|} \times \left(\frac{|L_1|\cdots|L_{k-1}|}{|L_0|\cdots|L_{k-2}|}\right)^2.$$

(3) *If $L_0 > L_1 < L_2$ and $L_{k-2} < L_{k-1} > L_k$*

$$\begin{tikzcd}[column sep=small]
L_0 \arrow[dr] & & L_2 \arrow[dl] \arrow[dr] & & \cdots & L_{k-1} \arrow[dl] \arrow[dr] & \\
& L_1 & & L_3 & L_{k-2} & & L_k
\end{tikzcd},$$

$$w_G(\Delta) = |L_0||L_k| \times \left(\frac{|L_2|\cdots|L_{k-1}|}{|L_1|\cdots|L_{k-2}||L_k|}\right)^2 = \frac{|L_0|}{|L_k|} \times \left(\frac{|L_2|\cdots|L_{k-1}|}{|L_1|\cdots|L_{k-2}|}\right)^2.$$

(4) *If* $L_0 < L_1 > L_2$ *and* $L_{k-2} > L_{k-1} < L_k$

$$\begin{tikzcd}
 & L_1 \arrow[dl] \arrow[dr] & & & & L_{k-2} \arrow[dl] \arrow[dr] & & L_k \arrow[dl] \\
L_0 & & L_2 & \cdots & L_{k-3} & & L_{k-1} &
\end{tikzcd},$$

$$w_G(\Delta) = |L_0||L_k| \times \left(\frac{|L_1|\cdots|L_{k-2}|}{|L_0|\cdots|L_{k-3}||L_{k-1}|}\right)^2 = \frac{|L_0|}{|L_k|} \times \left(\frac{|L_1|\cdots|L_{k-2}||L_k|}{|L_0|\cdots|L_{k-3}||L_{k-1}|}\right)^2.$$

PROOF. Straightforward. □

LEMMA 4.11. *Let* $\Delta = (L_0 - L_1 - \cdots - L_k)$ $(k \geq 2)$ *be a path in* Q_G^{ud} *such that* $G = L_0L_k$. *Suppose that, for any* $1 \leq i \leq k-1$, $L_{i-1} < L_i > L_{i+1}$ *or* $L_{i-1} > L_i < L_{i+1}$. *Let* $\mathcal{G} = G/L_0 \times \cdots \times G/L_k$ *be the direct product of families* G/L_i $(0 \leq i \leq k)$ *of the left cosets of* L_i *in* G. *Set*

$$\mathcal{F} = \left\{ (g_0L_0, \ldots, g_kL_k) \in \mathcal{G} \;\middle|\; \begin{array}{l} g_iL_i \subset g_{i+1}L_{i+1} \text{ or} \\ g_iL_i \supset g_{i+1}L_{i+1} \ (0 \leq \forall i \leq k-1) \end{array} \right\}$$

Then we have that $\sqrt{w_G(\Delta)/|G : L_0 \cap L_k|} = |\mathcal{F}|/|G : L_0 \cap L_k|$, *and this is an integer. In particular,* $w_G(\Delta)$ *is divisible by* $|G : L_0 \cap L_k|$.

PROOF. Given a coset $g_iL_i \in G/L_i$ $(0 \leq i \leq k-1)$. We will determine the next $g_{i+1}L_{i+1} \in G/L_{i+1}$ satisfying the condition of elements in $\mathcal{F}$. Suppose that $g_iL_i \subset g_{i+1}L_{i+1}$. Then since $g_i \in g_{i+1}L_{i+1}$, we have that $g_{i+1}L_{i+1}$ is uniquely determined as g_iL_{i+1}. Suppose next that $g_iL_i \supset g_{i+1}L_{i+1}$. Then since $g_i^{-1}g_{i+1}L_{i+1}$ lies in $L_i/L_{i+1} := \{y_1L_{i+1}, \ldots, y_mL_{i+1}\}$, $g_{i+1}L_{i+1}$ must be one of $m = |L_i : L_{i+1}|$ cosets $g_iy_jL_{i+1}$ $(1 \leq j \leq m)$. Using these facts, we find the value $|\mathcal{F}|$. On the other hand, $w_G(\Delta)$ is obtained in Lemma 4.10. Furthermore since $G = L_0L_k$ by our assumption, we have that

$$|G : L_0 \cap L_k| = \frac{|L_0||L_k|}{|L_0 \cap L_k|^2} \quad \text{and} \quad |G/L_0| = \frac{|L_k|}{|L_0 \cap L_k|}.$$

Then by direct calculation, we can see that $\sqrt{w_G(\Delta)/|G : L_0 \cap L_k|} = |\mathcal{F}|/|G : L_0 \cap L_k|$ as desired.

Now, G acts on $\mathcal{F}$ via $aF := (ag_0L_0, \ldots, ag_kL_k)$ for $F = (g_0L_0, \ldots, g_kL_k) \in \mathcal{F}$ and $a \in G$. Let S be the stabilizer in G of $F = (g_0L_0, \ldots, g_kL_k) \in \mathcal{F}$, that is,

$$S = \{a \in G \mid aF = F\} = \bigcap_{i=0}^{k} (L_i)^{g_i^{-1}} \leq (L_0)^{g_0^{-1}} \cap (L_k)^{g_k^{-1}} \cong L_0 \cap (L_k)^{g_k^{-1}g_0}.$$

Since $g := g_k^{-1}g_0 \in G = L_kL_0$ by our assumption, we have that $g = xy$ for some $x \in L_k$ and $y \in L_0$, and thus $L_0 \cap (L_k)^g = (L_0 \cap L_k)^y$. It follows that $|S|$ divides $|L_0 \cap L_k|$, and thus the length $|G : S|$ of the G-orbit of F is divisible by $|G : L_0 \cap L_k|$. Therefore $|\mathcal{F}| \equiv 0 \pmod{|G : L_0 \cap L_k|}$. The proof is complete. □

PROPOSITION 4.12. (1) *For $A, B \in \mathrm{Sgp}(G)$ with $A > B$, we have that $\mathfrak{s}_{A,B} = |A : B|$.*

(2) *For any path $\Delta = (H_0 \to H_1 \to \cdots \to H_k)$ in Q_G, we have that*

$$\prod_{i=0}^{k-1} \mathfrak{s}_{H_i,H_{i+1}} = |H_0 : H_k|\,.$$

PROOF. (1) Let $\Delta = (A =: L_0 - L_1 - \cdots - L_k := B)$ be a path from A to B in Q_G^{ud}. It suffices to show that $w_G(\Delta)$ is divisible by $|A : B|$. We proceed by induction on the length $k = \ell(\Delta)$ of Δ.

The first case where $\Delta = (A - B)$ is trivial. So we may assume that $k \geq 2$. Suppose that $L_{i-1} > L_i > L_{i+1}$ or $L_{i-1} < L_i < L_{i+1}$ for some $1 \leq i \leq k-1$. Put $\Delta' := (L_0 - \cdots - L_{i-1} - L_{i+1} - \cdots - L_k)$ just deleting L_i from Δ. Then since $w_G(\Delta) = w_G(\Delta')$ and $\ell(\Delta') = \ell(\Delta) - 1$, we have that $|A : B|$ divides $w_G(\Delta)$ by induction. Thus for any $1 \leq i \leq k-1$, we may assume that $L_{i-1} < L_i > L_{i+1}$ or $L_{i-1} > L_i < L_{i+1}$. Then by Lemma 4.10, $w_G(\Delta)$ is divisible by $|A : B|$.

(2) Straightforward from (1). □

PROPOSITION 4.13. *For $A, B \in \mathrm{Sgp}(G)$, the followings are equivalent.*

(1) $\mathfrak{s}_{A,B} = |G : A \cap B|$.

(2) $G = AB$.

PROOF. (1) ⇒ (2): By Proposition 4.5 (1), $\mathfrak{s}_{A,B} = |G : A \cap B|$ divides $|AB|/|A \cap B|$. Since $|AB| \leq |G|$, we have that $|AB| = |G|$.

(2) ⇒ (1): The proof is similar to that of Proposition 4.12. Let $\Delta = (A =: L_0 - L_1 - \cdots - L_k := B)$ be a path from A to B in Q_G^{ud}. It suffices to show that $w_G(\Delta)$ is divisible by $|G : A \cap B|$. We proceed by induction on the length $k = \ell(\Delta)$ of Δ.

If $k = 1$ then $\Delta = (A - B)$, and we may assume that $A > B$. By our assumption, $G = AB = A$, so that $w_G(\Delta) = |A : B| = |G : A \cap B|$ as desired. Hence we may assume that $k \geq 2$. Suppose that $L_{i-1} > L_i > L_{i+1}$ or $L_{i-1} < L_i < L_{i+1}$ for some $1 \leq i \leq k-1$. Put $\Delta' := (L_0 - \cdots - L_{i-1} - L_{i+1} - \cdots - L_k)$ just deleting L_i from Δ. Then since $w_G(\Delta) = w_G(\Delta')$ and $\ell(\Delta') = \ell(\Delta) - 1$, we have that $|G : A \cap B|$ divides $w_G(\Delta)$ by induction. Thus for any $1 \leq i \leq k-1$, we may assume that $L_{i-1} < L_i > L_{i+1}$ or $L_{i-1} > L_i < L_{i+1}$. Then by Lemma 4.11, $w_G(\Delta)$ is divisible by $|G : A \cap B|$. The proof is complete. □

THEOREM 4.14. *For $A, B \in \mathrm{Sgp}(G)$, the followings are equivalent.*

(1) $\mathfrak{s}_{A,B} = |G|$.

(2) $G = AB$ *and* $A \cap B = \{e\}$.

PROOF. (1) $\Rightarrow$ (2): By Proposition 4.5 (1), $\mathfrak{s}_{A,B} = |G|$ divides $|AB|/|A \cap B|$. Since $|AB| \leq |G|$, we have that $|AB| = |G|$ and $|A \cap B| = 1$, and thus the assertion holds.

(2) $\Rightarrow$ (1): This is clear from Proposition 4.13. □

Recall that a finite group which is the product of two nilpotent subgroups is solvable (see [5, 13.2.9]). So applying this fact, the following is a consequence of Theorem 4.14.

COROLLARY 4.15. *If* $\mathfrak{s}_{A,B} = |G|$ *for some nilpotent subgroups* $A, B \leq G$ *then* G *is solvable.*

5. Group characters

In this section, we apply the results in Section 3 on path algebras to group characters. Let G be a finite group. For a subgroup H of G, denote by $\mathrm{Irr}(H)$ the totality of irreducible complex characters of H. The set of all pairs of subgroups $H \leq G$ and irreducible characters χ of H forms a poset with an ordering defined by the multiplicity of characters (see Definition 5.1). Then, by Definition 4.1, we have an associated quiver Q_G^{ch}. So the UD-algebra $\mathsf{UD}(Q_G^{\mathrm{ch}}, w_G^{\mathrm{ch}})$ over $\mathbf{Z}$ can be considered where a weight function w_G^{ch} is defined by the multiplicity of characters. We first define some elements in $\mathsf{UD}(Q_G^{\mathrm{ch}}, w_G^{\mathrm{ch}})$ corresponding to Bratteli diagrams (see [2]), and examine their properties. Next we see that the group G is characterized by a weight function w_G^{ch} which behaves in a special way. Finally we investigate the case where all of the generating constants of $\mathsf{UD}(Q_G^{\mathrm{ch}}, w_G^{\mathrm{ch}})$ are trivial. For character theory of finite groups, we refer to [3].

5.1. Our setting and Bratteli operators. In this section, we define a quiver Q_G^{ch} and a weight function w_G^{ch} associated to group characters. And then we consider elements $B_{\downarrow}(K, H)$ and $B_{\uparrow}(K, H)$ in $\mathsf{UD}(Q_G^{\mathrm{ch}}, w_G^{\mathrm{ch}})$ corresponding to the Bratteli diagram of subgroups K and H of G.

DEFINITION 5.1. (1) Let $\mathfrak{C}_G$ be the set of all pairs of subgroups $H \leq G$ and characters $\chi \in \mathrm{Irr}(H)$, namely

$$\mathfrak{C}_G := \bigcup_{H \in \mathrm{Sgp}(G)} \{(H, \chi) \mid \chi \in \mathrm{Irr}(H)\}.$$

Then $\mathfrak{C}_G$ is a poset under an ordering $\preceq$ defined by $(H, \chi) \preceq (K, \theta)$ precisely when $H \leq K$ and $(\chi, \theta|_H)_H \neq 0$. Denote by

$$Q_G^{\mathrm{ch}} := Q_{(\mathfrak{C}_G, \preceq)}$$

a quiver associated to a poset $(\mathfrak{C}_G, \preceq)$ (see Definition 4.1). In this case, a weight function w_G^{ch} of Q_G^{ch} is defined by

$$w_G^{\mathrm{ch}}\big((K,\theta) \to (H,\chi)\big) := (\chi, \theta|_H)_H \in \mathbf{Z}.$$

(2) For a generalized character $\xi = \sum_{\chi \in \mathrm{Irr}(H)} m_\chi \chi \in \mathbf{Z}[\mathrm{Irr}(H)]$ of H, we write (H, ξ) for an element $\sum_{\chi \in \mathrm{Irr}(H)} m_\chi (H, \chi)$ in a $\mathbf{Z}$-subalgebra $\mathbf{Z}(Q_G^{\mathrm{ch}})_0 \subseteq \mathbf{Z}Q_G^{\mathrm{ch}}$ (see Section 2).

REMARK 5.2. An arrow $\alpha = \big((K,\theta) \to (H,\chi)\big)$ of weight $m := (\chi, \theta|_H)_H \neq 0$ is defined by the restriction $\theta|_H$ of θ. Then, by Frobenius reciprocity (cf. [3, page 62]), we think that the opposite arrow ${}^t\alpha = \big((K,\theta) \leftarrow (H,\chi)\big)$ of weight $m = \big(\theta, \chi^K\big)_K$ is defined by the induction χ^K of χ.

DEFINITION 5.3. For subgroups $H < K \leq G$, put

$$(\mathsf{P}_G^{\mathrm{ch}})_{K,H} := \big\{\big((K,\theta) \to (H,\chi)\big) \in \mathsf{P}\big(Q_G^{\mathrm{ch}}\big) \mid \theta \in \mathrm{Irr}(K),\, \chi \in \mathrm{Irr}(H)\big\}.$$

Define an element $\beta(K,H)$ in the path algebra $\mathbf{Z}Q_G^{\mathrm{ch}}$ as follows:

$$\beta(K,H) := \sum_{\Delta \in (\mathsf{P}_G^{\mathrm{ch}})_{K,H}} \Delta \quad \in \mathbf{Z}Q_G^{\mathrm{ch}}.$$

Furthermore, define an element $\beta(H,H) := \sum_{\chi \in \mathrm{Irr}(H)} (H,\chi) \in \mathbf{Z}(Q_G^{\mathrm{ch}})_0$ to be the sum of all trivial paths corresponding to vertices (H,χ) for all $\chi \in \mathrm{Irr}(H)$.

REMARK 5.4. It is worth mentioning that the above $\beta(K,H)$ $(H < K)$ together with weights w_G^{ch} can be thought of the "Bratteli diagram" of K and H (see [2] for example). This is a graph whose vertex set is $\mathrm{Irr}(K) \cup \mathrm{Irr}(H)$, and two distinct $\theta, \chi \in \mathrm{Irr}(K) \cup \mathrm{Irr}(H)$ are joined by m edges if and only if $(\chi, \theta_H)_H = m$ for $\theta \in \mathrm{Irr}(K)$ and $\chi \in \mathrm{Irr}(H)$.

DEFINITION 5.5 (Bratteli operators). For subgroups $H \leq K \leq G$, define two elements of $\mathsf{UD}(Q_G^{\mathrm{ch}}, w_G^{\mathrm{ch}}) \subseteq \mathrm{End}(\mathbf{Z}(Q_G^{\mathrm{ch}})_0)$ corresponding to $\beta(K,H)$ as follows:

$$B_\downarrow(K,H) := \rho_{w_G^{\mathrm{ch}}}\big(\beta(K,H)\big) = \sum_{\Delta \in (\mathsf{P}_G^{\mathrm{ch}})_{K,H}} \rho_{w_G^{\mathrm{ch}}}(\Delta) : \mathbf{Z}(Q_G^{\mathrm{ch}})_0 \longrightarrow \mathbf{Z}(Q_G^{\mathrm{ch}})_0$$

$$B_\uparrow(K,H) := \lambda_{w_G^{\mathrm{ch}}}\big(\beta(K,H)\big) = \sum_{\Delta \in (\mathsf{P}_G^{\mathrm{ch}})_{K,H}} \lambda_{w_G^{\mathrm{ch}}}(\Delta) : \mathbf{Z}(Q_G^{\mathrm{ch}})_0 \longrightarrow \mathbf{Z}(Q_G^{\mathrm{ch}})_0$$

In particular, we have that $B_\downarrow(H,H) = B_\uparrow(H,H)$, and that for $(L,\eta) \in (Q_G^{\mathrm{ch}})_0 = \mathfrak{C}_G$,

$$(L,\eta)^{B_\downarrow(K,H)} = \sum_{\Delta \in (\mathsf{P}_G^{\mathrm{ch}})_{K,H}} (L,\eta)\Delta = \sum_{\Delta \in (\mathsf{P}_G^{\mathrm{ch}})_{K,H}} w_G^{\mathrm{ch}}(\Delta)\big(\delta_{(L,\eta),s(\Delta)} r(\Delta)\big) \in \mathbf{Z}(Q_G^{\mathrm{ch}})_0$$

$$(L,\eta)^{B_\uparrow(K,H)} = \sum_{\Delta\in(\mathsf{P}_G^{\mathrm{ch}})_{K,H}} \Delta(L,\eta) = \sum_{\Delta\in(\mathsf{P}_G^{\mathrm{ch}})_{K,H}} w_G^{\mathrm{ch}}(\Delta)\big(\delta_{r(\Delta),(L,\eta)}s(\Delta)\big) \in \mathbf{Z}(Q_G^{\mathrm{ch}})_0$$

EXAMPLE 5.6. For subgroups $H < K \leq G$, suppose that $\mathrm{Irr}(K) = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$ and $\mathrm{Irr}(H) = \{\chi_1, \chi_2, \chi_3\}$, and that $\beta(K,H) \in \mathbf{Z}Q_G^{\mathrm{ch}}$ with weights m_i is given as follows:

$$\begin{tikzcd}
\mathrm{Irr}(K) & \theta_1 \arrow[dr, "m_1"'] & \theta_2 \arrow[d, "m_2"'] \arrow[dr, "m_3"] & \theta_3 \arrow[d, "m_4"'] & \theta_4 \arrow[dl, "m_5"'] \arrow[d, "m_6"] & \theta_5 \arrow[dl, "m_7"'] \\
\mathrm{Irr}(H) & & \chi_1 & \chi_2 & \chi_3 &
\end{tikzcd}$$

In this case, for example, $(K,\theta_2)^{B_\downarrow(K,H)}$ and $(H,\chi_2)^{B_\uparrow(K,H)}$ are calculated as follows:

$$\begin{aligned}
(K,\theta_2)^{B_\downarrow(K,H)} &= m_2(H,\chi_1) + m_3(H,\chi_2) = (H, m_2\chi_1 + m_3\chi_2) = (H, \theta_2|_H) \\
(H,\chi_2)^{B_\uparrow(K,H)} &= m_3(K,\theta_2) + m_4(K,\theta_3) + m_5(K,\theta_4) \\
&= (K, m_3\theta_2 + m_4\theta_3 + m_5\theta_4) = \big(K, (\chi_2)^K\big)
\end{aligned}$$

In general, we get the next.

PROPOSITION 5.7. *For subgroups $H, K \leq G$, and $(L,\eta) \in (Q_G^{\mathrm{ch}})_0 = \mathfrak{C}_G$, we have that*

(1) $(L,\eta)^{B_\downarrow(K,H)} = \delta_{K,L}(H, \eta|_H)$ *and* $(L,\eta)^{B_\uparrow(K,H)} = \delta_{H,L}\big(K, \eta^K\big)$ *for* $H \leq K$.

(2) (*Mackey Decomposition*) *For a set* $\{x_1, \ldots, x_r\}$ *of* (K,H)*-double coset representatives in* G,

$$(L,\eta)^{B_\uparrow(G,H)\circ B_\downarrow(G,K)} = \delta_{H,L}\sum_{i=1}^{r}(H_i, \eta^{x_i})^{B_\downarrow(H_i, H_i\cap K)\circ B_\uparrow(H_i\cap K, K)}$$

where $H_i := x_i H x_i^{-1}$.

PROPOSITION 5.8. *As in the following, let $\mathcal{B}$ be a $\mathbf{Z}$-subalgebra of $\mathbf{Z}Q_G^{\mathrm{ch}}$, and let $\mathsf{B}_\downarrow$ and $\mathsf{B}_\uparrow$ be $\mathbf{Z}$-subalgebras of* $\mathsf{UD}(Q_G^{\mathrm{ch}}, w_G^{\mathrm{ch}}) \leq \mathrm{End}\big(\mathbf{Z}(Q_G^{\mathrm{ch}})_0\big)$.

$$\begin{aligned}
\mathcal{B} &:= \big\langle\, \beta(K,H) \mid H \leq K \leq G \,\big\rangle \leq \mathbf{Z}Q_G^{\mathrm{ch}} \\
\mathsf{B}_\downarrow &:= \big\langle\, B_\downarrow(K,H) \mid H \leq K \leq G \,\big\rangle \leq \mathsf{UD}(Q_G^{\mathrm{ch}}, w_G^{\mathrm{ch}}) \\
\mathsf{B}_\uparrow &:= \big\langle\, B_\uparrow(K,H) \mid H \leq K \leq G \,\big\rangle \leq \mathsf{UD}(Q_G^{\mathrm{ch}}, w_G^{\mathrm{ch}})
\end{aligned}$$

(1) *The $\mathbf{Z}$-algebra $\mathcal{B}$ is isomorphic to the path algebra $\mathbf{Z}Q_G$ defined by the lattice* $(\mathrm{Sgp}(G), \leq)$.

(2) *For a $\mathbf{Z}$-submodule M of $\mathbf{Z}(Q_G^{\mathrm{ch}})_0$, M is a right* (*resp. left*) *$\mathbf{Z}Q_G$-module if and only if M is invariant under* $\mathsf{B}_\downarrow$ (*resp.* $\mathsf{B}_\uparrow$).

PROOF. (1) It is clear from the fact that generators $\beta(K,H)$ and $\beta(H,H)$ $(H < K \leq G)$ of $\mathcal{B}$ correspond to a path $(K \to H)$ of length 1 and the trivial path H in the quiver Q_G (see Definition 4.2).

(2) From the definition, M is invariant under $\mathsf{B}_\downarrow$ (resp. $\mathsf{B}_\uparrow$) if and only if M is a right (resp. left) $\mathcal{B}$-module. So the assertion follows from (1). □

EXAMPLE 5.9. Let M_{tri} and M_{reg} be $\mathbf{Z}$-submodules of $\mathbf{Z}(Q_G^{\text{ch}})_0$ as follows:

$$M_{\text{tri}} := \sum_{K \in \text{Sgp}(G)} (K, 1_K)\mathbf{Z}, \quad M_{\text{reg}} := \sum_{H \in \text{Sgp}(G)} (H, \rho_H)\mathbf{Z}$$

where 1_H and ρ_H are the trivial and regular characters of H. Since, for $H \leq K \leq G$,

$$(K, 1_K)^{B_\downarrow(K,H)} = (H, 1_H),\ (H, \rho_H)^{B_\uparrow(K,H)} = (K, \rho_K)\,,$$
$$(K, \rho_K)^{B_\downarrow(K,H)} = |K : H|(H, \rho_H)\,,$$

we have from Proposition 5.8 (2) and Proposition 3.2 that

$$\begin{aligned} M_{\text{tri}} &\cong \big(\Phi_w, \mathbf{Z}(Q_G)_0\big) && \text{as right } \mathbf{Z}Q_G\text{-modules where } w = 1 \\ M_{\text{reg}} &\cong \big(\Psi_w, \mathbf{Z}(Q_G)_0\big) && \text{as left } \mathbf{Z}Q_G\text{-modules where } w = 1 \\ M_{\text{reg}} &\cong \big(\Phi_w, \mathbf{Z}(Q_G)_0\big) && \text{as right } \mathbf{Z}Q_G\text{-modules where } w = w_G \end{aligned}$$

where w_G is defined by indices of subgroups of G (see Definition 4.2).

5.2. A characterization of G by weights. In this section, we see that the group G is characterized by a weight function w_G^{ch} which behaves in a special way. For any element $x := (H, \chi) \in \mathfrak{C}_G$, define $\chi_x := \chi$.

LEMMA 5.10. *For any path Δ in Q_G^{ch}, we have that $w_G^{\text{ch}}(\Delta) \leq \left[\frac{\chi_{s(\Delta)}(1)}{\chi_{r(\Delta)}(1)}\right]$.*

PROOF. For any path $\Delta := \big((H_0, \chi_0) \to (H_1, \chi_1) \to \cdots \to (H_k, \chi_k)\big)$ in Q_G^{ch}, we set $m_i := (\chi_i|_{H_{i+1}}, \chi_{i+1})_{H_{i+1}}$ $(0 \leq i \leq k-1)$. Then $w_G^{\text{ch}}(\Delta) = \prod_{i=0}^{k-1} m_i$. Furthermore set $m := (\chi_0|_{H_k}, \chi_k)_{H_k}$. Then we have that

$$\prod_{i=0}^{k-1} m_i \leq m \leq \left[\frac{\chi_0(1)}{\chi_k(1)}\right] = \left[\frac{\chi_{s(\Delta)}(1)}{\chi_{r(\Delta)}(1)}\right]$$

as desired. □

PROPOSITION 5.11. *The followings are equivalent.*

(1) *For any path Δ in Q_G^{ch}, we have that $w_G^{\text{ch}}(\Delta) = \left[\frac{\chi_{s(\Delta)}(1)}{\chi_{r(\Delta)}(1)}\right]$.*

(2) *G is abelian.*

PROOF. (2) $\Rightarrow$ (1): Since every irreducible character of a (finite) abelian group has degree 1, the assertion clearly holds.

(1) $\Rightarrow$ (2): First of all, we will show that any abelian subgroup $C \le G$ is normal in G. Take any $\psi \in \mathrm{Irr}(C)$, and express $\psi^G = \sum_{i=1}^t m_i\theta_i$ for some $m_i \ge 1$ and $\theta_i \in \mathrm{Irr}(G)$. Then $\Delta := \big((G, \theta_i) \to (C, \psi)\big)$ forms a path in Q_G^{ch}, and by our assumption $w_G^{\mathrm{ch}}(\Delta) = \left[\frac{\chi_{s(\Delta)}(1)}{\chi_{r(\Delta)}(1)}\right] = \left[\frac{\theta_i(1)}{\psi(1)}\right] = \theta_i(1)$. This implies that

$$\theta_i|_C = \theta_i(1)\psi, \quad (\psi^G)|_C = \left(\sum_{i=1}^t m_i\theta_i(1)\right)\psi\,. \tag{$*$}$$

In particular, in the case of the trivial character $\psi := 1_C$ of C, we have, by $(*)$, that $(1_C)^G(y) = (1_C)^G(1) = |G : C|$ for any $y \in C$. On the other hand, the definition of induced characters tells us that $(1_C)^G(y) = \frac{1}{|C|}\sum_{g\in G} 1_C^\circ(g^{-1}yg)$, and hence $g^{-1}yg \in C$ for any $g \in G$. It follows that C is a normal subgroup of G.

Now, we will show that G is abelian by induction on the order $|G|$ of G. Before doing this, it is worth mentioning that, for any proper subgroup $H < G$ and a non-trivial normal subgroup $\{1\} \ne N \trianglelefteq G$, H and G/N clearly satisfy the condition (1), so that by induction both H and G/N are abelian.

Suppose that $|\pi(G)| \ge 2$ where $\pi(G)$ is the set of primes dividing the order of G. Take any elements $x, y \in G$ whose orders are relatively prime. As shown in the above, abelian subgroups $\langle x\rangle$ and $\langle y\rangle$ are normal in G, so that $\langle x\rangle\langle y\rangle = \langle x\rangle \times \langle y\rangle$ and $[x, y] = 1$. This yields that G is the direct product of Sylow subgroups of G, namely it is nilpotent. Since $|\pi(G)| \ge 2$, each Sylow subgroup is abelian by induction, and thus G is abelian. So we may assume that G is a p-group for some prime p.

Suppose that there exist elements $x, y \in G$ such that $[x, y] \ne 1$. If $\langle x, y\rangle < G$ then by induction $[x, y] = 1$, a contradiction. So we get $G = \langle x, y\rangle$. Let $C_p \cong \langle a\rangle \le Z(G)$, and then by induction $G/\langle a\rangle$ is abelian, so that $[x, y] \in [G, G] \le \langle a\rangle$. Suppose further that there exists an element $b \in G$ of order p such that $\langle a\rangle \ne \langle b\rangle$. Since $\langle b\rangle \cap \langle x\rangle, \langle b\rangle \cap \langle y\rangle \le \langle b\rangle \cong C_p$, we obtain that $[b, x] = 1 = [b, y]$. This means that $b \in Z(G)$, and by induction $G/\langle b\rangle$ is abelian, so that $[x, y] \in [G, G] \le \langle b\rangle$. It follows that $[x, y] \in \langle a\rangle \cap \langle b\rangle = \{1\}$, a contradiction. Thus the p-group G possesses the unique subgroup of order p, which implies that G is isomorphic to a cyclic group or a generalized quaternion group (cf. [7, page 59]). But since $[x, y] \ne 1$, and since any proper subgroup of G is abelian by induction, we have that $G \cong Q_8 = \langle A, B \mid A^4 = 1, A^2 = B^2, B^{-1}AB = A^{-1}\rangle$ the quaternion group. Then there exists the unique irreducible character $\theta \in \mathrm{Irr}(G)$ of degree 2. Let $C := \langle A\rangle \cong C_4$ be an abelian subgroup of G, and then $\theta|_C = \psi_1 + \psi_2$ for certain distinct $\psi_1, \psi_2 \in \mathrm{Irr}(C)$. However, this contradicts $(*)$. The proof is complete. □

5.3. Trivial generating constants. In this section, we investigate the case where all of the generating constants of $\mathsf{UD}(Q_G^{\mathrm{ch}}, w_G^{\mathrm{ch}})$ are equal to 1. And indeed we finally show that every finite group has this property. For a subgroup $H \le G$, set

$$\mathsf{P}_H := \mathsf{P}\Big(\big(Q_H^{\mathrm{ch}}\big)^{\mathrm{ud}}\Big) \quad \text{and} \quad \mathsf{P} := \mathsf{P}\Big(\big(Q_G^{\mathrm{ch}}\big)^{\mathrm{ud}}\Big)\,.$$

For elements $u, v \in \mathfrak{C}_H \subseteq \mathfrak{C}_G$, we denote by $\mathfrak{s}^H_{u,v}$ the generating constant of $\mathsf{UD}(Q^{\mathrm{ch}}_H, w^{\mathrm{ch}}_H)$. Furthermore an element $(\{e\}, 1_{\{e\}})$ of $\mathfrak{C}_G$ for the trivial subgroup $\{e\}$ of G is denoted by just $\mathbf{1}$.

DEFINITION 5.12. We say that a finite group G is a TGC-group (Trivial Generating Constant) if $\mathfrak{s}_{x,y} = 1$ for any $x, y \in \mathfrak{C}_G$.

LEMMA 5.13. (1) *If* $\mathfrak{s}_{x,\mathbf{1}} = 1$ *for any* $x \in \mathfrak{C}_G$ *then* G *is a* TGC*-group.*

(2) *Let* H *be a subgroup of* G*. If* $\mathfrak{s}^H_{x,y} = 1$ *for some* $x, y \in \mathfrak{C}_H$ *then* $\mathfrak{s}_{x,y} = 1$.

PROOF. (1) For any $x, y \in \mathfrak{C}_G$, we have that $\mathfrak{s}_{x,\mathbf{1}} = 1$ and $\mathfrak{s}_{\mathbf{1},y} = \mathfrak{s}_{y,\mathbf{1}} = 1$ by our assumption. So there exist integers m_Δ and n_Γ such that

$$\sum_{\Delta \in \mathsf{P}_{x\Rightarrow \mathbf{1}}} m_\Delta w^{\mathrm{ch}}_G(\Delta) = 1\,, \qquad \sum_{\Gamma \in \mathsf{P}_{\mathbf{1}\Rightarrow y}} n_\Gamma w^{\mathrm{ch}}_G(\Gamma) = 1\,.$$

It follows that

$$\begin{aligned}\mathfrak{s}_{x,y}\cdot \mathbf{Z} := \sum_{\Upsilon \in \mathsf{P}_{x\Rightarrow y}} w^{\mathrm{ch}}_G(\Upsilon)\cdot \mathbf{Z} &\ni \sum_{\Delta \in \mathsf{P}_{x\Rightarrow \mathbf{1}}} \sum_{\Gamma \in \mathsf{P}_{\mathbf{1}\Rightarrow y}} m_\Delta n_\Gamma w^{\mathrm{ch}}_G(\Delta\Gamma) \\ &= \sum_{\Delta \in \mathsf{P}_{x\Rightarrow \mathbf{1}}} \sum_{\Gamma \in \mathsf{P}_{\mathbf{1}\Rightarrow y}} m_\Delta n_\Gamma w^{\mathrm{ch}}_G(\Delta) w^{\mathrm{ch}}_G(\Gamma) \\ &= \left(\sum_{\Delta \in \mathsf{P}_{x\Rightarrow \mathbf{1}}} m_\Delta w^{\mathrm{ch}}_G(\Delta)\right) \times \left(\sum_{\Gamma \in \mathsf{P}_{\mathbf{1}\Rightarrow y}} n_\Gamma w^{\mathrm{ch}}_G(\Gamma)\right) = 1\,.\end{aligned}$$

(2) By our assumption $\mathfrak{s}^H_{x,y} = 1$, there exist integers m_Δ such that

$$\sum_{\Delta \in (\mathsf{P}_H)_{x\Rightarrow y}} m_\Delta w^{\mathrm{ch}}_H(\Delta) = 1.$$

But since $(\mathsf{P}_H)_{x\Rightarrow y} \subseteq \mathsf{P}_{x\Rightarrow y}$, we have that $1 \in \mathfrak{s}_{x,y}\cdot \mathbf{Z}$. □

LEMMA 5.14. *Let* H *and* K *be* TGC*-groups. Set* $G := H \times K$*. Then for any* $x := (G, \chi) \in \mathfrak{C}_G$*, we have that* $\mathfrak{s}_{x,\mathbf{1}} = 1$.

PROOF. The irreducible characters of G can be obtained as $\mathrm{Irr}(G) = \{\theta \times \psi \mid \theta \in \mathrm{Irr}(H),\ \psi \in \mathrm{Irr}(K)\}$ where $(\theta \times \psi)(hk) := \theta(h)\psi(k)$ for $hk \in G = H \times K$ (cf. [3, page 59]). So we may assume that $\chi = \theta \times \psi$ for some $\theta \in \mathrm{Irr}(H)$ and $\psi \in \mathrm{Irr}(K)$. Let $y := (H, \theta) \in \mathfrak{C}_H$ and $z := (K, \psi) \in \mathfrak{C}_K$. Then $\mathfrak{s}^H_{y,\mathbf{1}} = \mathfrak{s}^K_{z,\mathbf{1}} = 1$ by our assumption, so there exist integers m_Δ and n_Γ such that

$$\sum_{\Delta \in (\mathsf{P}_H)_{y\Rightarrow \mathbf{1}}} m_\Delta w^{\mathrm{ch}}_H(\Delta) = 1\,, \qquad \sum_{\Gamma \in (\mathsf{P}_K)_{z\Rightarrow \mathbf{1}}} n_\Gamma w^{\mathrm{ch}}_K(\Gamma) = 1\,.$$

Now for paths

$$\Delta = \big(y =: (H_1, \theta_1) - (H_2, \theta_2) - \cdots - (H_s, \theta_s) := \mathbf{1}\big) \in (\mathsf{P}_H)_{y \Rightarrow \mathbf{1}}\,,$$
$$\Gamma = \big(z =: (K_1, \psi_1) - (K_2, \psi_2) - \cdots - (K_t, \psi_t) := \mathbf{1}\big) \in (\mathsf{P}_K)_{z \Rightarrow \mathbf{1}}\,,$$

define $p_{i,j} := (H_i \times K_j, \theta_i \times \psi_j) \in \mathfrak{C}_G$ $(1 \le i \le s,\ 1 \le j \le t)$. Then it is straightforward to check that

$$\Delta \sharp \Gamma := \big(p_{1,1} - p_{2,1} - \cdots - p_{s-1,1} - p_{s,1} - p_{s,2} - \cdots - p_{s,t-1} - p_{s,t}\big)$$

forms a path in $\mathsf{P}_{x \Rightarrow \mathbf{1}}$ with the property that

$$w_G^{\mathrm{ch}}(p_{i,1} - p_{i+1,1}) = w_H^{\mathrm{ch}}\big(\,(H_i, \theta_i) - (H_{i+1}, \theta_{i+1})\,\big) \quad (1 \le i \le s-1)\,,$$
$$w_G^{\mathrm{ch}}(p_{s,j} - p_{s,j+1}) = w_K^{\mathrm{ch}}\big(\,(K_j, \psi_j) - (K_{j+1}, \psi_{j+1})\,\big) \quad (1 \le j \le t-1)\,.$$

This tells us that $w_G^{\mathrm{ch}}(\Delta \sharp \Gamma) = w_H^{\mathrm{ch}}(\Delta) \times w_K^{\mathrm{ch}}(\Gamma)$. It follows that

$$\begin{aligned}\mathfrak{s}_{x,\mathbf{1}} \cdot \mathbf{Z} := \sum_{\Upsilon \in \mathsf{P}_{x \Rightarrow \mathbf{1}}} w_G^{\mathrm{ch}}(\Upsilon) \cdot \mathbf{Z} &\ni \sum_{\Delta \in (\mathsf{P}_H)_{y \Rightarrow \mathbf{1}}} \sum_{\Gamma \in (\mathsf{P}_K)_{z \Rightarrow \mathbf{1}}} m_\Delta n_\Gamma w_G^{\mathrm{ch}}(\Delta \sharp \Gamma) \\ &= \Bigg(\sum_{\Delta \in (\mathsf{P}_H)_{y \Rightarrow \mathbf{1}}} m_\Delta w_H^{\mathrm{ch}}(\Delta)\Bigg) \times \Bigg(\sum_{\Gamma \in (\mathsf{P}_K)_{z \Rightarrow \mathbf{1}}} n_\Gamma w_K^{\mathrm{ch}}(\Gamma)\Bigg) = 1\,.\end{aligned}$$

The proof is complete. □

DEFINITION 5.15 (cf. page 127 in [3]). A finite group E is said to be elementary or p-elementary (where p is a prime) if E is the direct product of a cyclic group and a p-group.

THEOREM 5.16. *If any elementary subgroups of G are TGC-groups then G is a TGC-group.*

PROOF. By Lemma 5.13, it suffices to show that $\mathfrak{s}_{x,\mathbf{1}} = 1$ for any $x := (K, \chi) \in \mathfrak{C}_G$. Now using Brauer's characterization of characters (cf. [3, Theorem 8.4]), $\chi \in \mathrm{Irr}(K)$ can be expressed as

$$\chi = \sum_{\lambda \in \Lambda} m_\lambda (\psi_\lambda)^K$$

where Λ is an index set, m_λ is an integer, and ψ_λ is a linear character of an elementary subgroup E_λ of K. Let $\{\lambda_1, \ldots, \lambda_s\} := \{\lambda \in \Lambda \mid 0 \ne \alpha_\lambda := \big(\chi, (\psi_\lambda)^K\big)_K\}$. Then we have that $1 = \sum_{i=1}^s m_{\lambda_i} \alpha_{\lambda_i}$. By our assumption, E_λ is a TGC-group, and thus $\mathfrak{s}_{x_i,\mathbf{1}}^{E_{\lambda_i}} = 1$ for $x_i := (E_{\lambda_i}, \psi_{\lambda_i})$. This implies that

$$\sum_{\Delta \in (\mathsf{P}_{E_{\lambda_i}})_{x_i \Rightarrow \mathbf{1}}} n_\Delta w_{E_{\lambda_i}}^{\mathrm{ch}}(\Delta) = 1$$

for some integers n_Δ. On the other hand, set $\Gamma_i := \big(x \to x_i\big)$ $(1 \le i \le s)$ an arrow of weight α_{λ_i}. Then $\Gamma_i\Delta$ is a member of $\mathsf{P}_{x\Rightarrow\mathbf{1}}$ for any $\Delta \in (\mathsf{P}_{E_{\lambda_i}})_{x_i\Rightarrow\mathbf{1}}$. Hence we have that

$$\mathfrak{s}_{x,\mathbf{1}} \cdot \mathbf{Z} := \sum_{\Upsilon\in\mathsf{P}_{x\Rightarrow\mathbf{1}}} w_G^{\mathrm{ch}}(\Upsilon)\cdot\mathbf{Z} \ni \sum_{i=1}^{s}\Bigg(\sum_{\Delta\in(\mathsf{P}_{E_{\lambda_i}})_{x_i\Rightarrow\mathbf{1}}} m_{\lambda_i} n_\Delta w_G^{\mathrm{ch}}(\Gamma_i\Delta)\Bigg)$$
$$= \sum_{i=1}^{s} m_{\lambda_i}\alpha_{\lambda_i}\Bigg(\sum_{\Delta\in(\mathsf{P}_{E_{\lambda_i}})_{x_i\Rightarrow\mathbf{1}}} n_\Delta w_G^{\mathrm{ch}}(\Delta)\Bigg) = \sum_{i=1}^{s} m_{\lambda_i}\alpha_{\lambda_i} = 1\,.$$

The proof is complete. □

LEMMA 5.17 (cf. Corollary 6.19 in [3]). *Let P be a p-group and let H its subgroup whose index in P is p. Suppose $\chi \in \mathrm{Irr}(P)$. Then one of the following holds*:

(1) *$\chi|_H$ is irreducible.*
(2) *$\chi|_H$ is a sum of distinct irreducible characters of H.*

LEMMA 5.18. (1) *An abelian group A is a TGC-group.*
(2) *A p-group P is a TGC-group.*

PROOF. (1) Let $\Delta = (x \to y)$ be an arrow in Q_A^{ch} where $x := (H,\chi)$ and $y := (K,\psi) \in \mathfrak{C}_A$. Put $0 \ne m := w_A^{\mathrm{ch}}(\Delta) = (\chi|_K, \psi)_K$. Then since $\chi(1) \ge m\psi(1)$, and since every irreducible character of a (finite) abelian group has degree 1, we have that $m = 1$. This implies that A is a TGC-group.

(2) We proceed by induction on the order $|P|$ of P. By Lemma 5.13, it suffices to show that $\mathfrak{s}_{x,\mathbf{1}} = 1$ for any $x := (K,\chi) \in \mathfrak{C}_P$. Suppose that $K < P$. Then since K is a TGC-group by induction, we have that $\mathfrak{s}^K_{x,\mathbf{1}} = 1$, and thus $\mathfrak{s}_{x,\mathbf{1}} = 1$ by Lemma 5.13.

Suppose next that $K = P$. Let H be a maximal subgroup of P, then $|P : H| = p$. By Lemma 5.17, for $\chi \in \mathrm{Irr}(P)$, there exists $\theta \in \mathrm{Irr}(H)$ such that $(\chi|_H, \theta)_H = 1$. Let $y := (H,\theta) \in \mathfrak{C}_H \subseteq \mathfrak{C}_P$, and set $\Gamma := \big(x \to y\big)$ an arrow of weight 1. Then $\Gamma\Delta$ is in $\mathsf{P}_{x\Rightarrow\mathbf{1}}$ for any $\Delta \in (\mathsf{P}_H)_{y\Rightarrow\mathbf{1}}$. Furthermore, since H is a TGC-group by induction, we have that $\mathfrak{s}^H_{y,\mathbf{1}} = 1$. So there exist integers n_Δ such that

$$\sum_{\Delta\in(\mathsf{P}_H)_{y\Rightarrow\mathbf{1}}} n_\Delta w_H^{\mathrm{ch}}(\Delta) = 1\,.$$

Therefore

$$\mathfrak{s}_{x,\mathbf{1}} \cdot \mathbf{Z} := \sum_{\Upsilon\in\mathsf{P}_{x\Rightarrow\mathbf{1}}} w_P^{\mathrm{ch}}(\Upsilon)\cdot\mathbf{Z} \ni \sum_{\Delta\in(\mathsf{P}_H)_{y\Rightarrow\mathbf{1}}} n_\Delta w_H^{\mathrm{ch}}(\Gamma\Delta)$$
$$= \sum_{\Delta\in(\mathsf{P}_H)_{y\Rightarrow\mathbf{1}}} n_\Delta w_H^{\mathrm{ch}}(\Delta) = 1\,.$$

The proof is complete. □

LEMMA 5.19. *Every p-elementary group $E = P \times C$ where P is a p-group and C is a p'-cyclic is a TGC-group.*

PROOF. We proceed by induction on the order $|E|$ of E. By Lemma 5.13, it suffices to show that $\mathfrak{s}_{x,\mathbf{1}} = 1$ for any $x := (S, \chi) \in \mathfrak{C}_E$. Suppose that $S < E$. Then since $S = (S \cap P) \times (S \cap C)$ is a p-elementary group whose order $|S|$ is less than $|E|$, S is a TGC-group by induction. It follows that $\mathfrak{s}^S_{x,\mathbf{1}} = 1$, and thus $\mathfrak{s}_{x,\mathbf{1}} = 1$ by Lemma 5.13. So we may assume that $S = E = P \times C$. Note that, by Lemma 5.18, P and C are both TGC-groups. Therefore Lemma 5.14 tells us that $\mathfrak{s}_{x,\mathbf{1}} = 1$. The proof is complete. □

THEOREM 5.20. *Every finite group is a TGC-group.*

PROOF. Straightforward from Theorem 5.16 and Lemma 5.19. □

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