TOKYO J. MATH.
VOL. 27, NO. 1, 2004

On the Moduli Space of Pointed Algebraic Curves of Low Genus —A Computational Approach—

Tetsuo NAKANO and Tatsuji MORI

Tokyo Denki University

(Communicated by S. Miyoshi)

Abstract. We compute explicitly the moduli space $\mathcal{M}_{g,1}^N$ of pointed algebraic curves of genus g with a given numerical semigroup N when g is small ($2 \le g \le 5$). It is known that such a moduli space $\mathcal{M}_{g,1}^N$ is non-empty for $g \le 7$. The main results obtained in this note are the irreducibility and the determination of the dimension of $\mathcal{M}_{g,1}^N$ for $g \le 5$ except a few cases. In particular, it turns out that many of these moduli spaces are unirational.

1. Introduction

Let X be a nonsingular projective curve of genus g defined over the complex number field $\mathbf{C}$ and $P \in X$ a point of X. Let $\mathbf{N}_0 := \{0, 1, 2, \cdots\}$ be the additive semigroup of nonnegative integers and set

$N_P := \{n \in \mathbf{N}_0 \mid$ there exists a rational function f on X such that f is regular on $X - \{P\}$ and has a pole of order exactly n at $P\} \subset \mathbf{N}_0$.

N_P is a subsemigroup of $\mathbf{N}_0$ and we call N_P the semigroup of P. If $N_P = \{0, g+1, g+2, g+3, \cdots\}$, then $P \in X$ is an *ordinary* point. If $P \in X$ is not an ordinary point, then P is a *Weierstrass* point. There are only finitely many Weierstrass points on X. The following lemma is well-known (cf. [8; Lemma 6.4]):

LEMMA 1.1. *The semigroup N_P has the following property*: $\mathbf{N}_0 - N_P = \{j_1, j_2, \cdots, j_g\}$ ($j_1 < j_2 < \cdots < j_g$). *We call $\{j_1, j_2, \cdots, j_g\}$ the gap sequence at P.*

Let $S \subset \mathbf{N}_0$ be a subsemigroup. If S has the property in Lemma 1.1, we call S a *numerical semigroup of genus g*. Such a semigroup contains $\{2g, 2g+1, 2g+2, \cdots\}$. Numerical semigroups of genus g can be determined and classified by a simple computation, at least for low genus (see Proposition 3.1 below).

Consider a couple (X, P) of a nonsingular projective curve X of genus g and a point $P \in X$. We denote by $\mathcal{M}_{g,1}$ the set of all isomorphim classes of such pairs (X, P). It is known that $\mathcal{M}_{g,1}$ has a natural structure of scheme (coarse moduli scheme, cf. [13]) and we call $\mathcal{M}_{g,1}$ the moduli space of pointed algebraic curves of genus g. $\mathcal{M}_{g,1}$ may be thought of

Received May 20, 2003; revised October 14, 2003

as the moduli space of punctured Riemann surfaces of genus g or the moduli space of affine algebraic curves of genus g with one place at infinity.

Fix $g \geq 2$ and let $N_1, N_2, \cdots, N_l$ be all the numerical semigroups of genus g. Set

$$\mathcal{M}_{g,1}^{N_i} := \{(X, P) \in \mathcal{M}_{g,1} \mid N_P = N_i\} \subset \mathcal{M}_{g,1} .$$

Then $\mathcal{M}_{g,1}^{N_i}$ is a subscheme of $\mathcal{M}_{g,1}$ and we have a direct sum decomposition of $\mathcal{M}_{g,1}$:

$$\mathcal{M}_{g,1} = \mathcal{M}_{g,1}^{N_1} \cup \mathcal{M}_{g,1}^{N_2} \cup \cdots \mathcal{M}_{g,1}^{N_l} .$$

In this paper, we study the following problem:

PROBLEM 1.2. If $\mathcal{M}_{g,1}^{N}$ is not empty, then compute $\mathcal{M}_{g,1}^{N}$ if possible and study its properties (dimension, irreducibility, singularity etc.).

According to Pinkham [14; p.110], M. Haure [6] claimed that $\mathcal{M}_{g,1}^{N}$ is not empty for any numerical semigroup N of genus $g \leq 7$. However, some parts of his proof are incorrect. The proof of this claim has been given in Komeda [9]. See the references in [9] for more information on the non-emptiness of $\mathcal{M}_{g,1}^{N}$.

By Pinkham [14], the moduli space $\mathcal{M}_{g,1}^{N}$ can be constructed from the miniversal deformation space of the monomial ring of the semigroup N. But the computation of miniversal deformations is very difficult and no approaches to compute $\mathcal{M}_{g,1}^{N}$ by this method seem to have been made yet, at least for $g \geq 3$. Recently, B. Martin [10], [11] implemented a procedure which computes the miniversal deformation of an isolated singularity to the computer algebra system SINGULAR. Thanks to this program, we can compute the miniversal deformation space of a monomial ring.

In this note, we compute the moduli space $\mathcal{M}_{g,1}^{N}$ explicitly for a numerical semigroup N of genus up to 5 using the computer algebra systems MAGMA and SINGULAR. From this computation, we find out that $\mathcal{M}_{g,1}^{N}$ is irreducible and determine the dimension of $\mathcal{M}_{g,1}^{N}$ except a few cases up to genus 5.

This note is organized as follows. In Section 2, we briefly review the deformation of monomial rings (curves) and the theorem of Pinkham, which constructs the moduli space $\mathcal{M}_{g,1}^{N}$ from the miniversal deformation space of the monomial ring of N. In Section 3, we first classify numerical semigroups up to genus 5 (Proposition 3.1) and we then calculate the moduli space $\mathcal{M}_{g,1}^{N}$ case by case using the computer algebra systems mentioned above. Our result is summarized in Theorem 3.3.

ACKNOWLEDGEMENT. We would like to express our appreciation to B. Martin for giving many suggestions and comments on his SINGULAR library "deform.lib". We also thank the referee for useful comments, especially for teaching us an important reference on the non-emptiness of $\mathcal{M}_{g,1}^{N}$.

2. Deformation of monomial rings (curves) and the theorem of Pinkham

In this section, we briefly review Pinkham's theory [14].

DEFINITION 2.1. Let $N \subset \mathbf{N}_0$ be a numerical semigroup of genus g. Then N is finitely generated as a semigroup and is expressed as

$$N = \langle a_1, a_2, \cdots, a_m \rangle = \sum_{i=1}^{m} \mathbf{N}_0 a_i \ \ (a_i \in \mathbf{N}_0) .$$

We then define the monomial ring $\mathbf{C}[t^N]$ of N as

$$\mathbf{C}[t^N] := \mathbf{C}[t^{a_1}, t^{a_2}, \cdots, t^{a_m}] \subset \mathbf{C}[t] ,$$

and call the affine algebraic curve Spec $\mathbf{C}[t^N]$ the monomial curve of N.

Given a numerical semigroup N of genus g, there always exists a miniversal deformation $\Phi : (\mathcal{X}, \Phi^{-1}(P)) \to (S, P)$ of $X_0 :=$ Spec $\mathbf{C}[t^N]$ ($\Phi : \mathcal{X} \to S$ is a flat morphism between schemes of finite type over $\mathbf{C}$ with $\Phi^{-1}(P) \cong X_0$, $P \in S$) and we may assume that the total space $\mathcal{X}$ and the base space S are both affine. We note the 1-dimensional algebraic torus $\mathbf{C}^\times$ acts naturally on the monomial curve X_0, which induces a natural algebraic $\mathbf{C}^\times$-action on $\mathcal{X}$ and S such that Φ is $\mathbf{C}^\times$-equivariant.

We may assume $\mathcal{X} =$ Spec $\mathbf{C}[x_1, x_2, \cdots, x_n, t_1, \cdots, t_r]/\mathbf{Fs}$, $S =$ Spec $\mathbf{C}[t_1, t_2, \cdots, t_r]/\mathbf{Js}$, where $\mathbf{Fs}$ and $\mathbf{Js}$ are ideals of these polynomial rings, and the morphism $\Phi : \mathcal{X} \to S$ is the natural projection. Further, we suppose $\mathbf{C}^\times$ acts on $\mathcal{X}$ and S by $\alpha \circ x_i := \alpha^{a_i} x_i$ and $\alpha \circ t_i := \alpha^{-e_i} t_i$ ($a_i, e_i \in \mathbf{Z}, \alpha \in \mathbf{C}^\times$).

The negative sign of the weight $-e_i$ of t_i is explained briefly as follows. Let $T^1_B = T^1(B/\mathbf{C}, B)$ be the first cohomology group of the cotangent complex of $B = \mathbf{C}[t^N]$. Let (S_0, m_0) be the local ring of S at the origin and $\mathbf{C}[\varepsilon]$ the ring of dual numbers. Then we have a canonical isomorphism:

$$\mathrm{Hom}_{\mathbf{C}-\mathrm{vec}}(m_0/{m_0}^2, \mathbf{C}) \cong \mathrm{Hom}_{\mathbf{C}-\mathrm{alg}}(S_0/{m_0}^2, \mathbf{C}[\varepsilon]) \cong T^1_B .$$

Choose a basis $\{s_1, \cdots, s_r\}$ of T^1_B with weights $\{e_1, \cdots, e_r\}$ respectively. Then $t_i \pmod{{m_0}^2}$ is the dual of s_i and thus has weight $-e_i$.

We will use the part of S with weight $e_i < 0$. Namely, let $\{i_1, \cdots, i_l\} \subset \{1, 2, \cdots, r\}$ be the integers with $e_{i_j} \geq 0$ and set $S' := \{t_{i_1} = \cdots = t_{i_l} = 0\} \cap S$. We restrict the miniversal deformation Φ to S' and get $\Phi' : \mathcal{X}' := \Phi^{-1}(S') \to S'$. Once again, we reset $\mathcal{X}' =$ Spec $\mathbf{C}[x_1, x_2, \cdots, x_n, t_1, \cdots, t_r]/\mathbf{Fs}$, $S' =$ Spec $\mathbf{C}[t_1, t_2, \cdots, t_r]/\mathbf{Js}$ so that $\mathbf{C}^\times$ acts on t_i with weight $-e_i > 0$ and on x_i with weight a_i.

We then projectivize each fiber of Φ' by adding one point. More precisely, take a set of generators of the defining ideal $\mathbf{Fs}$ of $\mathcal{X}'$: $\mathbf{Fs} = \langle F_i \mid 1 \leq i \leq m \rangle \subset \mathbf{C}[x_1, x_2, \cdots, x_n, t_1, \cdots, t_r]$. Introduce a new indeterminate x_{n+1} with weight 1 and substitute $t_i x_{n+1}^{-e_i}$ in the variable t_i of F_i to get a polynomial $F'_i \in \mathbf{C}[t_1, \cdots, t_r, x_1, \cdots, x_n, x_{n+1}]$. Set $\mathbf{Fs}' := \langle F'_i \rangle$ and we get a projective flat morphism $\pi : \mathrm{Proj}\, \mathbf{C}[x_1, \cdots, x_n, t_1, \cdots, t_r]/\mathbf{Fs}' \to S'$. Then any fiber of π is a projective algebraic curve and the equation $\{x_{n+1} = 0\}$ gives a section of π so that π is a flat family of projective pointed algebraic curves.

Now we can state the fundamental theorem of Pinkham ([14; Theorem (13.9)]):

THEOREM 2.2. *Set $U := \{x \in S' \mid$ the fiber $\Phi'^{-1}(x)$ is smooth$\} \subset S'$. If U is not empty, then there exists an isomorphism $U/\mathbf{C}^\times \cong \mathcal{M}^N_{g,1}$. This isomorphism is given by $U \ni x \mapsto \pi^{-1}(x) \in \mathcal{M}^N_{g,1}$.*

3. Main Results

We first classify the numerical semigroups of genus up to 5. The computation is easy and the result may be well-known. However, since we do not have a good reference for this, we include it here.

PROPOSITION 3.1. *Let g be an integer with $2 \le g \le 5$. Then the numerical semigroups of genus g are classified as follows. We use the notation $N = N(g)_n = N(\text{genus})_{\text{id number}}$.*

We need the following lemma (cf. [8; p. 204, Exc. 26]):

LEMMA 3.2. *Let N be a numerical semigroup of genus g and denote the gap sequence by $\{j_1, j_2, \cdots, j_g\} = \mathbf{N}_0 - N \quad (j_1 < \cdots < j_g)$. Then*:

(i) $j_i \le 2i - 1$.

(ii) *Let μ be the minimal element of N. Then $2 \le \mu \le g + 1$.*

PROOF OF PROPOSITION 3.1. We prove the $g = 4$ case. The other cases can be dealt with similarly.

From Lemma 1.1 and Lemma 3.2, we have the following information:

$$\begin{cases} \mathbf{N}_0 - N = \{j_1, j_2, j_3, j_4\} \\ j_1 \le 1,\ j_2 \le 3,\ j_3 \le 5,\ j_4 \le 7\,, \\ \{2g, 2g+1, \cdots\} = \{8, 9, 10, \cdots\} \subset N \\ 2 \le \mu \le g + 1 = 5 \end{cases}$$

(i) Assume $\mu = 2$. Then $j_1 = 1$, $j_2, j_2, j_3 \ne 2, 4, 6$ since $\{2, 4, 6\} \subset N$. Hence $\mathbf{N}_0 - N = \{1, 3, 5, 7\}$ and we have $N = N(4)_1 = \{0, 2, 4, 6, 8, \cdots\}$.

(ii) Assume $\mu = 3$. Then $j_1 = 1$, $j_2 = 2$, $j_3 \ne 3$, $j_4 \ne 6$. Hence we have three possibilities: $\{j_1, j_2, j_3, j_4\} = \{1, 2, 4, 5\}$ $\{1, 2, 4, 7\}$, $\{1, 2, 5, 7\}$. But $\mathbf{N}_0 - \{1, 2, 5, 7\} = \{0, 3, 4, 6, 8, \cdots\}$ is not a semigroup since $3 + 4 = 7 \notin \{0, 3, 4, 6, 8, \cdots\}$. Thus we have

$\mathbf{N}_0 - N(4)_2 = \{1, 2, 4, 5\}\,, \quad N(4)_2 = \{0, 3, 6, 7, 8, \cdots\}\,,$

$\mathbf{N}_0 - N(4)_3 = \{1, 2, 4, 7\}\,, \quad N(4)_3 = \{0, 3, 5, 6, 8, \cdots\}\,.$

(iii) Assume $\mu = 4$. Then $j_1 = 1$, $j_2 = 2$, $j_3 = 3$, $j_4 \ne 4$. Hence N is equal to one of the following three:

$\mathbf{N}_0 - N(4)_4 = \{1, 2, 3, 5\}\,, \quad N(4)_4 = \{0, 4, 6, 7, 8, \cdots\}\,,$

TABLE 1. Classification of numerical semigroups for $2 \leq g \leq 5$.

notation	elements of N	generators	gap sequence
$N(2)_1$	$\{0, 2, 4, \cdots\}$	$\langle 2, 5\rangle$	$\{1, 3\}$
$N(2)_2$	$\{0, 3, 4, \cdots\}$	$\langle 3, 4, 5\rangle$	$\{1, 2\}$
$N(3)_1$	$\{0, 2, 4, 6, \cdots\}$	$\langle 2, 7\rangle$	$\{1, 3, 5\}$
$N(3)_2$	$\{0, 3, 5, 6, \cdots\}$	$\langle 3, 5, 7\rangle$	$\{1, 2, 4\}$
$N(3)_3$	$\{0, 3, 4, 6, \cdots\}$	$\langle 3, 4\rangle$	$\{1, 2, 5\}$
$N(3)_4$	$\{0, 4, 5, 6, \cdots\}$	$\langle 4, 5, 6, 7\rangle$	$\{1, 2, 3\}$
$N(4)_1$	$\{0, 2, 4, 6, 8, \cdots\}$	$\langle 2, 9\rangle$	$\{1, 3, 5, 7\}$
$N(4)_2$	$\{0, 3, 6, 7, 8, \cdots\}$	$\langle 3, 7, 8\rangle$	$\{1, 2, 4, 5\}$
$N(4)_3$	$\{0, 3, 5, 6, 8, \cdots\}$	$\langle 3, 5\rangle$	$\{1, 2, 4, 7\}$
$N(4)_4$	$\{0, 4, 6, 7, 8, \cdots\}$	$\langle 4, 6, 7, 9\rangle$	$\{1, 2, 3, 5\}$
$N(4)_5$	$\{0, 4, 5, 7, 8, \cdots\}$	$\langle 4, 5, 7\rangle$	$\{1, 2, 3, 6\}$
$N(4)_6$	$\{0, 4, 5, 6, 8, \cdots\}$	$\langle 4, 5, 6\rangle$	$\{1, 2, 3, 7\}$
$N(4)_7$	$\{0, 5, 6, 7, 8, \cdots\}$	$\langle 5, 6, 7, 8, 9\rangle$	$\{1, 2, 3, 4\}$
$N(5)_1$	$\{0, 2, 4, 6, 8, 10, \cdots\}$	$\langle 2, 11\rangle$	$\{1, 3, 5, 7, 9\}$
$N(5)_2$	$\{0, 3, 6, 8, 9, 10, \cdots\}$	$\langle 3, 8, 10\rangle$	$\{1, 2, 4, 5, 7\}$
$N(5)_3$	$\{0, 3, 6, 7, 9, 10, \cdots\}$	$\langle 3, 7, 11\rangle$	$\{1, 2, 4, 5, 8\}$
$N(5)_4$	$\{0, 4, 7, 8, 9, 10, \cdots\}$	$\langle 4, 7, 9, 10\rangle$	$\{1, 2, 3, 5, 6\}$
$N(5)_5$	$\{0, 4, 6, 8, 9, 10, \cdots\}$	$\langle 4, 6, 9, 11\rangle$	$\{1, 2, 3, 5, 7\}$
$N(5)_6$	$\{0, 4, 6, 7, 8, 10, \cdots\}$	$\langle 4, 6, 7\rangle$	$\{1, 2, 3, 5, 9\}$
$N(5)_7$	$\{0, 4, 5, 8, 9, 10, \cdots\}$	$\langle 4, 5, 11\rangle$	$\{1, 2, 3, 6, 7\}$
$N(5)_8$	$\{0, 5, 7, 8, 9, 10, \cdots\}$	$\langle 5, 7, 8, 9, 11\rangle$	$\{1, 2, 3, 4, 6\}$
$N(5)_9$	$\{0, 5, 6, 8, 9, 10, \cdots\}$	$\langle 5, 6, 8, 9\rangle$	$\{1, 2, 3, 4, 7\}$
$N(5)_{10}$	$\{0, 5, 6, 7, 9, 10, \cdots\}$	$\langle 5, 6, 7, 9\rangle$	$\{1, 2, 3, 4, 8\}$
$N(5)_{11}$	$\{0, 5, 6, 7, 8, 10, \cdots\}$	$\langle 5, 6, 7, 8\rangle$	$\{1, 2, 3, 4, 9\}$
$N(5)_{12}$	$\{0, 6, 7, 8, 9, 10, \cdots\}$	$\langle 6, 7, 8, 9, 10, 11\rangle$	$\{1, 2, 3, 4, 5\}$

$\mathbf{N}_0 - N(4)_5 = \{1, 2, 3, 6\}, \quad N(4)_5 = \{0, 4, 5, 7, 8, \cdots\},$
$\mathbf{N}_0 - N(4)_6 = \{1, 2, 3, 7\}, \quad N(4)_6 = \{0, 4, 5, 6, 8, \cdots\}.$

(iv) Assume $\mu = 5$. Then $j_1 = 1, j_2 = 2, j_3 = 3, j_4 = 4$. Hence
$\mathbf{N}_0 - N(4)_7 = \{1, 2, 3, 4\}, \quad N(4)_7 = \{0, 5, 6, 7, 8, \cdots\}.$

The generators of each $N(4)_n$ can be determined easily. For instance, let us see $N(4)_7 = \{0, 5, 6, 7, 8, \cdots\} = \langle 5, 6, 7, 8, 9\rangle$. Indeed, take any integer $c \geq 5$. Then according to $c \equiv 0, 1, 2, 3, 4 \pmod 5$, we have

$c = 5k \in \langle 5, 6, 7, 8, 9\rangle \ (k \geq 1)\,,$
$c = 5k + 1 = 5k + (6 - 5) = 5(k - 1) + 6 \in \langle 5, 6, 7, 8, 9\rangle \ (k \geq 1)\,,$
$c = 5k + 2 = 5k + (7 - 5) = 5(k - 1) + 7 \in \langle 5, 6, 7, 8, 9\rangle \ (k \geq 1)\,,$
$c = 5k + 3 = 5k + (8 - 5) = 5(k - 1) + 8 \in \langle 5, 6, 7, 8, 9\rangle \ (k \geq 1)\,,$
$c = 5k + 4 = 5k + (9 - 5) = 5(k - 1) + 9 \in \langle 5, 6, 7, 8, 9\rangle \ (k \geq 1)\,.$

The other cases can be checked similarly. □

The following theorem is the main result of this paper. Throughout this note, in the case of giving a lengthy polynomial, we use typewriter fonts and also use the monomial expression such as `3AB2C` instead of $3AB^2C$ (the reason for doing this is that we prefer just copying outputs of the computer algebra system rather than manipulating outputs and making mistakes).

THEOREM 3.3. *Let N be any numerical semigroup of genus g with $2 \leq g \leq 5$. Then we have:*

(i) *If $N \neq N(5)_5$, then the dimension of $\mathcal{M}_{g,1}^N$ is given as in the following table.*

(ii) *If $N \neq N(5)_5, N(5)_8$, then $\mathcal{M}_{g,1}^N$ is irreducible.*

More precisely, $\mathcal{M}_{g,1}^N$ is isomorphic to a non-empty Zariski open subset of a projective variety $\overline{\mathcal{M}_{g,1}^N}$ in the column "structure of $\overline{\mathcal{M}_{g,1}^N}$" in the following table. We denote by $\mathbf{P}^n_{(d_0,d_1,\cdots,d_n)}$ the n-dimensional weighted projective space with weights $(d_0, d_1.\cdots, d_n)$. For $N = N(3)_4, N(4)_4, N(5)_4, N(5)_9$, $\overline{\mathcal{M}_{g,1}^N}$ is described as follows:

Suppose $N = N(3)_4$. Then $\overline{\mathcal{M}_{g,1}^N} := \operatorname{Proj} \mathbf{C}[A, B, \cdots, K]/\mathbf{J_1}$, *where* $\mathbf{C}[A, B, \cdots, K]$ *is a weighted graded polynomial ring of* 11 *variables with weights* (7, 5, 6, 2, 3, 8, 6, 4, 5, 3, 4) *and* $\mathbf{J_1} = \langle J_1[1], \cdots, J_1[6]\rangle$ *is a homogeneous prime ideal generated by*

```
J_1[1]=-AC-FI-2H2I+AEJ+CHJ+HIK-CJK-EHJ2+DIJ2+EJ2K
J_1[2]=BC-CI+GI+FJ-AK+CDJ-BEJ+2H2J-IJ2+DIK-JK2-DEJ2-DJ3
J_1[3]=-BI+I2+AJ-DIJ-HJ2+J2K
J_1[4]=GI-AK-EIJ-IJ2+DIK+HJK-JK2
J_1[5]=GJ-BK+IK-EJ2-J3
J_1[6]=CG+FK-CEJ-CJ2+CDK-BEK+2H2K+EIK-HK2-DEJK-DJ2K
```

Suppose $N = N(4)_4$. Then $\overline{\mathcal{M}_{g,1}^N} := \operatorname{Proj} \mathbf{C}[A, B, \cdots, M]/\mathbf{J_2}$, *where* $\mathbf{C}[A, B, \cdots, M]$ *is a weighted graded polynomial ring of* 13 *variables with weights* (5, 7, 3, 6, 4, 6, 5, 1, 2, 8, 8, 4, 10) *and* $\mathbf{J_2} = \langle J_2[1], \cdots, J_2[6]\rangle$ *is a homogeneous prime ideal generated by*

```
J_2[1]=-BJ-GM+3EFG+BFI-FHJ-CJL+CIM-3CEFI+F2HI+CFIL
J_2[2]=-BC-AG-CFH+ACI-C2L
J_2[3]=-BD+GK-DFH-CIK-CDL
J_2[4]=-AD-CK
J_2[5]=-AJ+CM-3CEF+AFI
J_2[6]=-JK-DM+3DEF+FIK
```

Suppose $N = N(5)_4$. Then $\overline{\mathcal{M}_{g,1}^N} := \operatorname{Proj} \mathbf{C}[A, B, \cdots, O]/\mathbf{J_3}$, *where* $\mathbf{C}[A, B, \cdots, O]$ *is a weighted graded polynomial ring of* 15 *variables with weights* (1, 3, 6, 10, 7, 2, 5, 9, 7, 4, 8, 6, 4, 5, 8) *and* $\mathbf{J_3} = \langle J_3[1], \cdots, J_3[6]\rangle$ *is a homogeneous prime ideal generated by*

```
J_3[1]=-IK+HL+IO+FIL+BL2-IM2
J_3[2]=EL-IL+KN-NO+AJK-FLN+M2N-AJO-AFJL+AJM2
J_3[3]=-EI+I2-HN-AHJ-BLN-ABJL
J_3[4]=DN+EO-IO+ADJ-BEN+2BIN+CMN-ABEJ+2ABIJ+ACJM
J_3[5]=-DL+KO-O2-BIL-CLM-BKN-FLO+M2O+BNO-ABJK+BFLN-BM2N+ABJO+ABFJL
```

```
-ABJM2
J_3[6]=DI-HO+BI2+CIM+BHN-BLO+ABHJ+B2LN+AB2JL
```

Suppose $N = N(5)_9$. Then $\overline{\mathcal{M}_{g,1}^N}$ is described in the proof.

We note that the weighted projective space $\mathbf{P}^n_{(d_0,d_1,\cdots,d_n)}$ is unirational since there exists a finite morphism $\mathbf{P}^n \to \mathbf{P}^n_{(d_0,d_1,\cdots,d_n)}$. Thus we have the following corollary:

COROLLARY 3.4. *Let N be a numerical semigroup of genus g with $2 \le g \le 5$. If $N \ne N(3)_4, N(4)_4, N(4)_7, N(5)_4, N(5)_5, N(5)_8, N(5)_9, N(5)_{12}$, then $\mathcal{M}_{g,1}^N$ is unirational.*

PROOF OF THEOREM 3.3. The proof is done in four steps.

TABLE 2. Dimension and structure of $\overline{\mathcal{M}_{g,1}^N}$ for $2 \le g \le 5$.

semigroups	$\dim \mathcal{M}_{g,1}^N$	structure of $\overline{\mathcal{M}_{g,1}^N}$
$N(2)_1 = \langle 2, 5\rangle$	3	$\mathbf{P}^3_{(4,6,8,10)}$
$N(2)_2 = \langle 3, 4, 5\rangle$	4	$\mathbf{P}^4_{(5,2,3,6,4)}$
$N(3)_1 = \langle 2, 7\rangle$	5	$\mathbf{P}^5_{(4,6,8,10,12,14)}$
$N(3)_2 = \langle 3, 5, 7\rangle$	6	$\mathbf{P}^6_{(7,1,4,3,6,9,5)}$
$N(3)_3 = \langle 3, 4\rangle$	5	$\mathbf{P}^5_{(2,5,8,6,9,12)}$
$N(3)_4 = \langle 4, 5, 6, 7\rangle$	7	$\mathrm{Proj}\,\mathbf{C}[A, \cdots, K]/\mathbf{J_1}$
$N(4)_1 = \langle 2, 9\rangle$	7	$\mathbf{P}^7_{(4,6,8,10,12,14,16,18)}$
$N(4)_2 = \langle 3, 7, 8\rangle$	8	$\mathbf{P}^8_{(4,7,3,6,9,2,5,8,6)}$
$N(4)_3 = \langle 3, 5\rangle$	7	$\mathbf{P}^7_{(1,4,7,10,6,9,12,15)}$
$N(4)_4 = \langle 4, 6, 7, 9\rangle$	9	$\mathrm{Proj}\,\mathbf{C}[A, \cdots, M]/\mathbf{J_2}$
$N(4)_5 = \langle 4, 5, 7\rangle$	8	$\mathbf{P}^8_{(3,1,5,2,6,10,4,8,7)}$
$N(4)_6 = \langle 4, 5, 6\rangle$	7	$\mathbf{P}^7_{(10,2,6,3,7,4,8,12)}$
$N(4)_7 = \langle 5, 6, 7, 8, 9\rangle$	10	unknown
$N(5)_1 = \langle 2, 11\rangle$	9	$\mathbf{P}^9_{(4,6,8,10,12,14,16,18,20,22)}$
$N(5)_2 = \langle 3, 8, 10\rangle$	10	$\mathbf{P}^{10}_{(5,8,3,6,9,12,1,4,7,10,6)}$
$N(5)_3 = \langle 3, 7, 11\rangle$	9	$\mathbf{P}^9_{(11,2,5,8,3,6,9,12,15,7)}$
$N(5)_4 = \langle 4, 7, 9, 10\rangle$	11	$\mathrm{Proj}\,\mathbf{C}[A, \cdots, O]/\mathbf{J_3}$
$N(5)_5 = \langle 4, 6, 9, 11\rangle$	unknown	unknown
$N(5)_6 = \langle 4, 6, 7, 9\rangle$	9	$\mathbf{P}^9_{(4,8,2,6,10,14,1,5,8,12)}$
$N(5)_7 = \langle 4, 5, 11\rangle$	9	$\mathbf{P}^9_{(6,11,2,3,7,4,8,12,5,10)}$
$N(5)_8 = \langle 5, 7, 8, 9, 11\rangle$	12	unknown
$N(5)_9 = \langle 5, 6, 8, 9\rangle$	11	$\mathrm{Proj}\,\mathbf{C}[A, \cdots, O]/\mathbf{J_4}$
$N(5)_{10} = \langle 5, 6, 7, 9\rangle$	10	$\mathbf{P}^{10}_{(2,3,4,9,7,1,8,6,5,5,3)}$
$N(5)_{11} = \langle 5, 6, 7, 8\rangle$	9	$\mathbf{P}^9_{(8,7,2,3,4,5,10,6,9,4)}$
$N(5)_{12} = \langle 6, 7, 8, 9, 10, 11\rangle$	13	unknown

(i) Let $N = \langle a_1, \cdots, a_n\rangle$ be a numerical semigroup of genus g. Define a surjective homomorphism $f : \mathbf{C}[X_1, \cdots, X_n] \to \mathbf{C}[t^{a_1}, \cdots, t^{a_n}]$ by $f(X_i) := t^{a_i}$, where $\mathbf{C}[X_1, \cdots, X_n]$ is a polynomial ring of n-variables over $\mathbf{C}$. If we give a weight a_i to X_i, then f is a graded homomorphism. By Herzog [7], $I := \ker f$ is generated by homogeneous elements of the form $F = \prod_{i=1}^{n} X_i^{d_i} - \prod_{i=1}^{n} X_i^{e_i}$ with $d_i \cdot e_i = 0$, from which it is easy to compute the ideal $I = \ker f$ for a given numerical semigroup N. Or actually, we compute the ideal I using the computer algebra system MAGMA [2].

(ii) We calculate the miniversal deformation $\Phi : \mathcal{X} \to S$ of the monomial curve $\operatorname{Spec} \mathbf{C}[X]/I \cong \operatorname{Spec} \mathbf{C}[t^N]$ using a computer algebra system SINGULAR [5]. We note that the calculation of miniversal deformation of a given affine variety (with an isolated singularity) is very complicated. Recently B. Martin developed an algorithm for computing the miniversal deformations and implemented it to SINGULAR. See [10] for this algorithm and see [11] for the actual program.

(iii) We restrict $\Phi : \mathcal{X} \to S$ to a closed subscheme S' on which $\mathbf{C}^\times$ acts with weight $-e_i > 0$ and get $\Phi' : \mathcal{X}' \to S'$. We observe that, for $2 \le g \le 4$, the $\mathbf{C}^\times$-action on S has always weights $-e_i > 0$ and we do not need restriction.

(iv) Set $U := \{x \in S' \mid \Phi'^{-1}(x) \text{ is smooth}\} \subset S'$. By [9], we know $U \ne \phi$ and $U/\mathbf{C}^\times \cong \mathcal{M}_{g,1}^N$ by Theorem 2.2. To show the irreducibility of $\mathcal{M}_{g,1}^N$, it is enough to show S' is irreducible. In many cases, it turns our that $S' = S$ is a weighted projective space, which is irreducible. In case that $S' = S$ is of the form $\operatorname{Spec} \mathbf{C}[t_1, \cdots, t_r]/\mathbf{Js}$, we will show $\mathbf{Js}$ is a prime ideal using the SINGULAR library "primdec.lib", which computes the primary decomposition of a given ideal (cf. [3]).

We perform the computation described above case by case. Since it costs too much space to give all the cases, we pick up and discuss one typical case (that is, $N = N(5)_9$) and omit the rest of them. So suppose $N = N(5)_9 = \langle 5, 6, 8, 9\rangle$.

(i) The monomial ring $\mathbf{C}[t^5, t^6, t^8, t^9]$ of N is described as $\mathbf{C}[t^5, t^6, t^8, t^9] \cong \mathbf{C}[x, y, z, w]/I$, where $I = \langle x^3 - yw, x^2y - z^2, xy^2 - zw, y^3 - w^2, x^2z - w^2, xw - yz\rangle$. We reproduce the MAGMA session of this computation for readers' convenience. In the following, ">" is the Magma prompt.

```
> S := [t^5,t^6,t^8,t^9];
> R<x,y,z,w> := PolynomialRing(Q,[5,6,8,9]);
> I := RelationIdeal(S,R);
> print I;
Ideal of Graded Polynomial ring of rank 4 over Rational Field
Lexicographical Order
Variables: x, y, z, w
Variable weights: 5 6 8 9
Basis:
[  x^3 - y*w,x^2*y - z^2,x*y^2 - z*w,y^3 - w^2,x^2*z - w^2, x*z^2
- y^2*w, y*w^2 - z^3, x*w - y*z  ]
```

Since $x * z^2 - y^2 * w$, $y * w^2 - z^3$ are unnecessary, we omit them.

(ii) The miniversal deformation $\Phi : \mathcal{X} \to S$ of $X_0 = \mathrm{Spec}\,(\mathbf{C}[x, y, z, w]/I) \cong \mathrm{Spec}\,\mathbf{C}[t^5, t^6, t^8, t^9]$ is given by

$$\Phi : \mathcal{X} = \mathrm{Spec}\,\mathbf{C}[x, y, z, w, A, \cdots, O]/\mathbf{Fs} \to S = \mathrm{Spec}\,\mathbf{C}[A, \cdots, O]/\mathbf{Js},$$

where $\mathbf{C}^\times$ acts on $\mathcal{X}$ with weights (5, 6, 8, 9; 3, 9, 1, 2, 4, 10, 4, 5, 6, 5, 6, 7, 7, 2, 8), Φ is the natural projection and $\mathbf{Fs}$, $\mathbf{Js}$ are the following $\mathbf{C}^\times$-invariant ideals:

```
Fs=< F[1],F[2],...,F[6] >
Js=< J[1],J[2],...,J[6] >

F[1]=x3-yw+Ay2+By+Cxw+Dxz+Exy+Fx+Iw-Kw+Lz-CHw-DHz+EHy-GHy+CJw+DJz
+1/2FJ-GJy+AKy+BK-1/2LO+1/2C2Gw+ADGy+1/2CDGz+1/2CEGy-1/2CG2y+2ACHy
-ACJy+3/2H2J+3/2GIJ-3/2HJ2+1/2J3+EHK-GHK-1/2EJK-GJK-DIM+1/2CIO
-3/2CGHJ+3/4CGJ2+ADGK+1/2CEGK-1/2CG2K+2ACHK-ACJK+CDHM-1/2CDJM
+3/8C2G2J-1/2C2DGM

F[2]=x2y-z2+Gy2+Jxy+Mw+Nyz+Oz-AGw+EGz-G2z+2AHz-H2y-GIy-AJz+HJy+GKy
-AMy-BM+CMz-EMx+GMx-ANxy-BNx-INz+KNz-AOx+CGHy-1/2CGJy-EHM+GHM-AHNy
-1/2BJN-CMNy-1/2LMN-AHO-CMO-1/4C2G2y-1/2CEGM+1/2CG2M-2ACHM+ACJM
+1/2ACGNy-AHKN+1/2AJKN+1/2CIMN-CKMN+1/2ACGO+1/2ACGKN

F[3]=xy2-zw+Gxz+Hy2+Kxy+Mx2+Nyw+Ow+1/2CGy2+DGxy+EGw-G2w-GJz+HKy+GLy
+CMw+DMz+HMx-JMx-ANy2-BNy-GNxy-INw+KNw-AOy-BO-GOx-EG2x+G3x-2AGHx
-DGHy+AGJx+DGJy+1/2CGKy-1/2CGMx-HJM+GINx+GJNy-AKNy-BKN-GKNx+GJO
+1/2CDG2y+EG2J-G3J+2AGHJ-AGJ2-DEGM+DG2M+1/2CGJM-CDM2-ADGNy-GIJN
+GJKN+DIMN-ADGO-ADGKN

F[4]=y3-w2+Gxw+2Hxz-Jxz+2Ky2+Mxy+Nx2y+Ox2+DGy2+2CHy2+2DHxy+2EHw-2GHw
-CJy2-DJxy-EJw-2HJz+J2z+K2y+2HLy-JLy+DMw-JMy+KMx+CNyw+DNyz+ENy2+FNy
-GNy2-HNxy+KNx2+LNw+COw+DOz+EOy+FO-GOy-HOx-2EGHx+2G2Hx-4AH2x-2DH2y
+EGJx-G2Jx+4AHJx+3DHJy-AJ2x-DJ2y+DGKy+2CHKy-CJKy-JKM-1/2CGNxy+H2Ny
+GINy+CKNw+DKNz+EKNy+FKN-2GKNy-HKNx-GLNx-1/2CGOx+H2O+GIO-GKO+CDGHy
-1/2CDGJy+2EGHJ-2G2HJ+4AH2J-EGJ2+G2J2-4AHJ2+AJ3-2DEHM+2DGHM+DEJM
-DGJM-2ADHNy-CGHNy+ADJNy+CGJNy-1/2CGKNx+H2KN+GIKN-GK2N+GJLN-DLMN
-2ADHO-CGHO+ADJO+CGJO+1/4C2G2Ny-2ADHKN-CGHKN+ADJKN+CGJKN+1/4C2G2O
+1/4C2G2KN

F[5]=x2z-w2+Ayw+Bw+Czw+Dz2+Eyz+Fz+Hxz+Iy2-Jxz+Lxy-1/2CGxz-EGx2+G2x2
-2AHx2+2EHw-GHw+H2z+GIz+AJx2-EJw-GJw-2HJz+J2z+IKy+HLy-CMx2+IMx+INx2
+LNw+ADGw-CEGw-DEGz-E2Gy+1/2CG2w+DG2z+EG2y-2ADHz-2AEHy-CGHz-EGHx+G2Hx
-2AH2x+DGIy+ADJz+AEJy+CGJz+EGJx-G2Jx+3AHJx-AJ2x-1/2CGLy-C2Mw-CDMz-CEMy
-CHMx+CJMx-IJM-ELM+CINw+DINz+EINy+FIN-GINy-1/2IJNx-ALNy-BLN-1/2GLNx
+BCO-EIO+1/4C2G2z+1/2CEG2x-1/2CG3x+ACGHx-1/2ACGJx+EGHJ-G2HJ+2AH2J-AHJ2
-E2GK+EG2K-2AEHK+AEJK+1/2C2GMx+CHJM-CEKM-1/2CGINx+GI2N+1/4IJ2N-GIKN
-1/2GHLN+3/4GJLN-ACKO-1/2CEG2J+1/2CG3J-ACGHJ+1/2ACGJ2+CDEGM-CDG2M
+2ACDHM-ACDJM-1/2C2GJM+C2DM2+1/2CGIJN+1/2CG2LN+1/4C2G2IN

F[6]=-yz+xw-Hw+Jw-Kz+1/2CGw+EGy-G2y+2AHy-AJy+CMy+LM+IO+EGK-G2K+2AHK
```

```
-AJK+CKM

J[1]=-2HI+IJ-GL
J[2]=-2BH+BJ+LM+2AHK-AJK
J[3]=-BG-IM+AGK
J[4]=-2FH+FJ-LO-2H3+3H2J-3HJ2+J3+2EHK-EJK+G2L+2CGH2-2CGHJ+1/2CGJ2
-2CDHM+CDJM
     -1/2C2G2H+1/4C2G2J
J[5]=-FG+IO-GH2-G2I+GHJ-GJ2+EGK+CG2H-1/2CG2J-CDGM-1/4C2G3
J[6]=FM+BO+H2M+GIM-HJM+J2M-EKM-AKO-CGHM+1/2CGJM+CDM2+1/4C2G2M
```

The SINGULAR session of this computation is as follows. Again, ">" is the SINGULAR prompt and some explanatory comments by the author is given with the symbol "%%%".

```
> LIB "deform.lib"; %%% calls the Singular library "deform.lib"
            [10] %%%
// ** loaded /usr/local/Singular/2-0-3/LIB/deform.lib
            (1.25.2.1,2002/02/20)
// ** loaded /usr/local/Singular/2-0-3/LIB/sing.lib
            (1.24.2.3,2002/02/20)
// ** loaded /usr/local/Singular/2-0-3/LIB/primdec.lib
            (1.98.2.9,2002/03/06)
// ** loaded /usr/local/Singular/2-0-3/LIB/poly.lib
            (1.33.2.4,2002/03/06)
// ** loaded /usr/local/Singular/2-0-3/LIB/ring.lib
            (1.17.2.1,2002/02/20)
// ** loaded /usr/local/Singular/2-0-3/LIB/elim.lib
            (1.14.2.2,2002/02/20)
// ** loaded /usr/local/Singular/2-0-3/LIB/random.lib
            (1.16.2.1,2002/02/20)
// ** loaded /usr/local/Singular/2-0-3/LIB/homolog.lib
            (1.15.2.1,2002/02/20)
// ** loaded /usr/local/Singular/2-0-3/LIB/matrix.lib
            (1.26.2.1,2002/02/20)
// ** loaded /usr/local/Singular/2-0-3/LIB/general.lib
            (1.38.2.6,2002/03/06)
// ** loaded /usr/local/Singular/2-0-3/LIB/inout.lib
            (1.21.2.3,2002/02/20)
> ring r = 0,(x,y,z,w),dp; %%% r = Q[x,y,z,w] with degree reverse lex
            ordering %%%
> ideal I = x^3 - y*w,x^2*y - z^2,x*y^2 - z*w,y^3 - w^2,x^2*z - w^2,
            x*w - y*z; %%% this is our ideal %%%
> int p = printlevel;
> printlevel = 2;
> versal(I); %%% calculates the miniversal deformation of the quotient
            ring r/I %%%
// dim T_1 = 15
// dim T_2 = 6
// ready: T_1 and T_2
```

```
// T_1 is quasi-homogeneous represented with weight-vector
3,9,1,2,4,10,4,5,6,5,6,7,7,2,8
// infinitesimal deformation: Fs:
Fs[1,1],Fs[1,2],Fs[1,3],Fs[1,4],Fs[1,5],Fs[1,6]
// start computation in degree 2.
// next equation of base space:
-2HI+IJ-GL,
-2BH+BJ+LM,
-BG-IM,
-2FH+FJ-LO,
-FG+IO,
FM+BO
// start computation in degree 3.
// next equation of base space:
-2HI+IJ-GL,
-2BH+BJ+LM+2AHK-AJK,
-BG-IM+AGK,
-2FH+FJ-LO-2H3+3H2J-3HJ2+J3+2EHK-EJK+G2L,
-FG+IO-GH2-G2I+GHJ-GJ2+EGK,
FM+BO+H2M+GIM-HJM+J2M-EKM-AKO
// start computation in degree 4.
// next equation of base space:
-2HI+IJ-GL,
-2BH+BJ+LM+2AHK-AJK,
-BG-IM+AGK,
-2FH+FJ-LO-2H3+3H2J-3HJ2+J3+2EHK-EJK+G2L+2CGH2-2CGHJ+1/2CGJ2-2CDHM
+CDJM,
-FG+IO-GH2-G2I+GHJ-GJ2+EGK+CG2H-1/2CG2J-CDGM,
FM+BO+H2M+GIM-HJM+J2M-EKM-AKO-CGHM+1/2CGJM+CDM2
// start computation in degree 5.
// next equation of base space:
-2HI+IJ-GL,
-2BH+BJ+LM+2AHK-AJK,
-BG-IM+AGK,
-2FH+FJ-LO-2H3+3H2J-3HJ2+J3+2EHK-EJK+G2L+2CGH2-2CGHJ+1/2CGJ2-2CDHM
+CDJM-1/2C2G2H
  +1/4C2G2J,
-FG+IO-GH2-G2I+GHJ-GJ2+EGK+CG2H-1/2CG2J-CDGM-1/4C2G3,
FM+BO+H2M+GIM-HJM+J2M-EKM-AKO-CGHM+1/2CGJM+CDM2+1/4C2G2M
// start computation in degree 6.
// no obstruction
// finished   %%% computation is finished at degree 6 %%%
// quasi-homogeneous weights of miniversal base
1,3,1,6,6,8,3,4,5,4,2,6,1,1,6
// ___ Equations of miniversal base space ___
-2HI+IJ-GL,Js[1,2],-BG-IM+AGK,Js[1,4],Js[1,5],Js[1,6]
// ___ Equations of miniversal total space ___
Fs[1,1],Fs[1,2],Fs[1,3],Fs[1,4],Fs[1,5],Fs[1,6]
```

```
// Result belongs to ring Px.
// Equations of total space of miniversal deformation are
// given by Fs, equations of miniversal base space by Js.
// Make Px the basering and list objects defined in Px by typing:
   setring Px; show(Px);
   listvar(matrix);
// NOTE: rings Qx, Px, So are alive!
// (use 'kill_rings("");' to remove)
> setring Px; show(Px);
   listvar(matrix);
// ring: (0),(A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,x,y,z,w),(ds(15),dp(4),C);
// minpoly = 0
// objects belonging to this ring:
// Rs                   [0]  matrix 6 x 8
// Fs                   [0]  matrix 1 x 6
// Js                   [0]  matrix 1 x 6
// Rs                   [0]  matrix 6 x 8
// Fs                   [0]  matrix 1 x 6
// Js                   [0]  matrix 1 x 6
> Fs;
Fs[1,1]=...  %%% the equations of Fs are omitted since they are given
above %%%
> Js;
Js[1,1]=...  %%% the equations of Js are omitted since they are given
above %%%
```

(iii) Since all the weights of $A, \cdots, O$ are positive, we do not need to restrict Φ.

(iv) Thus we have $U/\mathbf{C}^{\times} \cong \mathcal{M}_{5,1}^{N(5)_9}$. Finally, we check that **Js** is a prime ideal and $\dim S = 12$ as follows:

```
> LIB "primdec.lib"; %%% calls the procedure "primdec.lib" %%%
// ** loaded /usr/local/Singular/2-0-3/LIB/primdec.lib
            (1.98.2.9,2002/03/06)
// ** loaded /usr/local/Singular/2-0-3/LIB/matrix.lib
            (1.26.2.1,2002/02/20)
// ** loaded /usr/local/Singular/2-0-3/LIB/ring.lib
            (1.17.2.1,2002/02/20)
// ** loaded /usr/local/Singular/2-0-3/LIB/inout.lib
            (1.21.2.3,2002/02/20)
// ** loaded /usr/local/Singular/2-0-3/LIB/random.lib
            (1.16.2.1,2002/02/20)
// ** loaded /usr/local/Singular/2-0-3/LIB/poly.lib
            (1.33.2.4,2002/03/06)
// ** loaded /usr/local/Singular/2-0-3/LIB/elim.lib
            (1.14.2.2,2002/02/20)
// ** loaded /usr/local/Singular/2-0-3/LIB/general.lib
            (1.38.2.6,2002/03/06)
> ring R = 0,(A,B,C,D,E,F,G,H,I,J,K,L,M,N,O),dp; %%% R = Q[A,B,...,O]
```

```
                              with degree reverse lex ordering %%%
> ideal j = -2HI+IJ-GL, 2AHK-AJK-2BH+BJ+LM, AGK-BG-IM, -1/2C2G2H
+1/4C2G2J+2CGH2-2CGHJ+1/2CGJ2-2CDHM+CDJM-2H3+3H2J-3HJ2+J3+2EHK
-EJK+G2L-2FH+FJ-LO, -1/4C2G3+CG2H-1/2CG2J-CDGM-GH2-G2I+GHJ-GJ2
+EGK-FG+IO,  1/4C2G2M-CGHM+1/2CGJM+CDM2+H2M+GIM-HJM+J2M-EKM-AKO
+FM+BO;     %%% ideal j := Js %%%
> primdecSY(j); %%% gives the primary decomposition of the
             ideal j %%%
[1]:
   [1]:
      _[1]=2HI-IJ+GL
      _[2]=2AHK-AJK-2BH+BJ+LM
      _[3]=AGK-BG-IM
      _[4]=C2G2M-4CGHM+2CGJM+4CDM2+4H2M+4GIM-4HJM+4J2M-4EKM-4AKO+4FM
           +4BO
      _[5]=2C2G2H-C2G2J-8CGH2+8CGHJ-2CGJ2+8CDHM-4CDJM+8H3-12H2J+12HJ2
           -4J3-8EHK+4EJK-4G2L+8FH-4FJ+4LO
      _[6]=C2G3-4CG2H+2CG2J+4CDGM+4GH2+4G2I-4GHJ+4GJ2-4EGK+4FG-4IO
      _[7]=C2GIM2+4ACDKM2+3AJ2KM-4AEK2M-4BCDM2+2CGLM2-4A2K2O-3BJ2M
           +4BEKM+4AFKM+4I2M2-2HLM2+JLM2+8ABKO-4BFM-4B2O
      _[8]=4A2CDK2M2+3A2J2K2M-4A2EK3M-8ABCDKM2+C2I2M3-4A3K3O-6ABJ2KM
           +8ABEK2M+4A2FK2M+4B2CDM2+4AI2KM2+2CILM3+12A2BK2O+3B2J2M
           -4B2EKM-8ABFKM-4BI2M2+L2M3-12AB2KO+4B2FM+4B3O
   [2]:
      _[1]=2HI-IJ+GL
      _[2]=2AHK-AJK-2BH+BJ+LM
      _[3]=AGK-BG-IM
      _[4]=C2G2M-4CGHM+2CGJM+4CDM2+4H2M+4GIM-4HJM+4J2M-4EKM-4AKO+4FM
           +4BO
      _[5]=2C2G2H-C2G2J-8CGH2+8CGHJ-2CGJ2+8CDHM-4CDJM+8H3-12H2J+12HJ2
           -4J3-8EHK+4EJK-4G2L+8FH-4FJ+4LO
      _[6]=C2G3-4CG2H+2CG2J+4CDGM+4GH2+4G2I-4GHJ+4GJ2-4EGK+4FG-4IO
      _[7]=C2GIM2+4ACDKM2+3AJ2KM-4AEK2M-4BCDM2+2CGLM2-4A2K2O-3BJ2M
           +4BEKM+4AFKM+4I2M2-2HLM2+JLM2+8ABKO-4BFM-4B2O
      _[8]=4A2CDK2M2+3A2J2K2M-4A2EK3M-8ABCDKM2+C2I2M3-4A3K3O-6ABJ2KM
           +8ABEK2M+4A2FK2M+4B2CDM2+4AI2KM2+2CILM3+12A2BK2O+3B2J2M
           -4B2EKM-8ABFKM-4BI2M2+L2M3-12AB2KO+4B2FM+4B3O
%%% there is only one primary ideal [1] whose radical (prime ideal)
           is [2] %%%
%%% since ideal [1] = ideal [2], the ideal j is prime %%%
> dim(std(j));
12  %%% the Krull dimension of the ring R/j is equal to 12 %%%
```

Thus U and $U/\mathbf{C}^{\times} \cong \mathcal{M}_{5,1}^{N(5)_9}$ is irreducible, and dim $\mathcal{M}_{5,1}^{N(5)_9} = 12 - 1 = 11$. This completes the computation in the case of $N = N(5)_9$.

If $N \neq N(4)_7, N(5)_5, N(5)_8, N(5)_{12}$, we can perform similar calculations as above and get the explicit descriptions of the projectivized moduli space $\overline{\mathcal{M}_{g,1}^{N}}$. On the other hand, if N

is one of these four semigroups, we cannot compute $\overline{\mathcal{M}_{g,1}^{N}}$ explicitly since the computations are beyond the capability of our computers (our system: Intel Pentium IV processor 1.5GHz, memory 654KB, OS Windows 2000).

However, in the case of $N = N(4)_7, N(5)_{12}$, which are the semigroups of ordinary points, we can determine the dimension and show the irreducibility of $\mathcal{M}_{g,1}^{N}$ by a general argument as follows. Let $N = \{0, g+1, g+2, \cdots\}$ be the semigroup of ordinary points of genus g and $\mathcal{M}_g$ the moduli space of nonsingular projective curves of genus g. Consider the natural surjective morphism $\alpha : \mathcal{M}_{g,1}^{N} \to \mathcal{M}_g$ defined by $\alpha([X, P]) := [X]$, where $[X, P] \in \mathcal{M}_{g,1}^{N}$ and $[X] \in \mathcal{M}_g$. Then for any point $[X] \in \mathcal{M}_g$, the fiber $\alpha^{-1}([X])$ is isomorphic to $(X - W_X)/\mathrm{Aut}(X)$, where W_X is the (finite) set of Weierstrass points of X and $\mathrm{Aut}(X)$ is the group of automorphisms of X, which is finite since $g \geq 2$. Thus all the fibers of α are irreducible. Since $\mathcal{M}_g$ is irreducible by [4], we conclude $\mathcal{M}_{g,1}^{N}$ is also irreducible and $\dim \mathcal{M}_{g,1}^{N} = \dim \mathcal{M}_g + 1 = 3g - 2$.

Finally, we can see the dimension of $\mathcal{M}_{5,1}^{N(5)_8}$ is equal to 12 in the following way. Generally, let $\Phi : \mathcal{X} \to S$ be the miniversal deformation of the monomial ring B of a numerical semigroup N of genus g and E a non-empty component of S such that the generic fiber over E is smooth. Then the dimension of E is given by the following formula (cf. [1; Theorem 4.1.1, Remark 4.1.2]):

$$\dim E = 2g - 1 + r\,,$$

where $r := \dim_{\mathbf{C}} \mathrm{Ext}_B^1(\mathbf{C}, B)$. We also have $r = \mathrm{card}(m^{-1}(N) - N)$, where $m^{-1}(N) := \{n \in \mathbf{N}_0 \mid n + m \in N \text{ for all } m \in N - \{0\}\}$ and $\mathrm{card}(Z)$ is the cardinality of a set Z.

Now suppose $N = N(5)_8$. Since $\Phi : \mathcal{X} \to S$ has a smooth fiber in this case, we can use the formula above. We note that all the weights of $T^1(B)$ are positive (i.e. $-e_i > 0$) in this case and we do not need to restrict S. Now we compute $m^{-1}(N(5)_8) = \{0, 2, 3, 4, 5, 6, 7, \cdots\}$ and $r = \mathrm{card}(m^{-1}(N(5)_8) - N(5)_8)) = \mathrm{card}\{2, 3, 4, 6\} = 4$. We thus have $\dim E = 2 \cdot 5 - 1 + 4 = 13$. It follows that $\dim \overline{\mathcal{M}_{5,1}^{N(5)_8}} = \dim S - 1 = 12$. □

REMARK 3.5. As for the dimension of $\mathcal{M}_{5,1}^{N(5)_5}$, since the miniversal deformation $\Phi : \mathcal{X} \to S$ has a smooth fiber, we can use the dimension formula above and see $\dim E = 12$. However, since $T^1(B)$ has a 1-dimensional subspace with negative weight (i.e., $-e_i < 0$), we only know $\dim \mathcal{M}_{5,1}^{N(5)_5} = 11$ or 10.

REMARK 3.6. As a concluding remark, we pose some problems to be discussed in the future.

(i) In the case of $g = 2$, we have a decomposition of the moduli space $\mathcal{M}_{2,1}$: $\mathcal{M}_{2,1} = \mathcal{M}_{2,1}^{N(2)_1} \cup \mathcal{M}_{2,1}^{N(2)_2}$. In [14; Chap. IV,15], how to glue these two subschemes to get the whole $\mathcal{M}_{2,1}$ is discussed. In the case of $g = 3$, we have a decomposition of $\mathcal{M}_{3,1}$ into four subschemes all of which are described explicitly. So it will be an interesting problem to understand how to glue these four subschemes to get the whole $\mathcal{M}_{3,1}$.

(ii) In this note, the moduli space $\mathcal{M}_{g,1}^N$ is described as a Zariski open subset of a projective variety $\overline{\mathcal{M}_{g,1}^N}$, many of which are weighted projective spaces. In other words, we get a compactification of $\mathcal{M}_{g,1}^N$ up to genus 5 (except four cases). Then the investigation of the boundary points of this compactifiction should be done next.

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Present Address:
DEPARTMENT OF MATHEMATICAL SCIENCES, TOKYO DENKI UNIVERSITY,
HATOYAMA, HIKI-GUN, SAITAMA-KEN, 350–0394 JAPAN.
e-mail: nakano@r.dendai.ac.jp