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Deforming Ricci Positive Metrics

David WRAITH

National University of Ireland, Maynooth

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Abstract. We find conditions under which a Ricci positive metric can be deformed in a tubular neighbourhood of some submanifold to agree with another given Ricci positive metric. We insist that this final metric has everywhere positive Ricci curvature.

1. Introduction.

This paper focuses on Riemannian manifolds with positive Ricci curvature. All metrics under consideration will be complete.

There are three commonly used measures of curvature for a Riemannian manifold: the sectional curvature, the Ricci curvature and the scalar curvature. The sectional curvature contains the most information. The Ricci curvature is an average of sectional curvatures and therefore contains less information. The scalar curvature is an average of Ricci curvatures, hence this is the weakest of the three measures of curvature.

Not surprisingly, the scalar curvature has proved the easiest to analyse. For example a classification has been given for simply-connected manifolds of dimension at least five which admit metrics of everywhere positive scalar curvature. (See [8] for more details.)

Much less is known about the Ricci curvature. Of particular interest are metrics with everywhere positive Ricci curvature. It is known [6] that every manifold of dimension at least three admits a metric of negative Ricci curvature. On the other hand positive Ricci curvature has strong topological implications for the manifold (for example the fundamental group must be finite). These implications are only partially understood.

Algebraically, the Ricci curvature is a smoothly varying quadratic form on the tangent bundle of the manifold. It is often convenient to work with the associated bilinear form. For the definition and details about the Ricci curvature, see [1].

One approach to a better understanding of the Ricci positive condition is to ask what kind of deformations one can make to a Ricci positive metric without introducing any non-positive curvature. Of course this is a very general question, but one might hope to say more in specific circumstances. For example, in [3] it was shown that a Ricci positive metric can be deformed in a neighbourhood of any point to a metric which has constant sectional curvature 1

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in some smaller region about that point, and which agrees with the original metric outside the neighbourhood of deformation. Moreover, this can be done so the final metric has everywhere positive Ricci curvature.

We must also mention the deformation result of Gao appearing in [2]. He showed that given two Ricci-negative metrics defined in (at least) a tubular neighbourhood of an embedded circle, one of the metrics can be deformed (within Ricci-negativity) to agree with the other metric in some smaller tubular neighbourhood, provided the 1-jets of the metric agree at all points on the circle. It is easily observed that this deformation result also holds for positive Ricci curvature, simply by reversing the inequality signs in his calculations.

The aim of this paper is to generalise Gao's deformation result. Instead of considering tubular neighbourhoods of embedded circles, we consider tubular neighbourhoods of arbitrary embedded submanifolds. Our main result (Theorem 1.10) asserts that the equality of 1-jets on the submanifold is again sufficient to perform the desired deformation.

Part of the motivation behind this paper concerns 'surgery'. Surgery is a technique developed to help with the diffeomorphism classification of manifolds. Understanding how metrics with positive scalar curvature behave under surgery (see for example [5]) led to the classification mentioned above. If a similar understanding could be developed for positive Ricci curvature, some kind of classification could be attempted for these manifolds too. Developing a surgery theorem for positive Ricci curvature would seem to be a very difficult proposal. However, some limited results do exist (see [7] and [10]) and these have had some nice applications (for example [9]). The problem with these Ricci-positive surgery results is that they make strong assumptions on the form the metric must take in a neighbourhood of the embedded sphere on which surgery is to be performed. If one could assert that an arbitrary Ricci positive metric could be deformed within Ricci positivity to some sort of standard form in a neighbourhood of where surgery is to take place, then there is a good chance that more powerful Ricci-positive surgery theorems could be developed. This paper could be viewed as a preliminary step in this direction.

2. Main results.

LEMMA 1.1. *Given Riemannian metrics g_0 and g_1 on a manifold X^n such that the* 1*-jet of g_0 is equal to the* 1*-jet of g_1 on an arbitrary submanifold Y^m, then at each point of Y we have*:

$$Ric(sg_0 + (1-s)g_1) = sRic(g_0) + (1-s)Ric(g_1)$$

for all $s \in [0, 1]$.

PROOF. Elementary calculation.

COROLLARY 1.2. *If g_0 and g_1 have positive Ricci curvature and Y is compact, then there exists an ε-tubular neighbourhood of Y for some $\varepsilon > 0$, in which $sg_0 + (1-s)g_1$ has positive Ricci curvature for all $s \in [0, 1]$.*

The calculations in this paper will be performed relative to *Fermi* coordinate systems. Fermi coordinate systems are defined on tubular neighbourhoods of submanifolds in the following way. Consider any coordinate patch on the submanifold. Above this patch, choose a collection of smooth linearly independent vector fields which span the normal fibres. (The vector fields we choose will always be orthonormal with respect to the ambient metric.)

Let $(y_1, \cdots, y_m)$ be the chosen local coordinates on the submanifold. Let $E_1, \cdots, E_{n-m}$ be the vector fields above this coordinate patch on Y. The exponential map sends a tubular neighbourhood of the normal bundle's zero-section diffeomorphically onto a tubular neighbourhood of the submanifold. Thus any point within the latter tubular neighbourhood is the image of some vector $\sum_{i=1}^{n-m} \lambda_i E_i$ originating at a point $(y_1, \cdots, y_m)$. We can thus represent the point by $(\lambda_1, \cdots, \lambda_{n-m}, y_1, \cdots, y_m)$ and in this way define a coordinate system in the tubular neighbourhood. Note that in the special case where the submanifold is a point, Fermi coordinate systems are just normal coordinate systems. For more details, see [4].

Our next task is to give some estimates relating to the collection of metrics $sg_0+(1-s)g_1$, with $s \in [0, 1]$. We do this in Lemmas 1.3 and 1.6. We will use these estimates to bound the Ricci curvature of these metrics from below. Lemma 1.4 and Corollary 1.5 are generalisations of Taylor's theorem which we will need in proving Lemma 1.6.

Given g_0, g_1, X and Y as in Lemma 1.1, let

$$g(\mathbf{x}, \mathbf{y}, s) = sg_0(\mathbf{x}, \mathbf{y}) + (1-s)g_1(\mathbf{x}, \mathbf{y})$$

where $\mathbf{y} = (y_1, \cdots, y_m)$ is the submanifold coordinate and $\mathbf{x} = (x_1, \cdots, x_{n-m})$ is the normal coordinate with respect to a Fermi coordinate system. (Of course, we always have such a system sufficiently close to Y.)

Let g_{ij} denote the components of the metric g. We denote $\frac{\partial g}{\partial s}$ by $\dot{g}$ and $\frac{\partial^2 g}{\partial s^2}$ by $\ddot{g}$.

LEMMA 1.3. *On Y we have for all $s \in [0, 1]$:*

$$\dot{g}_{ij}(\mathbf{0}, \mathbf{y}, s) = 0 \qquad (1) \qquad\qquad \dot{g}^{ij}(\mathbf{0}, \mathbf{y}, s) = 0 \qquad (2)$$

$$\frac{\partial \dot{g}_{ij}}{\partial y_k}(\mathbf{0}, \mathbf{y}, s) = 0 \qquad (3) \qquad\qquad \frac{\partial \dot{g}_{ij}}{\partial x_k}(\mathbf{0}, \mathbf{y}, s) = 0 \qquad (4)$$

$$\frac{\partial \dot{g}^{ij}}{\partial y_k}(\mathbf{0}, \mathbf{y}, s) = 0 \qquad (5) \qquad\qquad \frac{\partial \dot{g}^{ij}}{\partial x_k}(\mathbf{0}, \mathbf{y}, s) = 0 \qquad (6)$$

$$\ddot{g}_{ij}(\mathbf{0}, \mathbf{y}, s) = 0 \qquad (7) \qquad\qquad \frac{\partial \ddot{g}_{ij}}{\partial y_k}(\mathbf{0}, \mathbf{y}, s) = 0 \qquad (8)$$

$$\frac{\partial \ddot{g}_{ij}}{\partial x_k}(\mathbf{0}, \mathbf{y}, s) = 0 \qquad (9) \qquad\qquad \dot{\Gamma}_{ij}^k(\mathbf{0}, \mathbf{y}, s) = 0 \qquad (10)$$

PROOF. As the 0-jets of g_0 and g_1 are equal on Y, the metric g must be constant with respect to s on Y. (1) and (2) follow immediately.

Now notice that $\frac{\partial \dot{g}_{ij}}{\partial y_k}(\mathbf{0}, \mathbf{y}, s) = \frac{\partial}{\partial s}\frac{\partial g_{ij}}{\partial y_k}(\mathbf{0}, \mathbf{y}, s)$. As the 1-jets of g_0 and g_1 are equal on Y, this means $\frac{\partial g_{ij}}{\partial y_k}$ must be constant with respect to s. Hence $\frac{\partial}{\partial s}\frac{\partial g_{ij}}{\partial y_k} = \frac{\partial \dot{g}_{ij}}{\partial y_k} = 0$ on Y. This and analogous arguments establish (3) and (4).

The 1-jet of the matrix for g determines the 1-jet of the inverse matrix. Explicitly, if G is the matrix of g and D represents differentiation in any chosen direction, a simple calculation yields:

$$D(G^{-1}) = -G^{-1}(DG)G^{-1}$$

(5) and (6) now follow from the fact that the entries of both G and DG are constant with respect to s on Y.

(7), (8) and (9) are immediate consequences of (1), (3) and (4) respectively.

To see (10), note that

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{n} g^{kl}(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij})$$

and therefore the Christoffel symbols only depend on the 1-jet of g, that is, on the 1-jets of g_0 and g_1. Hence Γ^k_{ij} is constant with respect to s on Y and the result follows.

LEMMA 1.4. *Regard* $\mathbf{R}^n$ *as* $\mathbf{R}^{n-m} \times \mathbf{R}^m$ *with coordinates* $(\mathbf{x}, \mathbf{y})$. *Let* $f : \mathbf{R}^n \to \mathbf{R}$ *be smooth and suppose* $f(\mathbf{0}, \mathbf{y}) = 0$ *and* $\nabla f(\mathbf{0}, \mathbf{y}) = 0$ *for all* $\mathbf{y} \in \mathbf{R}^m$. *Let* B *be any compact domain in* $\mathbf{R}^n$ *which intersects* $\{\mathbf{0}\} \times \mathbf{R}^m \subset \mathbf{R}^n$. *Then there exists* $M > 0$ *such that* $|f(\mathbf{x}, \mathbf{y})| \leq M|\mathbf{x}|^2$ *for all* $(\mathbf{x}, \mathbf{y}) \in B$.

PROOF. We estimate $f(\mathbf{x}, \mathbf{y})$ by using a Taylor expansion about $(\mathbf{0}, \mathbf{b})$.

Taylor's Theorem gives:

$$f(\mathbf{x}, \mathbf{y}) = f(\mathbf{0}, \mathbf{b}) + \sum_{i=1}^{m}(y_i - b_i) f_{y_i}(\mathbf{0}, \mathbf{b}) + \sum_{j=1}^{n-m} x_j f_{x_j}(\mathbf{0}, \mathbf{b}) + E(\mathbf{x}, \mathbf{y})$$

where

$$E(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\Bigg[\sum_{i=1}^{m}\sum_{j=1}^{m}(y_i - b_i)(y_j - b_j)\frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{m}\sum_{j=1}^{n-m}(y_i - b_i)x_j\frac{\partial^2}{\partial y_i \partial x_j}$$
$$+ \sum_{i=1}^{n-m}\sum_{j=1}^{n-m} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j}\Bigg] f|_{(c\mathbf{x}, \mathbf{b}+c(\mathbf{y}-\mathbf{b}))}$$

for some $c \in [0, 1]$, depending on $\mathbf{x}$ and $\mathbf{y}$.

Since $f(\mathbf{0}, \mathbf{y}) = 0$ and $\nabla f(\mathbf{0}, \mathbf{y}) = \mathbf{0}$ for all $\mathbf{y} \in B \cap (\{\mathbf{0}\} \times \mathbf{R}^m)$ we have

$$f(\mathbf{x}, \mathbf{y}) = E(\mathbf{x}, \mathbf{y})\,.$$

When $\mathbf{x} = \mathbf{a}$ and $\mathbf{y} = \mathbf{b}$, $E(\mathbf{x}, \mathbf{y})$ collapses to

$$\frac{1}{2}\sum_{i=1}^{n-m}\sum_{j=1}^{n-m} a_i a_j f_{x_i x_j}(c\mathbf{a}, \mathbf{b})$$

Let

$$M^b_{ij} = \frac{1}{2}\max_{c\in[0,1]}\max_{(\mathbf{x},\mathbf{b})\in B}|f_{x_i x_j}(c\mathbf{x}, \mathbf{b})|$$

Define

$$M_0^b = \max_{i,j} M_{ij}^b$$

We have

$$\begin{aligned}|f(\mathbf{a},\mathbf{b})| = |E(\mathbf{a},\mathbf{b})| &\le M_0^b\Big(\sum_{i,j}|a_i a_j|\Big)\\ &\le 2(n-m)M_0^b|\mathbf{a}|^2\end{aligned}$$

where we have used the (easily established) numerical inequality

$$\sum_{i=1}^{s}\sum_{j=1}^{s} n_i n_j \le 2s\sum_{i=1}^{s} n_i^2 \tag{$*$}$$

which holds for any collection of positive numbers $n_1, \cdots, n_s$.

Finally, let

$$M = 2(n-m)\max_{\{\mathbf{b}\in\mathbf{R}^m : \exists(\mathbf{x},\mathbf{b})\in B\}} M_0^b$$

and the result follows.

REMARK. The only property of B that we use is its compactness. In particular it is not necessary to assume that either $(\mathbf{0},\mathbf{b})$ or $(c\mathbf{x}, \mathbf{b}+c(\mathbf{y}-\mathbf{b}))$ are points in B.

COROLLARY 1.5. *In the situation described by Lemma* 1.4, *if we remove the condition on* ∇f, *then there exists* $M > 0$ *such that* $|f(\mathbf{x},\mathbf{y})| \le M|\mathbf{x}|$ *for all* $(\mathbf{x},\mathbf{y}) \in B$.

PROOF. The proof is almost identical to that of Lemma 1.4. This time, however, we use the Taylor formula:

$$f(\mathbf{x},\mathbf{y}) = f(\mathbf{0},\mathbf{b}) + E'(\mathbf{x},\mathbf{y})$$

where $E'(\mathbf{x},\mathbf{y})$ is the error term:

$$E'(\mathbf{x},\mathbf{y}) = \Big(\sum_{i=1}^{m}(y_i - b_i)\frac{\partial}{\partial y_i} + \sum_{j=1}^{n-m} x_j\frac{\partial}{\partial x_j}\Big)f|_{(c\mathbf{x},\mathbf{b}+c(\mathbf{y}-\mathbf{b}))}$$

for some $c \in [0,1]$. Using the fact that $\frac{\partial f}{\partial y_i} = 0$ on $\{\mathbf{0}\}\times\mathbf{R}^m$ for all i, the error term for $f(\mathbf{a},\mathbf{b})$ again simplifies, and the inequality is obtained by taking square roots of both sides of $(*)$.

From now on, g_1 will be a Ricci positive metric on X, g_0 will be a Ricci positive metric defined (at least) in a neighbourhood of Y, and $g(s) = sg_0 + (1-s)g_1$ will therefore only be defined where g_0 is. $E_1, \cdots, E_{m-n}$ will denote linearly independent sections of $\nu(Y)$ which are orthonormal with respect to g_1.

Choose and fix a local coordinate patch $(y_1, \cdots, y_m)$ on Y and extend to a system of Fermi coordinates $(x_1, \cdots, x_{m-n}, y_1, \cdots, y_m)$ in some neighbourhood N of Y using $\{E_i\}$. Let $B \subset N$ be any compact domain of X which intersects Y.

LEMMA 1.6. *Measuring all vectors with respect to g_1, there exists $M > 0$ such that for all $(\mathbf{x}, \mathbf{y}) \in B$ and for all $s \in [0, 1]$:*

$$|g_{ij}(\mathbf{x}, \mathbf{y}, s)| \le M \quad (11) \qquad |g^{ij}(\mathbf{x}, \mathbf{y}, s)| \le M \quad (12)$$

$$|\dot{g}_{ij}(\mathbf{x}, \mathbf{y}, s)| \le M|\mathbf{x}|^2 \quad (13) \qquad |\dot{g}^{ij}(\mathbf{x}, \mathbf{y}, s)| \le M|\mathbf{x}|^2 \quad (14)$$

$$|\Gamma^k_{ij}(\mathbf{x}, \mathbf{y}, s)| \le M \quad (15) \qquad |\dot{\Gamma}^k_{ij}(\mathbf{x}, \mathbf{y}, s)| \le M|\mathbf{x}| \quad (16)$$

$$|\ddot{g}_{ij}(\mathbf{x}, \mathbf{y}, s)| \le M|\mathbf{x}|^2 \quad (17) \qquad \left| \frac{\partial g^{ij}}{\partial y_k}(\mathbf{x}, \mathbf{y}, s) \right| \le M \quad (18)$$

$$\left| \frac{\partial g^{ij}}{\partial x_k}(\mathbf{x}, \mathbf{y}, s) \right| \le M \quad (19) \qquad \left| \frac{\partial \dot{g}_{ij}}{\partial y_k}(\mathbf{x}, \mathbf{y}, s) \right| \le M|\mathbf{x}| \quad (20)$$

$$\left| \frac{\partial \dot{g}_{ij}}{\partial x_k}(\mathbf{x}, \mathbf{y}, s) \right| \le M|\mathbf{x}| \quad (21)$$

PROOF. (11), (12), (15), (18) and (19) follow from the compactness of B.
(13) follows from (1), (3) and (4) using Lemma 1.4.
(14) follows from (2), (5) and (6) using Lemma 1.4.
(16) follows from (10) using Corollary 1.5.
(17) follows from (7), (8) and (9) using Lemma 1.4.
(20) follows from (3) using Corollary 1.5.
(21) follows from (4) using Corollary 1.5.

REMARK. M depends on the choice of B.

We saw in Corollary 1.2 that there exists a neighbourhood of Y such that $Ric(g(s)) > 0$ at all points within the neighbourhood and for all $s \in [0, 1]$. In particular there exists $\varepsilon > 0$ such that $Ric(g(s)) > 0$ at all points of $N_\varepsilon(Y)$-the closed ε-tubular neighbourhood of Y measured with respect to g_1. (We assume that ε is sufficiently small so that we can cover $N_\varepsilon(Y)$ by Fermi coordinate patches.)

Our next task is to define a Ricci positive metric $\bar{g}$ on $N_\varepsilon(Y)$ which agrees with g_1 near the boundary of $N_\varepsilon(Y)$ and with g_0 near Y.

Let $f : \mathbf{R} \to \mathbf{R}$ be any smooth function satisfying:

$$f(r) = \begin{cases} 1 & r \le 1 \\ 0 & r \ge 2 \end{cases}$$

and $0 \le f \le 1$.

Define $\psi : N_\varepsilon(Y) \to \mathbf{R}$ by

$$\psi(x_1, \cdots, x_{m-n}, y_1, \cdots, y_m) = f\left(\frac{|\mathbf{x}|^\lambda}{\sqrt{\lambda}}\right)$$

for some fixed positive constant λ which we will determine later.

Notice that even though the defining equation uses coordinates for a particular patch, it is clear that ψ is well-defined on patch overlaps, hence:

1) ψ is well-defined throughout $N_\varepsilon(Y)$, and

2) ψ is smooth.

We are now in a position to define $\bar{g}$:

$$\bar{g}(\mathbf{x}, \mathbf{y}) = \psi(\mathbf{x}, \mathbf{y})g_0(\mathbf{x}, \mathbf{y}) + (1 - \psi(\mathbf{x}, \mathbf{y}))g_1(\mathbf{x}, \mathbf{y})\,.$$

The smoothness of ψ ensures the smoothness of $\bar{g}$.

Now note that as g_1 is defined globally on X, we can extend $\bar{g}$ to a metric on X in the obvious way.

It is clear that $\bar{g}$ agrees with g_0 sufficiently close to Y. It is also clear that $\bar{g}$ will agree with g_1 at a sufficient distance from Y. We need to ensure, however, that $\bar{g} = g_1$ in a neighbourhood of the boundary of $N_\varepsilon(Y)$.

We control $\bar{g}$ by altering λ. Explicitly: $\bar{g} = g_1$ when $|\mathbf{x}|^\lambda/\sqrt{\lambda} \geq 2$, that is, when $|\mathbf{x}| \geq (2\sqrt{\lambda})^{1/\lambda}$, so we must have $\varepsilon \geq (2\sqrt{\lambda})^{1/\lambda}$.

LEMMA 1.7. *For any $\varepsilon > 0$ we can choose λ such that*

$$\varepsilon \geq (2\sqrt{\lambda})^{\frac{1}{\lambda}}\,.$$

PROOF. Clearly $\lim_{\lambda\to 0}(2\sqrt{\lambda})^{1/\lambda} = \lim_{n\to\infty}(2/\sqrt{n})^n$.

For $n > 16$, $2/\sqrt{n} < 1/2$, and so

$$0 < \left(\frac{2}{\sqrt{n}}\right)^n < \left(\frac{1}{2}\right)^n.$$

Therefore

$$0 \leq \lim_{\lambda\to 0}(2\sqrt{\lambda})^{\frac{1}{\lambda}} = \lim_{n\to\infty}\left(\frac{2}{\sqrt{n}}\right)^n \leq \lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0\,.$$

Hence $\lim_{\lambda\to 0}(2\sqrt{\lambda})^{1/\lambda} = 0$.

Let $T > 0$ be such that $|f'(r)| \leq T$ and $|f''(r)| \leq T$ for all $r \in \mathbf{R}$.

LEMMA 1.8. *At any point $(\mathbf{x}, \mathbf{y})$ in $N_\varepsilon(Y)$ we have*:

(i) $$|\nabla\psi(\mathbf{x}, \mathbf{y})| \leq \sqrt{\lambda}T|\mathbf{x}|^{\lambda-1}$$

(ii) $$|\partial^2_{ij}\psi(\mathbf{x}, \mathbf{y})| \leq \lambda T|\mathbf{x}|^{2\lambda-2} + 3\sqrt{\lambda}T|\mathbf{x}|^{\lambda-2}$$

where ∂ denotes partial differentiation and the indices i and j range over all possible directions.

PROOF. Elementary calculation.

It remains to show that we can choose a value for λ so that $\bar{g}$ has positive Ricci curvature on $N_\varepsilon(Y)$ (and therefore over all X). We calculate the Ricci curvature locally, (with respect to a Fermi coordinate system).

LEMMA 1.9. *Consider vectors u_1 and u_2 at a point $(\mathbf{x}, \mathbf{y}) \in N_\varepsilon(Y)$ with unit g_1-length. Putting $s = \psi(\mathbf{x}, \mathbf{y})$ we have*:

$$Ric(\bar{g})(u_1, u_2) = Ric(sg_0 + (1 - s)g_1)(u_1, u_2) + A(u_1, u_2)$$

where

$$|A(u_1, u_2)| \leq C_1 \lambda^{\frac{1}{2}} |\mathbf{x}|^{\lambda}$$

and

$$C_1 = 2nMT + n^2 MT(6MT + 6M + 4) + 6n^3 M^3 T + \frac{9}{2} n^4 M^4 T^2 .$$

PROOF. Because of the way we have set things up, our calculations are identical to those of [2, Proposition 2.1], and we omit the details. Explicitly, $A(u_1, u_2)$ is a very complicated expression in the metric components, the inverse metric components, the Christoffel symbols, the function ψ and the derivatives of all these quantities. The estimates of Lemma 1.6 can then be used to produce the inequality above.

We are now in a position to state our main result:

THEOREM 1.10. *Let X^n be a manifold and Y^m an embedded compact submanifold. Let g_1 be a Ricci positive metric on X and g_0 a Ricci positive metric defined in (at least) a neighbourhood of Y. If the 1-jets of g_0 and g_1 are equal at every point in Y, then there exists a Ricci positive metric $\bar{g}$ on X and numbers ε, ε' with $0 < \varepsilon' < \varepsilon/2$, such that $\bar{g}|_{N_\varepsilon(Y) - N_{\varepsilon-\varepsilon'}(Y)}$ agrees with g_1 and $\bar{g}|_{N_{\varepsilon'}(Y)}$ agrees with g_0.*

PROOF. Lemma 1.7 shows we can choose λ as small as we wish. Lemma 1.9 shows that the maximum possible value of $|A(u_1, u_2)|$ on $N_\varepsilon(Y)$ is $C_1 \lambda^{1/2} \varepsilon^{\lambda}$. Now $N_\varepsilon(Y)$ is closed and therefore compact as a consequence of the compactness of Y. This means there exists $\delta > 0$ such that $Ric(sg_0 + (1-s)g_1)(u, u) > \delta$ for all $s \in [0, 1]$ and g_1-unit vectors u tangent to $N_\varepsilon(Y)$. Lemma 1.9 then says that by choosing λ sufficiently small we can guarantee $Ric(\bar{g}) > 0$ on $N_\varepsilon(Y)$, and therefore throughout X.

REMARK. It can be shown that Theorem 1.10 remains true if equality of 1-jets is replaced by equality of 0-jets, provided $Ric(sg_0 + (1-s)g_1) > 0$ at all points of Y and for all $s \in [0, 1]$.

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Present Address:
DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF IRELAND,
MAYNOOTH, CO. KILDARE, IRELAND.