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π_1-Equivalent Weak Zariski Pairs

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Abstract. Consider a moduli space $\mathcal{M}(\Sigma, d)$ of reduced curves in $\mathbf{CP}^2$ with a given degree d and having a prescribed configuration of singularities Σ. Let $C, C' \in \mathcal{M}(\Sigma, d)$. The pair of curves (C, C') is called a weak Zariski pair if the pairs of spaces $(\mathbf{CP}^2, C)$ and $(\mathbf{CP}^2, C')$ are not homeomorphic. There exists two classical ways to detect weak Zariski pairs: (i) showing that the generic Alexander polynomials $\Delta_C(t)$ and $\Delta_{C'}(t)$ of C and C' are different; (ii) showing that the fundamental groups $\pi_1(\mathbf{CP}^2 - C)$ and $\pi_1(\mathbf{CP}^2 - C')$ are not isomorphic. In this paper, we give the first example of a weak Zariski pair (C, C') such that $\pi_1(\mathbf{CP}^2 - C)$ and $\pi_1(\mathbf{CP}^2 - C')$ are isomorphic (notice that such an isomorphism automatically implies $\Delta_C(t) = \Delta_{C'}(t)$). We shall call such a pair a π_1-equivalent weak Zariski pair. The members C and C' of our pair are reducible sextics with the following configuration of singularities $\{D_{10} + A_5 + A_4\}$. By the way, we determine the fundamental group $\pi_1(\mathbf{CP}^2 - D)$ for any curve D in the moduli space $\mathcal{M}(\{D_{10} + A_5 + A_4\}, 6)$. As an application, we give a new weak Zariski 4-ple (we recall that a 4-ple (D_1, D_2, D_3, D_4) of curves in $\mathcal{M}(\Sigma, d)$ is called a weak Zariski 4-ple if for any $1 \le i < j \le 4$ the pairs of spaces $(\mathbf{CP}^2, D_i)$ and $(\mathbf{CP}^2, D_j)$ are not homeomorphic).

Introduction

Consider a moduli space $\mathcal{M}(\Sigma, d)$ of reduced curves in $\mathbf{CP}^2$ with a given degree d and having a prescribed configuration of singularities Σ. In general, it is not easy to see if $\mathcal{M}(\Sigma, d)$ can be endowed with an algebraic structure and, in the case where it has such a structure, if it is irreducible or not. There are very few examples for which the irreducibility or the numbers of irreducible components is known (for examples see [H]). If $\mathcal{M}(\Sigma, d)$ has a weak Zariski pair (C, C'), then it is *not* irreducible since in this case the curves C and C' of the pair necessarily belong to different irreducible components.

The notion of weak Zariski pair is used, in particular, in [P] and [O7]. The precise definition is as follows.

DEFINITION 0.1. Let C, C' be two curves in $\mathcal{M}(\Sigma, d)$. One says that the pair (C, C') is a *weak Zariski pair* if the pairs of spaces $(\mathbf{CP}^2, C)$ and $(\mathbf{CP}^2, C')$ are not homeomorphic.

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This definition is weaker than the definition of Zariski pairs introduced by Artal Bartolo in [A]: a pair (C, C') of curves in $\mathcal{M}(\Sigma, d)$ is called a *Zariski pair* if there exist regular neighbourhoods $T(C)$ and $T(C')$ of C and C', respectively, such that the pairs $(T(C), C)$ and $(T(C'), C')$ are homeomorphic while the pairs $(\mathbf{CP}^2, C)$ and $(\mathbf{CP}^2, C')$ are not homeomorphic. A Zariski pair is always a weak Zariski pair. In the case of *irreducible* curves the two notions coincide. The first example of Zariski pair appears in the works by Zariski [Z1,2,3]; the members C and C' of the pair are irreducible 6-cuspidal sextics such that the cusps of C are on a conic while those of C' are not on a conic. Then, several other examples were found by Artal Bartolo [A], Artal Bartolo - Carmona Ruber [AC], Oka [O1,2,4,6], Shimada [Sh], Tokunaga [T], Pho [P] (Pho considered weak Zariski pairs).

Given two curves C and C' in $\mathcal{M}(\Sigma, d)$, there are two classical ways to detect if (C, C') is a weak Zariski pair: (i) showing that the generic Alexander polynomials $\Delta_C(t)$ and $\Delta_{C'}(t)$ of C and C' are different; (ii) showing that the fundamental groups $\pi_1(\mathbf{CP}^2 - C)$ and $\pi_1(\mathbf{CP}^2 - C')$ are not isomorphic. The computation of the Alexander polynomials is generally easier than those of the fundamental groups. However, there exist weak Zariski pairs (C, C') such that the Alexander polynomials $\Delta_C(t)$ and $\Delta_{C'}(t)$ coincide. These pairs are called *Alexander-equivalent weak Zariski pairs*. The first example of such a pair for reducible curves was given by Artal Bartolo - Carmona Ruber in [AC] (although they work with reducible curves their example in fact provides an Alexander-equivalent Zariski pair). The first example with irreducible curves is due to Oka [O4]; the members C and C' of the pair are curves of degree 12 with 27 cusps; C is a generic $(3, 3)$-covering of a 3-cuspidal quartic and C' is constructed using a 6-cuspidal non-conical sextic. Another example can be found in [O6]; here, the members C and C' of the pair are irreducible curves of degree 8 with 12 cusps.

In this paper, we give the first example of a weak Zariski pair (C, C') where C and C' are reducible curves such that $\pi_1(\mathbf{CP}^2 - C)$ and $\pi_1(\mathbf{CP}^2 - C')$ are isomorphic (notice that such an isomorphism automatically implies $\Delta_C(t) = \Delta_{C'}(t)$). We shall call such a pair a *π_1-equivalent weak Zariski pair*. The members C and C' of our pair are sextics with the following configuration of singularities $\{D_{10} + A_5 + A_4\}$ (cf. Theorem 2.1). By the way, we determine the fundamental group $\pi_1(\mathbf{CP}^2 - D)$ for any curve D in the moduli space $\mathcal{M}(\{D_{10} + A_5 + A_4\}, 6)$ (cf. Theorem 9.1). As an application, we give a new weak Zariski 4-ple (cf. Theorem 6.1); we recall that a weak Zariski k-ple is a k-ple $(D_1, \cdots, D_k)$ of curves in $\mathcal{M}(\Sigma, d)$ such that for any $1 \leq i < j \leq k$ the pairs of spaces $(\mathbf{CP}^2, D_i)$ and $(\mathbf{CP}^2, D_j)$ are not homeomorphic (cf. [O7]).

In [GLS], Greuel-Lossen-Shustin gave an example of a moduli $\mathcal{M}(\Sigma, d)$ with at least two irreducible components such that $\pi_1(\mathbf{CP}^2 - D) \simeq \mathbf{Z}/d\mathbf{Z}$ for any curve $D \in \mathcal{M}(\Sigma, d)$, but they do not discuss about the topology of the pair $(\mathbf{CP}^2, D)$.

This paper is organized as follows. In Section 1, we recall the Zariski-van Kampen pencil method which we use to compute the fundamental groups. In Section 2, we give an example of π_1-equivalent weak Zariski pair (cf. Theorem 2.1 and Corollary 2.2). In Sections 4, 5, 7, 8 and 9, we compute the fundamental groups $\pi_1(\mathbf{CP}^2 - D)$ for every curve D in the moduli space $\mathcal{M}(\{D_{10} + A_5 + A_4\}, 6)$ (cf. Theorems 4.1, 5.1, 7.1, 8.1, 9.1 and Corollaries 4.2, 5.2, 7.2, 8.2). Section 9 also contains a discussion about the connected components of $\mathcal{M}(\{D_{10} + A_5 + A_4\}, 6)$. In Section 6, we give an example of weak Zariski 4-ple (cf. Theorem 6.1 and Corollary 6.2). Notice that the proof of Theorem 2.1 (resp. Theorem 6.1) is an immediate consequence of Theorems 3.1, 4.1 and 5.1 (resp. Theorems 3.1, 4.1, 5.1, 7.1 and 8.1).

1. Zariski-van Kampen pencil method

In this section, we recall the classical Zariski-van Kampen theorem, and we give a non-generic version of it. We also recall a useful result on the first homology of the complement of a plane curve.

1.1. Classical Zariski-van Kampen theorem. Let $F(X, Y, Z)$ be a reduced homogeneous polynomial of degree d, and let

$$C := \{(X : Y : Z) \in \mathbf{CP}^2 \mid F(X, Y, Z) = 0\}$$

be the corresponding projective curve in $\mathbf{CP}^2$. We consider two independent linear forms $l_1(X, Y, Z)$ and $l_2(X, Y, Z)$, and for every point $\tau := (S : T)$ in $\mathbf{CP}^1$ we denote by L_τ the projective line of $\mathbf{CP}^2$ defined by

$$L_\tau := \{(X : Y : Z) \in \mathbf{CP}^2 \mid T\, l_1(X, Y, Z) - S\, l_2(X, Y, Z) = 0\} .$$

The family of lines $\mathcal{L} := (L_\tau)_{\tau \in \mathbf{CP}^1}$ is called the pencil generated by the linear forms l_1 and l_2. The point $b_0 := L_{(0:\,1)} \cap L_{(1:\,0)}$, which belongs to every line of the pencil, is called the axis of $\mathcal{L}$. The pencil is said *generic with respect to* C if $b_0 \notin C$. Hereafter, in Section 1, we shall assume that $\mathcal{L}$ is generic with respect to C.

A member L_τ of $\mathcal{L}$ is called a *generic line with respect to* C if it avoids the singularities of C and if it is transverse to the non-singular part of C; otherwise, it is called a *singular line*. If L_τ is generic, then it intersects C at exactly d points. If it is singular, then it intersects C at a singular point or it is tangent to C at some simple point. Notice that the set of singular lines is finite. If necessary, one may consider some generic lines of $\mathcal{L}$ as "singular" ones. Let Ξ be the set of parameters $\tau \in \mathbf{CP}^1$ corresponding to the singular lines, and let L_{τ_0} be a generic line (which we have not decided to consider as "singular").

As the base point for the fundamental group $\pi_1(\mathbf{CP}^2 - C)$ we take the point b_0. It is well-known that there is a canonical action, called *monodromy action*, of $\pi_1(\mathbf{CP}^1 - \Xi, \tau_0)$ on $\pi_1(L_{\tau_0} - C, b_0)$ (see e.g. [O2,5]). The relations $\xi = \xi^\sigma$, for $\sigma \in \pi_1(\mathbf{CP}^1 - \Xi, \tau_0)$ and $\xi \in \pi_1(L_{\tau_0} - C, b_0)$, are called the *monodromy relations*, where ξ^σ is the image of (σ, ξ) by the monodromy action. The classical Zariski-van Kampen theorem is as follows.

THEOREM 1.1.1 (cf. [Z1], [vK] and [C]). *The inclusion map* $L_{\tau_0} - C \hookrightarrow \mathbf{CP}^2 - C$ *induces an isomorphism*

$$\pi_1(L_{\tau_0} - C, b_0) \,/\, N \xrightarrow{\sim} \pi_1(\mathbf{CP}^2 - C, b_0) ,$$

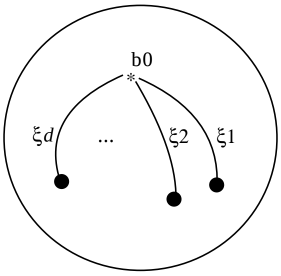

FIGURE 1. Generators of $\pi_1(L_{\tau_0} - C, b_0)$.

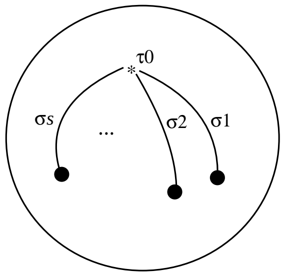

FIGURE 2. Generators of $\pi_1(\mathbf{CP}^1 - \Xi, \tau_0)$.

where N is the normal subgroup of $\pi_1(L_{\tau_0} - C, b_0)$ generated by

$$\{\xi^{-1}\xi^{\sigma} \mid \sigma \in \pi_1(\mathbf{CP}^1 - \Xi, \tau_0),\ \xi \in \pi_1(L_{\tau_0} - C, b_0)\}.$$

Theorem 1.1.1 can be rephrased in terms of a presentation by generators and relations as follows. We give a natural presentation of $\pi_1(L_{\tau_0} - C, b_0)$ (resp. $\pi_1(\mathbf{CP}^1 - \Xi, \tau_0)$) by d generators $\xi_1, \cdots, \xi_d$ as in Figure 1 and the relation $\xi_d \cdots \xi_1 = 1$ (resp. by s generators $\sigma_1, \cdots, \sigma_s$ as in Figure 2 and the relation $\sigma_s \cdots \sigma_1 = 1$, where s is the cardinality of Ξ); $\xi_1, \cdots, \xi_d$ are lassos around the d intersection points of L_{τ_0} with C; $\sigma_1, \cdots, \sigma_s$ are lassos around the s parameters corresponding to the singular lines of the pencil[1]. Then, the fundamental group $\pi_1(\mathbf{CP}^2 - C, b_0)$ is presented by the generators $\xi_1, \cdots, \xi_d$ and the relations $\xi_d \cdots \xi_1 = 1$ and $\xi_i = \xi_i^{\sigma_j}$ for all i and j.

1.2. A non-generic version of the Zariski-van Kampen theorem. We still use the notation and hypotheses of Section 1.1.

Let $\tau_e := (S_e : T_e) \in \mathbf{CP}^1 - \{\tau_0\}$. We consider the reduced homogeneous polynomial $F'(X, Y, Z)$ defined by

$$F'(X, Y, Z) := (T_e\, l_1(X, Y, Z) - S_e\, l_2(X, Y, Z))\, F(X, Y, Z),$$

and we denote by

$$C' := \{(X : Y : Z) \in \mathbf{CP}^2 \mid F'(X, Y, Z) = 0\}$$

the corresponding projective curve in $\mathbf{CP}^2$. We have $C' = C \cup L_{\tau_e}$. Obviously, the pencil $\mathcal{L}$ is *not* generic with respect to C'. A line L_τ of $\mathcal{L}$ is called a *generic line with respect to C'* if it is generic with respect to C and different from the line L_{τ_e}; otherwise, it is called a *singular*

[1] In the figures, for simplicity of drawing pictures, we shall denote a lasso oriented counter-clockwise just by a path ending with a black disk ——• as in [O3,4]. We recall that a lasso is defined as follows. Let D be a reduced curve in $\mathbf{CP}^2$, and let $(D_i)_i$ be the irreducible components of D. An element $\zeta \in \pi_1(\mathbf{CP}^2 - D, *)$ is called a lasso oriented counter-clockwise for D_i if it is represented by a loop written as $\varrho\,\omega\varrho^{-1}$, where ω is a loop running once counter-clockwise around the boundary circle of a small closed normal disk Δ of D at a simple point such that Δ does not intersect with D_j for $j \neq i$, and where ϱ is a simple path connecting the base point $*$ and the loop ω such that $im\varrho \cap \Delta$ is reduced to a single point (cf. [O2]).

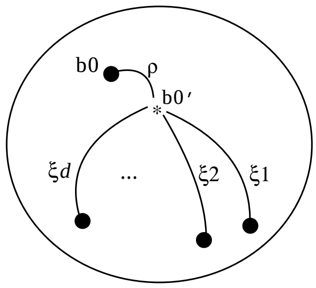

FIGURE 3. Generators of $\pi_1(L_{\tau_0} - C', b_0')$.

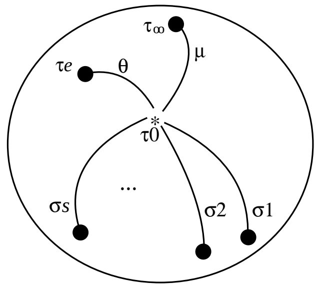

FIGURE 4. Generators of $\pi_1(\mathbf{CP}^1 - \Xi', \tau_0)$ if $\tau_e \notin \Xi$.

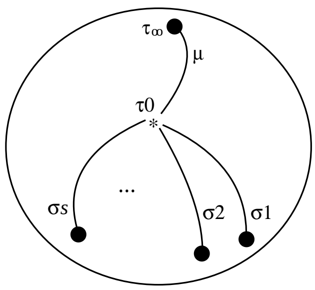

FIGURE 4′. Generators of $\pi_1(\mathbf{CP}^1 - \Xi', \tau_0)$ if $\tau_e \in \Xi$.

line. As the base point for the fundamental group $\pi_1(\mathbf{CP}^2 - C')$ we take a point b_0' on the generic line L_{τ_0} sufficiently close to b_0 but $b_0' \neq b_0$.

Let $\tau_\infty \in \mathbf{CP}^1 - (\Xi \cup \{\tau_e, \tau_0\})$, and let $\Xi' := \Xi \cup \{\tau_e, \tau_\infty\}$. So, Ξ' is the set of parameters $\tau \in \mathbf{CP}^1$ such that L_τ is singular with respect to C' or $\tau = \tau_\infty$.

We give a natural presentation of the fundamental group $\pi_1(L_{\tau_0} - C', b_0')$ by $d+1$ generators $\rho, \xi_1, \cdots, \xi_d$ as in Figure 3 and the relation $\rho\xi_d \cdots \xi_1 = 1$. The generators $\xi_1, \cdots, \xi_d$ are lassos around the d intersection points of L_{τ_0} with C while the generator ρ is a lasso around the intersection point of L_{τ_0} with L_{τ_e} (i.e., the axis b_0 of the pencil). Similarly, we give a natural presentation of $\pi_1(\mathbf{CP}^1 - \Xi', \tau_0)$ by $s + 2$ generators $\theta, \mu, \sigma_1, \cdots, \sigma_s$ as in Figure 4 and the relation $\mu\theta\sigma_s \cdots \sigma_1 = 1$, if $\tau_e \notin \Xi$, or by $s + 1$ generators $\mu, \sigma_1, \cdots, \sigma_s$ as in Figure 4′ and the relation $\mu\sigma_s \cdots \sigma_1 = 1$, if $\tau_e \in \Xi$. The generators $\sigma_1, \cdots, \sigma_s$ are lassos around the s points of Ξ, the generator μ is a lasso around τ_∞. If $\tau_e \notin \Xi$, then θ is a lasso around τ_e. If $\tau_e \in \Xi$, then there is j_0, $1 \leq j_0 \leq s$, such that σ_{j_0} is a lasso around τ_e.

There is an action of $\pi_1(\mathbf{CP}^1 - \Xi', \tau_0)$ on $\pi_1(L_{\tau_0} - C', b_0')$. In this action, for each j, $1 \leq j \leq s$, and each $\xi \in \pi_1(L_{\tau_0} - C', b_0')$, the image of (σ_j, ξ) (resp. (θ, ξ)), denoted by ξ^{σ_j} (resp. ξ^θ), is just the image of ξ by the local monodromy along σ_j (resp. along θ), as in the usual monodromy action. The image of (μ, ξ) is more complicated (it is defined by the action of $(\theta\sigma_s \cdots \sigma_1)^{-1}$ if $\tau_e \notin \Xi$ or by the action of $(\sigma_s \cdots \sigma_1)^{-1}$ if $\tau_e \in \Xi$) but we do not need it for our purpose.

A non-generic version of the Zariski-van Kampen theorem is as follows.

THEOREM 1.2.1. *The inclusion map* $L_{\tau_0} - C' \hookrightarrow \mathbf{CP}^2 - C'$ *induces an isomorphism*

$$\pi_1(L_{\tau_0} - C', b'_0)/N' \xrightarrow{\sim} \pi_1(\mathbf{CP}^2 - C', b'_0)\,,$$

where N' *is the normal subgroup of* $\pi_1(L_{\tau_0} - C', b'_0)$ *generated by*:

- $\{\xi^{-1}\xi^{\sigma_j},\ \rho^{-1}\xi^{-1}\rho\,\xi^{\theta} \mid j \in \{1, \cdots, s\},\ \xi \in \pi_1(L_{\tau_0} - C', b'_0)\}$, *if* $\tau_e \notin \Xi$;
- $\{\xi^{-1}\xi^{\sigma_j},\ \rho^{-1}\xi^{-1}\rho\,\xi^{\sigma_{j_0}} \mid j \in \{1, \cdots, s\} - \{j_0\},\ \xi \in \pi_1(L_{\tau_0} - C', b'_0)\big\}$, *if* $\tau_e \in \Xi$.

In other words, the fundamental group $\pi_1(\mathbf{CP}^2 - C', b'_0)$ is presented by the generators $\rho, \xi_1, \cdots, \xi_d$ and the following relations:

$$\bullet \text{ if } \tau_e \notin \Xi\,, \quad \begin{cases} \rho\xi_d \cdots \xi_1 = 1, \\ \xi_i = \xi_i^{\sigma_j}\,, \quad \text{for all } i \text{ and } j\,, \\ \rho^{-1}\xi_i\rho = \xi_i^{\theta}\,, \quad \text{for all } i\,; \end{cases}$$

$$\bullet \text{ if } \tau_e \in \Xi, \quad \begin{cases} \rho\xi_d \cdots \xi_1 = 1\,, \\ \xi_i = \xi_i^{\sigma_j}\,, \quad \text{for all } i \text{ and all } j \neq j_0\,, \\ \rho^{-1}\xi_i\rho = \xi_i^{\sigma_{j_0}}\,, \quad \text{for all } i\,. \end{cases}$$

Theorem 1.2.1 can be proved in the same way as the classical Zariski-van Kampen theorem. We thus omit the proof.

REMARK. There is also a discussion on non-generic Zariski-van Kampen theorems in [D, Remark (4.3.19)] but different from our setting.

1.3. Fundamental group and first homology. Let D be a reduced curve in $\mathbf{CP}^2$ with r irreducible components $D_1, \cdots, D_r$ of degree $d_1, \cdots, d_r$ respectively. By Lefschetz duality (cf. [Sp]), it is not difficult to see that the first integral homology group $H_1(\mathbf{CP}^2 - D; \mathbf{Z})$ is isomorphic to

$$\mathbf{Z}^{r-1} \times (\mathbf{Z}/d_0\mathbf{Z})\,,$$

where $d_0 := gcd(d_1, \cdots, d_r)$ (cf. [O5]). On the other hand, by the Hurewicz theorem (cf. [Sp]), $H_1(\mathbf{CP}^2 - D; \mathbf{Z})$ is isomorphic to the quotient of $\pi_1(\mathbf{CP}^2 - D)$ by the commutator subgroup. So, in the case where $\pi_1(\mathbf{CP}^2 - D)$ is abelian, we have the isomorphism

$$\pi_1(\mathbf{CP}^2 - D) \simeq \mathbf{Z}^{r-1} \times (\mathbf{Z}/d_0\mathbf{Z})\,.$$

1.4. Notation. For our purpose, we shall use only the pencil $\mathcal{L}_{X,Z}$ and $\mathcal{L}_{Y,Z}$ generated by the linear forms l_X, l_Z and l_Y, l_Z, respectively, where

$$l_X(X, Y, Z) = X\,, \quad l_Y(X, Y, Z) = Y\,, \quad l_Z(X, Y, Z) = Z\,.$$

Let $L_\infty = \{(X\colon Y\colon Z) \in \mathbf{CP}^2 \mid Z = 0\}$ be the line at infinity of $\mathbf{CP}^2$. We shall identify $\mathbf{CP}^2 - L_\infty$ with the affine space $\mathbf{C}^2$ and we shall consider on this space the affine coordinates $x := X/Z$ and $y := Y/Z$. In $\mathbf{C}^2$, the pencils $\mathcal{L}_{X,Z}$ and $\mathcal{L}_{Y,Z}$ are simply given by $\{x = \eta\}_{\eta\in\mathbf{C}}$ and $\{y = \eta\}_{\eta\in\mathbf{C}}$ respectively.

For any given parameter $\tau = (S : T) \in \mathbf{CP}^1 - \{\tau_\infty\} \simeq \mathbf{C}$, we shall also denote the line L_τ by L_η where $\eta = S/T$. In $\mathbf{C}^2$, the line L_η is simply defined by $x = \eta$ for the pencil $\mathcal{L}_{X,Z}$ and by $y = \eta$ for the pencil $\mathcal{L}_{Y,Z}$.

If $G(X, Y, Z)$ is a reduced homogeneous polynomial defining a curve D in $\mathbf{CP}^2$, then the affine equation of D is the equation $G(x, y, 1) = 0$.

Hereafter, we shall always assume that ε is a sufficiently small strictly positive number.

2. A π_1-equivalent weak Zariski pair

Consider the sextics C_1 and C_2 defined by following affine equations:

$$C_1 : \quad f_1(x, y) := f_1'(x, y)\, f_1''(x, y) = 0\,,$$
$$C_2 : \quad f_2(x, y) := f_2'(x, y)\, f_2''(x, y) = 0\,,$$

where f_1', f_1'' and f_2', f_2'' are given by

$$\begin{aligned} f_1'(x, y) &:= x\,,\\ f_1''(x, y) &:= 26556\, y^4 x + 19932\, y^2 x^3 - 14336\, x^3 y - 7255\, x^4 y - 38112\, y^3 x\\ &\quad - 31802\, y^3 x^2 + 13632\, y^2 x + 35120\, y^2 x^2 - 8192\, x^2 y - 12167\, y^5\\ &\quad + 25392\, y^4 + 704\, x^5 + 4096\, x^4 + 4096\, y^2 - 17664\, y^3\,, \end{aligned}$$

and

$$\begin{aligned} f_2'(x, y) &:= y\,,\\ f_2''(x, y) &:= \frac{34600}{1331} y^5 + \left(-\frac{6421}{121} x - \frac{81300}{1331}\right) y^4\\ &\quad + \left(-\frac{96402}{1331} x^2 + \frac{12963}{121} x + \frac{58800}{1331}\right) y^3\\ &\quad + \left(\frac{20127}{121} x^3 + \frac{116004}{1331} x^2 - \frac{6663}{121} x - \frac{100}{11}\right) y^2\\ &\quad + \left(\frac{65536}{1331} x^4 - \frac{20127}{121} x^3 - \frac{162}{11} x^2 + x\right) y - \frac{16000}{121} x^5\,. \end{aligned}$$

The curve C_1 has two irreducible components: a line C_1' defined by the equation $f_1'(x, y) = 0$ and a quintic C_1'' defined by the equation $f_1''(x, y) = 0$. The configuration of singularities of C_1 is $\{D_{10} + A_5 + A_4\}$: D_{10} at the origin, A_5 at $(0, 16/23)$ and A_4 at $(1, 1)$[1]. We show the real plane section of C_1 in Figure 5 below (in the figures, we do not respect the numerical scale).

The curve C_2 has also two irreducible components: a line C_2' defined by the equation $f_2'(x, y) = 0$ and a quintic C_2'' defined by the equation $f_2''(x, y) = 0$. The configuration of singularities of C_2 is also $\{D_{10}+A_5+A_4\}$: D_{10} at the origin, A_5 at $(0, 1)$ and A_4 at $(1, 2)$. We show the real plane section of C_2 in Figure 12 below. Observe that, after the analytic change of coordinates $(x, y) \mapsto (x, y + 1 + \frac{128}{75} x^2)$, the equation of C_2 near $(0, 1)$ takes the form

$$\frac{22500}{1331} y^2 - \frac{1377}{121} x^3 y + \frac{425984}{185625} x^6 + \text{higher terms} = 0\,.$$

So, as the leading term $\frac{22500}{1331} y^2 - \frac{1377}{121} x^3 y + \frac{425984}{185625} x^6$ has no real factorization, the point $(0, 1)$ is an isolated point of the real plane section of C_2.

Notice that the curves C_1 and C_2 are not of torus type (for the definition, see e.g. [O3]).

THEOREM 2.1. *The pair (C_1, C_2) is a π_1-equivalent weak Zariski pair.*

The proof of Theorem 2.1 follows immediately from Theorems 3.1, 4.1 and 5.1 below.

Let $\mathcal{M} := \mathcal{M}(\{D_{10} + A_5 + A_4\}, 6)$ be the moduli space of reduced sextics in $\mathbf{CP}^2$ with the configuration of singularities $\{D_{10} + A_5 + A_4\}$. Let $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) be the connected component of $\mathcal{M}$ containing the curve C_1 (resp. C_2). Since the topology of the pair $(\mathbf{CP}^2, D)$ is independent on the choice of D in $\mathcal{M}_1$ (resp. in $\mathcal{M}_2$) (cf. [Z4,5] and [LR]), Theorem 2.1 implies the following result.

COROLLARY 2.2. *Any pair (D_1, D_2), where $D_1 \in \mathcal{M}_1$ and $D_2 \in \mathcal{M}_2$, is a π_1-equivalent weak Zariski pair.*

3. Topology of C_1 and C_2

The notation is as in Section 2.

THEOREM 3.1. *The curves C_1 and C_2 are not homeomorphic. In particular (C_1, C_2) is a weak Zariski pair.*

PROOF. First, we observe that the quintics C_1'' and C_2'' are rational curves (i.e., curves with genus 0). This follows immediately from the genus formula which can be stated as follows. Given an irreducible curve D with degree d and singular locus $\Sigma(D)$, the genus

[1] We recall that a point p of a curve C is called a singularity of type A_n, where n is an integer ≥ 1, if the germ (C, p) is topologically equivalent to the germ $(\{x^2 + y^{n+1} = 0\}, O)$ as embedded germs (for the definition of "topologically equivalent", see e.g. [Di, Definition (1.4)]). It is called a singularity of type D_{10} if (C, p) is topologically equivalent to $(\{x^2 y + y^9 = 0\}, O)$.

$g(D)$ of D is given by the formula:

$$2g(D) = (d-1)(d-2) - \sum_{p \in \Sigma(D)} (\mu(D,p) + r(D,p) - 1)\,,$$

where $\mu(D,p)$ and $r(D,p)$ are the Milnor number of D at p and the number of local irreducible components of D at p respectively (cf. [M] and [BK]). In our case, notice that the configuration of singularities $\Sigma(C_1'')$ of C_1'' is $\{A_7 + A_4\}$ (A_7 at the origin, A_4 at $(1,1)$) while the configuration $\Sigma(C_2'')$ of C_2'' is $\{A_1 + A_5 + A_4\}$ (A_1 at the origin, A_5 at $(0,1)$, A_4 at $(1,2)$). The line C_1' intersects with C_1'' at two points: at the origin, transversally with the tangent cone of C_1'' at O so that the singularity of C_1 at O is D_{10}, and at $(0, 16/23)$ where C_1' is tangent to C_1'' with intersection multiplicity 3 so that the singularity of C_1 at this point is A_5. The line C_2' intersects C_2'' only at the origin and it is tangent to one of the branches of A_1 with intersection multiplicity 5 so that the singularity of C_2 at O is D_{10}.

So, topologically, C_1'' is the sphere $\mathbf{S}^2$ with two points identified while C_2'' is $\mathbf{S}^2$ with two pairs of points identified.

Now, observe that the line C_1' intersects C_1'' at the (unique) exceptional point of C_1'' (i.e., the point where C_1'' is not a topological manifold) and at an "ordinary" point (i.e., a point where one has a structure of topological manifold), while the line C_2' intersects C_2'' at only one point which is exceptional. So, the set of ordinary points of C_1 is homeomorphic to the disjoint union

$$(\mathbf{S}^2 - 2 \text{ points}) \sqcup (\mathbf{S}^2 - 3 \text{ points})\,,$$

while the set of ordinary points of C_2 is homeomorphic to the disjoint union

$$(\mathbf{S}^2 - 1 \text{ point}) \sqcup (\mathbf{S}^2 - 4 \text{ points})\,.$$

So, if there was a homeomorphism from C_1 onto C_2, then (for example) the connected component $\mathbf{S}^2 - \{2 \text{ points}\}$ of the set of ordinary points of C_1 would be sent homeomorphically onto one of the two connected components, $\mathbf{S}^2 - \{1 \text{ point}\}$ or $\mathbf{S}^2 - \{4 \text{ points}\}$, of the set of ordinary points of C_2. Of course, this is impossible.

REMARK. Notice that C_1 and C_2 have the same integral reduced homology:

$$\tilde{H}_q(C_i; \mathbf{Z}) = \begin{cases} \mathbf{Z}^2 & \text{if } q = 1 \text{ or } 2 \\ 0 & \text{otherwise} \end{cases} \qquad (i = 1, 2)\,.$$

4. Fundamental group of $\mathbf{CP}^2 - C_1$

Again, the notation is as in Section 2.

THEOREM 4.1. *The fundamental group $\pi_1(\mathbf{CP}^2 - C_1)$ is isomorphic to $\mathbf{Z}$.*

We still denote by $\mathcal{M}$ the moduli space of reduced sextics in $\mathbf{CP}^2$ with the configuration of singularities $\{D_{10} + A_5 + A_4\}$, and by $\mathcal{M}_1$ the connected component of $\mathcal{M}$ containing the curve C_1. Theorem 4.1 (together with [Z4,5] and [LR] as above) implies the following result.

COROLLARY 4.2. *For any curve $D \in \mathcal{M}_1$, we have $\pi_1(\mathbf{CP}^2 - D) \simeq \mathbf{Z}$.*

PROOF OF THEOREM 4.1. We use the classical Zariski-van Kampen theorem (cf. Theorem 1.1.1) with the pencil $\mathcal{L}_{Y,Z}$ (cf. Notation 1.4). We recall that in $\mathbf{C}^2 := \mathbf{CP}^2 - L_\infty$ this pencil is given by $\{y = \eta\}_{\eta \in \mathbf{C}}$. Observe that the point b_0 (i.e., the axis of the pencil) does not belong to the curve C_1. To prove Theorem 4.1, it suffices (cf. Section 1.3) to show that $\pi_1(\mathbf{CP}^2 - C_1, b_0)$ is abelian.

The discriminant $\Delta_x(f_1)$ of f_1 as a polynomial in x, which describes the singular lines of the pencil $\mathcal{L}_{Y,Z}$ (with respect to C_1), is a polynomial in y given by

$$\Delta_x(f_1)(y) = a_1 y^{14}(a_2 y^3 + a_3 y^2 + a_4 y + a_5)(23y - 16)^6 (y - 1)^7 .$$

Of course, we know the numbers a_i $(1 \leq i \leq 5)$ but we do not write them here because they are too big; we observe, nevertheless, that $\Delta_x(f_1)$ has six distinct real roots:

$$\eta_1 = 0\,, \quad \eta_2 = 0.683\cdots\,, \quad \eta_3 = 0.695\cdots\,, \quad \eta_4 = 0.754\cdots\,, \quad \eta_5 = 1\,, \quad \eta_6 = 1.085\cdots$$

The singular lines of the pencil are the lines $L_{\eta_1}, \cdots, L_{\eta_6}$ corresponding to these six roots.

We take generators $\xi_1, \cdots, \xi_6$ of the fundamental group $\pi_1(L_{\eta_5+\varepsilon} - C_1, b_0)$ (which are also generators of $\pi_1(\mathbf{CP}^2 - C_1, b_0)$) as in Figure 6; $\xi_1, \cdots, \xi_5$ are lassos for C_1'' and ξ_6 is a lasso for the line component C_1'.

We first look at the monodromy relations around L_{η_6} (obtained when y moves on the real axis from $y := \eta_5 + \varepsilon \to \eta_6 - \varepsilon$, then runs once counter-clockwise on the circle $|y - \eta_6| = \varepsilon$, and then comes back on the real axis from $y := \eta_6 - \varepsilon \to \eta_5 + \varepsilon$). In Figure 7 we show how the generators at $y = \eta_5 + \varepsilon$ are deformed when y moves on the real axis from $y := \eta_5 + \varepsilon \to \eta_6 - \varepsilon$. Then, to read the monodromy relations around L_{η_6}, it suffices to observe that the line L_{η_6} is tangent to the curve at the simple point p_0 (cf. Figure 5) and that the intersection multiplicity $I(L_{\eta_6}, C_1; p_0)$ of L_{η_6} with C_1 at p_0 is equal to 2. Thus, by the implicit functions

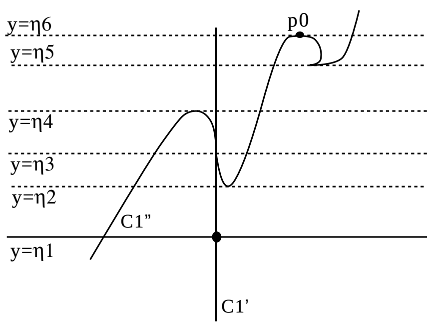

FIGURE 5. Real plane section of C_1.

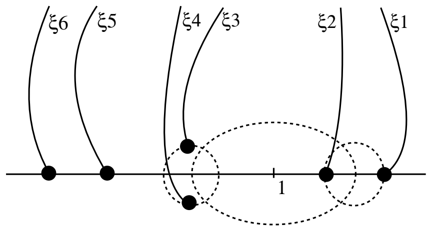

FIGURE 6. Generators at $y = \eta_5 + \varepsilon$.

theorem, the germ (C_1, p_0) is topologically equivalent to the germ $(\{y = -x^2\}, O)$. The monodromy relations around L_{η_6} thus give the relation

$$\xi_5 = \xi_3\xi_2\xi_3^{-1}\,. \tag{4.3}$$

To read the monodromy relations around L_{η_5}, we look at the Puiseux parametrization of C_1 at $(1, 1)$:

$$\begin{cases} y = 1 + t^4 \\ x = 1 + \frac{2}{19}\sqrt{38}\,t^2 + \frac{162}{361}\,38^{(1/4)}\sqrt{3}\,t^3 + \text{higher terms}\,. \end{cases}$$

When $y = 1 + \varepsilon\exp(i\theta)$ moves around $\eta_5 = 1 \in (\mathbf{C}, y)$ once counter-clockwise, the topological behavior of the four points near $1 \in (\mathbf{C}, x)$ (cf. Figure 6) looks like the movement of four satellites accompanying two planets, two satellites around each planet corresponding to $t = \varepsilon^{1/4}\exp(i\nu)$, $\nu = \theta/4,\ \theta/4+\pi/2,\ \theta/4+\pi,\ \theta/4+(3\pi)/2$. The movement of the planets is described by the term $\frac{2}{19}\sqrt{38}\,t^2$; each of them do $(1/2)$-turn counter-clockwise around the sun ($\approx 1 \in (\mathbf{C}, x)$). The movement of each satellite around its planet is described by the term $\frac{162}{361}\,38^{(1/4)}\sqrt{3}\,t^3$; each of them does $(3/4)$-turn counter-clockwise around its planet. So, the monodromy relations around L_{η_5} give the relations

$$\begin{aligned} \xi_1 &= \xi_4, \\ \xi_2 &= \xi_4\xi_3\xi_4^{-1}, \\ \xi_3 &= (\xi_4\xi_3\xi_2)\xi_1(\xi_4\xi_3\xi_2)^{-1} \\ &= (\xi_1\xi_3)^2\xi_1(\xi_1\xi_3)^{-2} \quad \text{(by the two previous relations)}\,. \end{aligned} \tag{4.4}$$

In order to read the monodromy relations around L_{η_4}, we first show in Figure 8 how the generators at $y = \eta_5 + \varepsilon$ are deformed when y does half-turn counter-clockwise on the circle $|y - \eta_5| = \varepsilon$. Now, we also need to know how the generators at $y = \eta_5 - \varepsilon$ are deformed when y moves on the real axis from $y := \eta_5 - \varepsilon \to \eta_4 + \varepsilon$. This is described by the following lemma.

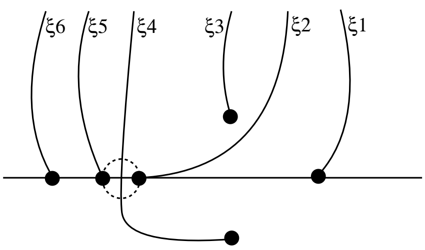

FIGURE 7. Generators at $y = \eta_6 - \varepsilon$.

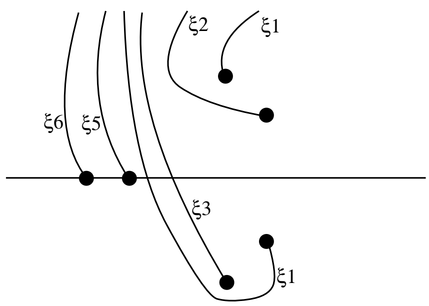

FIGURE 8. Generators at $y = \eta_5 - \varepsilon$.

LEMMA 4.5. *When y moves on the real axis from $y := \eta_5 - \varepsilon \to \eta_4 + \varepsilon$, the generators at $y = \eta_5 - \varepsilon$ (cf. Figure* 8) *are deformed as in Figure* 9.

PROOF. We consider the polynomial

$$h(u, v, y) := f_1(u + iv, y)$$

for u, v, y real. We denote by $f_{1e}(u, v, y)$ and $f_{1o}(u, v, y)$ the real and the imaginary part of $h(u, v, y)$ respectively. They have degree 6 and 5 respectively in v. Suppose that there exists an $y_0 \in [\eta_4 + \varepsilon, \eta_5 - \varepsilon]$ such that four complex solutions of the equation (in x) $f_1(x, y_0) = 0$ are on a same vertical line $u = u_0$ in the complex plane $(\mathbf{C}, x = u + iv)$. This implies that the equations (in v)

$$f_{1e}(u_0, v, y_0) = f_{1o}(u_0, v, y_0) = 0$$

have four common real solutions v_1, v_2, v_3, v_4. These solutions are not 0 since the equation (in y) $\Delta_x(f_1)(y) = 0$ has no solution on $[\eta_4 + \varepsilon, \eta_5 - \varepsilon]$. Thus, the equations (in v)

$$f_{1e}(u_0, v, y_0) = f_{1oo}(u_0, v, y_0) = 0\,,$$

where $f_{1oo}(u, v, y) := f_{1o}(u, v, y)/v$ (notice that v divides $f_{1o}(u, v, y)$, and thus $f_{1oo}(u, v, y)$ is a polynomial), have also v_1, v_2, v_3, v_4 as common solutions. As f_{1oo} has degree 4 in v, this implies that $f_{1oo}(u_0, v, y_0)$ divides $f_{1e}(u_0, v, y_0)$. Thus, the remainder $R(u, v, y)$ of f_{1e} by f_{1oo}, as a polynomial of v, must be identically 0 for $u = u_0$ and $y = y_0$ (of course, R is written as $R = R'/R''$, where R' is a polynomial in u, v, y, while R'' is a polynomial just depending on u and y). By an easy computation, we see that $R = (R'_2/R''_2)\, v^2 + (R'_0/R''_0)$, where R'_2, R''_2, R'_0 and R''_0 are polynomials in u and y. Thus, (u_0, y_0) is a common real solution of the equations

$$R'_2(u, y) = R'_0(u, y) = 0\,. \tag{4.6}$$

This implies that y_0 is a root of the resultant $Res(y)$ of the polynomials $u \mapsto R'_2(u, y)$ and $u \mapsto R'_0(u, y)$. Note that the condition $Res(y_0) = 0$ is necessary to have a real partner u_0 such that $R'_2(u_0, y_0) = R'_0(u_0, y_0) = 0$, but it is not sufficient since the possible partner u_0 might be not real. There are three real solutions y_{01}, y_{02}, y_{03} of the equation $Res(y) = 0$ on the interval $[\eta_4 + \varepsilon, \eta_5 - \varepsilon]$. Each of them gives a real number, say u_{0j} for y_{0j} $(1 \le j \le 3)$, such that (u_{0j}, y_{0j}) $(1 \le j \le 3)$ are three solutions of (4.6). But none of these three solutions gives four real roots v of the polynomial $v \mapsto f_{1oo}(u_0, v, y_0)$. Thus, we cannot have an overcrossing of the four (purely) complex roots on $[\eta_4 + \varepsilon, \eta_5 - \varepsilon]$. This completes the proof of Lemma (4.5).

The monodromy relations around L_{η_4} thus give the relation

$$\xi_1 = (\xi_6\xi_5)\xi_3(\xi_6\xi_5)^{-1}\,.$$

We shall not need the monodromy relations around L_{η_3} but in order to read the monodromy relations around L_{η_2}, we need to know how the generators at $y = \eta_4 + \varepsilon$ are deformed

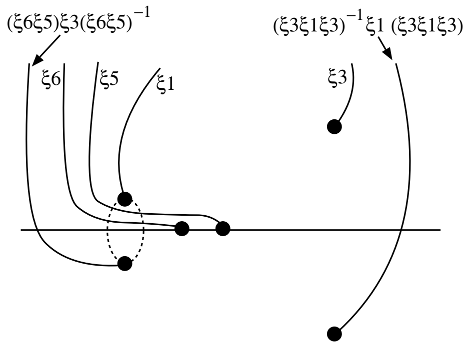

FIGURE 9. Generators at $y = \eta_4 + \varepsilon$.

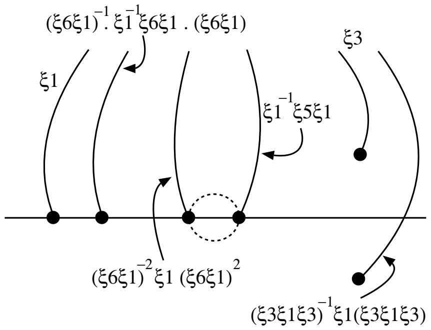

FIGURE 10. Generators at $y = \eta_2 + \varepsilon$.

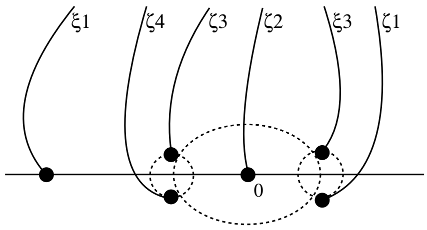

FIGURE 11. Generators at $y = \eta_1 + \varepsilon$.

when y moves as follows: half-turn counter-clockwise on the circle $|y - \eta_4| = \varepsilon$; on the real axis from $y := \eta_4 - \varepsilon \to \eta_3 + \varepsilon$; half-turn counter-clockwise on the circle $|y - \eta_3| = \varepsilon$; on the real axis from $y := \eta_3 - \varepsilon \to \eta_2 + \varepsilon$. This deformation is shown in Figure 10. To see the movement of the generators when y does half-turn counter-clockwise on the circle $|y - \eta_3| = \varepsilon$, just observe that near the A_5-singularity $(0, 16/23)$ the curve has two branches K_1 and K_2, corresponding to the line C_1' and the quintic C_1'' respectively, given by

$$K_1 : x = 0\,,$$

$$K_2 : x = -\frac{6436343}{15552}\left(y - \frac{16}{23}\right)^3 + \text{higher terms}\,.$$

The monodromy relations around L_{η_2} give the relations

$$\xi_1^{-1}\xi_5\xi_1 = (\xi_6\xi_1)^{-2}\xi_1(\xi_6\xi_1)^2\,.$$

To read the monodromy relations around L_{η_1}, we first need to know how the generators at $y = \eta_2+\varepsilon$ are deformed when y does half-turn counter-clockwise on the circle $|y-\eta_2| = \varepsilon$, then moves on the real axis from $y := \eta_2 - \varepsilon \to \eta_1 + \varepsilon$. This deformation is shown in Figure

11, where (use the previous relation)

$$\begin{aligned}
\zeta_1 &:= (\xi_3\xi_1\xi_3)^{-1}\xi_1(\xi_3\xi_1\xi_3)\,,\\
\zeta_2 &:= (\xi_6\xi_1)^{-2}\xi_1^{-1}(\xi_6\xi_1)^3\,,\\
\zeta_3 &:= (\xi_6\xi_1)^{-2}\xi_1(\xi_6\xi_1)^2\,,\\
\zeta_4 &:= (\xi_6\xi_1)^{-1}\xi_1(\xi_6\xi_1)\,.
\end{aligned}$$

To see the deformation when y moves on the real axis from $y := \eta_2 - \varepsilon \to \eta_1 + \varepsilon$, just proceed as in Lemma 4.5.

Now, we observe that near the origin the curve has three branches K_3, K_4 and K_5 given by

$$\begin{aligned}
&K_3 : y = x^2 - \frac{101}{64}x^3 + \frac{3}{256}\Big(\frac{11011}{32} + \frac{27}{32}i\sqrt{15}\Big)x^4 + \text{higher terms}\,,\\
&K_4 : y = x^2 - \frac{101}{64}x^3 + \frac{3}{256}\Big(\frac{11011}{32} - \frac{27}{32}i\sqrt{15}\Big)x^4 + \text{higher terms}\,,\\
&K_5 : x = 0\,.
\end{aligned}$$

An easy computation shows that the Puiseux parametrizations of K_3 and K_4 at the origin are given by

$$\begin{aligned}
&K_3 : y = t^2\,, \quad x = b_1\,t + b_2\,t^2 + b_3\,t^3 + \text{higher terms}\,,\\
&K_4 : y = t^2\,, \quad x = b_1'\,t + b_2'\,t^2 + b_3'\,t^3 + \text{higher terms}\,,
\end{aligned}$$

for some complex numbers b_i and b_i' such that $b_i = b_i'$ for $1 \le i \le 2$, the number $b_1 = b_1'$ is non-zero, and $b_3 \ne b_3'$. Put

$$\omega = \xi_3\zeta_1\,, \quad \omega' = \zeta_4\zeta_3\,, \quad \Omega = \omega'\zeta_2\omega\,.$$

The equations above show that the monodromy relations around L_{η_1} give the relations

$$\begin{aligned}
\zeta_1 &= \zeta_4\zeta_3\zeta_4^{-1}\,,\\
\xi_3 &= \omega'\zeta_4\omega'^{-1}\,,\\
\zeta_2 &= \omega'\zeta_2\omega'^{-1}\,,\\
\zeta_3 &= \Omega\xi_3\zeta_1\xi_3^{-1}\Omega^{-1}\,,\\
\zeta_4 &= (\Omega\omega)\xi_3(\Omega\omega)^{-1}\,.
\end{aligned} \tag{4.7}$$

Now, we have enough relations to conclude that the fundamental group $\pi_1(\mathbf{CP}^2 - C_1, b_0)$ is abelian. The first and the second relations of (4.7) imply

$$\omega = \omega'\,.$$

The vanishing relation at infinity, $\xi_1 \Omega = 1$, thus implies $\omega\xi_1 = (\omega\zeta_2)^{-1}$. Since $\omega\zeta_2 = \zeta_2\omega$ (third relation of (4.7)), one deduces

$$\xi_1\omega = \omega\xi_1 .$$

In other words, $\xi_3^{-1}(\xi_1\xi_3)^2 = \xi_1^{-1}\xi_3^{-1}(\xi_1\xi_3)^2\xi_1$. Now, by the third relation of (4.4), we have $\xi_3(\xi_1\xi_3)^2 = (\xi_1\xi_3)^2\xi_1$. So, the previous relation implies $\xi_1 = \xi_3$. Then, the first and the second relations of (4.4) imply $\xi_1 = \xi_2$, the relation (4.3) shows $\xi_1 = \xi_5$, and the vanishing relation at infinity gives $\xi_6 = \xi_1^{-5}$.

The fundamental group $\pi_1(\mathbf{CP}^2 - C_1, b_0)$ is thus generated by a single generator, and consequently it is abelian.

5. Fundamental group of $\mathbf{CP}^2 - C_2$

Again, the notation is as in Section 2.

THEOREM 5.1. *The fundamental group $\pi_1(\mathbf{CP}^2 - C_2)$ is isomorphic to $\mathbf{Z}$.*

We still denote by $\mathcal{M}$ the moduli space of reduced sextics in $\mathbf{CP}^2$ with the configuration of singularities $\{D_{10} + A_5 + A_4\}$, and by $\mathcal{M}_2$ the connected component of $\mathcal{M}$ containing the curve C_2. As above, Theorem 5.1 implies the following result.

COROLLARY 5.2. *For any curve $D \in \mathcal{M}_2$, we have $\pi_1(\mathbf{CP}^2 - D) \simeq \mathbf{Z}$.*

PROOF OF THEOREM 5.1. We use the non-generic version of the Zariski-van Kampen theorem (cf. Theorem 1.2.1) with the pencil $\mathcal{L}_{Y,Z}$ (cf. Notation 1.4). We recall that in $\mathbf{C}^2 := \mathbf{CP}^2 - L_\infty$ this pencil is given by $\{y = \eta\}_{\eta\in\mathbf{C}}$. Pencil $\mathcal{L}_{Y,Z}$ is not generic with respect to the curve C_2. Notice, nevertheless, that b_0 does not belong to the quintic C_2''. To prove Theorem 5.1, it suffices (cf. Section 1.3) to show that the fundamental group $\pi_1(\mathbf{CP}^2 - C_2)$ is abelian. We recall that as the base point for $\pi_1(\mathbf{CP}^2 - C_2)$ we take a point b_0' on a generic line such that b_0' is sufficiently close to b_0 but $b_0' \neq b_0$ (cf. Section 1.2 and below).

The discriminant $\Delta_x(f_2)$ of f_2 as a polynomial in x, which describes the singular lines of the pencil $\mathcal{L}_{Y,Z}$ (with respect to C_2), is a polynomial in y given by

$$\Delta_x(f_2)(y) = a_1 y^{13}(y-2)^5(y-1)^8(a_2y^2 + a_3y + a_4) .$$

Again, of course, we know the numbers a_i ($1 \le i \le 4$) but we do not write them here because they are too big; we observe, nevertheless, that $\Delta_x(f_2)$ has five distinct real roots:

$$\eta_1 = 0 , \quad \eta_2 = 0.001\cdots , \quad \eta_3 = 1 , \quad \eta_4 = 1.954\cdots , \quad \eta_5 = 2 .$$

The singular lines of the pencil are the lines $L_{\eta_1}, \cdots, L_{\eta_5}$ corresponding to these five roots.

We take generators $\rho, \xi_1, \cdots, \xi_5$ of the fundamental group $\pi_1(L_{\eta_2-\varepsilon} - C_2, b_0')$ as in Figure 13; $\xi_1, \cdots, \xi_5$ are lassos for C_2'', while ρ is a lasso for the line component C_2'.

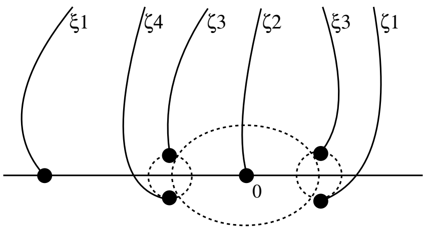

FIGURE 12. Real plane section of C_2.

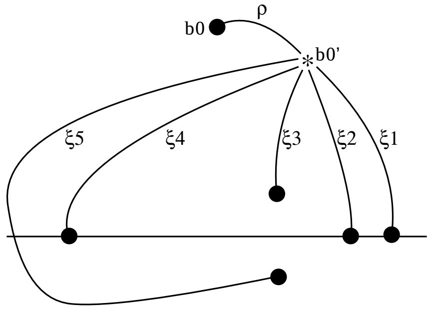

FIGURE 13. Generators at $y = \eta_2 - \varepsilon$.

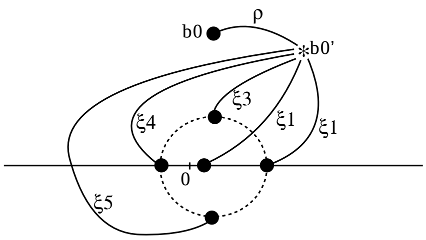

FIGURE 14. Generators at $y = \eta_1 + \varepsilon$.

As above, the monodromy relations around L_{η_2} give the relation

$$\xi_2 = \xi_1 \,.$$

To read the monodromy relations around L_{η_1}, we first show in Figure 14 how the generators at $y = \eta_2 - \varepsilon$ are deformed when y moves on the real axis from $y := \eta_2 - \varepsilon \to \eta_1 + \varepsilon$. Then we observe that, at the origin, the curve has three branches K_1, K_2 and K_3 given by

$$K_1 : y = 0 \,,$$

$$K_2 : y = t \,, \quad x = \frac{100}{11}\, t + \text{higher terms} \,,$$

$$K_3 : y = t^4 \,, \quad x = \left(\frac{121}{16000}\right)^{(1/4)} t + \text{higher terms} \,.$$

One deduces that the monodromy relations around L_{η_1} give the relations

$$
\begin{aligned}
&\rho^{-1}\xi_1\rho = \xi_3\xi_1\xi_3^{-1}\,,\\
&\rho^{-1}\xi_1\rho = \xi_3\,,\\
&\rho^{-1}\xi_3\rho = \xi_4\,,\\
&\rho^{-1}\xi_4\rho = \xi_5\,,\\
&\rho^{-1}\xi_5\rho = (\xi_5\xi_4\xi_3\xi_1)\xi_1(\xi_5\xi_4\xi_3\xi_1)^{-1}\,.
\end{aligned}
\tag{5.3}
$$

We can already conclude that the fundamental group $\pi_1(\mathbf{CP}^2 - C_2, b_0')$ is abelian. Indeed, the two first relations in (5.3) immediately imply $\xi_3 = \xi_1$. By the third relation, we then have $\xi_4 = \xi_3$. The fourth relation thus shows $\xi_4 = \xi_5$. And the big circle relation $\rho\xi_5\xi_4\xi_3\xi_2\xi_1 = 1$ then gives $\rho = \xi_1^{-5}$. So, the fundamental group $\pi_1(\mathbf{CP}^2 - C_2, b_0')$ is generated by a single generator, and consequently it is abelian.

6. A weak Zariski 4-ple

Consider the sextics C_3 and C_4 defined by following affine equations:

$$
\begin{aligned}
&C_3:\ f_3(x,y) := f_3'(x,y)\, f_3''(x,y) = 0\,,\\
&C_4:\ f_4(x,y) := f_4'(x,y)\, f_4''(x,y)\, f_4'''(x,y) = 0\,,
\end{aligned}
$$

where f_3', f_3'' and f_4', f_4'', f_4''' are given by

$$
\begin{aligned}
f_3'(x,y) &:= y^2 + y + \frac{128}{11}x^2\,,\\
f_3''(x,y) &:= -\frac{184}{33}y^4 + \left(\frac{347}{11}x - \frac{272}{33}\right)y^3 + \left(-\frac{24124}{363}x^2 + \frac{358}{11}x - \frac{8}{3}\right)y^2\\
&\quad + \left(\frac{6916}{121}x^3 - \frac{1076}{33}x^2 + x\right)y - \frac{6656}{363}x^4 + \frac{128}{11}x^3\,,
\end{aligned}
$$

and

$$
\begin{aligned}
f_4'(x,y) &:= x\,,\\
f_4''(x,y) &:= y\,,\\
f_4'''(x,y) &:= -x^4 - 8y + 36y^2 - 54y^3 + 27y^4 + 3xy - 8xy^3 - 6yx^2\\
&\qquad + 6y^2x^2 + 4yx^3\,.
\end{aligned}
$$

The curve C_3 has two irreducible components: a conic C_3' defined by the equation $f_3'(x,y) = 0$ and a quartic C_3'' defined by the equation $f_3''(x,y) = 0$. The configuration of singularities of C_3 is $\{D_{10} + A_5 + A_4\}$: D_{10} at the origin, A_5 at $(0,-1)$ and A_4 at $(1,1)$. We show the real plane section of C_3 in Figure 15 below.

The curve C_4 has three irreducible components: two lines C_4' and C_4'' defined by the equations $x = 0$ and $y = 0$ respectively, and a quartic C_4''' defined by the equation $f_4'''(x, y) = 0$. The configuration of singularities of C_4 is also $\{D_{10} + A_5 + A_4\}$: D_{10} at the origin, A_5 at $(0, 2/3)$ and A_4 at $(1/2, 1/2)$. We show the real plane section of C_4 in Figure 21 below.

Notice that the curves C_3 and C_4 are not of torus type.

THEOREM 6.1. *The 4-ple* (C_1, C_2, C_3, C_4), *where* C_1, C_2 *are the sextics given in Section 2 and* C_3, C_4 *the sextics defined above*, *is a weak Zariski 4-ple.*

The proof of Theorem 6.1 follows immediately from Theorems 3.1, 4.1 and 5.1 above and Theorems 7.1 and 8.1 below.

We still denote by $\mathcal{M}$ the moduli space of reduced sextics in $\mathbf{CP}^2$ with the configuration of singularities $\{D_{10} + A_5 + A_4\}$, and by $\mathcal{M}_1$ and $\mathcal{M}_2$ the connected component of $\mathcal{M}$ containing the curves C_1 and C_2 respectively. Let $\mathcal{M}_3$ and $\mathcal{M}_4$ be the connected component of $\mathcal{M}$ containing the curves C_3 and C_4 respectively. Theorem 6.1 has the following immediate corollary.

COROLLARY 6.2. *Any* 4*-ple* (D_1, D_2, D_3, D_4), *where* $D_i \in \mathcal{M}_i$ *for* $1 \le i \le 4$, *is a weak Zariski* 4*-ple.*

7. Fundamental group of $\mathbf{CP}^2 - C_3$

The notation is as in Section 6.

THEOREM 7.1. *The fundamental group* $\pi_1(\mathbf{CP}^2 - C_3)$ *is isomorphic to* $\mathbf{Z} \times (\mathbf{Z}/2\mathbf{Z})$.

We still denote by $\mathcal{M}$ the moduli space of reduced sextics in $\mathbf{CP}^2$ with the configuration of singularities $\{D_{10} + A_5 + A_4\}$, and by $\mathcal{M}_3$ the connected component of $\mathcal{M}$ containing the curve C_3. Theorem 7.1 implies the following result.

COROLLARY 7.2. *For any curve* $D \in \mathcal{M}_3$, *we have the following isomorphism*: $\pi_1(\mathbf{CP}^2 - D) \simeq \mathbf{Z} \times (\mathbf{Z}/2\mathbf{Z})$.

PROOF OF THEOREM 7.1. We use the classical Zariski-van Kampen theorem (cf. Theorem 1.1.1) with the pencil $\mathcal{L}_{X,Z}$ (cf. Notation 1.4). We recall that in $\mathbf{C}^2 := \mathbf{CP}^2 - L_\infty$ this pencil is given by $\{x = \eta\}_{\eta \in \mathbf{C}}$. Observe that b_0 does not belong to the curve C_3. To prove Theorem 7.1, it suffices (cf. Section 1.3) to show that $\pi_1(\mathbf{CP}^2 - C_3, b_0)$ is abelian.

The discriminant $\Delta_y(f_3)$ of f_3 as a polynomial in y, which describes the singular lines of the pencil $\mathcal{L}_{X,Z}$ (with respect to C_3), is a polynomial in x given by

$$\Delta_y(f_3)(x) = a_1 x^{18}(a_2 x^2 + a_3)(a_4 x^5 + a_5 x^4 + a_6 x^3 + a_7 x^2 + a_8 x + a_9)(x - 1)^5 .$$

The numbers a_i $(1 \le i \le 9)$ above are such that $\Delta_y(f_3)$ has seven distinct real roots:

$$\eta_1 = -0.146\cdots, \quad \eta_2 = -0.053\cdots, \quad \eta_3 = -0.015\cdots, \quad \eta_4 = 0,$$
$$\eta_5 = 0.054\cdots, \quad \eta_6 = 0.146\cdots, \quad \eta_7 = 1 .$$

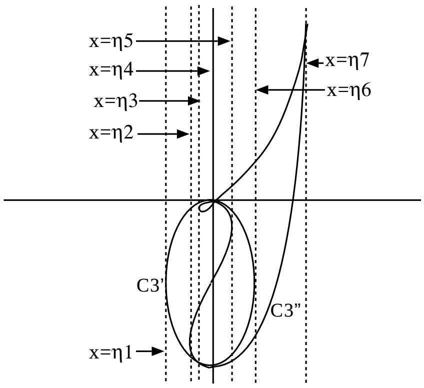

FIGURE 15. Real plane section of C_3.

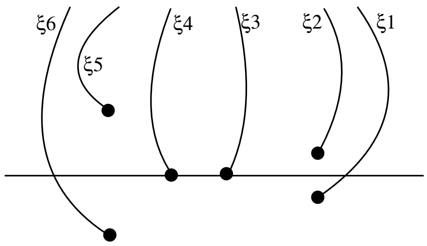

FIGURE 16. Generators at $x = \eta_1 + \varepsilon$.

The lines $L_{\eta_1}, \cdots, L_{\eta_7}$ corresponding to these seven roots are thus some singular lines of the pencil.

We take generators $\xi_1, \cdots, \xi_6$ of the fundamental group $\pi_1(L_{\eta_1+\varepsilon} - C_1, b_0)$ as in Figure 16; $\xi_1, \xi_2, \xi_5, \xi_6$ are lassos for the quartic C_3'' and ξ_3, ξ_4 are lassos for the conic component C_3'.

As above, the monodromy relations around L_{η_1} give the relation

$$\xi_4 = \xi_3. \tag{7.3}$$

To read the monodromy relations around L_{η_2}, we first show in Figure 17 how the generators at $x = \eta_1 + \varepsilon$ are deformed when x moves on the real axis from $x := \eta_1 + \varepsilon \to \eta_2 - \varepsilon$ (the proof is similar to Lemma 4.5). Then, as above, the monodromy relations around L_{η_2} give the relation

$$\xi_6 = (\xi_5\xi_3)\xi_5(\xi_5\xi_3)^{-1}. \tag{7.4}$$

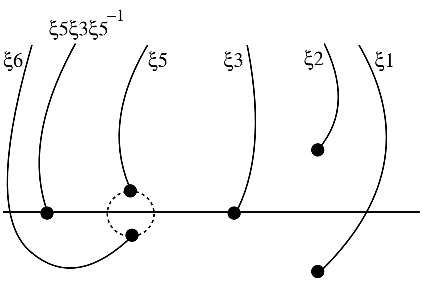

FIGURE 17. Generators at $x = \eta_2 - \varepsilon$.

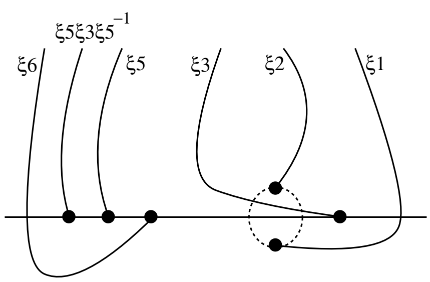

FIGURE 18. Generators at $x = \eta_3 - \varepsilon$.

To read the monodromy relations around L_{η_3}, we first show in Figure 18 how the generators at $x = \eta_2 - \varepsilon$ are deformed when x does half-turn counter-clockwise on the circle $|x - \eta_2| = \varepsilon$, then moves on the real axis from $x := \eta_2 + \varepsilon \to \eta_3 - \varepsilon$. Then, as above, it is easy to see that the monodromy relations around L_{η_3} give the relation

$$\xi_1 = (\xi_3\xi_2)^{-1}\xi_2(\xi_3\xi_2)\,. \tag{7.5}$$

Now, we look at the monodromy relations around L_{η_4}. For this purpose, we show in Figure 19 how the generators at $x = \eta_3 - \varepsilon$ are deformed when x does half-turn counter-clockwise on the circle $|x - \eta_3| = \varepsilon$, then moves on the real axis from $x := \eta_3 + \varepsilon \to \eta_4 - \varepsilon$. We look at the contribution of the origin. At $(0, 0)$, the curve C_3 has three branches: two branches K_1 and K_2 corresponding to the quartic C_3'', and another one K_3 corresponding to the conic C_3':

$$K_1 : \; y = -\frac{128}{11}x^2 - \frac{4980224}{1331}x^4 + \text{higher terms}\,,$$
$$K_2 : \; y = \frac{3}{8}x + \text{higher terms}\,,$$
$$K_3 : \; y = -\frac{128}{11}x^2 - \frac{16384}{121}x^4 + \text{higher terms}\,.$$

The monodromy relations around L_{η_4} (contribution of the origin) thus give the relation

$$\xi_2 = (\xi_3\xi_2\xi_1)\xi_2(\xi_3\xi_2\xi_1)^{-1}\,.$$

The latter, together with (7.5), implies $\xi_1 = \xi_2$. Notice that we thus have

$$\xi_1\xi_3 = \xi_3\xi_1\,. \tag{7.6}$$

We shall not use the contribution of $(0, -1)$.

Now, in order to read the monodromy relations around L_{η_5}, we first show in Figure 20 how the generators at $x = \eta_4 - \varepsilon$ are deformed when x does half-turn counter-clockwise

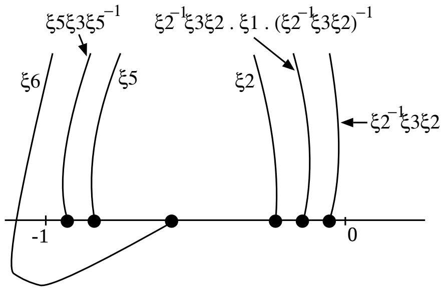

FIGURE 19. Generators at $x = \eta_4 - \varepsilon$.

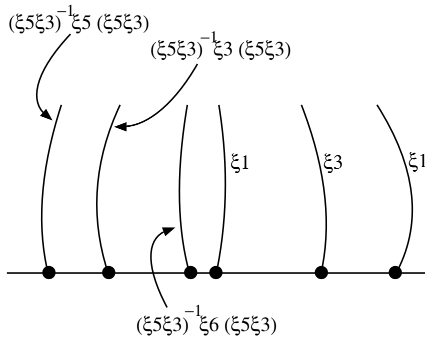

FIGURE 20. Generators at $x = \eta_5 - \varepsilon$.

on the circle $|x - \eta_4| = \varepsilon$, then moves on the real axis from $x := \eta_4 + \varepsilon \to \eta_5 - \varepsilon$. To see the movement of the generators near -1, when x does half-turn counter-clockwise on the circle $|x - \eta_4| = \varepsilon$, just observe that near $(0, -1)$ the curve has two branches K_4 and K_5, corresponding to the quartic C_3'' and the conic C_3' respectively, given by

$$K_4: \; y = -1 + \frac{128}{11} x^2 - \frac{9375}{88} x^3 + \text{higher terms},$$
$$K_5: \; y = -1 + \frac{128}{11} x^2 + \frac{16384}{121} x^4 + \text{higher terms}.$$

The monodromy relations around L_{η_5} thus give the relation

$$\xi_1 = (\xi_5\xi_3)^{-1}\xi_6(\xi_5\xi_3),$$

that is $\xi_1 = \xi_5$ by (7.4). One deduces immediately $\xi_1 = \xi_6$.

We can now conclude that the fundamental group $\pi_1(\mathbf{CP}^2 - C_3, b_0)$ is abelian. Indeed, we have seen that $\xi_1 = \xi_2 = \xi_5 = \xi_6$ and $\xi_4 = \xi_3$, that is that there is only two generators ξ_1 and ξ_3. Theorem 7.1 thus follows immediately from the relation (7.6) which asserts that these generators commute.

8. Fundamental group of $\mathbf{CP}^2 - C_4$

We still use the notation of Section 6.

THEOREM 8.1. *The fundamental group $\pi_1(\mathbf{CP}^2 - C_4)$ is isomorphic to $\mathbf{Z}^2$.*

We still denote by $\mathcal{M}$ the moduli space of reduced sextics in $\mathbf{CP}^2$ with the configuration of singularities $\{D_{10} + A_5 + A_4\}$, and by $\mathcal{M}_4$ the connected component of $\mathcal{M}$ containing the curve C_4. Theorem 8.1 implies the following result.

COROLLARY 8.2. *For any curve $D \in \mathcal{M}_4$, we have: $\pi_1(\mathbf{CP}^2 - D) \simeq \mathbf{Z}^2$.*

PROOF OF THEOREM 8.1. We use the non-generic version of the Zariski-van Kampen theorem (cf. Theorem 1.2.1) with the pencil $\mathcal{L}_{Y,Z}$ (cf. Notation 1.4). We recall that in $\mathbf{C}^2 := \mathbf{CP}^2 - L_\infty$ this pencil is given by $\{y = \eta\}_{\eta \in \mathbf{C}}$. Pencil $\mathcal{L}_{Y,Z}$ is not generic with respect to the curve C_4. Notice, nevertheless, that b_0 does not belong to the curve $C_4' \cup C_4'''$. To prove Theorem 8.1, it suffices (cf. Section 1.3) to show that the fundamental group $\pi_1(\mathbf{CP}^2 - C_4)$ is abelian. We recall that as the base point for $\pi_1(\mathbf{CP}^2 - C_4)$ we take a point b_0' on a generic line such that b_0' is sufficiently close to b_0 but $b_0' \neq b_0$ (cf. Section 1.2 and below).

The discriminant $\Delta_x(f_4)$ of f_4 as a polynomial in x, which describes the singular lines of the pencil $\mathcal{L}_{Y,Z}$ (with respect to C_4), is a polynomial in y given by

$$\Delta_x(f_4)(y) = -y^{13}(171542\, y^2 - 316811\, y + 131072)(3y - 2)^6(2y - 1)^7.$$

This polynomial has five distinct real roots:

$$\eta_1 = 0, \quad \eta_2 = 0.5, \quad \eta_3 = 0.625\cdots, \quad \eta_4 = 0.666\cdots, \quad \eta_5 = 1.221\cdots$$

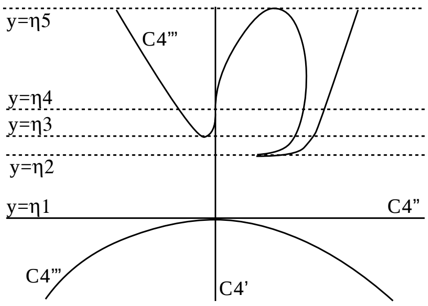

FIGURE 21. Real plane section of C_4.

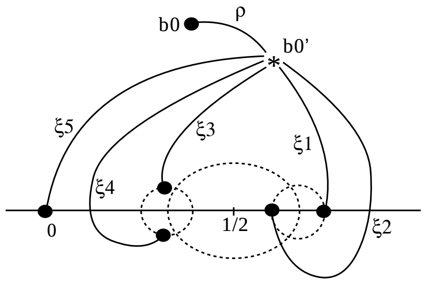

FIGURE 22. Generators at $y = \eta_2 + \varepsilon$.

The singular lines of the pencil are the lines $L_{\eta_1}, \cdots, L_{\eta_5}$ corresponding to these five roots.

We take generators $\rho, \xi_1, \cdots, \xi_5$ of the fundamental group $\pi_1(L_{\eta_2+\varepsilon} - C_4, b_0')$ as in Figure 22; $\xi_1, \cdots, \xi_5$ are lassos around the intersection points of the generic line $L_{\eta_2+\varepsilon}$ with $C_4' \cup C_4'''$, while ρ is a lasso around the intersection point of $L_{\eta_2+\varepsilon}$ with C_4'' (i.e., around the axis b_0 of the pencil).

To read the monodromy relations around L_{η_2}, we look at the Puiseux parametrization of C_4 at $(1/2, 1/2)$:

$$\begin{cases} y = \dfrac{1}{2} + t^4 \\ x = \dfrac{1}{2} + \sqrt{3}\, t^2 - \dfrac{1}{2}\sqrt{2}\, 3^{(1/4)}\, t^3 + \text{higher terms}\,. \end{cases}$$

As above, these equations show that the monodromy relations around L_{η_2} give the relations

$$\begin{aligned} \xi_1 &= \xi_4\,, \\ \xi_2 &= \xi_3\,, \\ \xi_3 &= (\xi_4\xi_3\xi_1\xi_2)\xi_1(\xi_4\xi_3\xi_1\xi_2)^{-1} \\ &= (\xi_1\xi_2)^2\xi_1(\xi_1\xi_2)^{-2}\,. \end{aligned} \tag{8.3}$$

To read the monodromy relations around L_{η_3}, we first show in Figure 23 how the generators at $y = \eta_2 + \varepsilon$ are deformed when y moves on the real axis from $y := \eta_2 + \varepsilon \to \eta_3 - \varepsilon$. The monodromy relations around L_{η_3} thus give the relation

$$\xi_2 = \xi_5\xi_1\xi_5^{-1}\,. \tag{8.4}$$

To read the monodromy relations around L_{η_4}, we first show in Figure 24 how the generators at $y = \eta_3 - \varepsilon$ are deformed when y does half-turn counter-clockwise on the circle $|y - \eta_3| = \varepsilon$, then moves on the real axis from $y := \eta_3 + \varepsilon \to \eta_4 - \varepsilon$. Then, we observe that

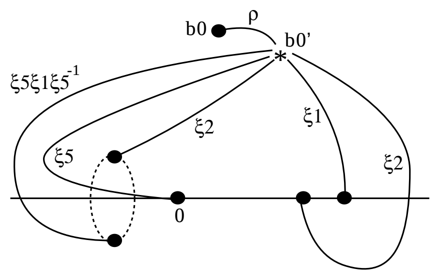

FIGURE 23. Generators at $y = \eta_3 - \varepsilon$.

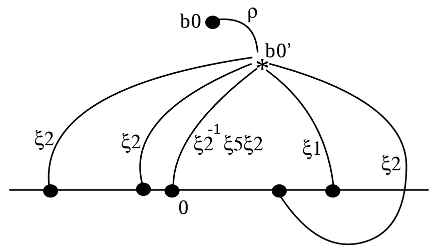

FIGURE 24. Generators at $y = \eta_4 - \varepsilon$.

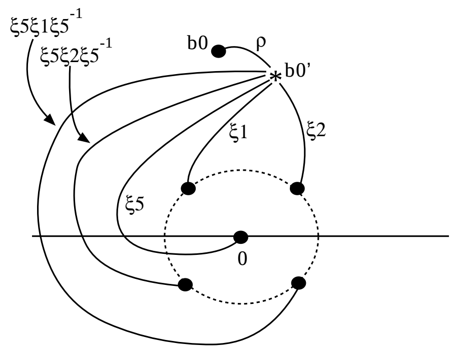

FIGURE 25. Generators at $y = \eta_1 + \varepsilon$

near the point $(0, 2/3)$ the curve has two branches K_1 and K_2 given by

$$K_1 : \ x = 0 ,$$

$$K_2 : \ x = \frac{243}{5}\left(y - \frac{2}{3}\right)^3 + \text{higher terms} .$$

The monodromy relations around L_{η_4} thus give the relation

$$\xi_2 = (\xi_5\xi_2)^3\xi_2(\xi_5\xi_2)^{-3} . \tag{8.5}$$

We shall not need the monodromy relations around L_{η_5}, but we shall use the monodromy relations around L_{η_1}. To determine them, we first show in Figure 25 how the generators at $y = \eta_2 + \varepsilon$ are deformed when y does half-turn counter-clockwise on the circle $|y - \eta_2| = \varepsilon$, then moves on the real axis from $y := \eta_2 - \varepsilon \to \eta_1 + \varepsilon$ (the proof is similar to Lemma 4.5). Then, we observe that near the origin the curve has three branches K_3, K_4 and K_5 given by

$$K_3 : \ x = 0 ,$$

$$K_4 : \ y = 0 ,$$

$$K_5 : \ y = -\frac{1}{8}x^4 + \text{higher terms}\,.$$

The monodromy relations around L_{η_1} thus give the relations

$$(8.6)\qquad \begin{aligned} &\rho^{-1}\xi_2\rho = \xi_1\,,\\ &\rho^{-1}\xi_1\rho = \xi_5\xi_2\xi_5^{-1}\,,\\ &\rho^{-1}\xi_5\rho = (\xi_5\xi_2)\xi_5(\xi_5\xi_2)^{-1}\,,\\ &\rho^{-1}\xi_5\xi_2\xi_5^{-1}\rho = \xi_5\xi_1\xi_5^{-1}\,,\\ &\rho^{-1}\xi_5\xi_1\xi_5^{-1}\rho = (\xi_5\xi_1\xi_2\xi_1)\xi_2(\xi_5\xi_1\xi_2\xi_1)^{-1}\,. \end{aligned}$$

Now we are ready to prove that $\pi_1(\mathbf{CP}^2 - C_4, b_0')$ is abelian. By the second and the third relation of (8.6), we have:

$$\begin{aligned} \rho^{-1}\xi_5\xi_1\xi_5^{-1}\rho &= \rho^{-1}\xi_5\rho\cdot\rho^{-1}\xi_1\rho\cdot(\rho^{-1}\xi_5\rho)^{-1}\\ &= (\xi_5\xi_2)\xi_5(\xi_5\xi_2)^{-1}\cdot\xi_5\xi_2\xi_5^{-1}\cdot((\xi_5\xi_2)\xi_5(\xi_5\xi_2)^{-1})^{-1}\\ &= (\xi_5\xi_2)^2\xi_5^{-1}(\xi_5\xi_2)^{-1}\\ &= (\xi_5\xi_2)^2\xi_2(\xi_5\xi_2)^{-2}\,. \end{aligned}$$

But, by (8.4), $\xi_5\xi_1\xi_5^{-1} = \xi_2$. We thus have

$$\rho^{-1}\xi_2\rho = (\xi_5\xi_2)^2\xi_2(\xi_5\xi_2)^{-2}\,,$$

that is, using the first relation of (8.6),

$$\xi_1 = (\xi_5\xi_2)^2\xi_2(\xi_5\xi_2)^{-2}\,.$$

This relation is equivalent to the following one:

$$(\xi_5\xi_2)\xi_1(\xi_5\xi_2)^{-1} = (\xi_5\xi_2)^3\xi_2(\xi_5\xi_2)^{-3}\,.$$

The relation (8.5) thus implies

$$(\xi_5\xi_2)\xi_1(\xi_5\xi_2)^{-1} = \xi_2\,,$$

that is $\xi_5\xi_2\xi_1 = \xi_2\xi_5\xi_2$. But, by (8.4), $\xi_2\xi_5 = \xi_5\xi_1$. So, we have $\xi_5\xi_2\xi_1 = \xi_5\xi_1\xi_2$, that is

$$(8.7)\qquad \xi_2\xi_1 = \xi_1\xi_2\,.$$

Now, by (8.3), $\xi_2 = (\xi_1\xi_2)^2\xi_1(\xi_1\xi_2)^{-2}$. So, the relation (8.7) implies $\xi_1 = \xi_2$. The big circle relation $\rho\xi_5\xi_4\xi_3\xi_1\xi_2 = 1$ is thus written as $\rho = (\xi_5\xi_1^4)^{-1}$.

So, we have proved that $\xi_1 = \xi_2 = \xi_3 = \xi_4$ and $\rho = (\xi_5\xi_1^4)^{-1}$, that is there is only two generators ξ_1 and ξ_5, and since they commute, by (8.4), the fundamental group $\pi_1(\mathbf{CP}^2 - C_4, b_0')$ is abelian.

9. On the moduli space $\mathcal{M} := \mathcal{M}(\{D_{10} + A_5 + A_4\}, 6)$ and concluding remarks

We still denote by $\mathcal{M}$ the moduli space of reduced sextics in $\mathbf{CP}^2$ with the configuration of singularities $\{D_{10} + A_5 + A_4\}$. It is known that the moduli of sub-lattices with this configuration in $K3$ surfaces has four irreducible components (cf. [Y]). We shall see below that each of them gives exactly one irreducible component in $\mathcal{M}$ (in general, this is not true for arbitrary moduli !) so that $\mathcal{M}$ has exactly four connected components which are necessarily (by Theorem 6.1 or Corollary 6.2) the connected components $\mathcal{M}_i$ $(1 \le i \le 4)$ defined above. So, in view of our previous results (cf. Corollaries 4.2, 5.2, 7.2 and 8.2), we have the following theorem.

THEOREM 9.1. *Let D be a curve in* $\mathcal{M}$. *Then,*

$$\pi_1(\mathbf{CP}^2 - D) \simeq \begin{cases} \mathbf{Z} & \text{if } D \in \mathcal{M}_1 or \mathcal{M}_2, \\ \mathbf{Z} \times (\mathbf{Z}/2\mathbf{Z}) & \text{if } D \in \mathcal{M}_3, \\ \mathbf{Z}^2 & \text{if } D \in \mathcal{M}_4\,. \end{cases}$$

Let us now explain why each irreducible component of the moduli of sub-lattices in $K3$ surfaces gives exactly one irreducible component in $\mathcal{M}$. Consider a curve $C \in \mathcal{M}$. First, using the genus formula mentioned in Section 3, we can see that C cannot be irreducible as the right side of the formula is -2. As A_4 is an irreducible singularity, and thus can appear only on curves of degree greater than or equal to 4, the possible component types of C (cf. [O7]) are:

(a) a quintic and a line;
(b) a quartic and a conic;
(c) a quartic and two lines.

On the other hand, the configuration of singularities $\{D_{10} + A_4\}$ cannot appear on an irreducible component of degree less than or equal to 5. Thus, for our curve C, the singularity D_{10} must be an intersection singularity (see [O7] for definition). Suppose that the origin $O \in C$ is a D_{10}-singularity. A D_{10}-singularity locally consists of three smooth branches, say K_1, K_2, K_3, so that their local intersection numbers are given by $I(K_1, K_2; O) = I(K_1, K_3; O) = 1$ and $I(K_2, K_3; O) = 4$ (if C is defined by the normal form $y^2x - x^9 = 0$, K_1 is just the line $x = 0$, and K_2, K_3 are defined by $y \pm x^4 = 0$). Thus, if D_{10} is an intersection singularity of two irreducible components, the possibilities are:

(d-1) one smooth component and one component with an A_1-singularity intersecting with intersection number 5 and so that the smooth component is tangent to one of the branch of A_1;

(d-2) one smooth component and one component with an A_7-singularity intersecting with intersection number 2 and so that the smooth component is transverse to the tangent cone direction of A_7.

If D_{10} is not an intersection singularity of two components, then:

(d-3) D_{10} is given as an intersection singularity of three irreducible components.

In the case (d-1), the possibilities for the component types are the following:

($\sharp 1$) C is a union of a line L and a quintic B_5 such that the configuration of singularities of B_5 is $\Sigma(B_5) := \{A_4 + A_5 + A_1\}$ and the line L is passing at the A_1-singularity. Let us denote by $\mathcal{M}'_2$ the subspace of sextics in $\mathcal{M}$ corresponding to this possibility;

($\sharp 2$) C is a union of a quartic B_4 and a conic B_2 such that: $\Sigma(B_4) = \{A_4 + A_1\}$; $B_2 \cap B_4 = \{O, P\}$; the singularity of B_4 at O is A_1; B_2, B_4 are non-singular at P and tangent with intersection multiplicity 3. We denote by $\mathcal{M}'_3$ the subspace of sextics in $\mathcal{M}$ corresponding to this possibility.

The case (d-2) is possible if and only if the components of C are a line L and a quintic B_5 such that: $\Sigma(B_5) = \{A_7 + A_4\}$; $L \cap B_5 = \{O, P\}$; the singularity of B_5 at O is A_7, and at this point L intersects transversely the tangent cone of the singularity; B_5 is non-singular at P and $I(L, B_5; P) = 3$ (to make A_5). Notice that P is a flex point of B_5. We denote by $\mathcal{M}'_1$ the subspace of $\mathcal{M}$ consisting of sextics which correspond to this possibility.

The case (d-3) takes place when C has two line components L_1, L_2 and a quartic component B_4 such that: $\Sigma(B_4) = \{A_4\}$; L_1 is the tangent line of a flex point O of B_4 of order 2 (i.e., $I(L_1, B_4; O) = 4$); L_2 is transverse to B_4 at O and intersects B_4 at another flex point P to make A_5. Let us denote by $\mathcal{M}'_4$ the subspace of $\mathcal{M}$ consisting of sextics which correspond to this possibility.

By definition of $\mathcal{M}'_1, \cdots, \mathcal{M}'_4$, we see that:

$$\mathcal{M} = \bigcup_{i=1}^{4} \mathcal{M}'_i \quad \text{and} \quad \mathcal{M}_i \subset \mathcal{M}'_i \ (i = 1, \cdots, 4)\,.$$

We assert that, for each $1 \le i \le 4$, the subspace $\mathcal{M}'_i$ is irreducible, and thus $\mathcal{M}'_i = \mathcal{M}_i$. The proof is done by a direct computation using a suitable slice condition as in [OP]. For example, consider the case of $\mathcal{M}'_2$. Take a sextic $C = L \cup B_5 \in \mathcal{M}'_2$. The quintic B_5 has the configuration of singularities $\{A_1 + A_5 + A_4\}$. First, it is not difficult to see that the three singularities are not colinear and strongly generic in the sense that the line defined by the tangent cone (or cones) at any one of the singular points does not pass through the other singularities. Thus, by the action of $PGL(3, \mathbf{C})$, we may consider the following slice condition:

($\mathcal{S}_2$) A_5 is at $(0, 1)$ with the tangent cone defined by $y = 1$; A_4 is at $(1, 0)$ with tangent cone $x = 1$; A_1 is at the origin O and one of the tangent lines at A_1 intersects the line L with intersection number 5 at O.

Under this slice condition, the computation using Maple can be carried out exactly as in [OP], and we can show that, in fact, the quintic (and the line component tangent at A_1) is uniquely determined by this slice condition. As the computation in detail is boring and heavy, we omit the proof.

The irreducibility of the other $\mathcal{M}'_i$ ($i = 1$, 3 and 4) can be shown using the following slice conditions:

Slice condition for $\mathcal{M}'_1$: the quintic B_5 has A_7 at O with $y = 0$ as the tangent cone,

A_4 at $(0, 1)$ with $y = 1$ as the tangent cone, and $(1, -1)$ is a flex point with the tangent line $y + x = 0$;

Slice condition for $\mathcal{M}'_3$: the quartic B_4 has A_1 at O, A_4 at $(1, 0)$ with $x = 1$ as the tangent line. The conic B_2 passes through O and $P := (0, 1)$ so that the singularity of $B_2 \cup B_4$ at O is D_{10}, $I(B_2, B_4; P) = 3$ and the singularity of $B_2 \cup B_4$ at P is A_5.

Slice condition for $\mathcal{M}'_4$: the quartic B_4 has A_4 at $(1, 0)$ with $x = 1$ as the tangent line, the two line components are $x = 0$ and $y - x = 0$, the origin is a flex point of B_4 of order 2 with $y - x = 0$ the flex tangent, and the line $x = 0$ intersects B_4 at another flex point P so that $x = 0$ is the flex tangent line at P.

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