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Stable Rank for C^*-tensor Products with the Jiang-Su Algebra

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Abstract. We estimate stable rank for C^*-tensor products with the Jiang-Su algebra. We also estimate real rank for them as well.

Introduction

Jiang and Su [4] first show that there exists a simple unital inductive limit of dimension drop algebras such that it has a unique tracial state and the same K-theory as $\mathbf{C}$. This remarkable C^*-algebra of Jiang-Su (we call it Jiang-Su algebra) has played an important key role in the classification program of Elliott for nuclear simple C^*-algebras (for instance, see Toms [13] and [8] and also [5]). Also, the Jiang-Su algebra is regarded as an analogue to the Cuntz algebra O_∞ generated by countably infinite, mutually orthogonal isometries. Indeed, K-theory of O_∞ is the same as that of $\mathbf{C}$, i.e., K_0-group $\mathbf{Z}$ and K_1-group trivial. It is shown by Kirchberg that $\mathfrak{A} \otimes O_\infty \cong \mathfrak{A}$ for any purely infinite, separable unital nuclear C^*-algebra $\mathfrak{A}$ (so called O_∞-absorbed (or absorbing) property) (see [11]), while it is shown by Jiang and Su that a similar, Jiang-Su algebra absorbed property holds for separable unital simple AF algebras and purely infinite, separable, unital nuclear C^*-algebras. On the other hand, it is shown by Rørdam [10] that any simple unital finite C^*-algebra absorbing the Jiang-Su algebra has stable rank 1, and the real rank zero for such C^*-algebras that are exact is determined in terms of K-theory.

In this paper we consider stable rank and real rank for tensor products of (general) C^*-algebras with the Jiang-Su algebra. It is found out that in those ranks point of view, the Jiang-Su algebra is rather near to $\mathbf{K}$ the C^*-algebra of compact operators on an infinite dimensional Hilbert space, which also has the same K-theory as $\mathbf{C}$, however not always, and it is not near to $\mathbf{C}$ in general. This point of view should be interesting. Since we deal with general C^*-algebras, it is natural that sharpness of estimates for those ranks of them is not much expected. In Section 1 we consider the stable rank case and do the real rank case in Section 2. We use

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some known results on these ranks, which are helpful for this research. Refer to Rieffel [9] for stable rank for C^*-algebras and to Brown-Pedersen [3] for real rank.

1. Stable rank

The stable rank of a unital C^*-algebra $\mathfrak{A}$ is defined to be the smallest positive integer $n = \mathrm{sr}(\mathfrak{A})$ such that any element of $\mathfrak{A}^n$ (n-direct sum) is arbitrarily approximated by an element $(b_j) \in \mathfrak{A}^n$ such that $\mathfrak{A}b_1 + \mathfrak{A}b_2 + \cdots + \mathfrak{A}b_n = \mathfrak{A}$. For a non-unital C^*-algebra, its stable rank is defined to be that of its unitization by $\mathbf{C}$.

Similarly, the real rank of a unital C^*-algebra $\mathfrak{A}$ is defined to be the smallest non-negative integer $n = \mathrm{RR}(\mathfrak{A})$ such that any self-adjoint element of $\mathfrak{A}^{n+1}$ is arbitrarily approximated by a self-adjoint element $(b_j) \in \mathfrak{A}^{n+1}$ such that $\mathfrak{A}b_1 + \mathfrak{A}b_2 + \cdots + \mathfrak{A}b_{n+1} = \mathfrak{A}$. For a non-unital C^*-algebra, its real rank is defined to be that of its unitization by $\mathbf{C}$.

Recall from Jiang-Su [4] that there exists an infinite dimensional, simple unital inductive limit of dimension drop algebras, that we denote it by $\mathcal{Z}$, such that $\mathcal{Z}$ has a unique tracial state, $K_0(\mathcal{Z}) \cong K_0(\mathbf{C}) \cong \mathbf{Z}$ and $K_1(\mathcal{Z}) \cong K_1(\mathbf{C}) \cong 0$, where a dimension drop algebra is a C^*-algebra defined by

$$\{f \in C([0,1], M_m(\mathbf{C})) \mid f(0) \in M_l(\mathbf{C}) \otimes \mathbf{C}1_{m/l},\ f(1) \in \mathbf{C}1_{m/n} \otimes M_n(\mathbf{C})\},$$

denoted by $D[l,m,n]$, where $C([0,1], M_m(\mathbf{C}))$ is the C^*-algebra of $m \times m$ matrix algebra $M_m(\mathbf{C})$ valued continuous functions on the closed interval $[0,1]$, and l, n are positive integers such that $m = l \times (m/l)$, $m = n \times (m/n)$ with m/l, m/n integers, and $1_{m/l}$, $1_{m/n}$ are $(m/l) \times (m/l)$ and $(m/n) \times (m/n)$ identity matrices respectively. Note that $D[l,m,n]$ is projectionless if and only if l and n are relatively prime, and if in addition $m = ln$, $D[l,m,n]$ is called prime. A prime dimension drop algebra has the same K-theory as $\mathbf{C}$, and the Jiang-Su algebra $\mathcal{Z}$ is an inductive limit of prime dimension drop algebras, i.e., $\mathcal{Z} = \varinjlim D[l_j, m_j, n_j]$ for some $m_j = l_j n_j$.

THEOREM 1.1. *Let $\mathfrak{A}$ be a C^*-algebra. If $\mathrm{sr}(\mathfrak{A})$ is finite, then*

$$\mathrm{sr}(\mathfrak{A} \otimes \mathcal{Z}) \le 2\,.$$

Furthermore, if $\mathrm{sr}(\mathfrak{A}) = 1$, $K_1(\mathfrak{A}) = 0$, and $\mathrm{RR}(\mathfrak{A}) = 0$, then

$$\mathrm{sr}(\mathfrak{A} \otimes \mathcal{Z}) = 1\,.$$

PROOF. We assume that $\mathfrak{A}$ is unital. If $\mathfrak{A}$ is non-unital, we replace it in the statement with its unitization $\mathfrak{A}^+$ by $\mathbf{C}$, which unchanges stable rank. Since $\mathcal{Z} = \varinjlim D[l_j, m_j, n_j]$ for some $m_j = l_j n_j$, we consider the stable rank for C^*-tensor products $\mathfrak{A} \otimes D[l_j, m_j, n_j]$, each of which can be viewed as a continuous field C^*-algebra on $[0,1]$ with its fibers given by $\mathfrak{A} \otimes M_{l_j}(\mathbf{C}) \otimes \mathbf{C}1_{n_j}$ at 0, $\mathfrak{A} \otimes M_{m_j}(\mathbf{C})$ on the open interval $(0,1)$, and $\mathfrak{A} \otimes M_{n_j}(\mathbf{C}) \otimes \mathbf{C}1_{l_j}$ at 1, where 1_{x_j} is the $x_j \times x_j$ identity matrix. Using a nice and helpful result of [7], we obtain

$$\mathrm{sr}(\mathfrak{A} \otimes D[l_j, m_j, n_j]) \le \max\{\mathrm{sr}(\mathfrak{A} \otimes M_{l_j}(\mathbf{C}) \otimes C([0,1]))\,,$$
$$\mathrm{sr}(\mathfrak{A} \otimes M_{m_j}(\mathbf{C}) \otimes C([0,1])), \mathrm{sr}(\mathfrak{A} \otimes M_{n_j}(\mathbf{C}) \otimes C([0,1]))\}\,.$$

Furthermore, it follows from [9, Theorem 6.1] that

$$\mathrm{sr}(M_{x_j}(\mathbf{C}) \otimes \mathfrak{A} \otimes C([0,1])) = \lceil (\mathrm{sr}(\mathfrak{A} \otimes C([0,1])) - 1)/x_j \rceil + 1\,,$$

where $\lceil y \rceil$ is the smallest integer $\ge y$. Since $\mathrm{sr}(\mathfrak{A} \otimes C([0,1])) \le \mathrm{sr}(\mathfrak{A}) + 1 < \infty$ by [9, Corollary 7.2] or [6], and the integers l_j, m_j, n_j are taken large enough in inductive steps, we then get

$$\mathrm{sr}(\mathfrak{A} \otimes D[l_j, m_j, n_j]) \le 2\,.$$

By [9, Theorem 5.1], we obtain $\mathrm{sr}(\mathfrak{A} \otimes \mathcal{Z}) \le 2$.

It is shown in [6, Theorem 4.3] that the conditions $K_1(\mathfrak{A}) = 0$, $\mathrm{sr}(\mathfrak{A}) = 1$, and $\mathrm{RR}(\mathfrak{A}) = 0$ imply $\mathrm{sr}(\mathfrak{A} \otimes C([0,1])) = 1$. If so,

$$\mathrm{sr}(M_{x_j}(\mathbf{C}) \otimes \mathfrak{A} \otimes C([0,1])) = 1\,,$$

which implies $\mathrm{sr}(\mathfrak{A} \otimes D[l_j, m_j, n_j]) = 1$. Hence $\mathrm{sr}(\mathfrak{A} \otimes \mathcal{Z}) = 1$. □

REMARK. Let $\mathbf{K}$ be the C^*-algebra of compact operators on an infinite dimensional, separable Hilbert space. It is obtained by [9, Theorems 3.6 and 6.4] that

$$\mathrm{sr}(\mathfrak{A} \otimes \mathbf{K}) \le 2$$

for any C^*-algebra $\mathfrak{A}$, and $\mathrm{sr}(\mathfrak{A} \otimes \mathbf{K}) = 1$ if and only if $\mathrm{sr}(\mathfrak{A}) = 1$.

The following estimate:

$$\mathrm{sr}(\mathfrak{A} \otimes \mathcal{Z}) \le 2$$

for a C^*-algebra $\mathfrak{A}$ is not always true (cf. [13]). In fact, it is shown in [4, Corollary 2.13] that if $\mathfrak{B}$ is a purely infinite, simple unital nuclear C^*-algebra, then $\mathfrak{B} \cong \mathfrak{B} \otimes \mathcal{Z}$. In this case, we have

$$\mathrm{sr}(\mathfrak{B} \otimes \mathcal{Z}) = \mathrm{sr}(\mathfrak{B}) = \infty\,.$$

It is also known that a purely infinite, separable simple C^*-algebra is either unital or stable (Zhang's dichotomy). If $\mathfrak{A}$ is such a stable one, then $\mathrm{sr}(\mathfrak{A} \otimes \mathcal{Z}) \le 2$.

Also, it is shown in [4, Corollary 6.3] that if $\mathfrak{A}$ is a unital simple AF algebra, then $\mathfrak{A} \cong \mathfrak{A} \otimes \mathcal{Z}$. Then

$$\mathrm{sr}(\mathfrak{A} \otimes \mathcal{Z}) = \mathrm{sr}(\mathfrak{A}) = 1$$

and $K_1(\mathfrak{A}) = 0$ and $\mathrm{RR}(\mathfrak{A}) = 0$.

By [6, Proposition 5.2], the condition $\mathrm{sr}(\mathfrak{A} \otimes C([0,1])) = 1$ implies $K_1(\mathfrak{A})$ and $\mathrm{sr}(\mathfrak{A}) = 1$, but $\mathrm{RR}(\mathfrak{A})$ may not be zero.

On the other hand, we always have

$$\mathrm{sr}(\mathfrak{A} \otimes O_\infty) = \infty = \mathrm{sr}(O_\infty)$$

for any unital C^*-algebra $\mathfrak{A}$ (see [9, Proposition 6.5]).

As a note, since $\mathfrak{A} \otimes \mathbf{C} \cong \mathfrak{A}$ for any C^*-algebra $\mathfrak{A}$, we always have $\mathrm{sr}(\mathfrak{A} \otimes \mathbf{C}) = \mathrm{sr}(\mathfrak{A})$. For $\mathcal{Z}$-absorbed C^*-algebra $\mathfrak{A}$, we have $\mathrm{sr}(\mathfrak{A} \otimes \mathcal{Z}) = \mathrm{sr}(\mathfrak{A})$.

COROLLARY 1.2. *The Jiang-Su algebra has stable rank* 1.

PROOF. Note that $\mathbf{C} \otimes \mathcal{Z} \cong \mathcal{Z}$ and $\mathrm{sr}(\mathbf{C}) = 1$, $K_1(\mathbf{C}) = 0$, and $\mathrm{RR}(\mathbf{C}) = 0$. Hence $\mathrm{sr}(\mathcal{Z}) = \mathrm{sr}(\mathbf{C} \otimes \mathcal{Z}) = 1$. □

Slightly generally,

THEOREM 1.3. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{D}$ be an inductive limit of dimension drop algebras $D[l_j, m_j, n_j]$ such that l_j, m_j, n_j go to ∞ in inductive steps. If* $\mathrm{sr}(\mathfrak{A})$ *is finite, then*

$$\mathrm{sr}(\mathfrak{A} \otimes \mathfrak{D}) \leq 2 .$$

Furthermore, if $\mathrm{sr}(\mathfrak{A}) = 1$, $K_1(\mathfrak{A}) = 0$, *and* $\mathrm{RR}(\mathfrak{A}) = 0$, *then*

$$\mathrm{sr}(\mathfrak{A} \otimes \mathfrak{D}) = 1 .$$

In particular, $\mathfrak{D}$ has stable rank 1.

EXAMPLE 1.4. Let $\otimes^\infty C(\mathbf{T})$ be the infinite tensor product of $C(\mathbf{T})$ the C^*-algebra of continuous functions on the torus $\mathbf{T}$. This is an inductive limit of tensor products $\otimes^n C(\mathbf{T})$ $(n \geq 1)$. There exists a canonical quotient map from $\otimes^\infty C(\mathbf{T})$ to $\otimes^n C(\mathbf{T}) \cong C(\mathbf{T}^n)$ for any $n \geq 1$. Since $\mathrm{sr}(C(\mathbf{T}^n)) = \lfloor n/2 \rfloor + 1$, where $\lfloor y \rfloor$ means the largest integer $\leq y$, by [9, Theorem 4.3], we have $\mathrm{sr}(\otimes^\infty C(\mathbf{T})) = \infty$. Since

$$(\otimes^\infty C(\mathbf{T})) \otimes \mathcal{Z} \cong \varinjlim (\otimes^n C(\mathbf{T})) \otimes \mathcal{Z} \cong \varinjlim C(\mathbf{T}^n) \otimes \mathcal{Z} ,$$

we obtain

$$\mathrm{sr}((\otimes^\infty C(\mathbf{T})) \otimes \mathcal{Z}) \leq 2$$

while $\mathrm{sr}(\otimes^\infty C(\mathbf{T})) = \infty$.

By the same way as above,

PROPOSITION 1.5. *Let $\otimes^\infty \mathfrak{A}_j$ be an infinite tensor product of C^*-algebras $\mathfrak{A}_j$ such that each $\otimes^n \mathfrak{A}_j$ has stable rank finite. Then*

$$\mathrm{sr}((\otimes^\infty \mathfrak{A}_j) \otimes \mathcal{Z}) \leq 2 .$$

More generally, if $\mathfrak{B}$ is an inductive limit of C^*-algebras with stable rank finite, then

$$\mathrm{sr}(\mathfrak{B} \otimes \mathcal{Z}) \leq 2 .$$

REMARK. We can take $\mathfrak{B}$ as (certain) AH algebras, i.e., inductive limits of homogeneous C^*-algebras (with stable rank finite). See Lin [5] for AH algebras and their classification theorems.

2. Real rank

At this moment, the following is the best:

THEOREM 2.1. *Let $\mathfrak{A}$ be a C^*-algebra. Then*

$$\mathrm{RR}(\mathfrak{A} \otimes \mathcal{Z}) \le \mathrm{RR}(C^b((0,1), \mathfrak{A})),$$

where $C^b((0,1),\mathfrak{A})$ is the C^*-algebra of bounded continuous $\mathfrak{A}$-valued functions on the open interval $(0,1)$.

Also, using another estimate, if $\mathrm{sr}(\mathfrak{A})$ *and* $\mathrm{sr}(C^b((0,1),\mathfrak{A})$ *are finite, then*

$$\mathrm{RR}(\mathfrak{A} \otimes \mathcal{Z}) \le 3.$$

If $\mathrm{sr}(\mathfrak{A}) = 1$ *and* $\mathrm{sr}(C^b((0,1),\mathfrak{A})) = 1$, *then*

$$\mathrm{RR}(\mathfrak{A} \otimes \mathcal{Z}) \le 1.$$

If $\mathfrak{A}$ is commutative, then $\mathrm{RR}(\mathfrak{A} \otimes \mathcal{Z}) \le 1$.

Furthermore, the Jiang-Su algebra $\mathcal{Z}$ can be replaced with an inductive limit of dimension drop algebras $D[l_j, m_j, n_j]$, where in the second, third, and fourth estimates above, l_j, m_j, n_j go to ∞ in inductive steps.

PROOF. We assume that $\mathfrak{A}$ is unital. If $\mathfrak{A}$ is non-unital, we replace it in the statement with its unitization $\mathfrak{A}^+$ by $\mathbf{C}$, which unchanges real rank. Since $\mathcal{Z} = \varinjlim D[l_j, m_j, n_j]$ for some $m_j = l_j n_j$, we consider the real rank for C^*-tensor products $\mathfrak{A} \otimes D[l_j, m_j, n_j]$, each of which can be viewed as a continuous field C^*-algebra on $[0,1]$ with its fibers given by $\mathfrak{A} \otimes M_{l_j}(\mathbf{C}) \otimes \mathbf{C}1_{n_j}$ at 0, $\mathfrak{A} \otimes M_{m_j}(\mathbf{C})$ on the open interval $(0,1)$, and $\mathfrak{A} \otimes M_{n_j}(\mathbf{C}) \otimes \mathbf{C}1_{l_j}$ at 1. Furthermore, we consider the following exact sequence:

$$\begin{aligned} 0 \to C_0((0,1), \mathfrak{A} \otimes M_{m_j}(\mathbf{C})) &\to \mathfrak{A} \otimes D[l_j, m_j, n_j] \\ &\to (\mathfrak{A} \otimes M_{l_j}(\mathbf{C})) \oplus (\mathfrak{A} \otimes M_{n_j}(\mathbf{C})) \to 0. \end{aligned}$$

Using a nice and helpful result of [6, Proposition 1.6], we obtain

$$\begin{aligned} \mathrm{RR}(\mathfrak{A} \otimes D[l_j, m_j, n_j]) \le \max\{&\mathrm{RR}(M(C_0((0,1), \mathfrak{A} \otimes M_{m_j}(\mathbf{C})))), \\ &\mathrm{RR}(\mathfrak{A} \otimes M_{l_j}(\mathbf{C})), \mathrm{RR}(\mathfrak{A} \otimes M_{n_j}(\mathbf{C}))\}, \end{aligned}$$

where $M(C_0((0,1), \mathfrak{A} \otimes M_{m_j}(\mathbf{C})))$ is the multiplier algebra of $C_0((0,1), \mathfrak{A} \otimes M_{m_j}(\mathbf{C}))$, and note that $\mathfrak{A} \otimes D[l_j, m_j, n_j]$ can be viewed as the pullback C^*-algebra associated with the above exact sequence, involving the multiplier algebra. Moreover, it follows by [1] that

$$M(C_0((0,1), \mathfrak{A} \otimes M_{m_j}(\mathbf{C}))) \cong C^b((0,1), \mathfrak{A} \otimes M_{m_j}(\mathbf{C})),$$

which is the C^*-algebra of bounded continuous $\mathfrak{A} \otimes M_{m_j}(\mathbf{C})$ valued functions on $(0,1)$. This is also identified with $C^b((0,1), \mathfrak{A}) \otimes M_{m_j}(\mathbf{C})$. Note that for a C^*-algebra $\mathfrak{B}$, we have

$M(M_k(\mathfrak{B})) \cong M_k(M(\mathfrak{B}))$ for any $k \geq 1$, which is easily deduced from the strict topology argument ([14, p. 50]). Using a result of [12] we obtain

$$\mathrm{RR}(\mathfrak{A} \otimes M_k(\mathbf{C})) \leq \mathrm{RR}(\mathfrak{A})$$

for any $k \geq 1$ and

$$\mathrm{RR}(C^b((0,1), \mathfrak{A}) \otimes M_{m_j}(\mathbf{C})) \leq \mathrm{RR}(C^b((0,1), \mathfrak{A}))\,.$$

Also, we have $\mathrm{RR}(C^b((0,1), \mathfrak{A})) \geq \mathrm{RR}(\mathfrak{A})$. Therefore, the first estimate in the statement is obtained.

On the other hand, it is shown by [3, Proposition 1.2] that

$$\mathrm{RR}(\mathfrak{B}) \leq 2\,\mathrm{sr}(\mathfrak{B}) - 1$$

for any C^*-algebra $\mathfrak{B}$. Using this estimate we have

$$\mathrm{RR}(\mathfrak{A} \otimes M_k(\mathbf{C})) \leq 2\,\mathrm{sr}(\mathfrak{A} \otimes M_k(\mathbf{C})) - 1\,,$$

$$\mathrm{RR}(C^b((0,1), \mathfrak{A}) \otimes M_{m_j}(\mathbf{C})) \leq 2\,\mathrm{sr}(C^b((0,1), \mathfrak{A}) \otimes M_{m_j}(\mathbf{C})) - 1\,.$$

If $\mathrm{sr}(\mathfrak{A})$ and $\mathrm{sr}(C^b((0,1), \mathfrak{A}))$ are finite, we can estimate as

$$\mathrm{sr}(\mathfrak{A} \otimes M_k(\mathbf{C})) \leq 2\,, \quad \mathrm{sr}(C^b((0,1), \mathfrak{A}) \otimes M_{m_j}(\mathbf{C})) \leq 2$$

for k, m_j large enough, as before. Therefore, we then obtain $\mathrm{RR}(\mathfrak{A} \otimes D[l_j, m_j, n_j]) \leq 3$, which implies $\mathrm{RR}(\mathfrak{A} \otimes \mathcal{Z}) \leq 3$.

If $\mathrm{sr}(\mathfrak{A}) = 1$ and $\mathrm{sr}(C^b((0,1), \mathfrak{A})) = 1$, then

$$\mathrm{sr}(\mathfrak{A} \otimes M_k(\mathbf{C})) = 1\,, \quad \mathrm{sr}(C^b((0,1), \mathfrak{A}) \otimes M_{m_j}(\mathbf{C})) = 1\,.$$

Then $\mathrm{RR}(\mathfrak{A} \otimes D[l_j, m_j, n_j]) \leq 1$. Therefore, $\mathrm{RR}(\mathfrak{A} \otimes \mathcal{Z}) \leq 1$.

If $\mathfrak{A}$ is commutative, then $\mathfrak{A} \cong C(X)$ for a compact Hausdorff space X. It is shown in [2] that

$$\mathrm{RR}(C(X) \otimes M_n(\mathbf{C})) = \lceil \dim X/(2n-1) \rceil\,.$$

Therefore, $\mathrm{RR}(\mathfrak{A} \otimes D[l_j, m_j, n_j]) \leq 1$ for l_j, m_j, n_j large enough. Hence, $\mathrm{RR}(\mathfrak{A} \otimes \mathcal{Z}) \leq 1$. □

REMARK. Those estimates that we obtained this time might be better improved when we know more about real rank estimates that have been unknown. It would be preferable to replace the conditions on stable rank with other ones on real rank.

It is shown in [2] that

$$\mathrm{RR}(\mathfrak{A} \otimes \mathbf{K}) \leq 1$$

for any C^*-algebra $\mathfrak{A}$. On the other hand, it is shown in [3] that if $\mathrm{RR}(\mathfrak{A}) = 0$, then $\mathrm{RR}(\mathfrak{A} \otimes \mathbf{K}) = 0$.

As a note, we always have

$$\mathrm{RR}(\mathfrak{A} \otimes D[l_j, m_j, n_j]) \geq 1$$

for any C^*-algebra $\mathfrak{A}$ by using [6, Proposition 5.1] and considering quotients. However, if $\mathfrak{A}$ is a unital simple AF algebra, then $\mathfrak{A} \otimes \mathcal{Z} \cong \mathfrak{A}$. Hence,

$$\mathrm{RR}(\mathfrak{A} \otimes \mathcal{Z}) = \mathrm{RR}(\mathfrak{A}) = 0\,.$$

Also, if $\mathfrak{B}$ is a purely infinite, simple unital nuclear C^*-algebra, then $\mathfrak{B} \otimes \mathcal{Z} \cong \mathfrak{B}$. Thus,

$$\mathrm{RR}(\mathfrak{B} \otimes \mathcal{Z}) = \mathrm{RR}(\mathfrak{B}) = 0$$

by [3, Proposition 3.9].

COROLLARY 2.2. *The Jiang-Su algebra has real rank* 1.

PROOF. We have shown that $\mathrm{RR}(\mathcal{Z}) \leq 1$. Since $\mathcal{Z}$ is an inductive limit of projectionless dimension drop algebras, it also has no nontrivial projections. It follows that $\mathrm{RR}(\mathcal{Z}) \geq 1$. □

As a note,

LEMMA 2.3. *Let $\mathfrak{A}$ be a C^*-algebra. Then we obtain*

$$\mathrm{RR}(C^b((0,1), \mathfrak{A})) \leq \mathrm{RR}(\mathfrak{A}) + 1\,.$$

PROOF. We always have $C^b((0,1)) \otimes \mathfrak{A} \subset C^b((0,1), \mathfrak{A})$ (see [1]). This inclusion is strict, but it is deduced from [6] that

$$\mathrm{RR}(C^b((0,1)) \otimes \mathfrak{A}) \leq \mathrm{RR}(C^b((0,1))) + \mathrm{RR}(\mathfrak{A}) = 1 + \mathrm{RR}(\mathfrak{A})\,,$$

where $C^b((0,1)) \cong C(\beta(0,1))$ and $\dim \beta(0,1) = \dim(0,1) = 1$, where $\beta(0,1)$ is the Stone-Čech compactification of $(0,1)$. Furthermore, since any element of $C^b((0,1), \mathfrak{A})$ can be locally approximated by elements of the tensor product $C^b((0,1)) \otimes \mathfrak{A}$, it is not hard to see that $C^b((0,1), \mathfrak{A})$ is an inductive limit of C^*-algebras of the form $C^b((s,t), \mathfrak{A})$ for $0 \leq s < t \leq 1$, where the intervals (s,t) are disjoint in each inductive step and each $|s-t|$ goes to 0. Therefore, we obtain the conclusion. □

Using the same argument for stable rank,

LEMMA 2.4. *Let $\mathfrak{A}$ be a C^*-algebra. Then we obtain*

$$\mathrm{sr}(C^b((0,1), \mathfrak{A})) \leq \mathrm{sr}(\mathfrak{A}) + 1\,.$$

A part of Theorem 2.1 can be rewritten as:

COROLLARY 2.5. *Let $\mathfrak{A}$ be a C^*-algebra. Then*

$$\mathrm{RR}(\mathfrak{A} \otimes \mathcal{Z}) \leq \mathrm{RR}(\mathfrak{A}) + 1\,.$$

Also, if $\mathrm{sr}(\mathfrak{A})$ *is finite, then* $\mathrm{RR}(\mathfrak{A} \otimes \mathcal{Z}) \leq 3$.

Furthermore, if $\mathfrak{B}$ is an inductive limit of C^*-algebras with stable rank finite,

$$\mathrm{RR}(\mathfrak{B} \otimes \mathcal{Z}) \leq 3\,.$$

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