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Excellent Extensions and Global Cotorsion Dimensions

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Abstract. In this paper, we mainly investigate the global cotorsion dimension under the excellent extension of rings. We show that if S is an excellent extension of R, then $\mathrm{cot.D}(S) = \mathrm{cot.D}(R)$. Furthermore, some known results, such as Corollary 3.8 and 3.12, can be also obtained as direct corollaries of our theorem.

1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary.

Let R be a ring. A right R-module C is called cotorsion (Enochs, 1984) if $\mathrm{Ext}_R^1(F, C) = 0$ for all flat right R-module F. The class of cotorsion R-modules contains all pure-injective (and hence injective) R-modules. The cotorsion dimension of an R-module M, denoted by $\mathrm{cd}_R(M)$, is the least positive integer n for which $\mathrm{Ext}_R^{n+1}(F, M) = 0$ for all flat R-module F. Namely, the modules of cotorsion dimension 0 are the known cotorsion modules. The global cotorsion dimension of R, denoted by $\mathrm{cot.D}(R)$, is the quantity: $\mathrm{cot.D}(R) = \sup\{\mathrm{cd}_R(M) \mid M \textit{ is any } R\textit{-module}\}$. Obviously, $\mathrm{cd}_R(M)$ and $\mathrm{cot.D}(R)$ measure how far a module M is from being cotorsion, and how far a ring R is from being perfect, respectively. As we know, cotorsion modules have turned out to be useful in characterizing rings. For example, in [10] the author showed that a ring R is right perfect if and only if every flat right R-module has a cotorsion envelope with the unique mapping property, and R is von Neumann regular ring if and only if every cotorsion right R-module has a flat cover with the unique mapping property. Bass's Theorem [1] characterized a left perfect ring R as a ring such that every flat left R-module is projective. Enochs et al. [6] introduced n-perfect rings as an extension of perfect rings. These new rings appear as those for which every flat left module has projective dimension at most n, i.e., n-perfectness measures how far a ring is from being perfect.

The extension of rings and modules is a classical research field in algebra. There are lots of papers considering various properties which are shared by R and S (please see [8, 9, 11, 12, 14, 15]). For example, if S is an excellent extension of R, then (1) $\mathrm{gldim}(S) = \mathrm{gldim}(R)$, and $\mathrm{wdim}(S) = \mathrm{wdim}(R)$ [9, Theorem 3]; (2) S is right perfect if and only if R is right perfect

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[12, Corollary 7]; (3) S is QF (resp. semisimple) if and only if R is QF (resp. semisimple) [14, Theorem 2 and Theorem 3]; (4) S is left coherent (resp. left semihereditary) if and only if R is left coherent (resp. left semihereditary) [15, Theorem 1.10].

As the dual notions of perfect rings and n-perfect rings, we introduce coperfect rings and n-coperfect rings and show some equivalent characterizations of these rings in this paper. Our main results are given in section 3. First, we investigate the relation between the global cotorsion dimensions of R and S under excellent extension of rings, and we prove that if S is an excellent extension of R, then $\mathrm{cot.D}(S) = \mathrm{cot.D}(R)$ (Theorem 3.7). Secondly, it seems natural to generalize descent of perfectness to coperfectness when S is an excellent extension of R, and we show that if S is an excellent extension of R, then S is left coperfect (resp. left n-perfect, left n-coperfect) if and only if R is left coperfect (resp. left n-perfect, left n-coperfect) (Corollary 3.10, 3.12 and 3.14). Furthermore, some known results can be obtained as direct corollaries of our theorem.

In what follows, $\mathrm{pd}_R(M)$ (resp. $\mathrm{id}_R(M), \mathrm{fd}_R(M)$) denotes the usual projective (resp. injective, flat) dimensions of M. We use $\mathrm{gldim}(R)$ (resp. $\mathrm{wdim}(R)$) to denote the classical global (resp. weak) dimensions of R.

2. Preliminaries

In this section, we recall some known definitions and facts needed in this article. For background on almost excellent extension and excellent extension we refer the reader to [8, 9, 11, 14, 15].

Let R be a subring of a ring S, and R and S have the same identity. The ring S is said to be an almost excellent extension of R [14, 15], if the following conditions are satisfied:

(1) S is right R-projective, that is, if N_S is a submodule of M_S, then $N_R \mid M_R$ implies $N_S \mid M_S$, where $N_R \mid M_R$ implies N_R is a direct summand of M_R.

(2) S is a finite normalizing extension of R, that is, there is a finite set $\{s_1, s_2, \ldots, s_n\} \subseteq S$, such that $S = \sum_{i=1}^{n} s_i R$ and $Rs_i = s_i R$ for all $i = 1, 2, \ldots, n$.

(3) ${}_RS$ is flat and S_R is projective.

Further, an almost excellent extension $S \geq R$ is called an excellent extension in case both ${}_RS$ and S_R are free modules with basis $\{s_1 = 1, s_2, \ldots, s_n\}$.

The concept of excellent extension was introduced by Passman and named by Bonami. Examples of excellent extensions include $n \times n$ matrix rings $M_n(R)$ and crossed product $R * G$, where G is a finite group with $|G|^{-1} \in R$.

3. Main results

We begin with the following Lemmas.

LEMMA 3.1 [15, Lemma 1.1]. *Let $S \geq R$ be a ring extension such that S is right R-projective. If M_S is a right S-module, then*

(1) *M_S is isomorphic to a summand of $(M \otimes_R S)_S$.*

(2) *M_S is isomorphic to a summand of* $\mathrm{Hom}_R(S, M)_S$.

LEMMA 3.2 [15, Lemma 1.2]. *Let $S \geq R$ be an almost excellent extension. If M_S is a right S-module, then*

(1) *M_S is projective if and only if M_R is projective.*
(2) *M_S is injective if and only if M_R is injective.*
(3) *M_S is flat if and only if M_R is flat.*

Recall that a ring R is called perfect, if every flat R-module is projective. A ring R is said to be n-perfect, if every flat R-module has projective dimension at most n.

DEFINITION 3.3. A ring R is called coperfect, if every cotorsion R-module is injective.

A ring R is called n-coperfect, if every cotorsion R-module has injective dimension at most n.

In the following we give some equivalent characterizations of n-coperfect rings.

PROPOSITION 3.4. *For any ring R and any integer $n \geq 0$, the following are equivalent*:

(1) *R is left n-coperfect (i.e. For any cotorsion left R-module M, $\mathrm{id}_R(M) \leq n$);*
(2) *For any pure injective left R-module N, $\mathrm{id}_R(N) \leq n$;*
(3) *For any cotorsion right R-module M, $\mathrm{fd}_R(M) \leq n$;*
(4) *For any pure injective right R-module N, $\mathrm{fd}_R(N) \leq n$;*
(5) $\mathrm{wdim}(R) \leq n$.

In particular, if $n = 0$, then we get the equivalent characterizations of coperfect rings.

PROOF. The proof follows from Definition 3.3 and [16, Theorem 3.3.2].

Note that the equivalences of (1) and (5) of Proposition 3.4 link the classical weak dimension of R with (n-)coperfect rings. Moreover, it is similar to the characterization of n-perfect rings (Lemma 3.6). In this sense, we list the following easy observation.

COROLLARY 3.5

(1) $\mathrm{wdim}(R) = 0$ *(i.e., R is von Neumann regular) if and only if R is coperfect*;
(2) $\mathrm{wdim}(R) \leq n$ *if and only if R is n-coperfect*.

LEMMA 3.6 [3, Proposition 1.2]. *For a positive integer n, $\mathrm{cot.D}(R) \leq n$ if and only if R is n-perfect.*

The next theorem is an interesting result establishing the relation between the global cotorsion dimensions of R and S when S is an excellent extension of R.

THEOREM 3.7. *If S is an excellent extension of R, then* $\mathrm{cot.D}(R) = \mathrm{cot.D}(S)$.

PROOF. First, we prove the inequality $\mathrm{cot.D}(R) \leq \mathrm{cot.D}(S)$. Without loss of generality, we may assume that $\mathrm{cot.D}(S) = n < \infty$. Let F be a flat right R-module and

N an right R-module. Then $F \otimes_R S$ is flat S-module. Since ${}_RS$ is flat, by [11, Theorem 11.65], we have $\mathrm{Ext}_R^{n+1}(F, N \otimes_R S) \cong \mathrm{Ext}_S^{n+1}(F \otimes_R S, N \otimes_R S) = 0$, and so $\mathrm{Ext}_R^{n+1}(F, N \otimes_R S) = 0$. Since S is an excellent extension of R, R is an R-bimodule direct summand of S. Let ${}_RS_R = R \oplus T$. Then $N \otimes_R S = N \otimes_R (R \oplus T) \cong N_R \oplus (N \otimes_R T)$, that is, $N \mid N \otimes_R S$, and hence $\mathrm{Ext}_R^{n+1}(F, N) \mid \mathrm{Ext}_R^{n+1}(F, N \otimes_R S)$, thus, $\mathrm{Ext}_R^{n+1}(F, N) = 0$, and so $\mathrm{cot.D}(R) \leq n = \mathrm{cot.D}(S)$, as desired.

Now we prove that $\mathrm{cot.D}(S) \leq \mathrm{cot.D}(R)$. For that we may assume that $\mathrm{cot.D}(R) = n < \infty$. Then R is n-perfect by Lemma 3.6. Let F be a flat right S-module. Then F is a flat right R-module by Lemma 3.2(3), and so $\mathrm{pd}_R(F) \leq n$. Thus, there exists an exact sequence of right R-modules $0 \to P_n \to \cdots \to P_0 \to F \to 0$, where each P_i is projective, $i = 0, 1, \ldots, n$. Then we have the exact sequence of right S-modules:

$$0 \to P_n \otimes_R S \to \cdots \to P_0 \otimes_R S \to F \otimes_R S \cong F \to 0$$

where each $P_i \otimes_R S$ is projective. This implies that S is n-perfect and thus $\mathrm{cot.D}(S) \leq n$ by Lemma 3.6. So we have the desired equality $\mathrm{cot.D}(R) = \mathrm{cot.D}(S)$.

According to this theorem, we have the following corollary.

COROLLARY 3.8 [8, Theorem 1.5]. *If S is an excellent extension of R, then S is left perfect if and only if R is left perfect.*

PROOF. It is clear by Lemma 3.6 and Theorem 3.7. □

COROLLARY 3.9. *The following are equivalent*:

(1) *R is left perfect.*

(2) *The matrix ring $M_n(R)$ is left perfect.*

(3) *The crossed product $R * G$ is left perfect, where G is a finite group with $|G|^{-1} \in R$.*

COROLLARY 3.10. *Let S be an excellent extension of R. Then S is left n-perfect if and only if R is left n-perfect.*

Now we study the transfer of cotorsion property of modules under excellent extension of rings.

PROPOSITION 3.11. *Let S be an excellent extension of R. Then the following are equivalent for any right S-module M_S*:

(1) *M_R is cotorsion.*

(2) *$\mathrm{Hom}_R(S, M)_S$ is cotorsion.*

(3) *M_S is cotorsion.*

PROOF. (1) $\Rightarrow$ (2) Let F be any flat right S-module. Then F is flat right R-module by Lemma 3.2, and so $\mathrm{Ext}_R^1(F, M) = 0$. Since $\mathrm{Ext}_S^1(F, \mathrm{Hom}_R(S, M)) \cong \mathrm{Ext}_R^1(F \otimes_S S, M)$ $\cong \mathrm{Ext}_R^1(F, M)$, we have $\mathrm{Ext}_S^1(F, \mathrm{Hom}_R(S, M)) = 0$, therefore, $\mathrm{Hom}_R(S, M)_S$ is cotorsion.

(2) $\Rightarrow$ (3) Note that M_S is isomorphic to a summand of $\mathrm{Hom}_R(S, M)_S$ by Lemma 3.1, so M_S is cotorsion.

(3) $\Rightarrow$ (1) Let F be any flat right R-module, since ${}_RS$ is free, $F \otimes_R S$ is flat right S-module, we have $\mathrm{Ext}^1_R(F, M) \cong \mathrm{Ext}^1_R(F, \mathrm{Hom}_S(S, M)) \cong \mathrm{Ext}^1_S(F \otimes_R S, M) = 0$, therefore, M_R is cotorsion.

COROLLARY 3.12 [14, Theorem 5]. *Let S be an excellent extension of R. Then S is left coperfect* (*i.e. von Neumann regular*) *if and only if R is left coperfect* (*i.e. von Neumann regular*).

PROOF. It is clear by Lemma 3.2 and Proposition 3.11.

COROLLARY 3.13. *The following are equivalent*:

(1) *R is left coperfect.*

(2) *The matrix ring $M_n(R)$ is left coperfect, for some $n > 1$.*

(3) *The crossed product $R * G$ is left coperfect, where G is a finite group with $|G|^{-1} \in R$.*

COROLLARY 3.14. *Let S be an excellent extension of R. Then S is left n-coperfect if and only if R is left n-coperfect.*

PROOF. It is obvious by [9, Theorem 3] and Corollary 3.5.

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