TOKYO J. MATH.
VOL. 35, NO. 2, 2012

Positive Solutions for Non-cooperative Singular p-Laplacian Systems

D. D. HAI

Mississippi State University

(Communicated by F. Nakano)

Abstract. We prove the existence of positive solutions for the p-Laplacian system

$$\begin{cases} -\Delta_p u_1 = \lambda f_1(u_2) & \text{in } \Omega\,, \\ -\Delta_p u_2 = \lambda f_2(u_1) & \text{in } \Omega\,, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$, $f_i : (0, \infty) \to \mathbf{R}$ are possibly singular at 0 and are not required to be positive or nondecreasing, and λ is a large parameter.

1. Introduction

Consider the system

$$\begin{cases} -\Delta_p u_1 = \lambda f_1(u_2) & \text{in } \Omega\,, \\ -\Delta_p u_2 = \lambda f_2(u_1) & \text{in } \Omega\,, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{I}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$, $f_i : (0, \infty) \to \mathbf{R}$, $i = 1, 2$, and λ is a positive parameter.

The system (I) with f_i nonsingular has been studied extensively in recent year (see e.g. [1, 3, 9, 11] and the references therein). In this paper, we are interested in obtaining positive solutions of (I) when f_i are possibly singular at 0 and are not required to be nonnegative, nondecreasing, or bounded away from 0 at infinity. Such nonlinearities have not been considered in the literature to the best of our knowledge. Our approach is based on the method of sub- and supersolutions.

2. Main results

We make the following assumptions:

(B.1) $f_i : (0, \infty) \to \mathbf{R}$ are continuous, $i = 1, 2$.

Received December 6, 2010
2000 *Mathematics Subject Classification*: 35J25, 35J70

(B.2) There exist numbers $a, b, c, A > 0, \alpha_i, \beta_i \in (0,1)$ with $\beta_i < p-1$ and $\alpha_i \geq \beta_i$ such that

$$-\frac{b}{t^{\alpha_i}} \leq f_i(t) \leq \frac{c}{t^{\beta_i}}$$

for $t > 0$, and

$$f_i(t) \geq \frac{a}{t^{\beta_i}}$$

for $t > A$.

(B.3) There exist numbers $L, A > 0$ such that

$$f_i(t) \geq L$$

for $t > A, i = 1, 2$, and

$$\lim_{t\to\infty} \frac{f_1^{\frac{1}{p-1}}\left(c f_2^{\frac{1}{p-1}}(t)\right)}{t} = 0$$

for each $c > 0$.

(B.4) There exists a number $\delta \in (0,1)$ such that

$$\limsup_{t\to 0^+} t^{\delta} |f_i(t)| < \infty$$

for $i = 1, 2$.

By a solution of (I), we mean a pair $(u, v) \in C^{1,\alpha}(\bar{\Omega}) \times C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$ that satisfies (I) in the weak sense.

THEOREM 2.1. *Let* (B.1)–(B.2) *hold. Then problem* (I) *has a positive solution* $u = (u_{1,\lambda}, u_{2,\lambda})$ *for* λ *large. Furthermore* $\|u_{i,\lambda}\|_\infty \to \infty$ *as* $\lambda \to \infty, i = 1, 2$.

THEOREM 2.2. *Let* (B.1), (B.3)*, and* (B.4) *hold. Then problem* (I) *has a positive solution* $u = (u_{1,\lambda}, u_{2,\lambda})$ *for* λ *large. Furthermore* $\|u_{i,\lambda}\|_\infty \to \infty$ *as* $\lambda \to \infty, i = 1, 2$.

REMARK 2.1. *A result similar to Theorem* 2.2 *was obtained in Theorem* 2.2 *of* [8]. *However, the theorem in* [8]*, when applied to* (B.4)*, requires that* $\delta < 1/n$. *Theorem* 2.2 *also improves Theorem A in* [11], *where* f_i *are assumed to be nondecreasing, nonsingular, and unbounded*

EXAMPLE 2.1. *Let* $f_1(u_2) = -\frac{b_1}{u_2^{\alpha_1}} + \frac{c_1}{u_2^{\beta_1}}$, $f_2(u_1) = -\frac{b_2}{u_1^{\alpha_2}} + \frac{c_2}{u_1^{\beta_2}}$, where $b_i, c_i > 0$, $p \geq 2$, $\alpha_i, \beta_i \in (0,1)$ *and* $\alpha_i > \beta_i$. *Then* f_i *satisfy* (B.1),(B.2) *and therefore* (I) *has a positive solution for* λ *large, by Theorem* 2.1. *Note that the nonlinearities* $f_i(t)$ *decay to* 0 as $t \to \infty$, *which do not seem to have been considered in the literature.*

3. Preliminary results

We shall denote the norms in $L^q(\Omega)$, $C^1(\bar{\Omega})$, and $C^{1,\alpha}(\bar{\Omega})$ by $\|\cdot\|_q$, $|\cdot|_1$, and $|\cdot|_{1,\alpha}$ respectively.

The following results were established in [10]. For convenience, we sketch the proofs. Let $d(x)$ denote the distance from x to the boundary of Ω.

LEMMA 3.1 [10]. *Let $h \in L^\infty_{loc}(\Omega)$ and suppose there exist numbers $\gamma \in (0,1)$ and $C > 0$ such that*

$$|h(x)| \leq \frac{C}{d^\gamma(x)} \tag{3.1}$$

for a.e. $x \in \Omega$. Let $u \in W_0^{1,p}(\Omega)$ be the solution of

$$\begin{cases} -\Delta_p u = h & \text{in } \Omega\,, \\ \quad u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3.2}$$

Then there exist constants $\alpha \in (0,1)$ and $M > 0$ depending only on C, γ, Ω such that $u \in C^{1,\alpha}(\bar{\Omega})$ and $|u|_{1,\alpha} < M$.

PROOF. Suppose $p = 2$. It follows from [5] that the problem

$$-\Delta v = \frac{1}{v^\gamma} \quad \text{in } \Omega\,, \quad v = 0 \quad \text{on } \partial\Omega\,,$$

has a positive solution v which is Lipschitz continuous in $\bar{\Omega}$. Let $C_1 > 0$ be such that $v(x) \leq C_1 d(x)$ in Ω. Then

$$-\Delta(CC_1^\gamma v) \geq \frac{C}{d^\gamma} \quad \text{in } \Omega\,.$$

Let $\tilde{u}$ be the solution of

$$-\Delta\tilde{u} = |h| \quad \text{in } \Omega\,, \quad \tilde{u} = 0 \quad \text{on } \partial\Omega\,,$$

and $\bar{u} = u + \tilde{u}$. Then

$$-\Delta\bar{u} = h + |h| \geq 0 \quad \text{in } \Omega\,.$$

By the maximum principle, $\tilde{u}(x) \leq CC_1^\gamma v(x) \leq C_2 d(x)$ and $u(x) \leq C_2 d(x)$ similarly, and thus one obtains $\bar{u}(x) \leq 2C_2 d(x)$ for $x \in \Omega$. Using the regularity result in [7, Theorem B.1], we conclude that there exist $\alpha \in (0,1)$ and $M_0 > 0$ such that $\tilde{u}, \bar{u} \in C^{1,\alpha}(\bar{\Omega})$ and $|\tilde{u}|_{1,\alpha}, |\bar{u}|_{1,\alpha} < M_0$. Since $u = \bar{u} - \tilde{u}$, Lemma 3.1 with $p = 2$ follows.

Now let u be the solution of (3.2) with $p > 1$. From Lemma 3.1, Theorem B.1, and the proof of Lemma A.7 in [7], it follows that the problem

$$\begin{cases} -\Delta_p v = \frac{C}{v^\gamma} & \text{in } \Omega\,, \\ \quad v = 0 & \text{on } \partial\Omega\,, \end{cases}$$

has a unique positive solution $v \in W_0^{1,p}(\Omega)$ with $v \le c_0 d$ in Ω. This implies

$$-\Delta_p \left(c_0^{\frac{\gamma}{p-1}} v \right) \ge \frac{C}{d^\gamma} \quad \text{in } \Omega ,$$

Since

$$-\Delta_p u \le \frac{C}{d^\gamma} \quad \text{and} \quad -\Delta_p(-u) \le \frac{C}{d^\gamma}$$

in Ω, the weak comparison principle (see e.g. [14]) implies

$$|u| \le c_0^{\frac{\gamma}{p-1}} v \le c_0^{\frac{\gamma}{p-1}+1} d \quad \text{in } \Omega.$$

Next, let $w \in C^{1,\alpha}(\bar{\Omega})$ be the solution of

$$-\Delta w = h \quad \text{in } \Omega , \quad w = 0 \quad \text{on } \partial\Omega .$$

Then

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u - \nabla w) = 0 \quad \text{in } \Omega ,$$

and Lemma 3.1 now follows from Lieberman's result [12, Theorem 1].

COROLLARY 3.1. *Let* $\varepsilon > 0$ *and* $h, \tilde{h} \in L^\infty_{loc}(\Omega)$ *satisfy (3.1) with* $h \ge 0, h \not\equiv 0$. *Let* $u, u_\varepsilon \in W_0^{1,p}(\Omega)$ *be, respectively, the solutions of*

$$\begin{cases} -\Delta_p u = h & \text{in } \Omega , \\ u = 0 & \text{on } \partial\Omega , \end{cases}$$

and

$$-\Delta_p u_\varepsilon = \begin{cases} h & \text{if } d(x) > \varepsilon , \\ \tilde{h} & \text{if } d(x) < \varepsilon . \end{cases}$$

Then for ε *small enough*,

$$u_\varepsilon \ge u/2 \quad \text{in } \Omega.$$

PROOF. By Lemma 3.1, there exist $M > 0$ and $\alpha \in (0, 1)$ so that $|u|_{1,\alpha}, |u_\varepsilon|_{1,\alpha} < M$. By the strong maximum principle [15], there exists $\kappa > 0$ such that $u \ge \kappa d$ in Ω. Multiplying the equation

$$-\Delta_p u - (-\Delta_p u_\varepsilon) = \begin{cases} 0 & \text{if } d(x) > \varepsilon , \\ h - \tilde{h} & \text{if } d(x) < \varepsilon \end{cases}$$

by $u - u_\varepsilon$ and integrating gives

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \right) \cdot \nabla(u - u_\varepsilon) dx \le 2M \int_{d<\varepsilon} |h - \tilde{h}| dx \qquad (3.3)$$

Note that for $x, y \in \mathbf{R}^n$,

$$(|x| + |y|)^r (|x|^{p-2}x - |y|^{p-2}y) \cdot (x - y) \geq C_0 |x - y|^{\max(p,2)},$$

where $r = 2 - \min(p, 2)$, $C_0 = (1/2)^{p-1}$, if $p \geq 2$, $C_0 = p - 1$, if $p < 2$ (see e.g. [13, Lemma 30.1]). Using this inequality with $x = \nabla u$, $y = \nabla u_\varepsilon$ in (3.3) and note that $|x| + |y| \leq 2M$, we obtain

$$\frac{C_0}{(2M)^r} \int_\Omega |\nabla(u - u_\varepsilon)|^{\max(p,2)} dx \leq 2M \int_{d<\varepsilon} |h - \tilde{h}| dx \leq 4MC \int_{d<\varepsilon} \frac{1}{d^\gamma(x)} dx$$

Hence $\|\nabla(u - u_\varepsilon)\|_2 \to 0$ as $\varepsilon \to 0$, and since $C^{1,\alpha}(\bar{\Omega})$ is compactly imbedded in $C^1(\bar{\Omega})$, we obtain $|u - u_\varepsilon|_1 \to 0$ as $\varepsilon \to 0$. Consequently, if ε is sufficiently small,

$$|u_\varepsilon - u|_1 \leq \kappa/2,$$

which implies

$$u_\varepsilon \geq u - (\kappa/2)d \geq u/2 \quad \text{in } \Omega,$$

which completes the proof.

4. Proofs of main results

PROOF OF THEOREM 2.1. Let z_i, $i = 1, 2$, be the solutions of

$$\begin{cases} -\Delta_p z_i = \frac{1}{z_i^{\beta_i}} & \text{in } \Omega, \\ z_i = 0 & \text{on } \partial\Omega, \end{cases}$$

and let $m > 0$ be such that $z_i \leq m z_j$ in Ω for $i \neq j$. Choose $\delta > 0$ so that

$$m\delta^{1 - \frac{\beta_i \beta_j}{(p-1)^2}} \leq \left(ac^{-\frac{\beta_i}{p-1}} m^{-\beta_i} / 2^{p-1}\right)^{\frac{1}{p-1}}, \quad i \neq j.$$

Let $\varepsilon > 0$ and u_i satisfy

$$-\Delta_p u_i = \begin{cases} a\left(\frac{\delta^{\beta_j}}{cm^{p-1}}\right)^{\frac{\beta_i}{p-1}} \frac{1}{z_i^{\beta_i}} & \text{if } d(x) > \varepsilon, \\ -\frac{b}{\delta^{\alpha_i} z_i^{\alpha_i}} & \text{if } d(x) < \varepsilon \end{cases}, \quad u_i = 0 \text{ on } \partial\Omega.$$

Using Corollary 3.1 with

$$u = \left[a\left(\frac{\delta^{\beta_j}}{cm^{p-1}}\right)^{\frac{\beta_i}{p-1}}\right]^{\frac{1}{p-1}} z_i, \quad h = a\left(\frac{\delta^{\beta_j}}{cm^{p-1}}\right)^{\frac{\beta_i}{p-1}} \frac{1}{z_i^{\beta_i}},$$

$u_\varepsilon = u_i$, $\tilde{h} = -\frac{b}{\delta^{\alpha_i} z_i^{\alpha_i}}$, and note that $h, \tilde{h}$ satisfy (3.1) with $\gamma = \max(\beta_i, \alpha_i)$, it follows that if $\varepsilon > 0$ is small enough then $u_\varepsilon \geq u/2$ in Ω, i.e.,

$$u_i \geq \frac{1}{2}\left[a\left(\frac{\delta^{\beta_j}}{cm^{p-1}}\right)^{\frac{\beta_i}{p-1}}\right]^{\frac{1}{p-1}} z_i \geq \delta m z_i \geq \delta z_j \tag{4.1}$$

in $\Omega,\ i = 1, 2,\ i \neq j$. Let $r_i = \frac{p-1-\beta_i}{(p-1)^2-\beta_i\beta_j}$ and note that $1 - r_j\beta_i = r_i(p-1)$ for $i \neq j$. Define

$$\Phi_i = \lambda^{r_i} u_i\,, \quad \Psi_i = \lambda^{r_i}\delta^{-\frac{\beta_i}{p-1}} c^{\frac{1}{p-1}} z_i\,,$$

$i = 1, 2$. By the comparison principle,

$$u_i \leq \left[a\left(\frac{\delta^{\beta_j}}{cm^{p-1}}\right)^{\frac{\beta_i}{p-1}}\right]^{\frac{1}{p-1}} z_i \quad \text{in } \Omega\,,$$

and so $\Phi_i \leq \Psi_i$ in Ω if δ is small enough. We shall verify that $\Phi = (\Phi_1, \Phi_2)$ and $\Psi = (\Psi_1, \Psi_2)$ form a system of sub- and supersolutions for (I) (see Appendix). For $\xi \in W_0^{1,p}(\Omega)$ with $\xi \geq 0$ and $v_j \in [\Phi_j, \Psi_j]$, we have from (4.1) that for $i \neq j$,

$$v_j \geq \lambda^{r_j}\delta z_i \quad \text{in } \Omega\,,$$

and thus

$$\lambda\int_\Omega f_i(v_j)\xi dx \leq \lambda c\int_\Omega \frac{\xi}{v_j^{\beta_i}}dx \leq \frac{\lambda^{1-r_j\beta_i}c}{\delta^{\beta_i}}\int_\Omega \frac{\xi}{z_i^{\beta_i}}dx = \frac{\lambda^{r_i(p-1)}c}{\delta^{\beta_i}}\int_\Omega \frac{\xi}{z_i^{\beta_i}}dx$$

$$= \int_\Omega |\nabla\Psi_i|^{p-2}\nabla\Psi_i \cdot \nabla\xi dx\,. \tag{4.2}$$

Next, we have

$$\int_\Omega |\nabla\Phi_i|^{p-2}\nabla\Phi_i \cdot \nabla\xi dx = \lambda^{r_i(p-1)} a\left(\frac{\delta^{\beta_j}}{cm^{p-1}}\right)^{\frac{\beta_i}{p-1}}\int_{d>\varepsilon}\frac{\xi}{z_i^{\beta_i}}dx$$

$$-\frac{\lambda^{r_i(p-1)}b}{\delta^{\alpha_i}}\int_{d<\varepsilon}\frac{\xi}{z_i^{\alpha_i}}dx\,. \tag{4.3}$$

Since there exists $m_0 > 0$ so that $z_i \geq m_0 d$ in Ω, $i = 1, 2$, it follows that

$$v_j(x) \geq \lambda^{r_j}\delta z_i(x) \geq \lambda^{r_j}\delta m_0\varepsilon > A\,,$$

if $d(x) > \varepsilon$ and $\lambda \gg 1$. Hence

$$\lambda \int_{d>\varepsilon} f_i(v_j)\xi dx \geq \lambda a \int_{d>\varepsilon} \frac{\xi}{v_j^{\beta_i}} dx \geq \lambda^{1-r_j\beta_i} a \left(\frac{\delta^{\beta_j}}{c}\right)^{\frac{\beta_i}{p-1}} \int_{d>\varepsilon} \frac{\xi}{z_j^{\beta_i}} dx$$

$$\geq \lambda^{r_i(p-1)} a \left(\frac{\delta^{\beta_j}}{cm^{p-1}}\right)^{\frac{\beta_i}{p-1}} \int_{d>\varepsilon} \frac{\xi}{z_i^{\beta_i}} dx\,. \tag{4.4}$$

On the other hand,

$$\lambda \int_{d<\varepsilon} f_i(v_j)\xi dx \geq -\lambda b \int_{d<\varepsilon} \frac{\xi}{v_j^{\alpha_i}} dx \geq -\frac{\lambda^{1-r_j\alpha_i} b}{\delta^{\alpha_i}} \int_{d<\varepsilon} \frac{\xi}{z_i^{\alpha_i}} dx$$

$$\geq -\frac{\lambda^{r_i(p-1)} b}{\delta^{\alpha_i}} \int_{d<\varepsilon} \frac{\xi}{z_i^{\alpha_i}} dx\,, \tag{4.5}$$

where we have used the fact that $1 - r_j\alpha_i \leq 1 - r_j\beta_i$ and $\lambda > 1$. Combining (4.3)–(4.5), we get

$$\lambda \int_{\Omega} f_i(v_j)\xi dx \geq \int_{\Omega} |\nabla \Phi_i|^{p-2} \nabla \Phi_i \cdot \nabla \xi dx\,,$$

which, together with (4.2), shows that $\{\Phi, \Psi\}$ is a system of sub- and supersolutions of (I). Theorem 2.1 now follows from Lemma A in the Appendix.

PROOF OF THEOREM 2.2. Let $\varepsilon, \lambda > 0$ and $z, \psi, \psi_\varepsilon$ satisfy

$$\begin{cases} -\Delta_p z = \frac{1}{z^\delta} & \text{in } \Omega\,, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \quad , \quad \begin{cases} -\Delta_p \psi = 1 & \text{in } \Omega\,, \\ \psi = 0 & \text{on } \partial\Omega\,, \end{cases}$$

and

$$-\Delta_p \psi_\varepsilon = \begin{cases} L & \text{if } d(x) > \varepsilon\,, \\ -\frac{1}{z^\delta} & \text{if } d(x) < \varepsilon \end{cases} \quad , \quad \psi_\varepsilon = 0 \quad \text{on } \partial\Omega\,,$$

respectively. Then, by Corollary 3.1,

$$\psi_\varepsilon \geq (L^{\frac{1}{p-1}}/2)\psi \quad \text{in} \quad \Omega$$

if ε is small enough, which we shall assume. By (B.3) and (B.4), there exists $b > 0$ such that

$$|f_i(t)| \leq \frac{b}{t^\delta}$$

for $t < A$, and

$$f_i(t) \geq -\frac{b}{t^\delta}$$

for $t > 0$. Define

$$\tilde{f}_i(t) = \begin{cases} \sup_{A \leq s \leq t} f_i(s) & \text{if } t \geq A\,, \\ f_i(A) & \text{if } t < A\,. \end{cases}$$

Then $\tilde{f}_i$ are nondecreasing and

$$\lim_{t \to \infty} \frac{\tilde{f}_1^{\frac{1}{p-1}}\left(c\tilde{f}_2^{\frac{1}{p-1}}(t)\right)}{t} = 0$$

for each $c > 0$. Hence there exists $M \gg 1$ so that

$$\lambda\left[b + \|z\|_\infty^\delta \tilde{f}_1\left(\lambda^{\frac{1}{p-1}}\|z\|_\infty\big(b + \|z\|_\infty^\delta \tilde{f}_2(M\|z\|_\infty)\big)^{\frac{1}{p-1}}\right)\right] \leq M^{p-1}. \tag{4.6}$$

Define

$$\Phi_i = \lambda^{\frac{1}{p-1}}\psi_\varepsilon\,, \quad i = 1, 2, \;\; \Psi_1 = Mz,\; \Psi_2 = \lambda^{\frac{1}{p-1}}\big(b + \|z\|_\infty^\delta \tilde{f}_2(M\|z\|_\infty)\big)^{\frac{1}{p-1}} z\,.$$

We shall verify that $\Phi = (\Phi_1, \Phi_2)$ and $\Psi = (\Psi_1, \Psi_2)$ form a system of sub- and supersolutions for (I) if λ is large enough.

By increasing b, we can assume that

$$\psi_\varepsilon \leq b^{\frac{1}{p-1}} z \quad \text{in } \Omega\,.$$

Next, take $\lambda > 0$ large enough so that

$$\lambda^{\frac{1}{p-1}}(L^{\frac{1}{p-1}}/2)\psi(x) > A$$

for $d(x) > \varepsilon$, and

$$\Phi_i \geq \max(1, b^{1/\delta})z \quad \text{in } \Omega\,.$$

Then, for $M \gg \lambda^{\frac{1}{p-1}}$, we have $\Phi_i \leq \Psi_i$ in Ω, $i = 1, 2$. Let $\xi \in W_0^{1,p}(\Omega)$ with $\xi \geq 0$. Then we have

$$\int_\Omega |\nabla\Phi_i|^{p-2}\nabla\Phi_i.\nabla\xi\,dx = \lambda L\int_{d>\varepsilon}\xi\,dx - \lambda\int_{d<\varepsilon}\frac{\xi}{z^\delta}dx\,. \tag{4.7}$$

For $v_j \in [\Phi_j, \Psi_j]$ and $d(x) > \varepsilon$, we have

$$v_j(x) \geq \lambda^{\frac{1}{p-1}}(L^{\frac{1}{p-1}}/2)\psi(x) > A\,,$$

which implies

$$\lambda \int_{d>\varepsilon} f_i(v_j)\xi dx \geq \lambda L \int_{d>\varepsilon} \xi dx\,. \tag{4.8}$$

On the other hand,

$$\lambda \int_{d<\varepsilon} f_i(v_j)\xi dx \geq -\lambda b \int_{d<\varepsilon} \frac{\xi}{v_j^\delta} \geq -\lambda \int_{d<\varepsilon} \frac{\xi}{z^\delta} dx\,. \tag{4.9}$$

Combining (4.7)-(4.9), we get

$$\int_\Omega |\nabla \Phi_i|^{p-2} \nabla \Phi_i \cdot \nabla \xi dx \leq \lambda \int_\Omega f_i(v_j)\xi dx \tag{4.10}$$

for $i \neq j$. Next, since

$$f_i(t) \leq \frac{b}{t^\delta} + \tilde{f}_i(t)$$

for $t > 0$, we deduce from (4.6) that

$$\begin{aligned} &c\lambda \int_\Omega f_1(v_2)\xi dx \\ &\leq \lambda \int_\Omega \left(\frac{b}{z^\delta} + \tilde{f}_1 \left(\lambda^{\frac{1}{p-1}} \|z\|_\infty \left(b + \|z\|_\infty^\delta \tilde{f}_2(M\|z\|_\infty) \right)^{\frac{1}{p-1}} \right) \right) \xi dx \\ &\leq M^{p-1} \int_\Omega \frac{\xi}{z^\delta} dx = \int_\Omega |\nabla \Psi_1|^{p-2} \nabla \Psi_1 \cdot \nabla \xi dx. \end{aligned} \tag{4.11}$$

Similarly,

$$\begin{aligned} &c\lambda \int_\Omega f_2(v_1)\xi dx \leq \lambda \int_\Omega \left(\frac{b}{z^\delta} + \tilde{f}_2(v_1) \right) \xi dx \\ &\leq \lambda \int_\Omega \left(\frac{b + \|z\|_\infty^\delta \tilde{f}_2(M\|z\|_\infty)}{z^\delta} \right) \xi dx = \int_\Omega |\nabla \Psi_2|^{p-2} \nabla \Psi_2 \cdot \nabla \xi dx\,. \end{aligned} \tag{4.12}$$

From (4.10)–(4.12), we see that Φ and Ψ form a system of sub- and supersolutions for (I), which completes the proof of Theorem 2.2.

Appendix

We shall present some results needed above concerning sub- and supersolutions for singular boundary value problems. Related results can be found in [4, 6, 9]. Consider the system

$$\begin{cases} -\Delta_p u_1 = h_1(x, u_1, u_2) & \text{in } \Omega\,, \\ -\Delta_p u_2 = h_2(x, u_1, u_2) & \text{in } \Omega\,, \\ \quad u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $h_i : \Omega \times (0,\infty) \times (0,\infty) \to \mathbf{R}$ are continuous, $i = 1, 2$. Let $\Phi = (\Phi_1, \Phi_2)$, $\Psi = (\Psi_1, \Psi_2)$, where $\Phi_i, \Psi_i \in C^1(\bar{\Omega})$, $\Phi_i \le \Psi_i$ in Ω. Suppose there exist $l, C > 0$, $\gamma \in (0, 1)$, such that $\Phi_i, \Psi_i \ge ld$ in Ω and

$$|h_i(x, w_1, w_2)| \le \frac{C}{d^\gamma(x)}$$

for a.e. $x \in \Omega$ and all $w_i \in C(\bar{\Omega})$ with $\Phi_i \le w_i \le \Psi_i$ in $\Omega, i = 1, 2$. We say that $\{\Phi, \Psi\}$ forms a system of sub- and supersolutions for (1) if $\Phi_i \le 0 \le \Psi_i$ on $\partial\Omega$ and for all $\xi \in W_0^{1,p}(\Omega)$ with $\xi \ge 0$,

$$\int_\Omega |\nabla\Phi_i|^{p-2}\nabla\Phi_i \cdot \nabla\xi dx \le \int_\Omega h_i(x, \tilde{u}_1, \tilde{u}_2)\xi dx\,,$$

where $\tilde{u}_j = \Phi_i$ if $j = i$, $\tilde{u}_j \in [\Phi_j, \Psi_j]$ if $j \ne i$, and

$$\int_\Omega |\nabla\Psi_i|^{p-2}\nabla\Psi_i \cdot \nabla\xi dx \ge \int_\Omega h_i(x, \tilde{v}_1, \tilde{v}_2)\xi dx\,,$$

where $\tilde{v}_j = \Psi_i$ if $j = i$, $\tilde{v}_j \in [\Phi_j, \Psi_j]$ if $j \ne i$. Here $[\Phi_j, \Psi_j] = \{u \in C(\bar{\Omega}) : \Phi_j \le u_j \le \Psi_j \text{ in } \Omega\}$.

Note that the integrals on the right-hand side are defined by virtue of Hardy's inequality (see e.g. [2]).

LEMMA A. *Under the above assumptions, there exists* $\alpha \in (0, 1)$ *such that* (1) *has a solution* $(u_1, u_2) \in C^{1,\alpha}(\bar{\Omega}) \times C^{1,\alpha}(\bar{\Omega})$, $i = 1, 2$.

PROOF. For $(v_1, v_2) \in C(\bar{\Omega}) \times C(\bar{\Omega})$, define $T(v_1, v_2) = (u_1, u_2)$, where u_i satisfy

$$-\Delta_p u_i = \tilde{h}_i(x, v_1, v_2) \quad \text{in } \Omega\,, \quad u_i = 0 \text{ on } \partial\Omega\,,$$

where $\tilde{h}_i(x, v_1, v_2) = h_i(x, \tilde{v}_1, \tilde{v}_2)$, $\tilde{v}_i = \min(\max(v_i, \Phi_i), \Psi_i)$, $i = 1, 2$. Note that $\Phi_i \le \tilde{v}_i \le \Psi_i$ in Ω. Since

$$|\tilde{h}_i(x, v_1, v_2)| \le \frac{C}{d^\gamma(x)}$$

for a.e. $x \in \Omega$ and all $v_1, v_2 \in C(\bar{\Omega})$, Lemma 3.1 implies the existence of $\alpha \in (0, 1)$ such that $u_i \in C^{1,\alpha}(\bar{\Omega})$ and $|u_i|_{1,\alpha} < \tilde{C}$, $i = 1, 2$, where $\tilde{C}$ is independent of v_i, $i = 1, 2$. It is easy to see that T is a compact operator. Since $T\left(C(\bar{\Omega}) \times C(\bar{\Omega})\right)$ is relatively compact in $C(\bar{\Omega}) \times C(\bar{\Omega})$, it follows from the Schauder Fixed Point Theorem that T has a fixed point $u = (u_1, u_2)$ with $u_i \in C^{1,\alpha}(\bar{\Omega})$, $i = 1, 2$, for some $\alpha \in (0, 1)$. Using standard arguments, we see that $\Phi_i \le u_i \le \Psi_i$ in Ω, $i = 1, 2$, which concludes the proof.

References

[1] C. AZIZIEH, PH. CLEMENT and E. MITIDIERI, Existence and a priori estimates for positive solutions for p-Laplace systems, J. Differential Equations **184** (2002), 422–442.

[2] H. BREZIS, *Analyse fonctionnelle, théorie et applications*, second edition, Masson, Paris (1983).

[3] PH. CLEMENT, J. FLECKINGER, E. MITIDIERI and F. DE THELIN, Existence of positive solutions for a nonvariational quasilinear elliptic system, J. Differential Equations **166** (2000), 455–477.

[4] A. CANADA, P. DRABEK and J. L. GAMEZ, Existence of positive solutions for some problems with nonlinear diffusion, Trans. Amer. Math. Soc. **349** (1997), 4231–4249.

[5] M. G. CRANDALL, P. H. RABINOWITZ and L. TARTAR, On a Dirichlet problem with a singular nonlinearity, Comm. Partial Differential Equations **2** (1977), 193–222.

[6] C. DECOSTER and S. NICAISE, Lower and upper solutions for elliptic problems in nonsmooth domains, J. Differential Equations **244** (2008), 599–629.

[7] J. GIACOMONI, I. SCHINDLER and P. TAKÁČ, Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) **6** (2007), no. 1, 117–158.

[8] D. D. HAI, Singular boundary value problems for the p-Laplacian, Nonlinear Anal. **73** (2010), 2876–2881.

[9] D. D. HAI, On positive solutions for p-Laplacian systems with sign-changing nonlinearities, Hokaiddo Math. J. **39** (2010), 67–84.

[10] D. D. HAI, On a class of singular p-Laplacian boundary value problems, J. Math. Anal. Appl. **383** (2011), 619–626.

[11] D. D. HAI and R. SHIVAJI, An existence result on positive solutions for a class of p-Laplacian systems, Nonlinear Anal. **56** (2004), 1007–1010.

[12] G. M. LIEBERMAN, Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Anal. **12** (1988), 1203–1219.

[13] T. ODEN, *Qualitative Methods in Nonlinear Mechanics*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1986).

[14] S. SAKAGUCHI, Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4) **14** (1987), 403–421.

[15] J. L. VAZQUEZ, A strong maximum principle for some quasilinear elliptic equations, Appl. Math. Optim. **12** (1984), 191–202.

Present Address:
DEPARTMENT OF MATHEMATICS,
MISSISSIPPI STATE UNIVERSITY,
MISSISSIPPI STATE, MS 39762, USA.
e-mail: dang@math.msstate.edu