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A Weighted Inequality for the Kakeya Maximal Operator with a Special Base

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Abstract. In this paper we shall give a weighted version of Igari's estimate on the Kakeya maximal operator with a special base.

1. Introduction and theorems.

Let $M_{a,N}$ be the Kakeya maximal operator in d-dimensional Euclidean space with the base $\mathcal{B}_{a,N}$, which is the set of all cylinders with the side length Na and the bottom of diameter a. Recently, S. Igari proved in [Ig] that if we restrict the base $\mathcal{B}_{a,N}$ to cylinders of which axes intersect a fixed line, then Córdoba's conjecture is true for general functions. In this note we shall prove a weighted version of this restricted maximal operator by using Igari's approach and ideas coming from [MS]. As in the unweighted case (see [Ig]) our result implies, as a corollary, the weighted estimate for $M_{a,N}$ on functions of radial type (the unweighted version was proved in [CHS]). We shall recall the definitions.

Fix $N \gg 1$. For a real number $a > 0$ let $\mathcal{B}_{a,N}$ be the family of all cylinders in the d-dimensional Euclidean space $\mathbf{R}^d$, $d \geq 2$, which are congruent to

$$\left\{ x = (x_1, \cdots, x_d) \in \mathbf{R}^d \;\middle|\; |x_1| < \frac{Na}{2},\ (x_2^2 + \cdots + x_d^2)^{1/2} < \frac{a}{2} \right\}$$

but with arbitrary direction and center. The so-called small Kakeya maximal operator $M_{a,N}$ is defined on locally integrable functions f on $\mathbf{R}^d$ by

$$(M_{a,N} f)(x) = \sup_{x \in R \in \mathcal{B}_{a,N}} \frac{1}{|R|} \int_R |f(y)| dy,$$

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where $|A|$ represents the Lebesgue measure of a set A. We define the Kakeya maximal operator K_N by putting

$$(K_N f)(x) = \sup_{a>0}(M_{a,N} f)(x)\,.$$

A weight w is a positive locally integrable function on $\mathbf{R}^d$ and we will represent the norm of the function space $L^p(\mathbf{R}^d, w)$ as

$$\|f\|_{L^p(\mathbf{R}^d,w)} = \left(\int_{\mathbf{R}^d} |f(x)|^p w(x)dx\right)^{1/p}\,.$$

For $w \equiv 1$ this norm is written simply as $\|f\|_p$.

If $d = 2$, then for $f \in L^p(\mathbf{R}^d, K_N w)$ the weighted inequality

$$\|K_N f\|_{L^p(\mathbf{R}^d,w)} \le C_{N,p}\|f\|_{L^p(\mathbf{R}^d,K_N w)} \tag{1}$$

holds with

$$C_{N,p} = \begin{cases} O(N^{d/p-1}(\log N)^{\alpha_p}), & 1 < p < d\,, \\ O((\log N)^{\alpha_p}), & d \le p < \infty\,, \end{cases}$$

for some constant $\alpha_p > 0$ (Müller and Soria [MS]). For $d \ge 3$ this inequality is known to be true only for the range $1 < p \le (d+1)/2$. (Vargas [Va].)

Hereafter notations partly follow those in [Ig].

For $R \in \mathcal{B}_{a,N}$ let $l(R)$ be the axis of R. Let L be a line in $\mathbf{R}^d$. We denote by $\mathcal{B}^L_{a,N,x}$ the family of $R \in \mathcal{B}_{2a,N}$ which has center at x and whose axis $l(R)$ intersects L. Put

$$(M^L_{a,N} f)(x) = \sup_{R\in\mathcal{B}^L_{a,N,x}} \frac{1}{|R|}\int_R |f(y)|dy$$

$$(K^L_N f)(x) = \sup_{a>0}(M^L_{a,N} f)(x)\,.$$

Then for $d \ge 3$ (1) holds good for K^L_N. Namely, we have the following

THEOREM 1. *Let $d \ge 3$. Let L be any line in $\mathbf{R}^d$. Then for every weight w on $\mathbf{R}^d$ we have the inequality*

$$\|K^L_N f\|_{L^p(\mathbf{R}^d,w)} \le C_{N,p}\|f\|_{L^p(\mathbf{R}^d,K_N w)} \tag{2}$$

such that for every f in $L^p(\mathbf{R}^d, K_N w)$, where $C_{N,p}$ is a constant independent of L with

$$C_{N,p} = \begin{cases} O(N^{d/p-1}(\log N)^{(d+2)/d}), & 1 < p < d\,, \\ O((\log N)^{(d+3)/d}), & d \le p < \infty\,. \end{cases}$$

Igari showed in [Ig] that

$$\|M^L_{a,N} f\|_d \le C(\log N)^{(d+1)/d}\|f\|_d\,. \tag{3}$$

We remark that the unweighted version ($w \equiv 1$) of (2) can be derived from (3) and the arguments we will use in Section 3, but without arguments in Section 2.

REMARK 2. Let L be the x_d-axis. If f is a radial function, then it can be seen that $(K_N^L f)(x) = (K_N f)(x)$ (See [Ig, Remark 2.1]). Therefore, (2) contains as a special case a weighted inequality for K_N on functions of radial type.

Theorem 1 follows from Theorem 3 by the sieve arguments and by the three-point lemma (see Section 3).

THEOREM 3. *Let $d \geq 3$. Let L be a line in $\mathbf{R}^d$. Then for every weight w on $\mathbf{R}^d$ there exists a constant C such that*

$$(w(\{x \in \mathbf{R}^d \mid (M_{a,N}^L f)(x) > \lambda\}))^{1/d} \leq C \frac{(\log N)^{(d+1)/d}}{\lambda} \|f\|_{L^d(\mathbf{R}^d, K_N w)}$$

holds for every f in $L^d(\mathbf{R}^d, K_N w)$ and for every $\lambda > 0$. Here $w(A)$ will denote the measure of the set A with respect to $w(x)dx$.

In the following C's will denote constants independent of f, N and L. It will be different in each occasion.

2. Proof of Theorem 3.

Fix $\lambda > 0$. We may assume that $f \geq 0$ and N is a positive integer. By translation and rotation we may assume that L is the x_d-axis (See [Ig, Proposition 2.1]). By dilation invariance it suffices to consider only the case $a = 1$. We write $M_{1,N}^L$ as M_N^L. We will linearize the problem first. We divide $\mathbf{R}^d$ into open unit cubes Q_p (and their boundaries) which have center at lattice points $p \in \mathbf{Z}^d$ and whose sides are parallel to the axes. By the local integrability of f we can find for every cube Q_p a point $p' \in Q_p$ and $R_p \in \mathcal{B}_{1,N,p'}^L$ such that

$$(M_N^L f)(x) \leq \frac{C}{N} \int_{R_p} f(y)dy, \quad \forall x \in Q_p.$$

Put

$$(Sf)(x) = \sum_{p \in \mathbf{Z}^d} \frac{1}{N} \int_{R_p} f(y)dy \cdot \chi_{Q_p}(x).$$

Then it suffices for proving Theorem 3 to estimate the measure $w(\{x \in \mathbf{R}^d \mid (Sf)(x) > \lambda\})$.

First of all we note that

$$w(\{x \in \mathbf{R}^d \mid (Sf)(x) > \lambda\}) = \sum_{p \in \{p \in \mathbf{Z}^d \mid (1/N) \int_{R_p} f > \lambda\}} w(Q_p).$$

2.1. Notations. In the proof we will use the following notations. A denotes the set of all lattice points p such that

$$\frac{1}{N} \int_{R_p} f(y)dy > \lambda.$$

Let $N_1 = \left[\frac{\log 2N}{\log 2}\right] + 1$ and $N_2 = \left[\frac{\log 3N}{\log 2}\right] + 1$. Then put

$$D_l = \begin{cases} \{x \in \mathbf{R}^d \mid (x_1^2 + \cdots + x_{d-1}^2)^{1/2} < 1\}, & l = 0, \\ \{x \in \mathbf{R}^d \mid 2^{l-1} \le (x_1^2 + \cdots + x_{d-1}^2)^{1/2} < 2^l\}, & 1 \le l \le N_2, \end{cases}$$

and

$$A_k = \begin{cases} A \cap D_k, & 0 \le k \le N_1, \\ A - (\bigcup_{l=0}^{N_1} A_l), & k = \infty \end{cases}$$

for $l = 0, 1, \cdots, N_2$ and $k = 0, 1, \cdots, N_1, \infty$.

Let $P(p)$ be the plain spanned by $l(R_p)$ and x_d-axis, and

$$\tilde{P}(p) = \{q \in \mathbf{Z}^d \mid \mathrm{dist}(q, P(p)) \le 3\sqrt{d}\}$$

be the $3\sqrt{d}$-neighborfood of $P(p)$.

2.2. Preliminary propositions. The following Proposition 4 was proved in p. 472 of [MS]. Here we shall present another proof by using Remark 10 of [Ta2].

For $a > 0$ let $\mathcal{B}_{a,\le N}$ denote the class of all rectangles in $\mathbf{R}^d$ which satisfy

$$a \le (\text{the length of shortest sides}) \le (\text{the length of longest sides}) \le Na .$$

the corresponding maximal operator associated to this base is defined by $M_{a,\le N}$.

PROPOSITION 4. *Let $d = 2$. Let I_2 be the set of lattice points* $([0, 3N-1] \times [-1, N]) \cap \mathbf{Z}^2$. *Suppose that $z \in [-1/2, 3n - 1/2] \times [-1, 1]$, then for every weight w on $\mathbf{R}^2$ we have*

$$\sum_{q \in I_2} \frac{w(Q_q)}{|q_2| + 1} \le CN \log N \sup_{\alpha \in [3/N, 3\sqrt{2}]} (M_{\alpha,N} w)(z) ,$$

where $q = (q_1, q_2)$.

PROOF. We shall use the same methods as in the proof of Proposition 4 of [Ta2]. Let the sequence $\{a(j)\}$ be

$$a(j) = \begin{cases} 1, & j = -1, 0, 1, \\ \dfrac{1}{j}, & j = 2, 3, \cdots, N, \\ 1, & j = N + 1, \\ 0, & j > N + 1. \end{cases}$$

We note that $a(j) = \sum_{i \geq j} a(i)a(i+1)$ for $1 \leq j \leq N$. Then it follows from this equality that

$$\begin{aligned}\sum_{q \in I_2} \frac{w(Q_q)}{|q_2|+1} &\leq \sum_{q_2=-1}^{N} a(q_2) \sum_{q_1=0}^{3N-1} w(Q_q) \\ &= \sum_{q_2=-1}^{N} \sum_{p \geq \max(q_2,1)} a(p)a(p+1) \sum_{q_1=0}^{3N-1} w(Q_q) \\ &= \sum_{p=1}^{N} a(p+1) \frac{p+2}{p} \frac{3N}{3N(p+2)} \sum_{q_2=-1}^{p} \sum_{q_1=0}^{3N-1} w(Q_q) \\ &\leq CN \left(\sum_{p=1}^{N} a(p+1) \right) (M_{3,\leq N} w)(z) \leq CN \log N (M_{3,\leq N} w)(z) .\end{aligned}$$

Therefore, by Remark 10 of [Ta2] we obtain

$$\sum_{q \in I_2} \frac{w(Q_q)}{|q_2|+1} \leq CN \log N \sup_{\alpha \in [3/N, 3\sqrt{2}]} (M_{\alpha,N} w)(z) . \quad \square$$

In the proof of Propositions 7 and 9 we will use the following

PROPOSITION 5. *Let $d \geq 3$. Let I_d be the set of lattice points* $([0, 3N-1] \times [0, 3\sqrt{d}]^{d-2} \times [-1, N]) \cap \mathbf{Z}^d$. *Suppose that* $z = (z_1, \cdots, z_d) \in [-1/2, 3N-1/2] \times [-1, 1]^{d-1}$. *Then for every weight w on $\mathbf{R}^d$ we have*

$$\sum_{q \in I_d} \frac{w(Q_q)}{|q_d|+1} \leq CN \log N (K_N w)(z) .$$

PROOF. Let $\delta = [3\sqrt{d}]$. By the local integrability of w and by Fubini's theorem we can define a locally integrable function $\tilde{w}(x_1, x_d)$ by

$$\tilde{w}(x_1, x_d) = \int_{[-1/2, \delta+1/2]^{d-2}} w(x) dx_2 \cdots dx_{d-1} , \quad \text{a.e. } (x_1, x_d) \in \mathbf{R}^2 .$$

Put

$$I = \{(q_1, q_d) \in \mathbf{Z}^2 \mid 0 \leq q_1 \leq 3N-1, -1 \leq q_d \leq N\}$$

and

$$\tilde{w}_{q_1,q_d} = \int_{(q_1-1/2, q_1+1/2) \times (q_d-1/2, q_d+1/2)} \tilde{w}(x_1, x_d) dx_1 dx_d .$$

Then we have

$$\sum_{q \in I_d} \frac{w(Q_q)}{|q_d|+1} \leq \sum_{(q_1,q_d) \in I} \frac{\tilde{w}_{q_1,q_d}}{|q_d|+1} . \tag{4}$$

Applying Proposition 4 to the right hand side of (4) with (z_1, z_d), we obtain

$$\sum_{q \in I_d} \frac{w(Q_q)}{|q_d|+1} \leq CN \log N \sup_{\alpha \in [3/N, 3\sqrt{2}]} (M_{\alpha,N} \tilde{w})(z_1, z_d) . \tag{5}$$

We see easily that the rectangles, which are congruent to

$$(0, \alpha) \times (0, \delta + 1)^{d-2} \times (0, N\alpha), \quad \alpha \in \left[\frac{3}{N}, 3\sqrt{2}\right],$$

are contained in $\mathcal{B}_{\leq CN}$. Hence the right hand side of (5) is bounded by

$$CN \log N(K_{\leq CN} w)(z).$$

Thus, it follows that

$$\sum_{q \in I_d} \frac{w(Q_q)}{|q_d| + 1} \leq CN \log N(K_{CN} w)(z) \leq CN \log N(K_N w)(z)$$

(cf. [Ta2]). □

2.3. Main estimate for the shell $k = 0$ or $k = \infty$. First we assume that

$$\sum_{p \in A_k} w(Q_p) < \infty.$$

We apply the following argument to finite subsets of A_k and use a limiting argument. The finiteness of the above sum can also be proved directly.

LEMMA 6. *If $k = 0$ or $k = \infty$, then we have*

$$\sum_{p \in A_k} w(Q_p) \leq C(\log N)^{1/d} \frac{1}{\lambda} \|f\|_{L^d(\mathbf{R}^d, K_N w)} \left(\sum_{p \in A_k} w(Q_p)\right)^{(d-1)/d}. \tag{6}$$

PROOF. For $p \in A$ we have

$$\frac{1}{N\lambda} \int_{R_p} f > 1.$$

It follows from this inequality and from Hölder's inequality that

$$\begin{aligned} &\sum_{p \in A_k} w(Q_p) \\ &\leq \frac{1}{N\lambda} \sum_p w(Q_p) \int_{R_p} f = \frac{1}{N\lambda} \int_{\mathbf{R}^d} f \left(\sum_p w(Q_p) \chi_{R_p}\right) \\ &\leq \frac{1}{N\lambda} \left\{\int_{\mathbf{R}^d} f^d K_N w\right\}^{1/d} \left\{\int_{\mathbf{R}^d} \left(\sum_p w(Q_p) \chi_{R_p}\right)^{d/(d-1)} \left(\frac{1}{K_N w}\right)^{1/(d-1)}\right\}^{(d-1)/d}. \end{aligned} \tag{7}$$

We have

$$\begin{aligned} &\int_{\mathbf{R}^d} \left(\sum_p w(Q_p) \chi_{R_p}\right)^{d/(d-1)} \left(\frac{1}{K_N w}\right)^{1/(d-1)} \\ &\quad = \int_{\mathbf{R}^d} \left(\sum_p w(Q_p) \chi_{R_p}\right) \left(\left(\sum_{q \in A_k} w(Q_q) \chi_{R_q}\right) \left(\frac{1}{K_N w}\right)\right)^{1/(d-1)} \end{aligned}$$

$$= \sum_p w(Q_p) \int_{R_p} 1 \cdot \left(\left(\sum_q w(Q_q) \chi_{R_q} \right) \left(\frac{1}{K_N w} \right) \right)^{1/(d-1)} \tag{8}$$

$$\leq C N^{(d-2)/(d-1)} \sum_p w(Q_p) \left\{ \int_{R_p} \left(\sum_q w(Q_q) \chi_{R_q} \right) \left(\frac{1}{K_N w} \right) \right\}^{1/(d-1)} . \tag{9}$$

Therefore, estimates (6) is a consequence of the following

PROPOSITION 7. *Let $k = 0$ or $k = \infty$. Then it holds that*

$$\sum_{q \in A_k} w(Q_q) \int_{R_p \cap R_q} \frac{1}{K_N w} \leq C N^2 \log N \tag{10}$$

for every $p \in A_k$.

PROOF. The case $k = \infty$. If $q \in (A_\infty - \tilde{P}(p))$, then we see that $R_p \cap R_q = \emptyset$ by the facts that $l(R_p)$ and $l(R_q)$ intersect the x_d-axis and that the distance between p (or q) and the x_d-axis is bigger than $2N$. Without loss of generality it suffices to consider only the case that $l(R_p)$ agrees with the x_1-axis. Let $s = \inf_{y \in R_p} (K_N w)(y)$. We note that

$$|R_p \cap R_q| \leq C \frac{N}{|q_d| + 1}$$

for $q = (q_1, \cdots, q_d) \in \tilde{P}(p)$. Hence we have

$$\sum_{q \in A_\infty} w(Q_q) \int_{R_p \cap R_q} \frac{1}{K_N w} \leq C \frac{N}{s} \sum_{q \in \{q \in \tilde{P}(p) \mid R_p \cap R_q \neq \emptyset\}} \frac{w(Q_q)}{|q_d| + 1} . \tag{11}$$

Now for every $z \in R_p$ we see that

$$\sum_q \frac{w(Q_q)}{|q_d| + 1} \leq C N \log N (K_N w)(z)$$

by symmetry of the problem and by Proposition 5. Thus, (10) is proved for $k = \infty$.

PROOF. The case $k = 0$. If $l(R_p)$ agrees with the x_d-axis, then we have

$$\sum_{q \in A_0} w(Q_q) \int_{R_p \cap R_q} \frac{1}{K_N w}$$

$$\leq C \frac{1}{s} \sum_q |R_p \cap R_q| w(Q_q) \leq C \frac{N^2}{s} \left(\sum_q w(Q_q) \right) \Big/ (CN) \leq C N^2 .$$

If $l(R_p)$ does not agree with the x_d-axis, then we have

$$\sum_{q \in A_0} w(Q_q) \int_{R_p \cap R_q} \frac{1}{K_N w} \leq \sum_{q \in \{q \in \tilde{P}(p) \mid R_p \cap R_q \neq \emptyset\}} w(Q_q) \int_{R_p \cap R_q} \frac{1}{K_N w} .$$

By the similar argument as in the above case we obtain (10) for $k = 0$. □

2.4. Main estimate for other shells.

LEMMA 8. *If* $1 \le k \le N_1$, *then we have*

$$\sum_{p\in A_k} w(Q_p) \le C \log N \frac{1}{\lambda} \|f\|_{L^d(\mathbf{R}^d, K_N w)} \left(\sum_{p\in A_k} w(Q_p)\right)^{(d-1)/d} . \tag{12}$$

PROOF. We first note that

$$\begin{aligned}\sum_{p\in A_k} w(Q_p) &\le \frac{1}{N\lambda} \sum_p w(Q_p) \int_{R_p} f \\ &= \frac{1}{N\lambda} \int_{\mathbf{R}^d} f \left(\sum_p w(Q_p)\chi_{R_p}\right) = \frac{1}{N\lambda} \sum_{l=0}^{N_2} \int_{D_l} f \left(\sum_p w(Q_p)\chi_{R_p}\right) . \end{aligned} \tag{13}$$

PROPOSITION 9. *If* $1 \le k \le N_1$ *and* $0 \le l \le N_2$, *then it follows that*

$$\int_{D_l} f \left(\sum_{p\in A_k} w(Q_p)\chi_{R_p}\right) \le CN(\log N)^{1/d} \left(\int_{D_l} f^d K_N w\right)^{1/d} \left(\sum_{p\in A_k} w(Q_p)\right)^{(d-1)/d} . \tag{14}$$

If we assume temporarily Proposition 9, then we have

$$\begin{aligned}\sum_{p\in A_k} w(Q_p) &\le C(\log N)^{1/d} \frac{1}{\lambda} \left(\sum_p w(Q_p)\right)^{(d-1)/d} \left\{\sum_{l=0}^{N_2} 1 \cdot \left(\int_{D_l} f^d K_N w\right)^{1/d}\right\} \\ &\le C \log N \frac{1}{\lambda} \|f\|_{L^d(\mathbf{R}^d, K_N w)} \left(\sum_p w(Q_p)\right)^{(d-1)/d}\end{aligned}$$

by (13) and by Hölder's inequality. Thus Lemma 8 is obtained.

PROOF OF PROPOSITION 9. Similarly to the proof of Lemma 6 we have

$$\begin{aligned}&\int_{D_l} f \left(\sum_{p\in A_k} w(Q_p)\chi_{R_p}\right) \\ &\le \left(\int_{D_l} f^d K_N w\right)^{1/d} \left\{\int_{D_l} \left(\sum_{p\in A_k} w(Q_p)\chi_{R_p}\right)^{d/(d-1)} \left(\frac{1}{K_N w}\right)^{1/(d-1)}\right\}^{(d-1)/d} ,\end{aligned} \tag{15}$$

and $\int_{D_l} \left(\sum_{p\in A_k} w(Q_p)\chi_{R_p}\right)^{d/(d-1)} \left(\frac{1}{K_N w}\right)^{1/(d-1)}$ is equal to

$$\sum_{p\in A_k} w(Q_p) \int_{D_l\cap R_p} 1 \cdot \left(\left(\sum_{q\in A_k} w(Q_q)\chi_{R_q}\right)\left(\frac{1}{K_N w}\right)\right)^{1/(d-1)} ,$$

which does not exceed, by Hölder's inequality

$$C|D_l \cap R_p|^{(d-2)/(d-1)} \sum_p w(Q_p) \left\{\int_{D_l\cap R_p} \left(\sum_q w(Q_q)\chi_{R_q}\right)\left(\frac{1}{K_N w}\right)\right\}^{1/(d-1)} . \tag{16}$$

FIRST STEP. The case $k < l$.

If $q \in (A_k - \tilde{P}(p))$ for $p \in A_k$, then we have $R_p \cap R_q \cap D_l = \emptyset$ by the facts that $l(R_p)$ and $l(R_q)$ intersect the x_d-axis, and $k < l$. Therefore, by an argument similar to the proof of the first case of Proposition 7 we have

$$\sum_{q \in A_k} w(Q_q) \int_{D_l \cap R_p \cap R_q} \frac{1}{K_N w} \le C N^2 \log N .$$

By (15), (16) and $|D_l \cap R_p| \le CN$ we obtain (14) for this case.

SECOND STEP. The case $l \le k$.

Let

$$C_k = \{(x_1, \cdots, x_{d-1}, 0) \in \mathbf{Z}^d \mid 2^k - 1 \le (x_1^2 + \cdots + x_{d-1}^2)^{1/2} < 2^k\} .$$

For $\alpha \in C_k$ let $\Pi(\alpha)$ be the plain spanned by α and the x_d-axis, and

$$\tilde{\Pi}(\alpha) = \{p \in \mathbf{Z}^d \mid \operatorname{dist}(p, \Pi(\alpha)) \le 1\}$$

be 1-neighborfood of $\Pi(\alpha)$. Let

$$B_{\alpha,k} = \tilde{\Pi}(\alpha) \cap A_k .$$

Then we see that the number of α, $\alpha \in C_k$, such that

$$D_l \cap R_p \cap \left(\bigcup_{q \in B_{\alpha,k}} R_q \right) \ne \emptyset$$

is at most $C(2^{k-l})^{d-2}$ (see [Ig]). By Proposition 5 we see that

$$\sum_{q \in B_{\alpha,k}} w(Q_q) \int_{R_q \cap R_p} \frac{1}{K_N w} \le C N^2 \log N , \quad \forall \alpha \in C_k .$$

Thus, we obtain

$$\begin{aligned} &|D_l \cap R_p|^{(d-2)/(d-1)} \left\{ \int_{D_l \cap R_p} \left(\sum_q w(Q_q) \chi_{R_q} \right) \left(\frac{1}{K_N w} \right) \right\}^{1/(d-1)} \\ &\le C |D_l \cap R_p|^{(d-2)/(d-1)} \left\{ \int_{D_l \cap R_p} \left(\sum_{\alpha \in C_k} \sum_{q \in B_{\alpha,k}} w(Q_q) \chi_{R_q} \right) \left(\frac{1}{K_N w} \right) \right\}^{1/(d-1)} \\ &\le C (|D_l \cap R_p| 2^{k-l})^{(d-2)/(d-1)} \cdot N^{2/(d-1)} \cdot (\log N)^{1/(d-1)} . \end{aligned}$$

By a simple geometric consideration we have

$$|D_l \cap R_p| 2^{k-l} \le CN . \tag{17}$$

Thus, we obtain (14) from (17) for this case.

2.5. Proof of Theorem 3. By Lemmas 6 and 8 we have

$$\sum_{p\in A} w(Q_p) = \sum_{k=0,\cdots,N_1,\infty} \sum_{p\in A_k} w(Q_p)$$
$$\leq C\log N \frac{1}{\lambda}\|f\|_{L^d(\mathbf{R}^d,K_N w)} \sum_k \left(\sum_{p\in A_k} w(Q_p)\right)^{(d-1)/d}$$
$$\leq C(\log N)^{(d+1)/d}\|f\|_{L^d(\mathbf{R}^d,K_N w)} \left(\sum_{p\in A} w(Q_p)\right)^{(d-1)/d} . \quad \square$$

3. Proof of Theorem 1.

Theorem 1 follows from Theorem 3 by standard arguments (see [MS], [CHS]).

3.1. Sieve arguments.

PROPOSITION 10. *For every weight w on $\mathbf{R}^d$ the weak-type d inequality*

$$(w(\{x\in\mathbf{R}^d \mid (K_N^L f)(x) > \lambda\}))^{1/d} \leq C(\log N)^{(d+2)/d}\frac{1}{\lambda}\|f\|_{L^d(\mathbf{R}^d,K_N w)} \tag{18}$$

holds for $f\in L^d(\mathbf{R}^d, K_N w)$ and $\lambda > 0$.

PROOF. The following arguments are essentially the same as those in [MS] (pp. 474–476), but we shall repeat them for the completeness.

Define dimensions of the cylinder as $a\times b$ when the cylinder has the bottom of diameter a and the side length b. Define the class $\mathcal{R}_{N,k}$, $k\in\mathbf{Z}$, as the collection of all cylinders in $\mathbf{R}^d$ which have dimensions $a\times aN$ for $N^k \leq a < N^{k+1}$. If $k\geq k'+2$, $R\in\mathcal{R}_{N,k}$ has dimensions $a\times aN$, $R'\in\mathcal{R}_{N,k'}$ and $R\cap R'\neq\emptyset$, then we have $R'\subset R^*$ where R^* is the cylinder concentric with R and with dimensions $3a\times a(N+2)$.

Now, if

$$\{K_N^L f > \lambda\} = \bigcup\left\{R_\alpha \,\middle|\, \frac{1}{|R_\alpha|}\int_{R_\alpha}|f| > \lambda\right\} := \bigcup_{\alpha\in\mathcal{D}} R_\alpha ,$$

we just need to show that, for every finite subset $D\subset\mathcal{D}$,

$$w\left(\bigcup_{\alpha\in D} R_\alpha\right) \leq C\frac{(\log N)^{d+2}}{\lambda^d}\int |f|^d K_N w .$$

Let us write $D = \bigcup_{t=-\gamma}^{\gamma} D_t$, where every $\alpha\in D_t$ corresponds to a cylinder $R_\alpha\in\mathcal{R}_{N,t}$. Without loss of generality we may assume that $D_t=\emptyset$ for every $|t|$ odd.

We define $\bar{D}_\gamma = D_\gamma$ and, by induction, having defined $\bar{D}_\gamma,\cdots,\bar{D}_{t+1}$, we put

$$\bar{D}_t = \left\{\alpha\in D_t \,\middle|\, R_\alpha\cap\left(\bigcup_{\beta\in\bar{D}_{t+1}\cup\cdots\cup\bar{D}_\gamma} R_\beta\right) = \emptyset\right\} .$$

If $\alpha' \in D_t - \bar{D}_t$, then from the above observation and our assumptions we have

$$R_{\alpha'} \subset \bigcup_{\beta \in \bar{D}_{t+1} \cup \cdots \cup \bar{D}_\gamma} R^*_\beta .$$

Now set $E_t = \bigcup_{\alpha \in \bar{D}_t} R_\alpha$ and $E^*_t = \bigcup_{\alpha \in \bar{D}_t} R^*_\alpha$. The families E_t are mutually disjoint by construction. Put $f_t = f\chi_{E_t}$. Then, if $\alpha \in \bar{D}_t$, we have

$$\lambda < \frac{1}{|R_\alpha|} \int_{R_\alpha} |f_t| \le \frac{3^d}{|R^{**}_\alpha|} \int_{R^{**}_\alpha} |f_t| ,$$

where R^{**}_α is the cylinder concentric with R_α and has dimensions 3 times bigger than those of R_α.

Therefore, if N is sufficiently large, we obtain

$$E^*_t \subset \left\{ x \in \mathbf{R}^d \;\middle|\; \sup_{x \in R \in \mathcal{R}_{N,t} \cup \mathcal{R}_{N,t+1}} \frac{1}{|R|} \int_R |f_t| > \frac{\lambda}{3^d} \right\} .$$

Observe that for a fixed x

$$\sup_{x \in R \in \mathcal{R}_{N,t} \cup \mathcal{R}_{N,t+1}} \frac{1}{|R|} \int_R |f_t| \le C \sup_{m=0,1,\cdots,2[\log N]+1} (M_{N^t 2^m, N} f_t)(x) .$$

Using Theorem 3 we obtain

$$w(E^*_t) \le C \frac{(\log N)^{d+2}}{\lambda^d} \int |f_t|^d K_N w .$$

Therefore, we conclude

$$w\left(\bigcup_{\alpha \in D} R_\alpha \right) \le \sum_t w(E^*_t) \le C \frac{(\log N)^{d+2}}{\lambda^d} \int |f|^d K_N w . \quad \square$$

3.2. Three-point interpolation lemma. Let M be the Hardy-Littlewood maximal operator. Then we have

$$CMf \le K_N f \le N^{d-1}(Mf) ,$$

that

$$w(\{x \in \mathbf{R}^d \mid (K^L_N f)(x) > \lambda\}) \le C \frac{N^{d-1}}{\lambda} \int_{\mathbf{R}^d} |f| K_N w . \tag{19}$$

On the other hand, we have the obvious inequality

$$\|K^L_N f\|_{L^\infty(\mathbf{R}^d, w)} \le \|f\|_{L^\infty(\mathbf{R}^d, K_N w)} . \tag{20}$$

The Proposition 11 is a consequence of so-called the three-point interpolation lemma.

PROPOSITION 11. *For every weight w on $\mathbf{R}^d$ the strong-type d inequality*

$$\|K^L_N f\|_{L^d(\mathbf{R}^d, w)} \le C(\log N)^{(d+3)/d} \|f\|_{L^d(\mathbf{R}^d, K_N w)} \tag{21}$$

holds for $f \in L^d(\mathbf{R}^d, K_N w)$.

PROOF. The argument follows basically Proposition 5 in [CHS; p. 48].

Put $T = K_N^L$ and $u = K_N w$. Let f be a function on $\mathbf{R}^d$. For a given $\lambda > 0$ split f as follows.

$$f = f\chi_{\{|f|\leq\lambda/3\}} = f\chi_{\{\lambda/3<|f|\leq\alpha\lambda\}} + f\chi_{\{|f|>\alpha\lambda\}} := f_1 + f_2 + f_3$$

with $\alpha > 0$ to be chosen later. Then we have

$$(Tf)(x) \leq (Tf_1)(x) + (Tf_2)(x) + (Tf_3)(x)$$

and hence

$$\begin{aligned} &w(\{x \mid (Tf)(x) > \lambda\}) \\ &\quad \leq w(\{x \mid (Tf_1)(x) > \lambda/3\}) + w(\{x \mid (Tf_2)(x) > \lambda/3\}) + w(\{x \mid (Tf_3)(x) > \lambda/3\}) . \end{aligned}$$

By (20) we have

$$w(\{x \mid (Tf)(x) > \lambda\}) \leq w(\{x \mid (Tf_2)(x) > \lambda/3\}) + w(\{x \mid (Tf_3)(x) > \lambda/3\}) . \qquad (22)$$

Set $c_1 = CN^{d-1}$ and $c_d = C(\log N)^{d+2}$. Now, it follows from (22), (18) and (19) that

$$\begin{aligned} &\int (Tf)^d w dx \\ &\quad = d\int_0^\infty w(\{x \mid (Tf)(x) > \lambda\})\lambda^{d-1} d\lambda \\ &\quad \leq d\int_0^\infty w(\{x \mid (Tf_2)(x) > \lambda/3\})\lambda^{d-1} d\lambda + d\int_0^\infty w(\{x \mid (Tf_3)(x) > \lambda/3\}) \\ &\quad \leq Cc_d \int_0^\infty \int_{\lambda/3<|f|\leq\alpha\lambda} |f|^d u dx \frac{d\lambda}{\lambda} + Cc_1 \int_0^\infty \int_{|f|>\alpha\lambda} |f| u dx \lambda^{d-2} d\lambda . \qquad (23) \end{aligned}$$

We see that

$$\int_0^\infty \int_{\lambda/3<|f|\leq\alpha\lambda} |f|^d u dx \frac{d\lambda}{\lambda} = \int |f|^d u \int_{|f|/\alpha\leq\lambda<3|f|} \frac{1}{\lambda} d\lambda dx = \log 3\alpha \int |f|^d u \qquad (24)$$

and

$$\int_0^\infty \int_{|f|>\alpha\lambda} |f| u dx \lambda^{d-2} d\lambda = \int |f| u \int_{0\leq\lambda<|f|/\alpha} \lambda^{d-2} d\lambda dx = \frac{1}{d-1}\frac{1}{\alpha^{d-1}} \int |f|^d u . \qquad (25)$$

Choosing $\alpha = N$, we obtain (21) by (25), (24) and (23). □

Interpolation argument between (19) and (18), and between (21) and (20) give Theorem 1.

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