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A Formula for the A-Polynomials of $(-2, 3, 1+2n)$-Pretzel Knots

Naoko TAMURA and Yoshiyuki YOKOTA

Tokyo Metropolitan University

1. Introduction

Let M be a compact 3-manifold such that ∂M is a torus and $\{\lambda, \mu\}$ a basis of $\pi_1(\partial M)$. Then $R = \mathrm{Hom}(\pi_1(M), \mathrm{SL}(2, \mathbf{C}))$ is an affine algebraic variety. Let R_U be the set of representations $\rho \in R$ such that

$$\rho(\lambda) = \begin{pmatrix} l & * \\ 0 & 1/l \end{pmatrix} \qquad \rho(\mu) = \begin{pmatrix} m & * \\ 0 & 1/m \end{pmatrix}$$

for some $l, m \in \mathbf{C}$. Note that any element of R can be conjugated to such a representation because λ and μ are commutative and that the Zariski closure of the image of the eigenvalue map $\xi : R_U \to \mathbf{C}^2$ defined by $\xi(\rho) = (l, m)$ is an algebraic subset of $\mathbf{C}^2$. Let $C_1, C_2, \cdots, C_k$ be the one-dimensional components of the closure of $\xi(R_U)$ and $g_1(l, m), g_2(l, m), \cdots, g_k(l, m) \in \mathbf{Z}[l, m]$ their defining polynomials which are supposed to be reduced. Then, the A-polynomial of M is defined by

$$A_M(l, m) = g_1(l, m) g_2(l, m) \cdots g_k(l, m) .$$

When M is the complement of a knot K in S^3, we choose $\{\lambda, \mu\}$ as the pair of the preferred longitude and the meridian of K. Then, the A-polynomial always has a factor $l - 1$, and so we shall compute $A_K(l, m) = A_M(l, m)/(l - 1)$.

In the study of knot theory, the polynomial invariants, such as Alexander and Jones polynomials, are very much useful and have been evaluated for a large number of knots. However, the A-polynomials have been computed for only some simple knots, see [1]. In particular, except for torus knots, there had been no formulae for the A-polynomials of infinite series of knots until Hoste and Shanahan found formulae for two infinite families of 2-bridge knots, including twist knots, in [3].

Inspired by [3], in this paper, we will derive a formula for the A-polynomials of the $(-2, 3, 1+2n)$-pretzel knots. Let K_n denote the $(-2, 3, 1+2n)$-pretzel knot depicted in Figure 1, where n is the number of left-handed full twists contained in the box. Note that

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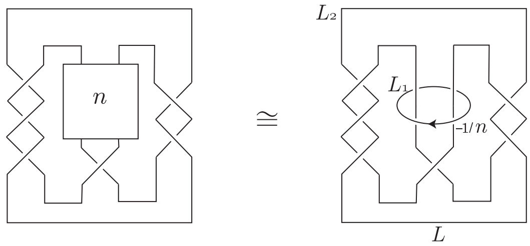


FIGURE 1. K_n.

K_0, K_1 and K_2 are respectively the torus knots 5_1, 8_{19} and 10_{124} in the notation of the table in [6] and K_3 is the famous $(-2, 3, 7)$-pretzel knot, and A_{K_0}, A_{K_1}, A_{K_2} and A_{K_3} are given by

$$A_{K_0}(l, m) = 1+lm^{10}, \quad A_{K_1}(lm^{-4}, m) = 1+lm^8, \quad A_{K_2}(lm^{-8}, m) = (1+lm^7)(1-lm^7),$$

$$A_{K_3}(lm^{-12}, m) = 1 - lm^4 + 2lm^6 - lm^8 - 2l^2m^{12} - l^2m^{14} + l^4m^{24} + 2l^4m^{26} + l^5m^{30} - 2l^5m^{32} + l^5m^{34} - l^6m^{38},$$

see [1] and [7].

MAIN THEOREM 1. *Put*

$$B_n = \begin{cases} -l^2(lm^8)^{3+n}(1-m^2)^n(1+lm^6)^{3+n} & (n > 3), \\ -(lm^8)^{-(2+n)}(1-m^2)^{-(1+n)}(1+lm^6)^{2-n} & (n < 0) \end{cases}$$

and define C_n recursively by

$$\alpha^2 C_n - \alpha\gamma\, C_{n-1} - (2\alpha^2 + 2\alpha\gamma - \beta^2)C_{n-2} - \alpha\gamma C_{n-3} + \alpha^2 C_{n-4} = 0,$$

where

$$\alpha = lm^8(1-m^2)(1+lm^6), \quad \beta = m^2 - lm^6 + 2lm^8 - 2l^2m^{16} + l^2m^{18} - l^3m^{22},$$
$$\gamma = -1 - m^4 - 2lm^8 - lm^{10} + lm^{12} - l^2m^{12} + l^2m^{14} + 2l^2m^{16} + l^3m^{20} + l^3m^{24},$$

with the initial conditions

$$C_0 = -\frac{lm^8\{A_{K_0}(l, m)\}^2}{(1+lm^6)^2}, \quad C_1 = \frac{m^4(1-lm^8)\{A_{K_1}(lm^{-4}, m)\}^2}{(1-m^2)(1+lm^6)},$$

$$C_2 = -\frac{\{A_{K_2}(lm^{-8}, m)\}^2}{l(1-m^2)^2}, \quad C_3 = \frac{A_{K_3}(lm^{-12}, m)}{l^2m^4(1-m^2)^3}.$$

Then, $A_{K_n}(lm^{-4n}, m)$ is a factor of $B_nC_n \in \mathbf{Z}[l, m]$ for $n > 3$ and $n < 0$.

REMARK. In fact, when $n \equiv 1 \bmod 3$, B_nC_n contains the factor $1 - lm^8$ but it is not a factor of the A-polynomial of K_n.

2. Proof of Main Theorem

Since K_n can be obtained from the link L depicted in Figure 1 by the $-1/n$ surgery along L_1, we first consider an ideal triangulation $\mathcal{S}$ of the complement of a hyperbolic link L and then apply the surgery along L_1.

Let D be an $(1, 1)$-tangle presentation of L depicted in Figure 2. Then, we prepare 4 ideal tetrahedra at each crossing of D as shown in Figure 3, where $\pm\infty$ denote the poles of S^3. We glue them along the edges of D as shown in Figure 4, and recover $\dot{M} = M \setminus \{\pm\infty\}$. In what follows, for $z \in \mathbf{C}\backslash\{0, 1\}$, we denote by $T(z)$ the ideal tetrahedron in 3-dimensional hyperbolic space $\mathbf{H}^3$ whose vertices are $0, 1, z, \infty$ in $\partial\mathbf{H}^3 = \mathbf{C} \cup \{\infty\}$ if it is not degenerate. We may use $T(z)$ as a symbol even if the corresponding ideal tetrahedron is degenerate. We assign complex numbers to the corners of D as shown in Figure 2 and identify $T(z)$ with the tetrahedron corresponding to the corner assigned z. Put

$$B = \{T(a_1) \cup T(d_1)\} \cap \{T(c_9) \cup T(d_9)\}.$$

As $\dot{M} \setminus B$ is homeomorphic to M, we can develop $\dot{M} \setminus B$ in $\mathbf{H}^3$, where each tetrahedron touching B can not specify distinct 4 points in $\partial\mathbf{H}^3$ and so is degenerate. In fact,

$$T(a_1),\ T(d_1),\ T(c_9),\ T(d_9)$$

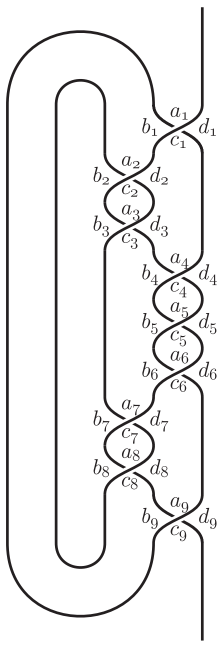


FIGURE 2.

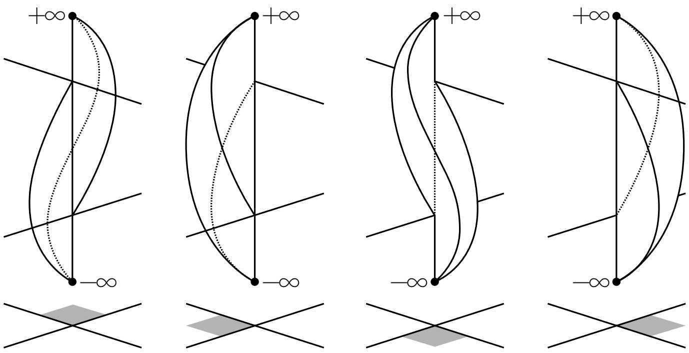


FIGURE 3.

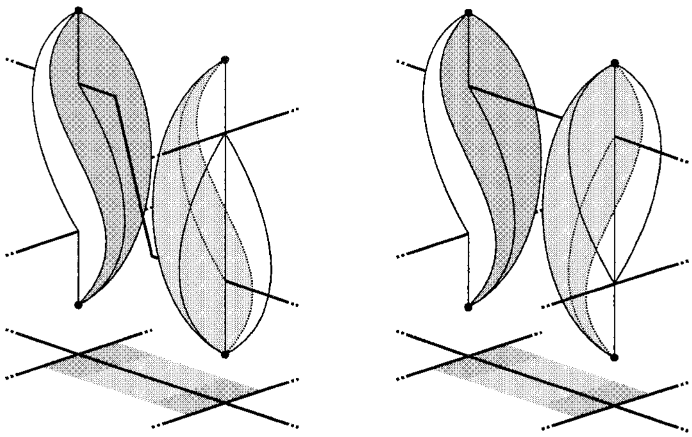


FIGURE 4.

are essentially *one*-dimensional objects and

$$T(b_1)\,,\,T(c_1)\,,\,T(a_2)\,,\,T(b_2)\,,\,T(c_2)T(d_2)\,,\,T(a_3)\,,\,T(b_3)\,,\,T(d_4)\,,\,T(d_5)\,,\,T(d_6)\,,$$
$$T(b_7)\,,\,T(c_7)\,,\,T(a_8)\,,\,T(b_8)\,,\,T(c_8)T(d_8)T(a_9)\,,\,T(b_9)$$

are essentially *two*-dimensional objects in $\dot{M} \setminus B$. Thus, we obtain an ideal triangulation $\mathcal{S}$ of M with

$$T(c_3)\,,\,T(d_3)\,,\,T(a_4)\,,\,T(b_4)\,,\,T(c_4)\,,\,T(a_5)\,,\,T(b_5)\,,\,T(c_5)\,,$$
$$T(a_6)\,,\,T(b_6)\,,\,T(c_6)\,,\,T(a_7)\,,\,T(d_7)\,,$$

see [5]. The triangulation of $\partial N(L_1)$, the boundary of a tubular neighbourhood of L_1 in S^3, induced by $\mathcal{S}$ is given by Figure 5, where the dotted edges should be contracted and the edges assigned the same number should be identified. Similarly, that of $\partial N(L_2)$ is given in Figure

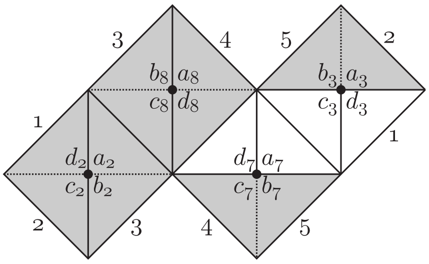


FIGURE 5.

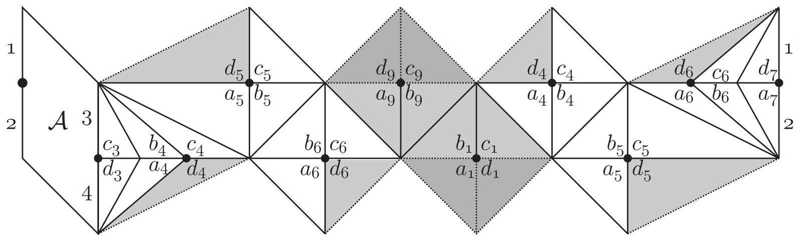


FIGURE 6.

6, where the triangulation of the annulus $\mathcal{A}$ is given in Figure 7. If $\mathcal{S}$ determines a hyperbolic structure of M, the product of the moduli around each edge in $\mathcal{S}$ should be 1. In fact, we can read

$$1 = a_4b_4c_4 = a_5b_5c_5 = a_6b_6c_6 \tag{1}$$

corresponsing to certain crossings of D,

$$1 = d_3a_4c_6d_7 = c_3b_4b_5b_6a_7 = c_4a_5 = c_5a_6 \tag{2}$$

corresponding to certain faces of D and

$$\frac{(1-1/d_3)(1-1/b_4)}{(1-c_3)(1-a_4)} = \frac{(1-1/b_6)(1-1/d_7)}{(1-c_6)(1-a_7)} = 1$$

corresponding to the non-alternating edges of D. Then, as explained in [4], the other equations should be generated by

$$\frac{1-a_7}{1-c_3} = \frac{1-1/d_7}{1-1/d_3} = t^2\,, \quad \frac{(1-1/b_6)(1-c_5)}{(1-a_6)(1-1/b_5)} = \frac{(1-1/b_5)(1-c_4)}{(1-a_5)(1-1/b_4)} = m^2\,,$$

where t, m denote the eigenvalues of the holonomy representations of the meridians of L_1, L_2, and

$$c_3d_3a_7d_7 = 1\,, \tag{3}$$

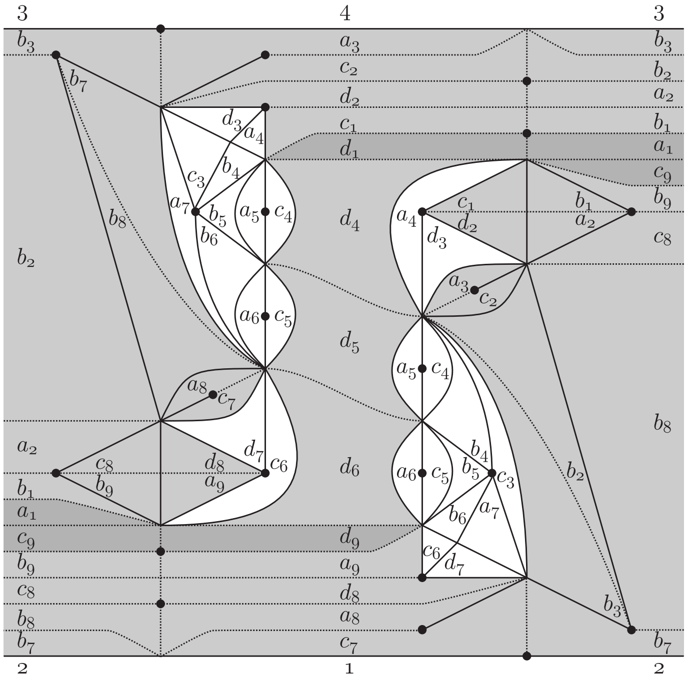


FIGURE 7.

where we have used

$$\frac{c_3 d_3 (1-b_6)(1-c_5)(1-c_4)(1-b_5)(1-a_6)(1-d_7)}{(1-1/a_7)(1-1/a_6)(1-1/a_5)(1-1/b_4)(1-1/c_5)(1-1/c_6)} = m^2 c_3 d_3 a_7 d_7$$

modulo the relations above. Furthermore, the left hand side of Figure 7 gives us a nice view of a fundamental domain of M in $\mathbf{H}^3$ from ∞ and we can read the eigenvalues s, l of the holonomy representations of the longitudes of L_1, L_2 as

$$s^2 = \frac{c_3 d_3 (1-1/d_3)(1-1/a_7)}{(1-c_3)(1-d_7)} = (c_3 d_3)^2 ,$$

which is the product of the moduli along a holizontal line in Figure 5, and $l^2 m^{10}$ is given by $m^2 P Q$, where

$$P = \frac{1}{c_3} \cdot \frac{1-c_3}{1-1/b_4} \cdot \frac{1}{b_4 c_4} \cdot \frac{(1-c_4)(1-b_5)}{(1-1/a_5)(1-1/b_6)} \cdot \frac{1}{b_6 c_6} \cdot (1-c_6)$$
$$\times \frac{1-b_4}{(1-1/a_4)(1-1/b_5)} \cdot \frac{1}{b_5 c_5} \cdot \frac{(1-c_5)(1-b_6)}{(1-1/a_6)(1-1/a_7)}$$

is the product of the moduli along a holizontal line in Figure 6. and

$$\begin{aligned} Q = {} & \frac{1-a_7}{1-1/c_3}\cdot\frac{1-c_3}{1-1/b_4}\cdot\frac{(1-b_4)(1-a_5)}{(1-1/c_4)(1-1/b_5)}\cdot\frac{(1-c_4)(1-b_5)}{(1-1/a_5)(1-1/b_6)} \\ & \times\frac{1-b_6}{1-1/a_7}\cdot R\cdot\frac{1-d_7}{1-1/c_6}\cdot(1-c_6)\cdot\frac{1-b_4}{(1-1/a_4)(1-1/b_5)}\cdot\frac{(1-b_5)(1-a_6)}{(1-1/b_6)(1-1/c_5)} \\ & \times\frac{(1-c_5)(1-b_6)}{(1-1/a_6)(1-1/a_7)}\cdot R\cdot(1-1/d_7)\cdot d_7c_6\cdot\frac{(1-1/a_6)(1-1/b_5)(1-1/c_6)}{(1-c_6)(1-c_5)(1-b_6)} \\ & \times c_6d_7\cdot\frac{1}{1-1/d_3}\cdot R \end{aligned}$$

with

$$R=\frac{(1-b_4)(1-a_5)(1-a_6)(1-b_5)(1-c_4)(1-d_3)}{(1-1/c_3)(1-1/c_4)(1-1/c_5)(1-1/b_6)(1-1/a_5)(1-1/a_4)}=\frac{c_3d_3}{m^2}$$

is the product of the moduli along the curve in $\mathcal{A}$ depicted in Figure 8(a). Note that

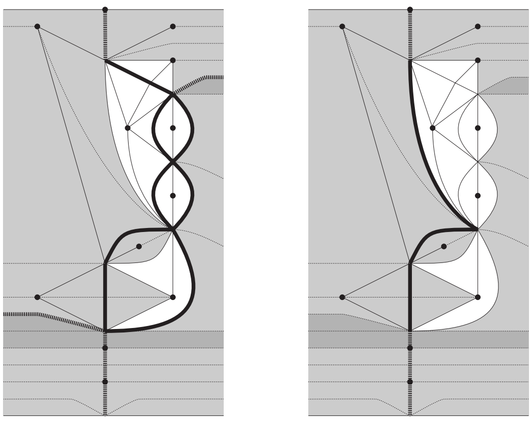
$$Q=P\cdot(c_3b_4c_4b_6c_6b_5c_5c_6d_7c_3d_3\cdot t^2\cdot m^{-6})^2$$

and m^2Q should be equal to the product of the moduli along the curve depicted in Figure 8(b), that is,

$$m^2Q=(c_6d_7c_3d_3\cdot t\cdot m^{-3})^2\,.$$

Therefore we have

$$l^2m^{10}=\left(\frac{c_6d_7c_3d_3}{c_3b_4c_4b_6c_6b_5c_5\cdot m}\right)^2.$$



(a) (b)

FIGURE 8.

Now, the equations (1), (2), (3) suggest putting

$$c_3 = a/x,\ d_3 = y/a,\ a_4 = a/m,\ b_4 = b/a,\ c_4 = m/b,\ a_5 = b/m,\ b_5 = c/b,$$
$$c_5 = m/c\,, a_6 = c/m,\ b_6 = d/c,\ c_6 = m/d,\ a_7 = x/d,\ d_7 = d/y\,.$$

Then the hyperbolicity equations for L are given by

$$\frac{(1-a/y)(1-a/b)}{(1-a/x)(1-a/m)} = \frac{(1-c/d)(1-y/d)}{(1-m/d)(1-x/d)} = 1\,,$$
$$\frac{(1-b/c)(1-m/b)}{(1-b/m)(1-a/b)} = \frac{(1-c/d)(1-m/c)}{(1-c/m)(1-b/c)} = m^2\,,$$
$$\frac{1-x/d}{1-a/x} = \frac{1-y/d}{1-a/y} = t^2, \quad lm^8 = -bc, \quad s = y/x\,.$$

Since the edges in $\mathcal{S}$ are nontrivial and M is hyperbolic, we have the following lemma.

LEMMA 1. *The moduli of the tetrahedra in $\mathcal{S}$ are in $\mathbf{C}\backslash\{0, 1\}$.*

From $\dfrac{(1-a/y)(1-a/b)}{(1-a/x)(1-a/m)} = 1$, we have

$$a\frac{\{a(-mx+by)+bmx-bmy-bxy+mxy\}}{by(a-m)(x-a)} = 0\,,$$

where $a, b, y, a-m, x-a \neq 0$ because of Lemma 1, and so we put

$$P_a = a(-mx+by)+bmx-bmy-bxy+mxy = 0\,.$$

Similarly we have

$$P_d = d(c-m-x+y)+mx-cy = 0\,,$$
$$P_b = b(1-cm)-c+acm = 0\,,$$
$$P_c = c(1-dm)-d+bdm = 0\,,$$
$$P_x = adt+x(d-dt)-x^2 = 0\,,$$
$$P_y = adt+y(d-dt)-y^2 = 0\,,$$
$$P_l = bc+lm^8 = 0\,, \quad P_s = sx-y = 0\,.$$

Suppose $-mx+by = 0$. Then from $P_a = 0$ we have

$$\frac{by(m-b)(y-m)}{m} = 0\,.$$

However $b, y, b-m, y-m \neq 0$ because of Lemma 1 and this is a contradiction. Thus we have $-mx+by \neq 0$ and hence

$$a = \frac{-bmx+bmy+bxy-mxy}{-mx+by}\,.$$

By substituting this equation for P_d, P_b, P_x and P_y, the variable a is eliminated. Similarly we can eliminate b, c, d, x, y and finally obtain the following two equations.

$$\begin{aligned} P_1 = &(-1-m^4-3lm^{10}+lm^{12}-l^2m^{12}+3l^2m^{14}+l^3m^{20}+l^3m^{24}) \\ &+(m^2-lm^6+2lm^8-2l^2m^{16}+l^2m^{18}-l^3m^{22})(s^{-1}+s) \\ &+(lm^8-lm^{10}+l^2m^{14}-l^2m^{16})(s^{-2}+s^2)=0\,, \\ P_2 = &(l^2m^{12}-2l^2m^{14}+l^2m^{16})(1+t^2s^7)+(2lm^6-2lm^8-2l^3m^{20}+2l^3m^{22})(1+t^2s^5)s \\ &+(1-lm^8+lm^{10}+l^2m^{12}-4l^2m^{14}+l^2m^{16}+l^3m^{18}-l^3m^{20}+l^4m^{28})(1+t^2s^3)s^2 \\ &+(-m^2+lm^6-lm^8+l^2m^{12}+l^2m^{16}-l^3m^{20}+l^3m^{22}-l^4m^{26})(1+t^2s)s^3 \\ &+(-lm^8-2l^2m^{14}-l^3m^{20})(1+t^2s^{-1})s^4=0\,. \end{aligned}$$

Then the A-polynomial $A_{K_n}(l,m)$ is obtained by eliminating s and t from P_1, P_2 and

$$t^2s^{-2n}=1\,. \tag{4}$$

Put $X=s+s^{-1}$ for simplicity. Then, P_1 becomes

$$P_1'=\alpha X^2+\beta X+\gamma\,,$$

where α, β and γ are given in Main Theorem, and P_2 becomes

$$\begin{aligned} f_n(X) = &(l^2m^{12}-2l^2m^{14}+l^2m^{16})a_{n+4}+(2lm^6-2lm^8-2l^3m^{20}+2l^3m^{22})a_{n+3} \\ &+(1-lm^8+lm^{10}+l^2m^{12}-4l^2m^{14}+l^2m^{16}+l^3m^{18}-l^3m^{20}+l^4m^{28})a_{n+2} \\ &+(-m^2+lm^6-lm^8+l^2m^{12}+l^2m^{16}-l^3m^{20}+l^3m^{22}-l^4m^{26})a_{n+1} \\ &+(-lm^8-2l^2m^{14}-l^3m^{20})a_n \end{aligned}$$

by using (4), where $a_n\in\mathbf{Z}[X]$ is defined by

$$a_n=Xa_{n-1}-a_{n-2}\,,\quad a_0=1\,,\quad a_1=1\,.$$

Then $f_n(X)$ obeys

$$f_n(X)=Xf_{n-1}(X)-f_{n-2}(X)\,. \tag{5}$$

Let X_1, X_2 be the solutions to $P_1'=0$ with respect to X. Then, the A-polynomial of K_n is a factor of

$$F_n=f_n(X_1)f_n(X_2)\,. \tag{6}$$

From now on, we evaluate a recursive formula for F_n. First of all, using (5), we can reduce (6) as

$$\begin{aligned} F_n &= \{X_1f_{n-1}(X_1)-f_{n-2}(X_1)\}\{X_2f_{n-1}(X_2)-f_{n-2}(X_2)\} \\ &= X_1X_2F_{n-1}+F_{n-2}-\{X_1f_{n-1}(X_1)f_{n-2}(X_2)+X_2f_{n-1}(X_2)f_{n-2}(X_1)\}\,, \end{aligned}$$

where

$$\begin{aligned} &X_1 f_{n-1}(X_1) f_{n-2}(X_2) + X_2 f_{n-1}(X_2) f_{n-2}(X_1) \\ &\quad = X_1\{X_1 f_{n-2}(X_1) - f_{n-3}(X_1)\} f_{n-2}(X_2) + X_2\{X_2 f_{n-2}(X_2) - f_{n-3}(X_2)\} f_{n-2}(X_1) \\ &\quad = (X_1^2 + X_2^2) F_{n-2} - \{X_1 f_{n-3}(X_1) f_{n-2}(X_2) + X_2 f_{n-3}(X_2) f_{n-2}(X_1)\} \end{aligned}$$

and

$$\begin{aligned} &X_1 f_{n-3}(X_1) f_{n-2}(X_2) + X_2 f_{n-3}(X_2) f_{n-2}(X_1) \\ &\quad = X_1 f_{n-3}(X_1)\{X_2 f_{n-3}(X_2) - f_{n-4}(X_2)\} + X_2 f_{n-3}(X_2)\{X_1 f_{n-3}(X_1) - f_{n-4}(X_1)\} \\ &\quad = 2X_1 X_2 F_{n-3} - \{X_1 f_{n-3}(X_1) f_{n-4}(X_2) + X_2 f_{n-3}(X_2) f_{n-4}(X_1)\}\,. \end{aligned}$$

Similarly we have

$$F_{n-2} = X_1 X_2 F_{n-3} + F_{n-4} - \{X_1 f_{n-3}(X_1) f_{n-4}(X_2) + X_2 f_{n-3}(X_2) f_{n-4}(X_1)\}$$

and so

$$F_n = X_1 X_2 F_{n-1} + (2 - X_1^2 - X_2^2) F_{n-2} + X_1 X_2 F_{n-3} - F_{n-4}\,.$$

From this equation, we obtain

$$\alpha^2 F_n - \alpha\gamma F_{n-1} - (2\alpha^2 + 2\alpha\gamma - \beta^2) F_{n-2} - \alpha\gamma F_{n-3} + \alpha^2 F_{n-4} = 0\,.$$

On the other hand, we can compute directly the initial conditions F_0, F_1, F_2 and F_3 from (6):

$$F_0 = -\frac{lm^8(1+lm^{10})^2}{(1+lm^6)^2} W\,, \quad F_1 = \frac{m^4(1-lm^8)(1+lm^8)^2}{(1-m^2)(1+lm^6)} W\,,$$

$$F_2 = -\frac{(1-lm^7)^2(1+lm^7)^2}{l(1-m^2)^2} W\,,$$

$$F_3 = \frac{1 - lm^4 + 2lm^6 - lm^8 - 2l^2m^{12} - l^2m^{14} + l^4m^{24} + 2l^4m^{26} + l^5m^{30} - 2l^5m^32 + l^5m^{34} - l^6m^{38}}{l^2m^4(1-m^2)^3} W\,,$$

where

$$W = (1+lm^4)^4(-1+m^2+lm^8)(-1-lm^6+lm^8)\,.$$

Note that $-1+m^2+lm^8$ and $-1-lm^6+lm^8$ are not a factor of $A_{K_n}(l,m)$ because the curve $(-1+m^2+lm^8)(-1-lm^6+lm^8) = 0$ does not pass through the points $(l,m) = (1,1), (1,-1), (-1,1)$ and $(-1,-1)$, see Section 2.8 in [1]. Then, we can complete the proof of the Main Theorem by the following two lemmas.

LEMMA 2. *None of* $1+lm^6,\ 1-m^2,\ 1+lm^4,\ 1-lm^8$ *are a factor of* $A_{K_n}(l,m)$.

PROOF. Otherwise, we have only finite points of (l, m) from $P_1' = 0$ and $f_n(X) = 0$. □

LEMMA 3. *For any integer* n, $\dfrac{B_n C_n}{W}$ *is a polynomial of* l, m.

PROOF. This is easily proved by induction on n. □

References

[1] D. COOPER, M. CULLER, H. GILLET, D. D. LONG and P. B. SHALEN, Plane curves associated to character varieties of 3-manifolds, Inventiones mathematicae **118** (1994), 47–84.

[2] D. COOPER and D. D. LONG, Remarks on the A-polynomial of a knot, Journal of Knot Theory and Its Ramifications. **5** (1996), 609–628.

[3] J. HOSTE and P. D. SHANAHAN, A formula for the A-polynomial of twist knots, preprint.

[4] Y. YOKOTA, From the Jones polynomial to the A-polynomial of hyperbolic knots, Interdisciplinary Information Sciences **9** (2003), 11–21.

[5] Y. YOKOTA, On the volume conjecture for hyperbolic knots, preprint, available from http://www.comp.metro-u.ac.jp/˜jojo.

[6] A. KAWAUCHI, *A survey of knot theory*, Birkhauser Verlag (1996).

[7] S. TILLMAN, Boundary slopes and the logarithmic limit set. preprint, available from arXiv:math.GT0306055.

Present Address:
DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY,
TOKYO, 192–0397 JAPAN.
e-mail: tamu@math.keio.ac.jp
jojo@math.metro-u.ac.jp