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On the Möbius and Allied Functions

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We define the arithmetical functions $\mu_k(n)$, $k=3, 4, \cdots$ as follows:

$\mu_k(n)=(-1)^\nu$, when n is k-free, i.e., not divisible by k-th power of any prime, and has ν prime factors, repeated factors being counted according to their multiplicity;

$\mu_k(n)=0$, when n is not k-free.

Thus $\mu_3(1)=1$, $\mu_3(2)=-1$, $\mu_3(3)=-1$, $\mu_3(4)=1$, $\mu_3(5)=-1$, $\mu_3(6)=1$, $\mu_3(7)=-1$, $\mu_3(8)=0$, $\mu_3(9)=1$, $\mu_3(10)=1, \cdots$, $\mu_3(100)=1, \cdots$, $\mu_3(1000)=0, \cdots$.

If we define $\mu_2(n)$ similarly, then the definition of $\mu_2(n)$ is just that of the Möbius function $\mu(n)$, and the Liouville function $\lambda(n)$ may be defined as

$$\lambda(n)=\lim_{k\to\infty}\mu_k(n) .$$

Thus the function $\mu_k(n)$, $k=3, 4, \cdots$ may be said to be situated between the functions $\mu(n)$ and $\lambda(n)$. We shall henceforth write $\mu_2(n)$ for $\mu(n)$, and $\mu_\infty(n)$ for $\lambda(n)$.

We put

$$f_2(s)=\frac{1}{\zeta(s)}\ , \quad f_k(s)=\frac{\zeta(2s)\zeta(ks)}{\zeta(s)\zeta(2ks)} \quad (k>2,\ \text{odd}) ,$$

$$f_k(s)=\frac{\zeta(2s)}{\zeta(s)\zeta(ks)} \quad (k>2,\ \text{even}) , \quad f_\infty(s)=\frac{\zeta(2s)}{\zeta(s)} ,$$

where $s=\sigma+ti$ is a complex variable, and $\zeta(s)$ is the Riemann zeta-function. Then we have

$$(1)\qquad f_k(s)=\sum_{n=1}^{\infty}\frac{\mu_k(n)}{n^s} \quad (\sigma>1,\ 2\leqq k\leqq\infty) .$$

The cases $k=2$, ∞ are well-known; the cases $2<k<\infty$ can be derived from

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$$\sum_{n=1}^{\infty}\frac{\mu_k(n)}{n^s}=\prod_p\left(1-\frac{1}{p^s}+\frac{1}{p^{2s}}-\cdots+\frac{(-1)^{k-1}}{p^{(k-1)s}}\right)$$
$$=\prod_p\left(1-\frac{(-1)^k}{p^{ks}}\right)\left(1+\frac{1}{p^s}\right)^{-1}\quad(\sigma>1)\,.$$

We put, for $x\geqq 1$,

$$M_k(x)=\sum_{n\leqq x}\mu_k(n)\quad(2\leqq k\leqq\infty)\,.$$

Then (1) can be rewritten as

$$(2)\qquad f_k(s)=s\int_1^{\infty}\frac{M_k(x)}{x^{s+1}}\,dx(\sigma>1,\ 2\leqq k\leqq\infty)\,.$$

Here we briefly mention that

$$M_k(x)=o(x)\quad(2\leqq k\leqq\infty)\,,$$

and, on the Riemann hypothesis,

$$M_k(x)=O(x^{1/2+\varepsilon})\quad(2\leqq k\leqq\infty)$$

with arbitrarily small positive ε. The case $k=2$ is well-known; the other cases can be obtained similarly.

Now we put

$$B_2=0\,,\quad B_k=\frac{\zeta\left(\frac{k}{2}\right)}{\zeta\left(\frac{1}{2}\right)\zeta(k)}\quad(k>2,\ \text{odd})\,,$$

$$B_k=\frac{1}{\zeta\left(\frac{1}{2}\right)\zeta\left(\frac{k}{2}\right)}\quad(k>2,\ \text{even})\,,\quad B_\infty=\frac{1}{\zeta\left(\frac{1}{2}\right)}$$

then we have the following theorem:

THEOREM.

$$M_k(x)-B_k\sqrt{x}=\Omega_{\pm}(\sqrt{x})\quad(2\leqq k\leqq\infty)\,.$$

This means that, for $2\leqq k\leqq\infty$,

$$\liminf_{x\to\infty}\frac{M_k(x)-B_k\sqrt{x}}{\sqrt{x}}<0\,,\quad \limsup_{x\to\infty}\frac{M_k(x)-B_k\sqrt{x}}{\sqrt{x}}>0\,.$$

Thus, for each k, $M_k(x)-B_k\sqrt{x}$ *changes its sign infinitely often as* x

tends to infinity.

PROOF. Let A be a real constant, the value of which will be assigned later on. Since we can write

$$\zeta(2s)=s\int_1^\infty \frac{[\sqrt{x}]}{x^{s+1}}dx \quad (\sigma>1)\ ,$$

we have, by (2),

$$f_k(s)+(A-B_k)\zeta(2s)=s\int_1^\infty \frac{M_k(x)+(A-B_k)[\sqrt{x}]}{x^{s+1}}dx\ . \tag{3}$$

Here we recall the following well known properties of $\zeta(s)$: it is regular for $\sigma>0$, $s\neq 1$; it has a simple pole at $s=1$ with residue 1; it does not vanish for $\sigma\geqq 1$, and for $s>0$; it has simple zeros for $1/2\leqq\sigma<1$, $t\neq 0$.

Now we assume that, for some k, $M_k(x)+(A-B_k)[\sqrt{x}]$ is of constant sign for all sufficiently large values of x, and for a while we shall consider this k. Since $f_k(s)+(A-B_k)\zeta(2s)$ is regular for $\sigma>1$, and for $s>1/2$, we can then conclude that the integral on the right hand side of (3) is convergent for $\sigma>1/2$ by similar argument as in Landau [2], § 197; thus $f_k(s)+(A-B_k)\zeta(2s)$ becomes regular for $\sigma>1/2$, and $\zeta(s)$ does not vanish for $\sigma>1/2$, and has simple zeros for $\sigma=1/2$, $t\neq 0$. We denote one of them by $\rho=1/2+\gamma i$, $\gamma\neq 0$; and, making $\sigma\to 1/2+0$, we have, from (3),

$$\begin{aligned} &2\lim_{\sigma\to 1/2}\left(\sigma-\frac{1}{2}\right)|f_k(\sigma)+(A-B_k)\zeta(2\sigma)| \\ &\qquad \geqq\frac{1}{|\rho|}\lim_{\sigma\to 1/2}\left(\sigma-\frac{1}{2}\right)|f_k(\sigma+\gamma i)+(A-B_k)\zeta(2\sigma+2\gamma i)|\ . \end{aligned} \tag{4}$$

On the other hand, since

$$\lim_{\sigma\to 1/2}\left(\sigma-\frac{1}{2}\right)\zeta(2\sigma)=\frac{1}{2}\ ,$$

and

$$\lim_{\sigma\to 1/2}\{f_k(\sigma)-B_k\zeta(2\sigma)\}<\infty\ ,$$

we have

$$\lim_{\sigma\to 1/2}\left(\sigma-\frac{1}{2}\right)\{f_k(\sigma)+(A-B_k)\zeta(2\sigma)\}=\frac{A}{2}\ .$$

Also

$$\lim_{\sigma\to 1/2}\left(\sigma-\frac{1}{2}\right)\{f_k(\sigma+\gamma i)+(A-B_k)\zeta(2\sigma+2\gamma i)\}$$
$$=\lim_{\sigma\to 1/2}\left(\sigma-\frac{1}{2}\right)f_k(\sigma+\gamma i)\,,$$

and this limit is finite and not zero. We denote this limit by C. Then (4) becomes

$$|A|\geqq\left|\frac{C}{\rho}\right|\,.$$

Thus, if we take A such that $|A|<|C/\rho|$, then $M_k(x)+(A-B_k)[\sqrt{x}]$ must change its sign infinitely often as x tends to infinity. The theorem now follows from this.

It is an open problem whether

$$\liminf_{x\to\infty}\frac{M_k(x)}{\sqrt{x}}=-\infty\,,\quad \limsup_{x\to\infty}\frac{M_k(x)}{\sqrt{x}}=+\infty$$

or not. In connexion with this, cf. Ingham [1].

As is easily seen, $B_k<0$ $(2<k\leqq\infty)$. Some numerical results concerning the behavior of $M_k(x)$, as x increases, obtained by the author will be reported elsewhere.

References

[1] A. E. INGHAM, On two conjectures in the theory of numbers, Amer. J. Math., **64** (1942), 313-319.

[2] E. LANDAU, Handbuch der Lehre von der Verteilung der Primzahlen II, Chelsea, N. Y., 1953.

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