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The Derived Category Analogue of the Hartshorne-Lichtenbaum Vanishing Theorem

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Abstract. Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and X a d-dimensional homologically bounded complex of R-modules whose all homology modules are finitely generated. We show that $H_{\mathfrak{a}}^d(X) = 0$ if and only if $\dim \widehat{R}/\mathfrak{a}\widehat{R} + \mathfrak{p} > 0$ for all prime ideals $\mathfrak{p}$ of $\hat{R}$ such that $\dim \hat{R}/\mathfrak{p} - \inf(X \otimes_R \hat{R})_{\mathfrak{p}} = d$.

1. Introduction

The Hartshorne-Lichtenbaum Vanishing Theorem is one of the most important results in the theory of local cohomology modules. There are several proofs known now of this result; see e.g. [BH], [CS] and [Sc]. Also, there are several generalizations of this result. The second named author, Naghipour and Tousi [DNT] have extended it to local cohomology with support in stable under specialization subsets. Takahashi, Yoshino and Yoshizawa [TYY] have extended it to local cohomology with respect to pairs of ideals. Also, more recently, the Hartshorne-Lichtenbaum Vanishing Theorem is extended to generalized local cohomology modules; see [DH]. Our aim in this paper is to establish a generalization of the Hartshorne-Lichtenbaum Vanishing Theorem which contains all of these generalizations. We do this by establishing the derived category analogue of the Hartshorne-Lichtenbaum Vanishing Theorem. For giving the precise statement of this result, we need to fix some notation.

Throughout, R is a commutative Noetherian ring with nonzero identity. The derived category of R-modules is denoted by $\mathcal{D}(R)$. We use the symbol $\simeq$ for denoting isomorphisms in $\mathcal{D}(R)$. For a complex $X \in \mathcal{D}(R)$, its supremum and infimum are defined, respectively, by $\sup X := \sup\{i \in \mathbb{Z} | H_i(X) \neq 0\}$ and $\inf X := \inf\{i \in \mathbb{Z} | H_i(X) \neq 0\}$, with the usual convention that $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. Also, amplitude of X is defined by $\operatorname{amp} X := \sup X - \inf X$. Recall that $\dim_R X$ is defined by $\dim_R X := \sup\{\dim R/\mathfrak{p} - \inf X_{\mathfrak{p}} | \mathfrak{p} \in \operatorname{Spec} R\}$

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and we define $\operatorname{Assh}_R X$ by

$$\operatorname{Assh}_R X := \{\mathfrak{p} \in \operatorname{Spec} R \mid \dim R/\mathfrak{p} - \inf X_{\mathfrak{p}} = \dim_R X\}.$$

Any R-module M can be considered as a complex having M in its 0-th spot and 0 in its other spots. We denote the full subcategory of homologically left bounded complexes by $\mathcal{D}_{\sqsubset}(R)$. Also, we denote the full subcategory of complexes with finitely generated homology modules that are homologically bounded (resp. homologically left bounded) by $\mathcal{D}^f_{\square}(R)$ (resp. $\mathcal{D}^f_{\sqsubset}(R)$).

Let $\mathfrak{a}$ be an ideal of R and $X \in \mathcal{D}_{\sqsubset}(R)$. A subset $\mathcal{Z}$ of $\operatorname{Spec} R$ is said to be *stable under specialization* if $\operatorname{V}(\mathfrak{p}) \subseteq \mathcal{Z}$ for all $\mathfrak{p} \in \mathcal{Z}$. For any R-module M, $\Gamma_{\mathcal{Z}}(M)$ is defined by

$$\Gamma_{\mathcal{Z}}(M) := \{x \in M \mid \operatorname{Supp}_R Rx \subseteq \mathcal{Z}\}.$$

The right derived functor of the functor $\Gamma_{\mathcal{Z}}(-)$ exists in $\mathcal{D}(R)$ and the complex $\mathbf{R}\Gamma_{\mathcal{Z}}(X)$ is defined by $\mathbf{R}\Gamma_{\mathcal{Z}}(X) := \Gamma_{\mathcal{Z}}(I)$, where I is any injective resolution of X. Also, for any integer i, the i-th local cohomology module of X with respect to $\mathcal{Z}$ is defined by $H^i_{\mathcal{Z}}(X) := H_{-i}(\mathbf{R}\Gamma_{\mathcal{Z}}(X))$. To comply with the usual notation, for $\mathcal{Z} := \operatorname{V}(\mathfrak{a})$, we denote $\mathbf{R}\Gamma_{\mathcal{Z}}(-)$ and $H^i_{\mathcal{Z}}(-)$ by $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$ and $H^i_{\mathfrak{a}}(-)$, respectively. By [F3, Corollary 3.7 and Proposition 3.14 d)], for any complex $X \in \mathcal{D}^f_{\square}(R)$, we know that

$$\sup\{i \in \mathbb{Z} \mid H^i_{\mathfrak{a}}(X) \neq 0\} \leq \dim_R X$$

with equality if R is local and $\mathfrak{a}$ is its maximal ideal. Denote the set of all ideals $\mathfrak{b}$ of R such that $\operatorname{V}(\mathfrak{b}) \subseteq \mathcal{Z}$ by $F(\mathcal{Z})$. Since for any R-module M, $\Gamma_{\mathcal{Z}}(M) = \bigcup_{\mathfrak{b} \in F(\mathcal{Z})} \Gamma_{\mathfrak{b}}(M)$, one can easily check that $H^i_{\mathcal{Z}}(X) \cong \varinjlim_{\mathfrak{b} \in F(\mathcal{Z})} H^i_{\mathfrak{b}}(X)$ for all integers i. Hence $H^i_{\mathcal{Z}}(X) = 0$ for all $i > \dim_R X$.

Let $(R, \mathfrak{m})$ be a local ring, $\mathcal{Z}$ a stable under specialization subset of $\operatorname{Spec} R$ and $X \in \mathcal{D}^f_{\square}(R)$. We prove that $H^{\dim_R X}_{\mathcal{Z}}(X) = 0$ if and only if for any $\mathfrak{p} \in \operatorname{Assh}_{\hat{R}}(X \otimes_R \hat{R})$, there is $\mathfrak{q} \in \mathcal{Z}$ such that $\dim \hat{R}/\mathfrak{q}\hat{R} + \mathfrak{p} > 0$. Yoshino and Yoshizawa [YY, Theorem 2.10] have showed that for any abstract local cohomology functor $\delta : \mathcal{D}_{\sqsubset}(R) \longrightarrow \mathcal{D}_{\sqsubset}(R)$, there is a stable under specialization subset $\mathcal{Z}$ of $\operatorname{Spec} R$ such that $\delta \cong \mathbf{R}\Gamma_{\mathcal{Z}}$. Thus our result may be considered as the largest generalization possible of the Hartshorne-Lichtenbaum Vanishing Theorem. In fact, we show that it includes all known generalizations of the Hartshorne-Lichtenbaum Vanishing Theorem.

2. Results

Let $\mathcal{Z}$ be a stable under specialization subset of $\operatorname{Spec} R$ and $X \in \mathcal{D}(R)$. The Propositions 2.1 and 2.3 below determine some situations where the local cohomology modules $H^i_{\mathcal{Z}}(X)$ are Artinian. Recall that $\operatorname{Supp}_R X$ is defined by $\operatorname{Supp}_R X := \{\mathfrak{p} \in \operatorname{Spec} R \mid X_{\mathfrak{p}} \not\simeq 0\}$ $(= \bigcup_{i \in \mathbb{Z}} \operatorname{Supp}_R H_i(X))$.

PROPOSITION 2.1. *Let $\mathcal{Z}$ be a stable under specialization subset of* Spec R *and* $X \in \mathcal{D}_{\sqsubset}^{f}(R)$. *Assume that* $\operatorname{Supp}_R X \cap \mathcal{Z}$ *consists only of finitely many maximal ideals. Then* $H_{\mathcal{Z}}^{i}(X)$ *is Artinian for all* $i \in \mathbb{Z}$.

PROOF. Let $\mathfrak{p}$ be a prime ideal and $E(R/\mathfrak{p})$ denote the injective envelope of $R/\mathfrak{p}$. Since, $\mathfrak{p}$ is the only associated prime ideal of $E(R/\mathfrak{p})$, it turns out

$$\Gamma_{\mathcal{Z}}(E(R/\mathfrak{p})) = \begin{cases} E(R/\mathfrak{p}), & \mathfrak{p} \in \mathcal{Z} \\ 0, & \mathfrak{p} \notin \mathcal{Z}. \end{cases}$$

For each integer i, the $R_{\mathfrak{p}}$-module $\operatorname{Ext}_{R_{\mathfrak{p}}}^{i}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, X_{\mathfrak{p}})$ is finitely generated, and so

$$\mu^{i}(\mathfrak{p}, X) := \operatorname{Vdim}_{R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}} \operatorname{Ext}_{R_{\mathfrak{p}}}^{i}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, X_{\mathfrak{p}}) < \infty.$$

By [F2, Proposition 3.18], X possesses an injective resolution I such that $I_i \cong \coprod_{\mathfrak{p} \in \operatorname{Spec} R} E(R/\mathfrak{p})^{(\mu^{i}(\mathfrak{p}, X))}$ for all integers i. Let $i \in \mathbb{Z}$. Then

$$\Gamma_{\mathcal{Z}}(I_i) = \coprod_{\mathfrak{p} \in \operatorname{Spec} R} \Gamma_{\mathcal{Z}}(E(R/\mathfrak{p}))^{(\mu^{i}(\mathfrak{p}, X))} = \coprod_{\mathfrak{p} \in \operatorname{Supp}_R X \cap \mathcal{Z}} E(R/\mathfrak{p})^{(\mu^{i}(\mathfrak{p}, X))}.$$

By the assumption, $\operatorname{Supp}_R X \cap \mathcal{Z}$ consists only of finitely many maximal ideals. This yields that $\Gamma_{\mathcal{Z}}(I_i)$ is an Artinian R-module, and so $H_{\mathcal{Z}}^{i}(X) = H_{-i}(\Gamma_{\mathcal{Z}}(I))$ is Artinian too. □

We record the following immediate corollary which extends [Z, Theorem 2.2]. We first recall some definitions. The left derived tensor product functor $- \otimes_R^{\mathbf{L}} \sim$ is computed by taking a projective resolution of the first argument or of the second one. Also, the right derived homomorphism functor $\mathbf{R}\operatorname{Hom}_R(-, \sim)$ is computed by taking a projective resolution of the first argument or by taking an injective resolution of the second one. Let $\mathfrak{a}$ be an ideal of R and M, N two R-modules. The notion of generalized local cohomology modules $H_{\mathfrak{a}}^{i}(M, N) := \varinjlim_n \operatorname{Ext}_R^{i}(M/\mathfrak{a}^n M, N)$ was introduced by Herzog in his Habilitationsschrift [He]. When M is finitely generated, [Y, Theorem 3.4] yields that $H_{\mathfrak{a}}^{i}(M, N) \cong H_{-i}(\mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\operatorname{Hom}_R(M, N)))$ for all integers i.

COROLLARY 2.2. *Let* $\mathfrak{a}$ *be an ideal of* R *and* M *and* N *two finitely generated* R*-modules. Assume that* $\operatorname{Supp}_R M \cap \operatorname{Supp}_R N \cap \operatorname{V}(\mathfrak{a})$ *consists only of finitely many maximal ideals. Then* $H_{\mathfrak{a}}^{i}(M, N)$ *is Artinian for all* $i \in \mathbb{Z}$.

PROPOSITION 2.3. *Let* $\mathcal{Z}$ *be a stable under specialization subset of* Spec R. *Assume that for any finitely generated* R*-module* M *of finite dimension,* $H_{\mathcal{Z}}^{\dim_R M}(M)$ *is Artinian. Then for any finite dimensional complex* $X \in \mathcal{D}_{\square}^{f}(R)$, $H_{\mathcal{Z}}^{\dim_R X}(X)$ *is Artinian.*

PROOF. Set $d := \dim_R X$ and $s := \sup X$. Clearly, we may assume that $X \not\simeq 0$, and so $n := \operatorname{amp} X$ is a non-negative integer. We argue by induction on n. Let $n = 0$. Then $X \simeq \Sigma^s H_s(X)$, and so

$$H^d_{\mathcal{Z}}(X) = H^d_{\mathcal{Z}}(\Sigma^s H_s(X)) = H^{d+s}_{\mathcal{Z}}(H_s(X)) .$$

On the other hand, by [F3, Proposition 3.5], $d = \sup\{\dim H_i(X) - i \mid i \in \mathbb{Z}\}$. Hence $\dim H_s(X) = d + s$, and so $H^{d+s}_{\mathcal{Z}}(H_s(X))$ is Artinian by our assumption. Now, assume that $n \geq 1$ and let $W := \tau_{\supset s} X$ and $Y := \tau_{s-1\subset} X$ be truncated complexes of X; see [C, A.1.14]. Since $\operatorname{amp} W = 0$ and $\operatorname{amp} Y \leq n - 1$, theses complexes satisfy the induction hypothesis. Next, one has

$$\begin{aligned}\dim_R X &= \sup\{\dim_R H_i(X) - i \mid i \in \mathbb{Z}\}\\ &= \max\{\sup\{\dim_R H_i(X) - i \mid i \in \mathbb{Z} - \{s\}\}, \dim_R H_s(X) - s\}\\ &= \max\{\dim_R Y, \dim_R W\} .\end{aligned}$$

Thus by Grothendieck's Vanishing Theorem and induction hypothesis, we deduce that $H^d_{\mathcal{Z}}(W)$ and $H^d_{\mathcal{Z}}(Y)$ are Artinian. Now, by [F1, Theorem 1.41], there is a short exact sequence

$$0 \longrightarrow W \longrightarrow X \longrightarrow Y \longrightarrow 0$$

of complexes which induces a long exact sequence

$$H^{d-1}_{\mathcal{Z}}(Y) \longrightarrow H^d_{\mathcal{Z}}(W) \stackrel{f}{\longrightarrow} H^d_{\mathcal{Z}}(X) \stackrel{g}{\longrightarrow} H^d_{\mathcal{Z}}(Y) \longrightarrow 0 .$$

It implies that $H^d_{\mathcal{Z}}(X)$ is Artinian. □

COROLLARY 2.4. *Let $\mathcal{Z}$ be a stable under specialization subset of* Spec R, $\mathfrak{a}$ *an ideal of R and $X \in \mathcal{D}^f_{\square}(R)$.*

i) *If R is local, then $H^{\dim_R X}_{\mathcal{Z}}(X)$ is Artinian.*

ii) *If* $\dim_R X$ *is finite, then $H^{\dim_R X}_{\mathfrak{a}}(X)$ is Artinian.*

PROOF. In view of the above proposition, i) follows by [DNT, Theorem 2.6 and Lemma 3.2] and ii) follows by [BS, Exercise 7.1.7]. □

Let A be an Artinian R-module. Recall that the set of attached prime ideals of A, $\operatorname{Att}_R A$, is the set of all prime ideals $\mathfrak{p}$ of R such that $\mathfrak{p} = \operatorname{Ann}_R L$ for some quotient L of A. Clearly, $A = 0$ if and only if $\operatorname{Att}_R A$ is empty. If R is local with the maximal ideal $\mathfrak{m}$, then $\operatorname{Att}_R A = \operatorname{Ass}_R(\operatorname{Hom}_R(A, E(R/\mathfrak{m})))$. Also, for an exact sequence $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ of Artinian R-modules, one can see $\operatorname{Att}_R W \subseteq \operatorname{Att}_R V \subseteq \operatorname{Att}_R U \cup \operatorname{Att}_R W$. For proving our theorem, we need to the following lemmas.

LEMMA 2.5. *Let $(R, \mathfrak{m})$ be a local ring and $X \in \mathcal{D}^f_{\square}(R)$. Then* $\operatorname{Att}_R(H^{\dim_R X}_{\mathfrak{m}}(X)) = \operatorname{Assh}_R X$.

PROOF. Set $d := \dim_R X$. By Proposition 2.1, $H^d_{\mathfrak{m}}(X)$ is an Artinian R-module. Hence, we have a natural isomorphism $H^d_{\mathfrak{m}}(X) \cong H^d_{\mathfrak{m}}(X) \otimes_R \hat{R}$, and so [L, Corollary 3.4.4] provides a natural $\hat{R}$-isomorphism $H^d_{\mathfrak{m}}(X) \cong H^d_{\mathfrak{m}\hat{R}}(X \otimes_R \hat{R})$. From the definition of attached prime ideals, it follows that

$$\mathrm{Att}_R(H^d_{\mathfrak{m}}(X)) = \{\mathfrak{q} \cap R | \mathfrak{q} \in \mathrm{Att}_{\hat{R}}(H^d_{\mathfrak{m}\hat{R}}(X \otimes_R \hat{R}))\}\,.$$

Let $\mathfrak{q}$ be a prime ideal of $\hat{R}$, $\mathfrak{p} := \mathfrak{q} \cap R$ and M an R-module. We have the natural isomorphism $M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (\hat{R})_{\mathfrak{q}} \cong (M \otimes_R \hat{R})_{\mathfrak{q}}$. Since, the natural ring homomorphism $R_{\mathfrak{p}} \longrightarrow (\hat{R})_{\mathfrak{q}}$ is faithfully flat, $M_{\mathfrak{p}} = 0$ if and only if $(M \otimes_R \hat{R})_{\mathfrak{q}} = 0$. This implies that $\inf X_{\mathfrak{p}} = \inf(X \otimes_R \hat{R})_{\mathfrak{q}}$. On the other hand, one can easily check that $\dim_R X = \dim_{\hat{R}}(X \otimes_R \hat{R})$. Thus, we can immediately verify that

$$\mathrm{Assh}_R X = \{\mathfrak{q} \cap R | \mathfrak{q} \in \mathrm{Assh}_{\hat{R}}(X \otimes_R \hat{R})\}\,.$$

Therefore, we may and do assume that R is complete, and so it possesses a normalized dualizing complex D. By [Ha, Chapter V, Theorem 6.2], there is a natural isomorphism

$$H^i_{\mathfrak{m}}(X) \cong \mathrm{Hom}_R(\mathrm{Ext}^{-i}_R(X, D), E(R/\mathfrak{m}))$$

for all integers i. Since all homology modules of X and of D are finitely generated, X is homologically bounded and the injective dimension of D is finite, it follows that $\mathbf{R}\,\mathrm{Hom}_R(X, D) \in \mathcal{D}^f_{\square}(R)$. In particular, $\mathrm{Ext}^{-i}_R(X, D)$ is a finitely generated R-module for all $i \in \mathbb{Z}$. Thus we have

$$\begin{aligned}\mathrm{Att}_R(H^d_{\mathfrak{m}}(X)) &= \mathrm{Att}_R(\mathrm{Hom}_R(\mathrm{Ext}^{-d}_R(X, D), E(R/\mathfrak{m})))\\ &= \mathrm{Ass}_R(\mathrm{Hom}_R(\mathrm{Hom}_R(\mathrm{Ext}^{-d}_R(X, D), E(R/\mathfrak{m}))), E(R/\mathfrak{m}))\\ &= \mathrm{Ass}_R(\mathrm{Ext}^{-d}_R(X, D))\\ &= \mathrm{Ass}_R(H_d(\mathbf{R}\,\mathrm{Hom}_R(X, D)))\,.\end{aligned}$$

[F1, Theorem 16.20] implies that $\sup(\mathbf{R}\,\mathrm{Hom}_R(X, D)) = d$. Let $\mathfrak{p} \in \mathrm{Spec}\, R$. By [F1, Theorem 12.26], $\mathfrak{p} \in \mathrm{Ass}_R(H_d(\mathbf{R}\,\mathrm{Hom}_R(X, D)))$ if and only if $\mathrm{depth}_{R_{\mathfrak{p}}} \mathbf{R}\,\mathrm{Hom}_R(X, D)_{\mathfrak{p}} = -d$. But, [C, Lemma A.6.4 and A.6.32] and [F1, Theorem 15.17], yield that

$$\begin{aligned}\mathrm{depth}_{R_{\mathfrak{p}}} \mathbf{R}\,\mathrm{Hom}_R(X, D)_{\mathfrak{p}} &= \mathrm{depth}_{R_{\mathfrak{p}}}(\mathbf{R}\,\mathrm{Hom}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}, D_{\mathfrak{p}}))\\ &= \mathrm{depth}_{R_{\mathfrak{p}}} D_{\mathfrak{p}} + \inf X_{\mathfrak{p}}\\ &= -\dim \frac{R}{\mathfrak{p}} + \inf X_{\mathfrak{p}}\,.\end{aligned}$$

Therefore, $\mathfrak{p} \in \mathrm{Ass}_R(H_d(\mathbf{R}\,\mathrm{Hom}_R(X, D)))$ if and only if $\dim \frac{R}{\mathfrak{p}} - \inf X_{\mathfrak{p}} = \dim_R X$. This means $\mathrm{Att}_R(H^d_{\mathfrak{m}}(X)) = \mathrm{Assh}_R X$, as desired. □

LEMMA 2.6. *Let* $(R, \mathfrak{m})$ *be a local ring*, $\mathcal{Z}$ *a stable under specialization subset of* Spec R *and* $X \in \mathcal{D}^f_\square(R)$. *Then* $H^{\dim_R X}_{\mathcal{Z}}(X)$ *is a homomorphic image of* $H^{\dim_R X}_{\mathfrak{m}}(X)$.

PROOF. Let $\mathfrak{a}$ be an ideal of R and $x \in \mathfrak{m}$. Let I be an injective resolution of X. Then I_x, the localization of I at x, provides an injective resolution of X_x in $\mathcal{D}^f_\square(R_x)$. Now, [BS, Lemma 8.1.1] yields the following exact sequence of complexes

$$0 \longrightarrow \Gamma_{\mathfrak{a}+(x)}(I) \longrightarrow \Gamma_{\mathfrak{a}}(I) \longrightarrow \Gamma_{\mathfrak{a}}(I_x) \longrightarrow 0\,,$$

where the maps are the natural ones. Set $d := \dim_R X$. We deduce the long exact sequence

$$\cdots \longrightarrow H^d_{\mathfrak{a}+(x)}(X) \longrightarrow H^d_{\mathfrak{a}}(X) \longrightarrow H^d_{\mathfrak{a}R_x}(X_x) \longrightarrow 0\,.$$

By Corollary 2.4, $H^d_{\mathfrak{a}}(X)$ is Artinian. Hence $H^d_{\mathfrak{a}}(X)$ is supported at most at $\mathfrak{m}$, and so

$$H^d_{\mathfrak{a}R_x}(X_x) \cong H^d_{\mathfrak{a}}(X)_x = 0\,.$$

Hence, the natural homomorphism $H^d_{\mathfrak{a}+(x)}(X) \longrightarrow H^d_{\mathfrak{a}}(X)$ is epic.

We may choose $x_1, x_2, \ldots, x_n \in R$ such that $\mathfrak{m} = \mathfrak{a} + (x_1, x_2, \ldots, x_n)$. Set $\mathfrak{a}_i := \mathfrak{a} + (x_1, \ldots, x_{i-1})$ for $i = 1, \ldots, n+1$. By the above argument, the natural homomorphism $H^d_{\mathfrak{a}_{i+1}}(X) \longrightarrow H^d_{\mathfrak{a}_i}(X)$ is epic for all $1 \leq i \leq n$. Hence $H^d_{\mathfrak{a}}(X)$ is a homomorphic image of $H^d_{\mathfrak{m}}(X)$. This completes the proof, because $H^d_{\mathcal{Z}}(X) \cong \varinjlim_{\mathfrak{b}} H^d_{\mathfrak{b}}(X)$, where the direct limit is over all ideals $\mathfrak{b}$ of R such that $\mathrm{V}(\mathfrak{b}) \subseteq \mathcal{Z}$. □

LEMMA 2.7. *Let M be a finitely generated R-module and* $X \in \mathcal{D}^f_\square(R)$.

i) $\dim_R(M \otimes^{\mathbf{L}}_R X) \leq \dim_R X$.

ii) *If* $\operatorname{Supp}_R M \cap \operatorname{Assh}_R X \neq \emptyset$, *then* $\dim_R(M \otimes^{\mathbf{L}}_R X) = \dim_R X$ *and*

$$\operatorname{Assh}_R(M \otimes^{\mathbf{L}}_R X) = \operatorname{Supp}_R M \cap \operatorname{Assh}_R X\,.$$

PROOF. For any Noetherian local ring S and any two complexes $V, W \in \mathcal{D}^f_\square(S)$, Nakayama's Lemma for complexes asserts that $\inf(V \otimes^{\mathbf{L}}_R W) = \inf V + \inf W$; see e.g. [C, Corollary A.4.16]. In particular, this yields that $\operatorname{Supp}_R(V \otimes^{\mathbf{L}}_R W) = \operatorname{Supp}_R V \cap \operatorname{Supp}_R W$. Now, by noting that for any complex $Y \in \mathcal{D}(R)$, we have

$$\dim_R Y = \sup\{\dim R/\mathfrak{p} - \inf Y_{\mathfrak{p}} \,|\, \mathfrak{p} \in \operatorname{Supp}_R Y\}\,,$$

both assertions follow immediately. □

Next, we conclude our theorem.

THEOREM 2.8. *Let* $(R, \mathfrak{m})$ *be a local ring*, $\mathcal{Z}$ *a stable under specialization subset of* Spec R *and* $X \in \mathcal{D}^f_\square(R)$. *Then* $\operatorname{Att}_{\hat{R}}(H^{\dim_R X}_{\mathcal{Z}}(X)) = \{\mathfrak{p} \in \operatorname{Assh}_{\hat{R}}(X \otimes_R \hat{R}) | \dim \hat{R}/\mathfrak{q}\hat{R} + \mathfrak{p} = 0$ *for all* $\mathfrak{q} \in \mathcal{Z}\}$.

PROOF. Set $d := \dim_R X$ and $s := \sup X$. We may assume that $n := \operatorname{amp} X$ is a non-negative integer. First, by induction on n, we prove the inclusion $\subseteq$. If $n = 0$, then $X \simeq \Sigma^s H_s(X)$, and so

$$H^d_{\mathcal{Z}}(X) = H^d_{\mathcal{Z}}(\Sigma^s H_s(X)) = H^{d+s}_{\mathcal{Z}}(H_s(X))\,.$$

In the proof of Proposition 2.3, we saw that $\dim H_s(X) = d + s$, hence [DNT, Corollary 2.7] implies that

$$\begin{aligned}\operatorname{Att}_{\hat{R}}(H^d_{\mathcal{Z}}(X)) &= \operatorname{Att}_{\hat{R}}(H^{d+s}_{\mathcal{Z}}(H_s(X)))\\ &= \{\mathfrak{p} \in \operatorname{Assh}_{\hat{R}}(H_s(X) \otimes_R \hat{R}) \mid \dim \hat{R}/\mathfrak{q}\hat{R} + \mathfrak{p} = 0 \ \text{ for all } \ \mathfrak{q} \in \mathcal{Z}\}\\ &= \{\mathfrak{p} \in \operatorname{Assh}_{\hat{R}}(X \otimes_R \hat{R}) \mid \dim \hat{R}/\mathfrak{q}\hat{R} + \mathfrak{p} = 0 \ \text{ for all } \ \mathfrak{q} \in \mathcal{Z}\}\,.\end{aligned}$$

Now, assume that $n \geq 1$ and $\mathfrak{p} \in \operatorname{Att}_{\hat{R}}(H^d_{\mathcal{Z}}(X))$. By Lemma 2.6, $H^d_{\mathcal{Z}}(X)$ is an homomorphic image of $H^d_{\mathfrak{m}}(X)$, and so Lemma 2.5 yields that

$$\operatorname{Att}_{\hat{R}}(H^d_{\mathcal{Z}}(X)) \subseteq \operatorname{Att}_{\hat{R}}(H^d_{\mathfrak{m}}(X)) = \operatorname{Assh}_{\hat{R}}(X \otimes_R \hat{R})\,.$$

Thus $\mathfrak{p} \in \operatorname{Assh}_{\hat{R}}(X \otimes_R \hat{R})$. Let $W := \tau_{\supset s} X$ and $Y := \tau_{s-1\subset} X$ be truncated complexes of X. We have a short exact sequence

$$0 \longrightarrow W \longrightarrow X \longrightarrow Y \longrightarrow 0$$

of complexes and from the proof of Proposition 2.3, we know that $\dim_R X = \max\{\dim_R W, \dim_R Y\}$. From the long exact sequence

$$\cdots \longrightarrow H^d_{\mathcal{Z}}(W) \longrightarrow H^d_{\mathcal{Z}}(X) \longrightarrow H^d_{\mathcal{Z}}(Y) \longrightarrow 0\,,$$

we deduce that

$$\operatorname{Att}_{\hat{R}}(H^d_{\mathcal{Z}}(X)) \subseteq \operatorname{Att}_{\hat{R}}(H^d_{\mathcal{Z}}(W)) \cup \operatorname{Att}_{\hat{R}}(H^d_{\mathcal{Z}}(Y))\,.$$

Thus, either $\mathfrak{p} \in \operatorname{Att}_{\hat{R}}(H^d_{\mathcal{Z}}(W))$ or $\mathfrak{p} \in \operatorname{Att}_{\hat{R}}(H^d_{\mathcal{Z}}(Y))$. By Grothendieck's Vanishing Theorem, the first case implies that $\dim_R W = d$ and the second case implies that $\dim_R Y = d$. Since $\operatorname{amp} W = 0$ and $\operatorname{amp} Y \leq n - 1$, in both cases, the induction hypothesis yields that $\dim \hat{R}/\mathfrak{q}\hat{R} + \mathfrak{p} = 0$ for all $\mathfrak{q} \in \mathcal{Z}$.

Now, we prove the inclusion $\supseteq$. Let $\mathfrak{p} \in \operatorname{Assh}_{\hat{R}}(X \otimes_R \hat{R})$ be such that $\dim \hat{R}/\mathfrak{q}\hat{R} + \mathfrak{p} = 0$ for all $\mathfrak{q} \in \mathcal{Z}$. We have to show that $\mathfrak{p} \in \operatorname{Att}_{\hat{R}}(H^d_{\mathcal{Z}}(X))$. Since $H^d_{\mathcal{Z}}(X)$ is an Artinian R-module, we have the natural isomorphism $H^d_{\mathcal{Z}}(X) \cong H^d_{\mathcal{Z}}(X) \otimes_R \hat{R}$. On the other hand, by [L, Corollary 3.4.4], for any ideal $\mathfrak{a}$ of R, there is a natural $\hat{R}$-isomorphism $H^d_{\mathfrak{a}}(X) \otimes_R \hat{R} \cong H^d_{\mathfrak{a}\hat{R}}(X \otimes_R \hat{R})$. Let $\hat{\mathcal{Z}} := \{\mathfrak{q} \in \operatorname{Spec} \hat{R} \mid \mathfrak{q} \cap R \in \mathcal{Z}\}$, which can be easily checked that is a stable under specialization subset of $\operatorname{Spec} \hat{R}$. It is straightforward to see that the two families $\{\mathfrak{a}\hat{R} \mid \mathfrak{a} \ \text{ is an ideal of } \ R \ \text{ with } \ \mathrm{V}(\mathfrak{a}) \subseteq \mathcal{Z}\}$ and $\{\mathfrak{b} \mid \mathfrak{b} \ \text{ is an ideal of } \ \hat{R} \ \text{ with } \ \mathrm{V}(\mathfrak{b}) \subseteq \hat{\mathcal{Z}}\}$ are cofinal. This implies that $H^d_{\mathcal{Z}}(X) \cong H^d_{\hat{\mathcal{Z}}}(X \otimes_R \hat{R})$. Also, we have $\dim_{\hat{R}}(X \otimes_R \hat{R}) = \dim_R X$ and $\dim \hat{R}/\mathfrak{q} + \mathfrak{p} = 0$ for all $\mathfrak{q} \in \hat{\mathcal{Z}}$. Therefore, we may and do assume that R is complete.

Since R is complete, there is a complete regular local ring $(T, \mathfrak{n})$ and a surjective ring homomorphism $f : T \longrightarrow R$. One can easily check that $X \in \mathcal{D}^f_{\square}(T)$ and $\dim_T X = \dim_R X$. Set $\bar{\mathcal{Z}} := \{f^{-1}(\mathfrak{q}) | \mathfrak{q} \in \mathcal{Z}\}$, which is clearly a stable under specialization subset of $\operatorname{Spec} T$. By [L, Corollary 3.4.3], for any ideal $\mathfrak{b}$ of T, there is a natural T-isomorphism $H^d_{\mathfrak{b}}(X) \cong H^d_{\mathfrak{b}R}(X)$. From this, we can conclude a natural T-isomorphism $H^d_{\bar{\mathcal{Z}}}(X) \cong H^d_{\mathcal{Z}}(X)$. For any Artinian R-module A and any $\mathfrak{q} \in \operatorname{Spec} R$, it turns out that A is also Artinian as a T-module and $\mathfrak{q} \in \operatorname{Att}_R A$ if and only if $f^{-1}(\mathfrak{q}) \in \operatorname{Att}_T A$. Finally, we have $\dim T/\bar{\mathfrak{q}} + f^{-1}(\mathfrak{p}) = 0$ for all $\bar{\mathfrak{q}} \in \bar{\mathcal{Z}}$ and $\operatorname{Assh}_T X = \{f^{-1}(\mathfrak{q}) | \mathfrak{q} \in \operatorname{Assh}_R X\}$. Thus from now on, we can assume that R is a complete regular local ring.

Lemma 2.7 yields that $\dim_R(\mathfrak{p} \otimes^{\mathbf{L}}_R X) \leq \dim_R X$ and $\dim_R(R/\mathfrak{p} \otimes^{\mathbf{L}}_R X) = \dim_R X$. Let P be a projective resolution of X. Applying $- \otimes_R P$ to the short exact sequence

$$0 \longrightarrow \mathfrak{p} \longrightarrow R \longrightarrow R/\mathfrak{p} \longrightarrow 0,$$

yields the following exact sequence of complexes

$$0 \longrightarrow \mathfrak{p} \otimes^{\mathbf{L}}_R X \longrightarrow X \longrightarrow R/\mathfrak{p} \otimes^{\mathbf{L}}_R X \longrightarrow 0.$$

It yields the following exact sequence

$$\cdots \longrightarrow H^d_{\mathcal{Z}}(\mathfrak{p} \otimes^{\mathbf{L}}_R X) \longrightarrow H^d_{\mathcal{Z}}(X) \longrightarrow H^d_{\mathcal{Z}}(R/\mathfrak{p} \otimes^{\mathbf{L}}_R X) \longrightarrow 0.$$

As R is regular, the projective dimension of any R-module is finite, and so for any finitely generated R-module M, one has $M \otimes^{\mathbf{L}}_R X \in \mathcal{D}^f_{\square}(R)$. Since $\dim R/\mathfrak{q} + \mathfrak{p} = 0$ for all $\mathfrak{q} \in \mathcal{Z}$, it follows that $\Gamma_{\mathcal{Z}}(\Gamma_{\mathfrak{p}}(M)) = \Gamma_{\mathfrak{m}}(M)$ for all R-modules M. Let I be an injective resolution of $R/\mathfrak{p} \otimes^{\mathbf{L}}_R X$. Since

$$\operatorname{Supp}_R I = \operatorname{Supp}_R(R/\mathfrak{p} \otimes^{\mathbf{L}}_R X) \subseteq \mathrm{V}(\mathfrak{p}),$$

by [L, Corollary 3.2.1], $\Gamma_{\mathfrak{p}}(I) \simeq I$, and so

$$\Gamma_{\mathcal{Z}}(I) \simeq \Gamma_{\mathcal{Z}}(\Gamma_{\mathfrak{p}}(I)) = \Gamma_{\mathfrak{m}}(I).$$

In particular, there is an isomorphism $H^d_{\mathcal{Z}}(R/\mathfrak{p} \otimes^{\mathbf{L}}_R X) \cong H^d_{\mathfrak{m}}(R/\mathfrak{p} \otimes^{\mathbf{L}}_R X)$. Therefore, by Lemmas 2.7 and 2.5, we deduce that $\mathfrak{p} \in \operatorname{Att}_R(H^d_{\mathcal{Z}}(R/\mathfrak{p} \otimes^{\mathbf{L}}_R X)) \subseteq \operatorname{Att}_R(H^d_{\mathcal{Z}}(X))$. □

Now, we are ready to establish the derived category analogue of the Hartshorne-Lichtenbaum Vanishing Theorem.

COROLLARY 2.9. *Let* $(R, \mathfrak{m})$ *be a local ring*, $\mathcal{Z}$ *a stable under specialization subset of* $\operatorname{Spec} R$ *and* $X \in \mathcal{D}^f_{\square}(R)$. *The following are equivalent*:

i) $H^{\dim_R X}_{\mathcal{Z}}(X) = 0$.

ii) *For any* $\mathfrak{p} \in \operatorname{Assh}_{\hat{R}}(X \otimes_R \hat{R})$, *there is* $\mathfrak{q} \in \mathcal{Z}$ *such that* $\dim \hat{R}/\mathfrak{q}\hat{R} + \mathfrak{p} > 0$.

COROLLARY 2.10. *Let* $\mathfrak{a}$ *be an ideal of the local ring* $(R, \mathfrak{m})$ *and* $X \in \mathcal{D}^f_\square(R)$.

1) $\operatorname{Att}_{\hat{R}}(H^{\dim_R X}_{\mathfrak{a}}(X)) = \{\mathfrak{p} \in \operatorname{Assh}_{\hat{R}}(X \otimes_R \hat{R}) \mid \dim \hat{R}/\mathfrak{a}\hat{R} + \mathfrak{p} = 0\}$.

2) *The following are equivalent*:

i) $H^{\dim_R X}_{\mathfrak{a}}(X) = 0$.

ii) $\dim \hat{R}/\mathfrak{a}\hat{R} + \mathfrak{p} > 0$ *for all* $\mathfrak{p} \in \operatorname{Assh}_{\hat{R}}(X \otimes_R \hat{R})$.

COROLLARY 2.11. *Let* $(R, \mathfrak{m})$ *be a local ring*, $\mathcal{Z}$ *a stable under specialization subset of* $\operatorname{Spec} R$ *and* M, N *two finitely generated* R*-modules. Assume that* $\mathbf{R}\operatorname{Hom}_R(M, N) \in \mathcal{D}^f_\square(R)$ *and set* $d := \dim_R(\mathbf{R}\operatorname{Hom}_R(M, N))$. *The following are equivalent*:

i) $H^d_{\mathcal{Z}}(M, N) = 0$.

ii) *For any* $\mathfrak{p} \in \operatorname{Assh}_{\hat{R}}(\mathbf{R}\operatorname{Hom}_{\hat{R}}(\hat{M}, \hat{N}))$, *there is* $\mathfrak{q} \in \mathcal{Z}$ *such that* $\dim \hat{R}/\mathfrak{q}\hat{R} + \mathfrak{p} > 0$.

PROOF. Note that $\mathbf{R}\operatorname{Hom}_R(M, N) \otimes_R \hat{R} \simeq \mathbf{R}\operatorname{Hom}_{\hat{R}}(\hat{M}, \hat{N})$, and so the result follows by Corollary 2.9. □

REMARK 2.12. Let $\mathcal{Z}$ be a stable under specialization subset of $\operatorname{Spec} R$ and $X \in \mathcal{D}^f_\square(R)$.

1) Suppose that dimension of X is finite. Then $H^{\dim_R X}_{\mathcal{Z}}(X)$ is supported only at maximal ideals. To realize this, in view of Corollary 2.4 ii), it is enough to notice that $H^{\dim_R X}_{\mathcal{Z}}(X) \cong \varinjlim_{\mathfrak{a}} H^{\dim_R X}_{\mathfrak{a}}(X)$, where the direct limit is over all ideals $\mathfrak{a}$ of R such that $\mathrm{V}(\mathfrak{a}) \subseteq \mathcal{Z}$. But, $H^{\dim_R X}_{\mathcal{Z}}(X)$ is not Artinian in general. To this end, let R be a finite dimensional Gorenstein ring such that the set $\mathcal{Z} := \{\mathfrak{m} \in \operatorname{Max} R \mid \operatorname{ht}\mathfrak{m} = \dim R\}$ is infinite. Clearly, $\mathcal{Z}$ is a stable under specialization subset of $\operatorname{Spec} R$. The minimal injective resolution of R has the form

$$0 \longrightarrow \bigsqcup_{\operatorname{ht}\mathfrak{p}=0} E(R/\mathfrak{p}) \longrightarrow \bigsqcup_{\operatorname{ht}\mathfrak{p}=1} E(R/\mathfrak{p}) \longrightarrow \cdots \longrightarrow \bigsqcup_{\operatorname{ht}\mathfrak{p}=\dim R} E(R/\mathfrak{p}) \longrightarrow 0\,.$$

Hence $H^{\dim R}_{\mathcal{Z}}(R) = \bigsqcup_{\mathfrak{m}\in\mathcal{Z}} E(R/\mathfrak{m})$, which is not Artinian.

2) Suppose that R is local with the maximal ideal $\mathfrak{m}$ and I, J two ideals of R. In [TYY], Takahashi, Yoshino and Yoshizawa considered the following stable under specialization subset of $\operatorname{Spec} R$

$$W(I, J) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I^n \subseteq \mathfrak{p} + J \ \text{ for a natural integer n}\}\,.$$

For each integer i, they called $H^i_{I,J}(-) := H^i_{W(I,J)}(-)$, i-th local cohomology functor with respect to (I, J). For the ring R itself, they extended the Hartshorne-Lichtenbaum Vanishing Theorem ; see [TYY, Theorem 4.9]. Namely, they showed that $H^{\dim R}_{I,J}(R) = 0$ if and only if for any prime ideal $\mathfrak{p} \in \operatorname{Assh}_{\hat{R}} \hat{R} \cap \mathrm{V}(J\hat{R})$, we

have $\dim \hat{R}/I\hat{R}+\mathfrak{p}>0$. On the other hand by [DNT, Theorem 2.8], $H^{\dim R}_{I,J}(R)=0$ if and only if for any prime ideal $\mathfrak{p}\in \operatorname{Assh}_{\hat{R}}\hat{R}$, there is $\mathfrak{q}\in W(I,J)$ such that $\dim \hat{R}/\mathfrak{q}\hat{R}+\mathfrak{p}>0$. Hence the following statements are equivalent:
i) For any prime ideal $\mathfrak{p}\in \operatorname{Assh}_{\hat{R}}\hat{R}\cap \operatorname{V}(J\hat{R})$, we have $\dim \hat{R}/I\hat{R}+\mathfrak{p}>0$.
ii) For any prime ideal $\mathfrak{p}\in \operatorname{Assh}_{\hat{R}}\hat{R}$, there is $\mathfrak{q}\in W(I,J)$ such that $\dim \hat{R}/\mathfrak{q}\hat{R}+\mathfrak{p}>0$.
As Takahashi, Yoshino and Yoshizawa [TYY, Remark 4.10] have mentioned, it is not so easy to check the equivalence of these statements directly. Here, we do this under the extra assumption that R is complete. (In fact this assumption is not needed for the implication $ii)\Longrightarrow i)$.) Suppose $ii)$ holds and let $\mathfrak{p}\in\operatorname{Assh}_R R\cap \operatorname{V}(J)$. By the assumption there is $\mathfrak{q}\in W(I,J)$ such that $\dim R/\mathfrak{q}+\mathfrak{p}>0$. Since $\mathfrak{q}\in W(I,J)$, there is a natural integer n, such that $I^n\subseteq \mathfrak{q}+J$. This yields that $I^n+\mathfrak{p}\subseteq\mathfrak{q}+\mathfrak{p}$, and so

$$\dim R/I+\mathfrak{p}=\dim R/I^n+\mathfrak{p}\geq \dim R/\mathfrak{q}+\mathfrak{p}>0\,.$$

Conversely, suppose that $i)$ holds and let $\mathfrak{p}\in\operatorname{Assh}_R R$. First, assume that $J\subseteq\mathfrak{p}$. Then by the assumption, $\dim R/I+\mathfrak{p}>0$, and so there is $\mathfrak{q}\in \operatorname{V}(I+\mathfrak{p})$ such that $\dim R/\mathfrak{q}>0$. Then $I+J\subseteq I+\mathfrak{p}\subseteq\mathfrak{q}$. Hence $\mathfrak{q}\in W(I,J)$ and $\dim R/\mathfrak{q}+\mathfrak{p}=\dim R/\mathfrak{q}>0$. Thus $ii)$ follows when $J\subseteq\mathfrak{p}$. Now, assume that $J\nsubseteq\mathfrak{p}$. By [TYY, Lemma 3.3],

$$\operatorname{V}(J)=\bigcap_{\mathfrak{q}\in W(\mathfrak{m},J)} W(\mathfrak{m},\mathfrak{q})\,.$$

So, there is $\mathfrak{q}\in W(\mathfrak{m},J)\subseteq W(I,J)$ such that $\mathfrak{p}\notin W(\mathfrak{m},\mathfrak{q})$. Since $\mathfrak{p}\notin W(\mathfrak{m},\mathfrak{q})$, it follows that $\mathfrak{q}+\mathfrak{p}$ is not $\mathfrak{m}$-primary, and so $\dim R/\mathfrak{q}+\mathfrak{p}>0$.

3) Suppose that R is local and $F(\mathcal{Z})$ denote the set of all ideals $\mathfrak{b}$ of R such that $\operatorname{V}(\mathfrak{b})\subseteq\mathcal{Z}$. As we mentioned in the introduction $H^i_{\mathcal{Z}}(X)\cong \varinjlim_{\mathfrak{b}\in F(\mathcal{Z})} H^i_{\mathfrak{b}}(X)$ for all integers i. The relationship between $H^{\dim_R X}_{\mathcal{Z}}(X)$ and $H^{\dim_R X}_{\mathfrak{b}}(X)'$s is more deeper. In fact by Theorem 2.8, we have

$$\operatorname{Att}_{\hat{R}}(H^{\dim_R X}_{\mathcal{Z}}(X))=\bigcap_{\mathfrak{b}\in F(\mathcal{Z})}\operatorname{Att}_{\hat{R}}(H^{\dim_R X}_{\mathfrak{b}}(X))\,.$$

This implies that $H^{\dim_R X}_{\mathcal{Z}}(X)=0$ if and only if $H^{\dim_R X}_{\mathfrak{b}}(X)=0$ for an ideal $\mathfrak{b}\in F(\mathcal{Z})$.

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