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Formulae for the Riemann Zeta Function at Half Integers

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Introduction

Let a denote a positive integer, $\zeta(s)$ the Riemann zeta function and B_k be the k-th Bernoulli number, respectively. Then it is well known that

$$(1) \qquad \zeta(2a)=\frac{(-1)^{a-1}B_{2a}(2\pi)^{2a}}{2(2a)!} .$$

But practically nothing is known about the numerical nature of $\zeta(2a+1)$ except for the irrationality of $\zeta(3)$ proved by R. Apéry (see [6]). There is the Ramanujan's formula, which shed light on this problem, proved by A. P. Guinand [4], E. Grosswald [2], [3], and others. Recently, Y. Matsuoka [5] formulated and proved the Ramanujan's formula for the values of $\zeta(s)$ at half integers. And he got interesting expressions for $\zeta(1/2)\zeta(2a-1/2)$ and $\zeta(-1/2)\zeta(2a+1/2)$, where a is greater than 1.

In the present paper, by a similar method used in [5], we shall give generalizations of Matsuoka's results.

§1. Notations and results.

From now on, we assume that a is an integer greater than 1 and b is a non-negative integer. As usual, N and Q denote the set of natural numbers and the field of rational numbers, respectively. For any positive integer n, we put

$$g_{a,b}(n)=\sum_{\substack{klm\mid n\\ (k,l,m)\in N^3}} k^{-b-1/2}l^{2a-1}m^{2a-b-3/2} .$$

Further we put

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$$\varepsilon(b)=\begin{cases} 1 & \text{if } b\equiv 0,\ 3 \pmod 4\ , \\ -1 & \text{if } b\equiv 1,\ 2 \pmod 4\ . \end{cases}$$

Then our formulae are formulated as follows.

THEOREM 1. *Assume* $2a\geqq b+1$, *and define, for* $x>0$,

$$\begin{aligned} G_{a,b}(x) &= x^{a-b/2-1/4}\left\{\sum_{j=0}^{b}\sum_{n=1}^{\infty}\frac{2^j(2b-j)!\,g_{a,b}(n)}{2^b(b-j)!\,j!}(4\sqrt{n\pi x})^j e^{-4\sqrt{n\pi x}}\right. \\ &\quad \left.+\frac{(-1)^{a-1}\varepsilon(b)(2b)!\,(4a-2b-2)!\,B_{2a}\zeta(b+1/2)\zeta(2a-b-1/2)}{a\cdot b!\,(2a-b-1)!\,2^{6a-2b-1}\pi^{2a-b-1}}\right\} \\ &\quad +x^{a+b/2+1/4}\frac{(-1)^{a+b}\varepsilon(b)(b+1)!\,(4a+2b)!\,B_{2a}\zeta(-b-1/2)\zeta(2a+b+1/2)}{a\cdot(2b+2)!\,(2a+b)!\,2^{6a-2b-1}\pi^{2a-1/2}}\ . \end{aligned}$$

Then for arbitrary positive numbers α, β *with* $\alpha\beta=\pi^2$, *we have*

$$G_{a,b}(\alpha)=G_{a,b}(\beta)\ . \tag{2}$$

THEOREM 2. *Assume* $2a\leqq b$, *and define, for* $x>0$,

$$\begin{aligned} G_{a,b}(x) &= x^{a-b/2-1/4}\left\{\sum_{j=0}^{b}\sum_{n=1}^{\infty}\frac{2^j(2b-j)!\,g_{a,b}(n)}{2^b(b-j)!\,j!}(4\sqrt{n\pi x})^j e^{-4\sqrt{n\pi x}}\right. \\ &\quad \left.+\frac{(-1)^{a+b}\varepsilon(b)(2b)!\,(b-2a+1)!\,B_{2a}\zeta(b+1/2)\zeta(2a-b-1/2)}{a\cdot b!\,(2b-4a+2)!\,2^{6a-2b-1}\pi^{2a-b-1}}\right\} \\ &\quad +x^{a+b/2+1/4}\frac{(-1)^{a+b}\varepsilon(b)(b+1)!\,(4a+2b)!\,B_{2a}\zeta(-b-1/2)\zeta(2a+b+1/2)}{a\cdot(2b+2)!\,(2a+b)!\,2^{6a-2b-1}\pi^{2a-1/2}}\ . \end{aligned}$$

Then for arbitrary positive numbers α, β *with* $\alpha\beta=\pi^2$, *we have*

$$G_{a,b}(\alpha)=G_{a,b}(\beta)\ . \tag{3}$$

Putting $b=0$ in Theorem 1, we deduce the main result of Matsuoka [5].

For brevity, we put, for $x>0$,

$$E_{a,b}(x)=\sum_{j=0}^{b}\sum_{n=1}^{\infty}\frac{2^j(2b-j)!\,g_{a,b}(n)}{2^b(b-j)!\,j!}(\pi\sqrt{nx})^j e^{-\pi\sqrt{nx}}\ .$$

Then, by setting $(\alpha,\beta)=(2\pi,\pi/2)$, $(4\pi,\pi/4)$ in (2), (3), we obtain the following corollaries.

COROLLARY 1.

$$\zeta\left(-b-\frac{1}{2}\right)\zeta\left(2a+b+\frac{1}{2}\right)$$
$$=\frac{(-1)^{a+b}\varepsilon(b)a\cdot(2b+2)!\,(2a+b)!\,\pi^{2a-b-1}}{(b+1)!\,(4a+2b)!\,B_{2a}C}\{c_1E_{a,b}(4)$$
$$+c_2E_{a,b}(8)+c_3E_{a,b}(32)+c_4E_{a,b}(64)\}\ ,$$

where C, $c_1, \cdots, c_4$ are numbers in $Q(\sqrt{2})$ defined by

$$C=(2^{2a+b+1/2}-1)(^{2a+b+1/2}-2^{2a}-2^{b+1/2}+1)\ ,$$
$$c_1=2^{6a}\ ,$$
$$c_2=-2^{6a}(2^{2a-b-1/2}+1)\ ,$$
$$c_3=2^{8a-b-1/2}(2^{2a-b-1/2}+1)\ ,$$
$$c_4=-2^{10a-2b-1}\ .$$

COROLLARY 2. *Assume $2a\geqq b+1$. Then we have*

$$\zeta\left(b+\frac{1}{2}\right)\zeta\left(2a-b-\frac{1}{2}\right)$$
$$=\frac{(-1)^{a-1}\varepsilon(b)a\cdot b!\,(2a-b-1)!\,\pi^{2a-b-1}}{(2b)!\,(4a-2b-2)!\,B_{2a}C'}\{c_1'E_{a,b}(4)$$
$$+c_2'E_{a,b}(8)+c_3'E_{a,b}(32)+c_4'E_{a,b}(64)\}\ ,$$

where C', $c_1', \cdots, c_4'$ are numbers in $Q(\sqrt{2})$ defined by

$$C'=(2^{2a}-2^{b+1/2})(2^{2a+b+1/2}-2^{2a}-2^{b+1/2}+1)\ ,$$
$$c_1'=-2^{6a}\ ,$$
$$c_2'=2^{6a-b-1/2}(2^{2a+b+1/2}+1)\ ,$$
$$c_3'=-2^{8a-2b-1}(2^{2a+b+1/2}+1)\ ,$$
$$c_4'=2^{10a-2b-1}\ .$$

COROLLARY 3. *Assume $2a\leqq b$. Then we have*

$$\zeta\left(b+\frac{1}{2}\right)\zeta\left(2a-b-\frac{1}{2}\right)$$
$$=\frac{(-1)^{a+b}\varepsilon(b)a\cdot b!\,(2b-4a+2)!\,\pi^{2a-b-1}}{(2b)!\,(b-2a+1)!\,B_{2a}C'}\{c_1'E_{a,b}(4)$$
$$+c_2'E_{a,b}(8)+c_3'E_{a,b}(32)+c_4'E_{a,b}(64)\}\ ,$$

where C', $c'_1, \cdots, c'_4$ are numbers in $Q(\sqrt{2})$ defined in Corollary 2.

§2. Legendre's duplication formula.

The aim of this section is to prove the following proposition, which is a generalization of Legendre's duplication formula for the gamma function.

PROPOSITION. *For any non-negative integer m, we have*

$$\Gamma(s)\Gamma\left(s+m+\frac{1}{2}\right)=2^{1-2m-2s}\pi^{1/2}\sum_{k=0}^{m}\frac{2^k(2m-k)!}{(m-k)!\,k!}\Gamma(2s+k)\ . \tag{4}$$

To prove this, we need the following lemma which is due to Professor M. Endo.

LEMMA. *For any positive integer n, we hare*

$$\prod_{k=1}^{n}(x+2k-1)=C_{n,0}+\sum_{k=1}^{n}C_{n,k}x(x+1)\cdots(x+k-1)\ , \tag{5}$$

where

$$C_{n,k}=\frac{2^k(2n-k)!}{2^n(n-k)!\,k!}\quad(k=0,\cdots,n)\ .$$

PROOF. From (5), we easily deduce that

$$C_{n,0}=\frac{(2n)!}{2^n n!}\ ,$$

$$C_{n,n}=1$$

and

$$C_{n,k}=C_{n-1,k-1}+(2n-k-1)C_{n-1,k}\quad(k\leqq n-1)\ .$$

Hence, by induction, we can easily obtain our assertion.

PROOF OF PROPOSITION. In the case of $m=0$, this is Legendre's duplication formula. So we may assume that $m\geqq 1$. Since

$$\Gamma(s+1)=s\Gamma(s)\ , \tag{6}$$

we know that

$$\Gamma\left(s+m+\frac{1}{2}\right)=\prod_{k=1}^{m}\left(s+k-\frac{1}{2}\right)\cdot\Gamma\left(s+\frac{1}{2}\right)\ .$$

From the above lemma, we have

$$\prod_{k=1}^{m}(2s+2k-1)=C_{m,0}+\sum_{k=1}^{m}C_{m,k}(2s)(2s+1)\cdots(2s+k-1)\ ,$$

where

$$C_{m,k}=\frac{2^k(2m-k)!}{2^m(m-k)!\,k!}\quad(k=0,\ \cdots,\ m)\ .$$

Therefore, from (6) and Legendre's duplication formula, we conclude that

$$\Gamma(s)\Gamma\left(s+m+\frac{1}{2}\right)=2^{1-m-2s}\pi^{1/2}\sum_{k=0}^{m}C_{m,k}\Gamma(2s+k)\ ,$$

which gives our assertion.

§3. Proofs of Theorem 1 and Theorem 2.

We put

$$\varphi_{a,b}(s)=\frac{2^b\Gamma(s)\Gamma(s+b+1/2)\zeta(s)\zeta(s+b+1/2)\zeta(s-2a+1)\zeta(s-2a+b+3/2)}{\pi^{1/2}(2\pi)^{2s}}\ .$$

Then, by noticing that

$$2\Gamma(s)\zeta(s)\cos\frac{\pi s}{2}=(2\pi)^s\zeta(1-s) \tag{7}$$

and

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin\pi s}\ , \tag{8}$$

we obtain the functional equation

$$\varphi_{a,b}(s)=\varphi_{a,b}\left(2a-b-\frac{1}{2}-s\right)\ . \tag{9}$$

Now, we define the function

$$F_{a,b}(t)=\sum_{j=0}^{b}C_{b,j}\left\{\sum_{n=1}^{\infty}g_{a,b}(n)(4\pi\sqrt{nt})^j e^{-4\pi\sqrt{nt}}\right\}\quad(t>0)\ ,$$

where

$$C_{b,j}=\frac{2^j(2b-j)!}{2^b(b-j)!\,j!}\ .$$

Hereafter, we usually write $s=\sigma+i\tau$, where σ and τ are real, and $i^2=-1$. The series

$$t^{s-1}\sum_{n=1}^{\infty} g_{a,b}(n)(4\pi\sqrt{nt})^j e^{-4\pi\sqrt{nt}} \quad (0\leqq j\leqq b)$$

converges absolutely in $t>0$ and uniformly in any interval $\delta\leqq t<\infty$ with $\delta>0$, since

(10)
$$|g_{a,b}(n)|\leqq n^{4a+1},$$
$$|(4\pi\sqrt{nt})^j e^{-2\pi\sqrt{nt}}|\leqq(2j)!$$

and

$$|t^{s-1}e^{-\pi\sqrt{nt}}|\leqq C^*,$$

so that

$$\left|t^{s-1}\sum_{n=1}^{\infty} g_{a,b}(n)(4\pi\sqrt{nt})^j e^{-4\pi\sqrt{nt}}\right|\leqq(2b)!\ C^*\sum_{n=1}^{\infty} n^{4a+1}e^{-\pi\sqrt{n\delta}}$$
$$<\infty \quad (0\leqq j\leqq b),$$

where C^* denotes a positive number depending only on σ and δ. Thus we get

$$\int_0^\infty F_{a,b}(t)t^{s-1}dt$$
$$=\sum_{j=0}^{b} C_{b,j}\left\{\sum_{n=1}^{\infty} g_{a,b}(n)(4\pi\sqrt{n})^j\int_0^\infty t^{s+j/2-1}e^{-4\pi\sqrt{nt}}dt\right\}.$$

Substituting $u=4\pi\sqrt{nt}$ in the above integral, we find that

(11)
$$\int_0^\infty F_{a,b}(t)t^{s-1}dt$$
$$=2(4\pi)^{-2s}\sum_{j=0}^{b} C_{b,j}\Gamma(2s+j)\sum_{n=1}^{\infty} n^{-s}g_{a,b}(n).$$

Since, by (10), the last series is absolutely convergent in the half-plane $\mathrm{Re}(s)>4a+2$, it follows that

(12)
$$\sum_{n=1}^{\infty} n^{-s}g_{a,b}(n)=\zeta(s)\zeta\left(s+b+\frac{1}{2}\right)\zeta(s-2a+1)\zeta\left(s-2a+b+\frac{3}{2}\right)$$

for $\mathrm{Re}(s)>4a+2$, and so for all s (by the theorem of identity). Thus we get from (4), (11) and (12),

$$\varphi_{a,b}(s)=\int_0^\infty F_{a,b}(t)t^{s-1}dt\ .$$

Since $\varphi_{a,b}(s)$ is regular in $\mathrm{Re}\,(s)>2a$, Mellin's inversion formula permits us to write

$$F_{a,b}(t)=\frac{1}{2\pi i}\int_{2a+1/2-i\infty}^{2a+1/2+i\infty}\varphi_{a,b}(s)t^{-s}ds\ . \tag{13}$$

We note here that

$$\varphi_{a,b}(\sigma+i\tau)=0(e^{-\pi|\tau|}|\tau|^{\Delta})(c\leqq\sigma\leqq d,\ |\tau|\geqq 1)\ , \tag{14}$$

where c and d are arbitrary fixed real numbers, and Δ is a positive constant independent of τ. To show this, we have only to recall the following estimates;

$$\Gamma(\sigma+i\tau)=0(e^{-\pi|\tau|/2}|\tau|^{\sigma-1/2})(c\leqq\sigma\leqq d,\ |\tau|\geqq 1)\ ,$$

$$\zeta(\sigma+i\tau)=0(|\tau|^{\varepsilon(\sigma)}\log|\tau|)\ ,$$

where

$$\varepsilon(\sigma)=\begin{cases}\frac{1}{2}-\sigma & (\sigma\leqq 0)\ ,\\ \frac{1}{2} & \left(0\leqq\sigma\leqq\frac{1}{2}\right)\ ,\\ 1-\sigma & \left(\frac{1}{2}\leqq\sigma\leqq 1\right)\ ,\\ 0 & (\sigma\geqq 1)\ .\end{cases}$$

(These estimates can be found in [7].) By (14), we can shift the line of integration in (13) to any position $(\sigma'-i\infty,\ \sigma'+i\infty)$. Taking $\sigma'=-(b+1)$, we get

$$F_{a,b}(t)=\frac{1}{2\pi i}\int_{-(b+1)-i\infty}^{-(b+1)+i\infty}\varphi_{a,b}(s)t^{-s}ds + \left\{\text{sum of residues of integrand at poles } s=-b-\frac{1}{2},\ 0,\ 2a-b-\frac{1}{2},\ 2a\right\}\ . \tag{15}$$

Substituting $s=2a-b-1/2-S$, it follows from (9) that

$$(16)\qquad \frac{1}{2\pi i}\int_{-(b+1)-i\infty}^{-(b+1)+i\infty}\varphi_{a,b}(s)t^{-s}ds=t^{-2a+b+1/2}\frac{1}{2\pi i}\int_{2a+1/2-i\infty}^{2a+1/2+i\infty}\varphi_{a,b}(S)\left(\frac{1}{t}\right)^{-S}dS$$
$$=t^{-2a+b+1/2}F_{a,b}\left(\frac{1}{t}\right).$$

Now we calculate the residues in the sum by using (1), (7), (8) and the following facts

$$\zeta(0)=-\frac{1}{2},$$
$$\zeta(1-2n)=-\frac{B_{2n}}{2n},$$
$$\Gamma\left(m+\frac{1}{2}\right)=\frac{(2m)!\,\pi^{1/2}}{m!\,2^{2m}},$$

where n is any positive integer and m is any non-negative integer.

$$\operatorname*{Res}_{s=-b-1/2}(\varphi_{a,b}(s)t^{-s})$$
$$=\frac{2^b\Gamma(-b-1/2)\zeta(-b-1/2)\zeta(0)\zeta(-2a-b+1/2)\zeta(1-2a)t^{b+1/2}}{\pi^{1/2}(2\pi)^{-2b-1}}$$
$$=\frac{(-1)^{a+b+1}\varepsilon(b)(b+1)!\,(4a+2b)!\,B_{2a}\zeta(-b-1/2)\zeta(2a+b+1/2)t^{b+1/2}}{a\cdot(2b+2)!\,(2a+b)!\,2^{6a-2b-1}\pi^{2a-b-1}}.$$

$$\operatorname*{Res}_{s=2a}(\varphi_{a,b}(s)t^{-s})$$
$$=\frac{(-1)^{a+b}\varepsilon(b)(b+1)!\,(4a+2b)!\,B_{2a}\zeta(-b-1/2)\zeta(2a+b+1/2)t^{-2a}}{a\cdot(2b+2)!\,(2a+b)!\,2^{6a-2b-1}\pi^{2a-b-1}}.$$

In the case of $2a\geqq b+1$, we have

$$\operatorname*{Res}_{s=0}(\varphi_{a,b}(s)t^{-s})$$
$$=\frac{(-1)^{a}\varepsilon(b)(2b)!\,(4a-2b-2)!\,B_{2a}\zeta(b+1/2)\zeta(2a-b-1/2)}{a\cdot b!\,(2a-b-1)!\,2^{6a-2b-1}\pi^{2a-b-1}},$$

$$\operatorname*{Res}_{s=2a-b-1/2}(\varphi_{a,b}(s)t^{-s})$$
$$=\frac{(-1)^{a-1}\varepsilon(b)(2b)!\,(4a-2b-2)!\,B_{2a}\zeta(b+1/2)\zeta(2a-b-1/2)t^{-2a+b+1/2}}{a\cdot b!(2a-b-1)!\,2^{6a-2b-1}\pi^{2a-b-1}}.$$

In the case of $2a\leqq b$, we have

$$\operatorname*{Res}_{s=0}(\varphi_{a,b}(s)t^{-s})$$
$$=\frac{(-1)^{a+b+1}\varepsilon(b)(2b)!\,(b-2a+1)!\,B_{2a}\zeta(b+1/2)\zeta(2a-b-1/2)}{a\cdot b!\,(2b-4a+2)!\,2^{6a-2b-1}\pi^{2a-b-1}},$$

$$\operatorname*{Res}_{s=2a-b-1/2}(\varphi_{a,b}(s)t^{-s})$$
$$=\frac{(-1)^{a+b}\varepsilon(b)(2b)!\,(b-2a+1)!\,B_{2a}\zeta(b+1/2)\zeta(2a-b-1/2)t^{-2a+b+1/2}}{a\cdot b!\,(2b-4a+2)!\,2^{6a-2b-1}\pi^{2a-b-1}}\,.$$

In the case of $2a\geqq b+1$, using (15) and (16), these calculations give the equality

$$(F_{a,b}(t)+(-1)^{a-1}A)+(-1)^{a+b}Bt^{b+1/2}$$
$$=t^{-2a+b+1/2}\left(F_{a,b}\left(\frac{1}{t}\right)+(-1)^{a-1}A\right)+(-1)^{a+b}Bt^{-2a}\,,$$

where

$$A=\frac{\varepsilon(b)(2b)!\,(4a-2b-2)!\,B_{2a}\zeta(b+1/2)\zeta(2a-b-1/2)}{a\cdot b!\,(2a-b-1)!\,2^{6a-2b-1}\pi^{2a-b-1}}$$

and

$$B=\frac{\varepsilon(b)(b+1)!\,(4a+2b)!\,B_{2a}\zeta(-b-1/2)\zeta(2a+b+1/2)}{a\cdot(2b+2)!\,(2a+b)!\,2^{6a-2b-1}\pi^{2a-b-1}}\,.$$

Setting $\pi t=\alpha$ and $\pi/t=\beta$, we obtain

$$\alpha^{a-b/2-1/4}\left(F_{a,b}\left(\frac{\alpha}{\pi}\right)+(-1)^{a-1}A\right)+\alpha^{a+b/2+1/4}\frac{(-1)^{a+b}B}{\pi^{b+1/2}}$$
$$=\beta^{a-b/2-1/4}\left(F_{a,b}\left(\frac{\beta}{\pi}\right)+(-1)^{a-1}A\right)+\beta^{a+b/2+1/4}\frac{(-1)^{a+b}B}{\pi^{b+1/2}}\,,$$

which yields Theorem 1.

If $2a\leqq b$, then we can easily obtain Theorem 2 by using the same way as above. So we omit the proof of it.

References

[1] E. M. EDWARDS, Riemann's Zeta Function, Academic Press, New York, 1974.

[2] E. GROSSWALD, Die Werte der Riemannschen Zeta-Funktion an ungeraden Argumentstellen, Nachr. Acad. Wiss. Göttingen, (1970), 9–13.

[3] E. GROSSWALD, Comments on some formulae of Ramanujan, Acta Arith., **21** (1972), 25–34.

[4] A. P. GUINAND, Functional equations and self-reciprocal functions connected with Lambert series, Quart. J. Math. Oxford Ser., **15** (1944), 11–23.

[5] Y. MATSUOKA, On the values of the Riemann zeta function at half integers, Tokyo J. Math., **2** (1979), 371–377.

[6] A. J. VAN DER POORTEN, A proof that Euler missed ... Apéry's proof of the irrationality of $\zeta(3)$, to appear.

[7] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, 4th. ed., Cambridge University Press, Cambridge, 1962.

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