TOKYO J. MATH.
VOL. 22, NO. 1, 1999

Critical Metrics of the Scalar Curvature Functional

Fumie YAGI

Nara Women's University

(Communicated by Y. Maeda)

1. Introduction.

Let M be a compact connected n-manifold and $\mathcal{M}(M)$ the space of Riemannian metrics on M. We study the critical metrics of the following functional;

$$\mathcal{S}^p : \mathcal{M}(M) \to \mathbf{R} ; \qquad g \mapsto \frac{\int_M R_g^p dv_g}{(\int_M dv_g)^{(n-2p)/n}} ,$$

where R_g is the scalar curvature of g and $p \in \mathbf{N}$.

The first variation formula for $\mathcal{S}^p$ is

$$\nabla^2 R_g^{p-1} = \frac{1}{n} \Delta R_g^{p-1} g + R_g^{p-1} \left(\mathrm{Ric}_g - \frac{R_g}{n} g \right) , \tag{i}$$

where Ric_g is the Ricci tensor of g. Taking the divergence of (i) with respect to g, we have

$$\Delta R_g^{p-1} = \frac{n-2p}{2p(n-1)} (R_g^p - \overline{R_g^p}) , \tag{ii}$$

where $\overline{R_g^p} = \int_M R_g^p dv_g / \int_M dv_g$. The equation (ii) is also the first variation formula for $\mathcal{S}^p|_C$, where C is a conformal class of $\mathcal{M}(M)$.

Obviously, if $R_g \equiv 0$ or g is an Einstein metric, the metric g satisfies the equation (i). A metric of constant scalar curvature satisfies the equation (ii). The question is whether the converses are true or not.

The case $p=1$ is well-known (e.g. [5]). When $n=2$, $\mathcal{S}^2|_C$ was studied by Calabi ([4], see also Section 3). If $n \geq 4$ and $p=n/2$, the answer is positive (e.g. [2]). If $n=3$, $p=2$ and R_g does not change the sign, then the metric which satisfies (i) is of constant scalar curvature ([1]). According to Anderson ([1]), the general case is an open question.

In this paper, we show the following results which are extentions of Anderson's

Received October 18, 1997
Revised March 11, 1998

result.

THEOREM 1. *Suppose g satisfies the equation* (ii). *If any of the following condition is satisfied, then R_g is constant*:

(i) *for* $p\geq 3$, $(p-n/2)\max R_g\leq 0$ *or* $(p-n/2)\min R_g\leq 0$;

(ii) *for* $p=2$, $(2-n/2)\max R_g\leq 0$ *and* $(2-n/2)\min R_g\leq 0$.

COROLLARY. *If g satisfies* (ii) *and* $(p-n/2)R_g\leq 0$ *then* R_g *is constant.*

THEOREM 2. *If g satisfies the equation* (i) *and* $R_g\geq 0$, *then g is an Einstein metric or* R_g *is identically* 0.

From Theorem 1 and Theorem 2, we obtain the affirmative answer to our question in case $p\geq\max\{3, n/2\}$, which implies a difficult part of the problem will be the case for relatively small p.

The author would like to thank Professor Minyo Katagiri for his valuable suggestions and the referee for helpful remarks.

2. Proof of Theorem 1.

It is easy to see that

$$\Delta R_g^p=\frac{p}{p-1}R_g\Delta R_g^{p-1}+pR_g^{p-2}|\nabla R_g|^2 .$$

Combining (ii) with this, we have

$$\Delta R_g^p(x)=\frac{n-2p}{2(n-1)(p-1)}R_g(x)(R_g^p(x)-\overline{R_g^p}) ,$$

if either x is a critical point of R_g or x is a critical point of R_g^p for $p\geq 3$. Thus we see that if R_g is not constant and if $p\geq 3$ then $(n-2p)R_g(x_1)\leq 0$ at a maximum point x_1 of R_g^p and $(n-2p)R_g(x_2)\leq 0$ at a minimum point x_2 of R_g^p. If R_g is not constant and p is even then $(n-2p)R_g(x_1)<0$. We then have $(n-2p)R_g\leq 0$ if p is odd, and $(n-2p)R_g<0$ if p is even and $R_g(x_2)\neq 0$. Again from (ii), if $p\geq 3$ and R_g takes 0 somewhere then $\overline{R_g^p}=0$. Now it is easy to see the assertion (i). The assertion (ii) is proved by a similar argument. □

3. The case of dimension 2.

In dimension 2, the equations (i) and (ii) become the following simple forms respectively:

$$\nabla^2 R_g^{p-1}=\frac{1}{2}\Delta R_g^{p-1}g , \tag{3.1}$$

$$\Delta R_g^{p-1} = \frac{1-p}{p}(R_g^p - \overline{R_g^p})\,. \tag{3.2}$$

In [4], Calabi introduced the functional $\mathcal{S}^2$ on a complex manifold with a fixed Kähler class. In dimension 2, a Kähler class is nothing but a conformal class of metrics. Thus the equation to be considered will be (3.2).

Following the method given by Xu ([6]), we answer to our question for general $\mathcal{S}^p$.

LEMMA 3.1. *For $p \geq 2$, g satisfies* (3.1) *if and only if g is of constant scalar curvature.*

PROOF. From (3.1), ∇R_g^{p-1} is a conformal vector field. For a conformal vector field X, the following formula is well-known (e.g. [3]):

$$\int_M XR_g dv_g = 0\,. \tag{3.3}$$

Hence we have

$$\int_M R_g^{p-2} |\nabla R_g|^2 dv_g = 0\,.$$

If p is even, this implies R_g is constant. If p is odd, R_g does not change a sign from Theorem 1. Thus R_g is constant. □

LEMMA 3.2. (3.2) *implies* (3.1).

PROOF. The Ricci identity shows that

$$\frac{1}{2}\Delta|\nabla f|^2 - \frac{1}{2}\operatorname{div}(\Delta f \nabla f) = \left|\nabla^2 f - \frac{1}{2}\Delta f g\right|^2 + \frac{1}{2}\nabla\Delta f \cdot \nabla f + \frac{R_g}{2}|\nabla f|^2\,,$$

where we have used $\mathrm{Ric}_g = (R_g/2)g$ because $n=2$. Taking the integrals of the both sides, we have

$$0 = \int_M \left|\nabla^2 f - \frac{1}{2}\Delta f g\right|^2 dv_g + \frac{1}{2}\int_M \nabla\Delta f \cdot \nabla f dv_g + \frac{1}{2}\int_M R_g |\nabla f|^2 dv_g\,.$$

We put $f = R_g^{p-1}$. Then it follows from (3.2) that

$$\begin{aligned} 0 &= \int_M \left|\nabla^2 R_g^{p-1} - \frac{1}{2}\Delta R_g^{p-1} g\right|^2 dv_g \\ &\quad - \frac{(p-1)^2}{2}\int_M R_g^{2p-3}|\nabla R_g|^2 dv_g + \frac{(p-1)^2}{2}\int_M R_g^{2p-3}|\nabla R_g|^2 dv_g \\ &= \int_M \left|\nabla^2 R_g^{p-1} - \frac{1}{2}\Delta R_g^{p-1} g\right|^2 dv_g\,. \end{aligned}$$

Hence (3.1) holds. □

REMARK. For $n\geq 3$, this lemma does not hold.

THEOREM. *For $p\geq 2$, g satisfies* (3.2) *if and only if g is of constant scalar curvature.*

4. Proof of Theorem 2.

In this section we consider the equation (i). Lemma 3.1 gives a complete answer in dimension 2. There the formula (3.3) plays an important role. This will be interpreted as follows: Since $\nabla^2 f=\frac{1}{2}\mathscr{L}_{\nabla f}g$ for any function f, the equation (i) gives us information on $\mathscr{L}_{\nabla R_g^{p-1}}g$. Naturally this leads to what will be $\mathscr{L}_{\nabla R_g^{p-1}}R_g$. We can regard the formula (3.3) as the integral of $\mathscr{L}_X R_g$ for $X=\nabla R_g^{p-1}$. Following this line, we proceed the argument in higher dimensional cases.

Recall that if a vector field X and a 2-tensor h satisfy $\mathscr{L}_X g=h$ then

$$\mathscr{L}_X R_g=-\Delta\,\mathrm{tr}\,h+h^{ij}{}_{;ij}-\langle h,\mathrm{Ric}_g\rangle\,.$$

Taking the integrals of the both sides, we obtain

$$\int_M XR_g dv_g=-\int_M\langle h,\mathrm{Ric}_g\rangle dv_g\,.$$

In view of the equation (i), we put

$$X=\nabla R_g^{p-1}\,,\qquad h=\frac{2}{n}\Delta R_g^{p-1}g+2R_g^{p-1}\left(\mathrm{Ric}_g-\frac{R_g}{n}g\right),$$

and we have

$$\int_M\nabla R_g^{p-1}R_g dv_g=-\int_M\left(\frac{2}{n}(\Delta R_g^{p-1})R_g+2R_g^{p-1}\left|\mathrm{Ric}_g-\frac{R_g}{n}g\right|^2\right)dv_g\,.$$

By integration by parts we have

$$(p-1)\left(1-\frac{2}{n}\right)\int_M R_g^{p-2}|\nabla R_g|^2 dv_g=-2\int_M R_g^{p-1}\left|\mathrm{Ric}_g-\frac{R_g}{n}g\right|^2 dv_g\,.$$

Consequently we get

$$\frac{(n-2)(p-1)}{2n}\int_M R_g^{p-2}|\nabla R_g|^2 dv_g=-\int_M R_g^{p-1}\left|\mathrm{Ric}_g-\frac{R_g}{n}g\right|^2 dv_g\,.$$

From our assumption $R_g\geq 0$, the left hand side is non-negative. Therefore the integrand of the right side vanishes. □

References

[1] M. T. ANDERSON, Extrema of curvature functionals on the space of metrics on 3-manifolds, Calc.

Var. Partial Differential Equations **5** (1997), 199–269.
[2] A. Besse, *Einstein Manifolds*, Springer (1987).
[3] J. P. Bourguignon and J. P. Ezin, Scalar curvature functions in a conformal class of metrics and conformal transformations, Trans. Amer. Math. Soc. **301** (1987), 723–736.
[4] E. Calabi, Extremal Kähler metrics, *Seminar on Differential Geometry*, Ann. of Math. Stud. **102** (1982), 259–290.
[5] O. Kobayashi, On the Yamabe problem, Sem. Math. Sci. **16** (1990), Dept. Math. Keio. Univ. (in Japanese).
[6] X. Xu, On the existence of extremal metrics, Pacific J. Math. **174** (1996), 555–568.

Present Address:
Department of Mathematics, Nara Women's University,
Nara, 630–8506 Japan.