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Some Arithmetic Fuchsian Groups with Signature $(0; e_1, e_2, e_3, e_4)$

Jun-ichi SUNAGA

Saitama University

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Abstract. We determine the arithmetic Fuchsian groups Γ with signature $(0; e_1, e_2, e_3, e_4)$ which are the subgroups of normalizer $\Gamma^*(A; O)$ of maximal orders O in quaternion algebras A over the rational number field $\mathbf{Q}$.

1. Introduction.

To begin with, we shall recall the definition of a Fuchsian group (cf. Beardon [1], Iversen [2]). The group $SL_2(\mathbf{R})$ acts on the upper half plane $H=\{z\in\mathbf{C} \mid \mathrm{Im}(z)>0\}$ as the group of fractional linear transformations. A finitely generated discrete subgroup Γ of this transformation group is called a Fuchsian group. In this paper, we shall consider only Fuchsian groups of the first kind. Let Γ be a Fuchsian group of the first kind. We denote by P_Γ the set of the parabolic points of Γ and put $H^*=H\cup P_\Gamma$. Then we can naturally introduce a structure of the compact Riemann surface on the quotient space H^*/Γ. Denote by g, r and s the genus of H^*/Γ, the number of the elliptic and parabolic points of H^*/Γ respectively, and by e_i $(1\leq i\leq r)$ the orders of the stabilizing groups of elliptic points of Γ. Then we call the symbol $(g; e_1, e_2, \cdots, e_r, e_{r+1}, \cdots, e_{r+s})$ $(e_i=\infty$ for $r+1\leq i\leq r+s)$ the signature of Γ. The following equality holds concerning the volume $\mathrm{vol}(H^*/\Gamma)$ of the quotient space H^*/Γ and the signature $(g; e_1, e_2, \cdots, e_r, e_{r+1}, \cdots, e_{r+s})$ of Γ (see Beardon [1]):

$$\mathrm{vol}(H^*/\Gamma)=\frac{1}{2\pi}\int_{D_\Gamma}\frac{dxdy}{y^2}=2g-2+\sum_{i=1}^{r+s}\left(1-\frac{1}{e_i}\right) \tag{1.1}$$

where D_Γ is a fundamental domain of Γ in H and $1/e_i=0$ for $r+1\leq i\leq r+s$.

Next we also recall the definition of an arithmetic Fuchsian group (cf. Shimura [7]). Let k be a totally real algebraic number field of degree n, φ_i $(1\leq i\leq n)$ be

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Q-isomorphisms of k into the real number field **R** and φ_1 be an identity map. Let A be a quaternion algebra which splits at the infinite place φ_1 and is ramified at all other infinite places φ_i $(2 \leq i \leq n)$. Then there exists an **R**-isomorphism

$$\rho : A \otimes_{\mathbf{Q}} \mathbf{R} \to M_2(\mathbf{R}) \oplus \mathbf{H}^{n-1}$$

where **H** is the Hamilton quaternion algebra over **R**. We denote by ρ_1 the composite of the restriction of ρ to A with the projection to $M_2(\mathbf{R})$. Let O be an order in A. Put

$$O^1 = \{x \in O \mid n(x) = 1\}$$

where $n(\ )$ denotes the reduced norm of A over k. If we put $\Gamma^{(1)}(A, O) = \rho_1(O^1)$, then $\Gamma^{(1)}(A, O)$ is a discrete subgroup of $SL_2(\mathbf{R})$. A discrete subgroup Γ of $SL_2(\mathbf{R})$ is called arithmetic if Γ is commensurable with some $\Gamma^{(1)}(A, O)$. Furthermore, we define the normalizer $N(O)$ of O:

$$N(O) = \{x \in A \mid xO = Ox,\ n(x) > 0\}\ .$$

Put

$$GL_2^+(\mathbf{R}) = \{g \in M_2(\mathbf{R})\ ;\ \det(g) > 0\}\ .$$

If we denote by $\Gamma^*(A, O)$ the image of $\rho_1(N(O))$ by the homomorphism

$$\psi : GL_2^+(\mathbf{R}) \ni g \to \det(g)^{-1/2} g \in SL_2(\mathbf{R}) \qquad (1.2)$$

then $\Gamma^*(A, O)$ is also a discrete subgroup of $SL_2(\mathbf{R})$.

We consider the problem to determine all arithmetic Fuchsian groups with given signature. It is proved that there exist only finitely many arithmetic Fuchsian groups with any given signature up to $SL_2(\mathbf{R})$-conjugation by K. Takeuchi (Takeuchi [11]). And he has determined explicitly all arithmetic Fuchsian groups with signature $(0; e_1, e_2, e_3)$ (i.e. the triangle groups) and signature $(1; e)$ (Takeuchi [10, 11]).

In this paper, we treat arithmetic Fuchsian groups with signature $(0; e_1, e_2, e_3, e_4)$. We shall determine all subgroups Γ of $\Gamma^*(A, O)$ with signature $(0; e_1, e_2, e_3, e_4)$ obtained from a quaternion algebra A over the rational number field **Q** up to $\Gamma^*(A, O)$-conjugation. Since Takeuchi has determined such groups in the case $A \cong M_2(\mathbf{Q})$ (in this case, it can be easily seen that $\Gamma^*(A, O) = \Gamma^{(1)}(A, O) = SL_2(\mathbf{Z})$), we shall deal with the remaining cases (i.e. $s = 0$). We make use of the homomorphisms of $\Gamma^*(A, O)$ into the symmetric group S_n of degree n (cf. Singerman [9]). This method is a generalization of the one used in Takeuchi [12]. In the main theorem (Theorem 6), we shall give the complete list of the groups Γ mentioned above and the corresponding homomorphisms.

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2. Signatures of $\Gamma^*(A, O)$, $\Gamma^{(1)}(A, O)$.

Let A be an indefinite quaternion algebra over **Q**, which means that A satisfies

(2.1) $$\rho : A \otimes_{\mathbf{Q}} \mathbf{R} \cong M_2(\mathbf{R}) .$$

From now on, we identify A with $\rho(A)$ by virtue of this isomorphism ρ and we regard A as a subring of $M_2(\mathbf{R})$. Then the reduced norm $n(x)$ coincides with $\det(x)$ and the reduced trace $\mathrm{tr}(x)$ coincides with $\mathrm{tr}(x)$ as a matrix x. As for the discriminant $D(A)$ of A, we have the following theorem (e.g. Shimura [7]).

THEOREM 1 (Hasse). *Let notations be as above. The number of the places of* $\mathbf{Q}$ *which are ramified in* A *is even.*

From this theorem, we can express the discriminant $D(A)$ of A as follows:

$$D(A)=p_1p_2\cdots p_{2m} ,$$

where p_i are distinct rational prime numbers. Let O be a maximal order in A. We note that there exists an element $\pi_i \in O$ such that $n(\pi_i)=p_i$ $(1 \le i \le 2m)$.

When we put $\Gamma^{(1)}(A, O)=\rho(O^1)$, $\Gamma^{(1)}(A, O)$ is a discrete subgroup of $SL_2(\mathbf{R})$ (see Shimizu [5]), and $\rho(N(O))$ is a subgroup of $GL_2^+(\mathbf{R})$. When we denote by $\Gamma^*(A, O)$ the image of $\rho(N(O))$ by the map ψ in (1.2), $\Gamma^*(A, O)$ is also a discrete subgroup of $SL_2(\mathbf{R})$.

We have (cf. Vignéras [13])

(2.2) $$\Gamma^*(A, O)/\Gamma^{(1)}(A, O) \cong (\mathbf{Z}/2\mathbf{Z})^{2m} .$$

The quotient spaces $H/\Gamma^*(A, O)$, $H/\Gamma^{(1)}(A, O)$, in our case, are compact Riemann surfaces. The volume of the Riemann surface $H/\Gamma^{(1)}(A, O)$ with respect to the $SL_2(\mathbf{R})$-invariant measure $dz=(1/y^2)dxdy$ $(x+iy \in \mathbf{C})$ on H is given by

(2.3) $$\mathrm{vol}(H/\Gamma^{(1)}(A, O))=\frac{1}{6}\prod_{p \mid D(A)} (p-1)$$

(Shimizu [6]). And we have

$$\mathrm{vol}(H/\Gamma^{(1)}(A, O))=[\Gamma^*(A, O) : \Gamma^{(1)}(A, O)]\, \mathrm{vol}(H/\Gamma^*(A, O)) ,$$

so by (2.2), we have

(2.4) $$\mathrm{vol}(H/\Gamma^*(A, O))=\frac{1}{2^{2m}}\mathrm{vol}(H/\Gamma^{(1)}(A, O)) .$$

On the other hand, if we denote by $(g^{(1)}; e_1, e_2, \cdots, e_r)$, $(g^*; e'_1, e'_2, \cdots, e'_{r'})$ the signatures of $\Gamma^{(1)}(A, O)$ and $\Gamma^*(A, O)$ respectively, by (1.1) we have

(2.5) $$2g^{(1)}-2=\mathrm{vol}(H/\Gamma^{(1)}(A, O))-\sum_{i=1}^{r}\left(1-\frac{1}{e_i}\right),$$

(2.6) $$2g^*-2=\mathrm{vol}(H/\Gamma^*(A, O))-\sum_{i=1}^{r'}\left(1-\frac{1}{e'_i}\right).$$

As for (2.5), for any elliptic element γ of $\Gamma^{(1)}(A, O)$, since $|\mathrm{tr}(\gamma)|<2$ and $\mathrm{tr}(\gamma)\in\mathbf{Z}$, we have $\mathrm{tr}(\gamma)=0, \pm 1$. Hence γ satisfies one of the equations $\gamma^2+1=0$, $\gamma^2\pm\gamma+1=0$. So we have $e_i=2, 3$. When we denote by $v_k^{(1)}$ the number of the elliptic points of $H/\Gamma^{(1)}(A, O)$ of order k, we have the following equality:

$$2g^{(1)}-2=\mathrm{vol}(H/\Gamma^{(1)}(A, O))-\frac{1}{2}v_2^{(1)}-\frac{2}{3}v_3^{(1)}. \tag{2.7}$$

By (2.2), $e_i'=2, 3, 4, 6$. Denote by v_k^* the number of elliptic points of $H/\Gamma^*(A, O)$ of order k. Then we have

$$2g^*-2=\mathrm{vol}(H/\Gamma^*(A, O))-\frac{1}{2}v_2^*-\frac{2}{3}v_3^*-\frac{3}{4}v_4^*-\frac{5}{6}v_6^*. \tag{2.8}$$

Now we have to calculate $v_k^{(1)}$, v_k^*.

Definition 1. Let $K=\mathbf{Q}(x)$ be a quadratic field, B its order, and p be a rational prime. We define the Artin symbol in the following way;

$$\left(\frac{K}{p}\right)=\begin{cases} 1 & \text{if } p \text{ slits in } K \\ -1 & \text{if } p \text{ is still a prime in } K \\ 0 & \text{if } p \text{ is ramified in } K. \end{cases}$$

We need the following theorems.

Theorem 2 (Vignéras [13]).

$$v_2^{(1)}=\prod_{p\mid D(A)}\left(1-\left(\frac{-4}{p}\right)\right), \qquad v_3^{(1)}=\prod_{p\mid D(A)}\left(1-\left(\frac{-3}{p}\right)\right)$$

where $\left(\frac{-d}{p}\right)$ *denotes the Artin symbol of quadratic field* $\mathbf{Q}(\sqrt{-d})$.

We denote by B_c the order of the quadratic imaginary field $\mathbf{Q}(\sqrt{-d})$ of conductor c $(c=1, 2)$. Let n_d^c be the number of $N_0(O)$-conjugate classes of maximal embeddings of B_c into A where $N_0(O)=N(O)\cup\varepsilon N(O)$ $(n(\varepsilon)=-1)$ (see Michon [4]).

Theorem 3 (Michon [4]).

(1)

$$v_2^*=\sum_{d\mid D(A)}(n_d^1+n_d^2)-\lambda(D)n_1^1-\mu(D)n_3^1,$$

$$v_3^*=(1-\mu(D))n_3^1, \quad v_4^*=\lambda(D)n_1^1, \quad v_6^*=\mu(D)n_3^1$$

where

$$\lambda(D)=\begin{cases} 1 & \text{if } D(A) \text{ is even} \\ 0 & \text{if } D(A) \text{ is odd} \end{cases}$$

$$\mu(D)=\begin{cases}1 & \textit{if}\quad D(A)\equiv 0 \pmod 3\\ 0 & \textit{if}\quad D(A)\not\equiv 0 \pmod 3\,.\end{cases}$$

(2) *n_d^c $(c=1, 2)$ is given as follows: $n_d^c=0$ if at least one $p_i|D(A)$ splits in $\mathbf{Q}(\sqrt{-d})$, or $c=2$ and $D(A)$ is even, or $c=2$ and $d\not\equiv 3$* (mod 4). *Otherwise*

$$n_d^c=\begin{cases}\dfrac{h(-d)}{r} & \textit{for}\quad c=1\\[2ex] \dfrac{h(-d)}{r\rho}\left(1-\left(\dfrac{-d}{2}\right)\right) & \textit{for}\quad c=2\end{cases}$$

where $\rho=[B_1^\times : B_2^\times]$, and $h(-d)$ is the class number of $\mathbf{Q}(\sqrt{-d})$ and r denotes the number of ideal classes of L generated by the prime ideals dividing p_i which do not split in B_c $(c=1, 2)$.

Now we shall determine the signatures of $\Gamma^*(A, O)$ which contains the subgroups Γ with signatures $(0; e_1, e_2, e_3, e_4)$. We give the conditions on the discriminant $D(A)$ of A and the index $n=[\Gamma^*(A, O) : \Gamma]$.

Put $D(A)=p_1p_2\cdots p_{2m}$, then by (2.2) we have that $[\Gamma^*(A, O) : \Gamma^{(1)}(A, O)]=2^{2m}$. And put $[\Gamma^*(A, O) : \Gamma]=n$. Hence it follows from (2.3), (2.4) that

$$\text{(2.9)}\qquad \operatorname{vol}(H/\Gamma^{(1)}(A, O))=\frac{1}{6}\prod_{i=1}^{2m}(p_i-1)\,,\quad \operatorname{vol}(H/\Gamma^*(A, O))=\frac{1}{2^{2m}}\operatorname{vol}(H/\Gamma^{(1)}(A, O))\,,$$

$$\text{(2.10)}\qquad \operatorname{vol}(H/\Gamma)=n\cdot\operatorname{vol}(H/\Gamma^*(A, O))\,.$$

Since the signature of Γ is $(0; e_1, e_2, e_3, e_4)$, we have

$$\operatorname{vol}(H/\Gamma)=2-\sum_{i=1}^{4}\frac{1}{e_i}\,.$$

We may assume that $e_i=2, 3, 4, 6$ $(1\le i\le 4)$, hence we have

$$\frac{1}{6}\le\operatorname{vol}(H/\Gamma)=2-\sum_{i=1}^{4}\frac{1}{e_i}\le\frac{4}{3}\,.$$

Then we see that the equalities (2.9), (2.10) lead to

$$\frac{1}{6}\le\frac{n}{6\cdot 2^{2m}}\prod_{i=1}^{2m}(p_1-1)\le\frac{4}{3}\,.$$

This implies that

$$\text{(2.11)}\qquad 1\le n\prod_{i=1}^{2m}\frac{p_i-1}{2}\le 8\,.$$

Since

$$\frac{1}{2} \leq \prod_{i=1}^{2m} \frac{p_i - 1}{2},$$

we have an upper bound on the index n: $n \leq 16$. And since

$$\prod_{i=1}^{2m} \frac{p_i - 1}{2} \leq \frac{8}{n} \leq 8,$$

we also have an upper bound on the discriminant $D(A)$: $D(A) \leq 2\cdot 3\cdot 5\cdot 17 = 510$. Considering these conditions, we obtain the following table for the pair $(D(A), n)$:

$D(A)$	n	$D(A)$	n	$D(A)$	n
$2\cdot 3$	$2\leq n\leq 16$	$3\cdot 11$	$n=1$	$2\cdot 29$	$n=1$
$2\cdot 5$	$1\leq n\leq 8$	$2\cdot 17$	$1\leq n\leq 2$	$2\cdot 31$	$n=1$
$2\cdot 7$	$1\leq n\leq 5$	$5\cdot 7$	$n=1$	$2\cdot 3\cdot 5\cdot 7$	$1\leq n\leq 2$
$3\cdot 5$	$1\leq n\leq 4$	$2\cdot 19$	$n=1$	$2\cdot 3\cdot 5\cdot 11$	$n=1$
$3\cdot 7$	$1\leq n\leq 2$	$3\cdot 13$	$n=1$	$2\cdot 3\cdot 5\cdot 13$	$n=1$
$2\cdot 11$	$1\leq n\leq 3$	$2\cdot 23$	$n=1$	$2\cdot 3\cdot 7\cdot 11$	$n=1$
$2\cdot 13$	$1\leq n\leq 2$	$3\cdot 17$	$n=1$	$2\cdot 3\cdot 5\cdot 17$	$n=1$

TABLE 1

We shall determine the signatures of $\Gamma^{(1)}(A, O)$, $\Gamma^*(A, O)$ for $D(A) < 100$, $D(A) = 210$, 330, 390, 462, 510 and give a table of these signatures together with $v^{(1)} = \mathrm{vol}(H/\Gamma^{(1)}(A, O))$, $v^* = \mathrm{vol}(H/\Gamma^*(A, O))$.

THEOREM 4. *Let the notations be as above. The data for* $\Gamma^{(1)}(A, O)$, $\Gamma^*(A, O)$ *is given as follows*:

$D(A)$	$v_2^{(1)}$	$v_3^{(1)}$	$g^{(1)}$	$v^{(1)}$	v_2^*	v_3^*	v_4^*	v_6^*	g^*	v^*
$2\cdot 3$	2	2	0	1/3	1	0	1	1	0	1/12
$2\cdot 5$	0	4	0	2/3	3	1	0	0	0	1/6
$2\cdot 7$	2	0	1	1	3	0	1	0	0	1/4
$3\cdot 5$	0	2	1	4/3	3	0	0	1	0	1/3
$3\cdot 7$	4	0	1	2	5	0	0	0	0	1/2
$2\cdot 11$	2	4	0	5/3	2	1	1	0	0	5/12
$2\cdot 13$	0	0	2	2	5	0	0	0	0	1/2
$3\cdot 11$	4	2	1	10/3	4	0	0	1	0	5/6
$2\cdot 17$	0	4	1	8/3	4	1	0	0	0	2/3
$5\cdot 7$	0	0	3	4	2	0	0	0	1	1
$2\cdot 19$	2	0	2	3	4	0	1	0	0	3/4
$3\cdot 13$	0	0	3	4	6	0	0	0	0	1
$2\cdot 23$	2	4	1	11/3	3	1	1	0	0	11/12
$3\cdot 17$	0	2	3	16/3	5	0	0	1	0	4/3
$5\cdot 11$	0	4	3	20/3	6	1	0	0	0	5/3
$3\cdot 19$	4	0	3	6	7	0	0	0	0	4/3

2·29	0	4	2	14/3	5	1	0	0	0	7/6
2·31	2	0	3	5	5	0	1	0	0	5/4
5·13	0	0	5	8	8	0	0	0	0	2
3·23	4	2	3	22/3	6	0	0	1	0	11/6
2·37	0	0	4	6	7	0	0	0	0	3/2
7·11	4	0	5	10	9	0	0	0	0	5/2
2·41	0	4	3	20/3	6	1	0	0	0	5/3
5·17	0	4	5	32/3	8	1	0	0	0	8/3
2·43	2	0	4	7	6	0	1	0	0	7/4
3·29	0	2	5	28/3	7	0	0	1	0	7/3
7·13	0	0	7	12	6	0	0	0	0	3
3·31	4	0	5	10	9	0	0	0	0	5/2
2·47	2	4	3	23/3	5	1	1	0	0	23/12
5·19	0	0	7	12	10	0	0	0	0	3
2·3·5·7	0	0	5	8	5	0	0	0	0	1/2
2·3·5·11	0	8	5	40/3	4	0	0	1	0	5/6
2·3·5·13	0	0	9	16	6	0	0	0	0	1
2·3·7·11	8	0	9	20	5	0	1	0	0	5/4
2·3·5·17	0	8	9	64/3	5	0	0	1	0	4/3

3. Main theorem.

Our main purpose in this paper is to determine all Fuchsian groups Γ with signature $(0; e_1, e_2, e_3, e_4)$ such that Γ is a subgroup of $\Gamma^*(A, O)$ of index n.

First in the case $n=1$, we have the following result directly from Theorem 4. We give the complete list of $\Gamma^*(A, O)$ with signature $(0; e_1, e_2, e_3, e_4)$ as follows:

$D(A)$	$(0; e_1, e_2, e_3, e_4)$
2·5	(0; 2, 2, 2, 3)
2·7	(0; 2, 2, 2, 4)
3·5	(0; 2, 2, 2, 6)
2·11	(0; 2, 2, 3, 4)

Hereafter, we assume that the index $n \geq 2$.

Using the signature of $\Gamma^*(A, O)$ and the equalities

$$\text{vol}(H/\Gamma)=2-\sum_{i=1}^{4}\frac{1}{e_i}=n\cdot\text{vol}(H/\Gamma^*(A, O)) \tag{3.1}$$

we have the necessary conditions on the signature of Γ for each pair $(D(A), n)$ listed in Table 1.

PROPOSITION 1. *The possible signatures* $(0; e_1, e_2, e_3, e_4)$ *of the subgroups* Γ *of* $\Gamma^*(A, O)$ *is as follows*:

$D(A)=2\cdot 3$	signature of $\Gamma^*(A, O)$: (0; 2, 4, 6)
n	signature of Γ
2	(0; 2, 2, 2, 3)
3	(0; 2, 2, 2, 4)
4	(0; 2, 2, 2, 6), (0; 2, 2, 3, 3)
5	(0; 2, 2, 3, 4)
6	(0; 2, 2, 3, 6), (0; 2, 2, 4, 4), (0; 2, 3, 3, 3)
7	(0; 2, 2, 4, 6), (0; 2, 3, 3, 4)
8	(0; 2, 2, 6, 6), (0; 2, 3, 3, 6), (0; 2, 3, 4, 4), (0; 3, 3, 3, 3)
9	(0; 2, 3, 4, 6), (0; 2, 4, 4, 4), (0; 3, 3, 3, 4)
10	(0; 2, 3, 6. 6), (0; 2, 4, 4, 6), (0; 3, 3, 3, 6), (0; 3, 3, 4, 4)
11	(0; 2, 4, 6, 6), (0; 3, 3, 4, 6), (0; 3, 4, 4, 4)
12	(0; 2, 6, 6, 6), (0; 3, 3, 6, 6), (0; 3, 4, 4, 6), (0; 4, 4, 4, 4)
13	(0; 3, 4, 6, 6), (0; 4, 4, 4, 6)
14	(0; 3, 6, 6, 6), (0; 4, 4, 6, 6)
15	(0; 4, 6, 6, 6)
16	(0; 6, 6, 6, 6)

$D(A)=2\cdot 5$	signature of $\Gamma^*(A, O)$: (0; 2, 2, 2, 3)
n	signature of Γ
2	(0; 2, 2, 2, 6), (0; 2, 2, 3, 3)
3	(0; 2, 2, 3, 6), (0; 2, 2, 4, 4), (0; 2, 3, 3, 3)
4	(0; 2, 2, 6, 6), (0; 2, 3, 3, 6), (0; 2, 3, 4, 4), (0; 3, 3, 3, 3)
5	(0; 2, 3, 6, 6), (0; 2, 4, 4, 6), (0; 3, 3, 3, 6), (0; 3, 3, 4, 4)
6	(0; 2, 6, 6, 6), (0; 3, 3, 6, 6), (0; 3, 4, 4, 6), (0; 4, 4, 4, 4)
7	(0; 3, 6, 6, 6), (0; 4, 4, 6, 6)
8	(0; 6, 6, 6, 6)

$D(A)=2\cdot 7$	signature of $\Gamma^*(A, O)$: (0; 2, 2, 2, 4)
n	signature of Γ
2	(0; 2, 2, 3, 6), (0; 2, 2, 4, 4), (0; 2, 3, 3, 3)
3	(0; 2, 3, 4, 6), (0; 2, 4, 4, 4), (0; 3, 3, 3, 4)
4	(0; 2, 6, 6, 6), (0; 3, 3, 6, 6), (0; 3, 4, 4, 6), (0; 4, 4, 4, 4)
5	(0; 4, 6, 6, 6)

$D(A)=3\cdot 5$	signature of $\Gamma^*(A, O)$: (0; 2, 2, 2, 6)
n	signature of Γ
2	(0; 2, 2, 6, 6), (0; 2, 3, 3, 6), (0; 2, 3, 4, 4), (0; 3, 3, 3, 3)
3	(0; 2, 6, 6, 6), (0; 3, 3, 6, 6), (0; 3, 4, 4, 6), (0; 4, 4, 4, 4)
4	(0; 6, 6, 6, 6)

$D(A)=3\cdot 7$	signature of $\Gamma^*(A, O)$: (0; 2, 2, 2, 2, 3)
n	signature of Γ
2	(0; 2, 6, 6, 6), (0; 3, 3, 6, 6), (0; 3, 4, 4, 6), (0; 4, 4, 4, 4)

$D(A)=2\cdot 11$	signature of $\Gamma^*(A, O)$: (0; 2, 2, 3, 4)
n	signature of Γ
2	(0; 2, 3, 6, 6), (0; 2, 4, 4, 6), (0; 3, 3, 3, 6), (0; 3, 3, 4, 4)
3	(0; 4, 6, 6, 6)

$D(A)=2\cdot 13$	signature of $\Gamma^*(A, O)$: (0; 2, 2, 2, 2, 2)
n	signature of Γ
2	(0; 2, 6, 6, 6), (0; 3, 3, 6, 6), (0; 3, 4, 4, 6), (0; 4, 4, 4, 4)

$D(A)=2\cdot 17$	signature of $\Gamma^*(A, O)$: (0; 2, 2, 2, 2, 3)
n	signature of Γ
2	(0; 6, 6, 6, 6)

$D(A)=2\cdot 3\cdot 5\cdot 7$	signature of $\Gamma^*(A, O)$: (0; 2, 2, 2, 2, 2)
n	signature of Γ
2	(0; 2, 6, 6, 6), (0; 3, 3, 6, 6), (0; 3, 4, 4, 6), (0; 4, 4, 4, 4)

PROOF. We get this result by solving the equation obtained from (3.1) and the data listed in Theorem 4. We note that $e_i=2, 3, 4, 6$. By virtue of this fact, we can find all solutions for the equation

$$2-\left(\frac{1}{e_1}+\frac{1}{e_2}+\frac{1}{e_3}+\frac{1}{e_4}\right)=n\cdot \mathrm{vol}(H/\Gamma^*(A, O)) . \qquad \text{Q.E.D.}$$

Now we need the following Theorem.

THEOREM 5 (Singerman [9]). *Let Γ be a Fuchsian group of the first kind with signature $(g; m_1, m_2, \cdots, m_r; s)$ which satisfies*

$$\Gamma=\left\langle a_1, b_1, \cdots, a_g, b_g, x_1, \cdots, x_r, p_1, \cdots, p_s \;\middle|\; \prod_{i=1}^{g}[a_i, b_i]\prod_{j=1}^{r}x_j\prod_{k=1}^{s}p_k=x_j^{m_j}=1\right\rangle .$$

Then Γ contains a subgroup Γ_1 of index N with signature $(g'\colon n_{11}, \cdots, n_{1\rho_1}, \cdots, n_{r\rho_r}; s')$ if and only if

(1) *There exist a permutation group G transitive on N letters and a surjective homomorphism $\theta : \Gamma \to G$ satisfying the following conditions:*

a) *The permutation $\theta(x_j)$ has precisely ρ_j cycles of lengths less than m_j, the lengths of these cycles being $m_j/n_{j1}, \cdots, m_j/n_{j\rho_j}$.*

b) *If we denote the number of cycles in the permutation $\theta(\gamma)$ by $\delta(\gamma)$ then*

$$s' = \sum_{k=1}^{s} \delta(p_k) .$$

(2)

$$\mathrm{vol}(H/\Gamma_1) = N \cdot \mathrm{vol}(H/\Gamma) .$$

By this theorem, we can determine the signature of Γ. Furthermore, in order to determine all Γ up to $\Gamma^*(A, O)$-conjugation, we need the following proposition.

PROPOSITION 2. *Let Γ^* be a Fuchsian group and θ_i $(i=1, 2)$ be an injective homomorphism from Γ^* to the symmetric group S_n of degree n, whose image $G_i = \theta_i(\Gamma^*)$ in S_n acts transitively. Let H_i be the stabilizing subgroup of G_i at 1, and put $\Gamma_i = \theta_i^{-1}(H_i)$. Then there exists an element $\gamma_0 \in \Gamma^*$ such that $\Gamma_2 = \gamma_0 \Gamma_1 \gamma_0^{-1}$ if and only if there exists an element $\sigma_0 \in S_n$ such that*

$$\theta_2(\gamma) = \sigma_0 \theta_1(\gamma) \sigma_0^{-1} \qquad \textit{for all } \gamma \in \Gamma^* .$$

PROOF. First we assume that $\Gamma_2 = \gamma_0 \Gamma_1 \gamma_0^{-1}$. For left coset decomposition $\Gamma^* = \bigcup_{l=1}^{n} \delta_l \Gamma_1$, suppose that an element $\gamma \in \Gamma^*$ transfers the left coset $\delta_j \Gamma_1$ to $\delta_k \Gamma_1$, i.e.

$$\gamma \delta_j \Gamma_1 = \delta_k \Gamma_1 .$$

This implies $\theta_1(\gamma)(j) = k (j, k \in \{1, 2, \cdots, n\})$. We can choose representatives $\{\delta_j'\}$ of left coset decomposition by Γ_2 such that $\delta_j' = \gamma_0 \delta_j \gamma_0^{-1}$. For this left coset decomposition, assume that

$$\gamma \delta_j' \Gamma_2 = \delta_m' \Gamma_2 .$$

Then we have $\gamma \gamma_0 \delta_j \Gamma_1 = \gamma_0 \delta_m \Gamma_1$. These imply $\theta_2(\gamma)(j) = m$, $\theta_1(\gamma_0^{-1} \gamma \gamma_0)(j) = m$. Hence we obtain

$$\theta_2(\gamma)(j) = \theta_1(\gamma_0^{-1} \gamma \gamma_0)(j) .$$

Since this equality holds for $1 \leq j \leq n$,

$$\theta_2(\gamma) = \theta_1(\gamma_0^{-1} \gamma \gamma_0) .$$

Therefore, putting $\sigma = \theta(\gamma_0)^{-1}$, we have

$$\theta_2(\gamma) = \sigma \theta_1(\gamma) \sigma^{-1} \qquad \text{for all} \quad \gamma \in \Gamma^* .$$

Conversely, suppose that $\theta_2(\gamma)=\sigma\theta(\gamma)\sigma^{-1}$, $\sigma\in S_n$, $\gamma\in\Gamma^*$. Let H_2 fix k and H_1 fix j. Since G_j acts transitively, there exists an element $\rho\in G_1$ such that $\rho\sigma^{-1}(k)=j$. By assumption, $\sigma^{-1}H_2\sigma\subset G_1$, so we obtain $\sigma^{-1}H_2\sigma=\rho^{-1}H_1\rho$. Therefore,

$$\begin{aligned}\Gamma_2=\theta_2^{-1}(H_2)&=\{\gamma\in\Gamma^* \mid \theta_2(\gamma)\in H_2\}=\{\gamma\in\Gamma^* \mid \sigma\theta_1(\gamma)\sigma^{-1}\in H_2\}\\&=\{\gamma\in\Gamma^* \mid \theta_1(\gamma)\in\sigma^{-1}H_2\sigma\}=\{\gamma\in\Gamma^* \mid \theta_1(\gamma)\in\rho^{-1}H_1\rho,\ \rho\in G_1\}\\&=\{\gamma\in\Gamma^* \mid \theta_1(\gamma_0)\theta_1(\gamma)\theta_1(\gamma_0)^{-1}\in H_1\}=\{\gamma\in\Gamma^* \mid \gamma_0\gamma\gamma_0^{-1}\in\theta_1^{-1}(H_1)\}\\&=\gamma_0^{-1}\theta_1^{-1}(H_1)\gamma_0=\gamma_0^{-1}\Gamma_1\gamma_0\end{aligned}$$

Q.E.D

By this proposition, we can classify the subgroups Γ of $\Gamma^*(A, O)$ up to $\Gamma^*(A, O)$-conjugation by giving the homomorphic images in S_n of the generators of $\Gamma^*(A, O)$. So we shall give the homomorphisms θ of $\Gamma^*(A, O)$ into S_n by determining the images of the generators of $\Gamma^*(A, O)$.

THEOREM 6. *Let notations be the same as before. The complete list of the subgroups Γ of $\Gamma^*(A, O)$ with signature $(0; e_1, e_2, e_3, e_4)$ up to $\Gamma^*(A, O)$-conjugation, and the homomorphisms $\theta : \Gamma^*(A, O)\to S_n$ is as follows*:

$D(A)=2\cdot 3$ |	$\Gamma^*(A, O)=\langle\gamma_1, \gamma_2, \gamma_3 \mid \gamma_1^2=\gamma_2^4=\gamma_3^6=\gamma_1\gamma_2\gamma_3=1\rangle$	
n	homomorphism $\theta: \Gamma^*(A, O)\to S_n$	signature of Γ
2	$\theta(\gamma_1)=(1)\ (2)$ $\theta(\gamma_2)=(1\ 2)$ $\theta(\gamma_3)=(1\ 2)$	(0; 2, 2, 2, 3)
3	$\theta(\gamma_1)=(1\ 2)\ (3)$ $\theta(\gamma_2)=(1\ 3)\ (2)$ $\theta(\gamma_3)=(1\ 2\ 3)$	(0; 2, 2, 2, 4)
4	$\theta(\gamma_1)=(1\ 2)\ (3)\ (4)$ $\theta(\gamma_2)=(1\ 2\ 3\ 4)$ $\theta(\gamma_3)=(1)\ (2\ 4\ 3)$	(0; 2, 2, 2, 6)
4	$\theta(\gamma_1)=(1\ 2)\ (3)\ (4)$ $\theta(\gamma_2)=(1\ 3\ 2\ 4)$ $\theta(\gamma_3)=(1\ 3)\ (2\ 4)$	(0; 2, 2, 3, 3)
	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)$ $\theta(\gamma_2)=(1\ 3)\ (2\ 4)$ $\theta(\gamma_3)=(1\ 4)\ (2\ 3)$	
5	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5)$ $\theta(\gamma_2)=(1\ 3\ 5\ 4)\ (2)$ $\theta(\gamma_3)=(1\ 2\ 4)\ (3\ 5)$	(0; 2, 2, 3, 4)

n	homomorphism $\theta: \Gamma^*(A, O) \to S_n$	signature of Γ
6	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)$ $\theta(\gamma_2)=(1)\ (3)\ (2\ 5\ 4\ 6)$ $\theta(\gamma_3)=(1\ 6\ 2)\ (3\ 5\ 4)$	(0; 2, 2, 4, 4)
	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)$ $\theta(\gamma_2)=(1)\ (3)\ (2\ 5)\ (4\ 6)$ $\theta(\gamma_3)=(1\ 5\ 4\ 3\ 6\ 2)$	
	$\theta(\gamma_1)=(1\ 2)(3\ 4)\ (5)\ (6)$ $\theta(\gamma_2)=(1)\ (3)\ (2\ 5\ 4\ 6)$ $\theta(\gamma_3)=(1\ 6\ 4\ 3\ 5\ 2)$	
	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5)\ (6)$ $\theta(\gamma_2)=(1)\ (3)\ (2\ 4\ 5\ 6)$ $\theta(\gamma_3)=(1\ 6\ 5\ 4\ 3\ 2)$	
6	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)$ $\theta(\gamma_2)=(1\ 2\ 3\ 5)\ (4\ 6)$ $\theta(\gamma_3)=(1)\ (2\ 5\ 4)\ (3\ 6)$	(0; 2, 2, 3, 6)
7	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7)$ $\theta(\gamma_2)=(1\ 2\ 3\ 5)\ (4\ 7)\ (6)$ $\theta(\gamma_3)=(1)\ (2\ 5\ 6\ 3\ 7\ 4)$	(0; 2, 2, 4, 6)
	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7)$ $\theta(\gamma_2)=(1\ 2\ 3\ 5)\ (6\ 7)\ (4)$ $\theta(\gamma_3)=(1)\ (2\ 5\ 7\ 6\ 3\ 4)$	
	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7)$ $\theta(\gamma_2)=(1\ 2\ 3\ 7)\ (4\ 5)\ (6)$ $\theta(\gamma_3)=(1)\ (2\ 7\ 3\ 5\ 6\ 4)$	
	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7)$ $\theta(\gamma_2)=(1\ 2\ 7\ 3)\ (4\ 5)\ (6)$ $\theta(\gamma_3)=(1)\ (2\ 3\ 5\ 6\ 4\ 7)$	
8	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7)\ (8)$ $\theta(\gamma_2)=(1\ 2\ 5\ 7)\ (3\ 4\ 6\ 8)$ $\theta(\gamma_3)=(1)\ (3)\ (2\ 7\ 5\ 4\ 8\ 6)$	(0; 2, 2, 6, 6)
	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)$ $\theta(\gamma_2)=(1\ 2\ 5\ 7)\ (3\ 4\ 6\ 8)$ $\theta(\gamma_3)=(1)\ (3)\ (2\ 7\ 6)\ (4\ 8\ 5)$	
	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7)\ (8)$ $\theta(\gamma_2)=(1\ 2\ 7\ 5)\ (3\ 4\ 8\ 6)$ $\theta(\gamma_3)=(1)\ (3)\ (2\ 5\ 8\ 4\ 6\ 7)$	

n	homomorphism $\theta: \Gamma^*(A, O) \to S_n$	signature of Γ
8	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7)\ (8)$ $\theta(\gamma_2)=(1\ 2\ 5\ 7)\ (3\ 4\ 8\ 6)$ $\theta(\gamma_3)=(1)\ (3)\ (2\ 7\ 5\ 8\ 4\ 6)$	(0; 2, 2, 6, 6)
8	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)$ $\theta(\gamma_2)=(1\ 3\ 5\ 7)\ (2\ 4)\ (6)\ (8)$ $\theta(\gamma_3)=(1\ 4)\ (2\ 7\ 8\ 5\ 6\ 3)$	(0; 2, 3, 4, 4)
8	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)$ $\theta(\gamma_2)=(1\ 3\ 5\ 7)\ (2\ 8\ 6\ 4)$ $\theta(\gamma_3)=(1\ 4)\ (2\ 7)\ (3\ 6)\ (5\ 8)$	(0; 3, 3, 3, 3)
9	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9)$ $\theta(\gamma_2)=(1\ 2\ 3\ 5)\ (4\ 7\ 9\ 6)\ (8)$ $\theta(\gamma_3)=(1)\ (2\ 5\ 9\ 7\ 8\ 4)\ (3\ 6)$	(0; 2, 3, 4, 6)
	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9)$ $\theta(\gamma_2)=(1\ 2\ 3\ 5)\ (4\ 9\ 7\ 6)\ (8)$ $\theta(\gamma_3)=(1)\ (2\ 5\ 7\ 8\ 9\ 4)\ (3\ 6)$	
	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9)$ $\theta(\gamma_2)=(1\ 2\ 3\ 7)\ (4\ 5\ 9\ 6)\ (8)$ $\theta(\gamma_3)=(1)\ (2\ 7\ 8\ 3\ 6\ 4)\ (5\ 9)$	
	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9)$ $\theta(\gamma_2)=(1\ 2\ 7\ 3)\ (4\ 5\ 9\ 6)\ (8)$ $\theta(\gamma_3)=(1)\ (2\ 3\ 6\ 4\ 7\ 8)\ (5\ 9)$	
10	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9\ 10)$ $\theta(\gamma_2)=(1\ 2\ 3\ 5)\ (4\ 6\ 7\ 9)\ (8)\ (10)$ $\theta(\gamma_3)=(1)\ (2\ 5\ 4)\ (3\ 9\ 10\ 7\ 8\ 6)$	(0; 2, 4, 4, 6)
	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9\ 10)$ $\theta(\gamma_2)=(1\ 2\ 3\ 7)\ (4\ 5\ 9\ 6)\ (8)\ (10)$ $\theta(\gamma_3)=(1)\ (2\ 7\ 8\ 3\ 6\ 4)\ (5\ 9\ 10)$	
	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9\ 10)$ $\theta(\gamma_2)=(1\ 2\ 7\ 3)\ (4\ 5\ 9\ 6)\ (8)\ (10)$ $\theta(\gamma_3)=(1)\ (2\ 3\ 6\ 4\ 7\ 8)\ (5\ 9\ 10)$	
10	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9\ 10)$ $\theta(\gamma_2)=(1\ 3\ 5\ 7)\ (2\ 9\ 6\ 4)\ (8)\ (10)$ $\theta(\gamma_3)=(1\ 4)\ (2\ 7\ 8\ 5\ 9\ 10)\ (3\ 6)$	(0; 3, 3, 4, 4)
12	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9\ 10)\ (11\ 12)$ $\theta(\gamma_2)=(1\ 2\ 7\ 9)\ (3\ 4\ 11\ 8)\ (5\ 6\ 10\ 12)$ $\theta(\gamma_3)=(1)\ (3)\ (5)\ (2\ 9\ 6\ 12\ 4\ 8)\ (7\ 11\ 10)$	(0; 2, 6, 6, 6)

n	homomorphism $\theta: \Gamma^*(A, O) \to S_n$	signature of Γ
12	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9\ 10)\ (11\ 12)$ $\theta(\gamma_2)=(1\ 2\ 5\ 7)\ (3\ 4\ 9\ 11)\ (6\ 12\ 10\ 8)$ $\theta(\gamma_3)=(1)\ (3)\ (2\ 7\ 10\ 4\ 11\ 6)\ (5\ 8)\ (9\ 12)$	(0; 3, 3, 6, 6)
12	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9\ 10)\ (11\ 12)$ $\theta(\gamma_2)=(1\ 3\ 5\ 7)\ (2\ 4\ 9\ 11)\ (6)\ (8)\ (10)\ (12)$ $\theta(\gamma_3)=(1\ 11\ 12\ 9\ 10\ 4)\ (2\ 7\ 8\ 5\ 6\ 3)$	(0; 4, 4, 4, 4)
	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9\ 10)\ (11\ 12)$ $\theta(\gamma_2)=(1\ 5\ 3\ 7)\ (2\ 9\ 4\ 11)\ (6)\ (8)\ (10)\ (12)$ $\theta(\gamma_3)=(1\ 11\ 12\ 4\ 5\ 6)\ (2\ 7\ 8\ 3\ 9\ 10)$	
14	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9\ 10)\ (11\ 12)\ (13\ 14)$ $\theta(\gamma_2)=(1\ 2\ 5\ 7)\ (3\ 4\ 6\ 9)\ (8\ 11\ 10\ 13)\ (12)\ (14)$ $\theta(\gamma_3)=(1)\ (2\ 7\ 13\ 14\ 10\ 6)\ (3)\ (4\ 9\ 11\ 12\ 8\ 5)$	(0; 4, 4, 6, 6)
	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9\ 10)\ (11\ 12)\ (13\ 14)$ $\theta(\gamma_2)=(1\ 2\ 5\ 7)\ (3\ 4\ 9\ 8)\ (6\ 11\ 10\ 13)\ (12)\ (14)$ $\theta(\gamma_3)=(1)\ (2\ 7\ 9\ 11\ 12\ 6)\ (3)\ (4\ 8\ 5\ 13\ 14\ 10)$	
	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9\ 10)\ (11\ 12)\ (13\ 14)$ $\theta(\gamma_2)=(1\ 2\ 5\ 11)\ (3\ 4\ 7\ 13)\ (6\ 9\ 8\ 10)\ (12)\ (14)$ $\theta(\gamma_3)=(1)\ (2\ 11\ 12\ 5\ 10\ 6)\ (3)\ (4\ 13\ 14\ 7\ 9\ 8)$	
	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9\ 10)\ (11\ 12)\ (13\ 14)$ $\theta(\gamma_2)=(1\ 2\ 11\ 5)\ (3\ 4\ 13\ 7)\ (6\ 9\ 8\ 10)\ (12)\ (14)$ $\theta(\gamma_3)=(1)\ (2\ 5\ 10\ 6\ 11\ 12)\ (3)\ (4\ 7\ 9\ 8\ 13\ 14)$	
	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9\ 10)\ (11\ 12)\ (13\ 14)$ $\theta(\gamma_2)=(1\ 2\ 5\ 11)\ (3\ 4\ 13\ 7)\ (6\ 9\ 8\ 10)\ (12)\ (14)$ $\theta(\gamma_3)=(1)\ (2\ 11\ 12\ 5\ 10\ 6)\ (3)\ (4\ 7\ 9\ 8\ 13\ 14)$	
16	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9\ 10)\ (11\ 12)\ (13\ 14)\ (15\ 16)$ $\theta(\gamma_2)=(1\ 2\ 9\ 11)\ (3\ 4\ 10\ 13)\ (5\ 6\ 12\ 15)\ (7\ 8\ 14\ 16)$ $\theta(\gamma_3)=(1)\ (2\ 11\ 6\ 15\ 14\ 10)\ (3)\ (4\ 13\ 8\ 16\ 12\ 9)\ (5)\ (7)$	(0; 6, 6, 6, 6)
	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ (9\ 10)\ (11\ 12)\ (13\ 14)\ (15\ 16)$ $\theta(\gamma_2)=(1\ 2\ 9\ 11)\ (3\ 4\ 10\ 13)\ (5\ 6\ 15\ 12)\ (7\ 8\ 16\ 14)$ $\theta(\gamma_3)=(1)\ (2\ 11\ 15\ 8\ 14\ 10)\ (3)\ (4\ 13\ 16\ 6\ 12\ 9)\ (5)\ (7)$	

$D(A)=2\cdot 5 \mid \Gamma^*(A, O)=\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \mid \gamma_1^2=\gamma_2^2=\gamma_3^2=\gamma_4^3=\gamma_1\gamma_2\gamma_3\gamma_4=1\rangle$

n	homomorphism $\theta: \Gamma^*(A, O) \to S_n$	signature of Γ
1		(0; 2, 2, 2, 3)
2	$\theta(\gamma_1)=(1)\ (2)$ $\theta(\gamma_2)=(1\ 2)$ $\theta(\gamma_3)=(1\ 2)$ $\theta(\gamma_4)=(1)\ (2)$	(0; 2, 2, 3, 3)

n	homomorphism $\theta: \Gamma^*(A, O) \to S_n$	signature of Γ
4	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)$ $\theta(\gamma_2)=(1\ 3)\ (2\ 4)$ $\theta(\gamma_3)=(1\ 4)\ (2\ 3)$ $\theta(\gamma_4)=(1)\ (2)\ (3)\ (4)$	(0; 3, 3, 3, 3)

$D(A)=2\cdot 7 \mid \Gamma^*(A, O)=\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \mid \gamma_1^2=\gamma_2^2=\gamma_3^2=\gamma_4^4=\gamma_1\gamma_2\gamma_3\gamma_4=1\rangle$

n	homomorphism $\theta: \Gamma^*(A, O) \to S_n$	signature of Γ
1		(0; 2, 2, 2, 4)
2	$\theta(\gamma_1)=(1)\ (2)$ $\theta(\gamma_2)=(1\ 2)$ $\theta(\gamma_3)=(1\ 2)$ $\theta(\gamma_4)=(1)\ (2)$	(0; 2, 2, 4, 4)
4	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)$ $\theta(\gamma_2)=(1\ 3)\ (2\ 4)$ $\theta(\gamma_3)=(1\ 4)\ (2\ 3)$ $\theta(\gamma_4)=(1)\ (2)\ (3)\ (4)$	(0; 4, 4, 4, 4)

$D(A)=3\cdot 5 \mid \Gamma^*(A, O)=\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \mid \gamma_1^2=\gamma_2^2=\gamma_3^2=\gamma_4^6=\gamma_1\gamma_2\gamma_3\gamma_4=1\rangle$

n	homomorphism $\theta: \Gamma^*(A, O) \to S_n$	signature of Γ
1		(0; 2, 2, 2, 6)
2	$\theta(\gamma_1)=(1)\ (2)$ $\theta(\gamma_2)=(1\ 2)$ $\theta(\gamma_3)=(1\ 2)$ $\theta(\gamma_4)=(1)\ (2)$	(0; 2, 2, 6, 6)
4	$\theta(\gamma_1)=(1\ 2)\ (3\ 4)$ $\theta(\gamma_2)=(1\ 3)\ (2\ 4)$ $\theta(\gamma_3)=(1\ 4)\ (2\ 3)$ $\theta(\gamma_4)=(1)\ (2)\ (3)\ (4)$	(0; 6, 6, 6, 6)

$D(A)=2\cdot 11 \mid \Gamma^*(A, O)=\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \mid \gamma_1^2=\gamma_2^2=\gamma_3^3=\gamma_4^4=\gamma_1\gamma_2\gamma_3\gamma_4=1\rangle$

n	homomorphism $\theta: \Gamma^*(A, O) \to S_n$	signature of Γ
1		(0; 2, 2, 3, 4)
2	$\theta(\gamma_1)=(1\ 2)$ $\theta(\gamma_2)=(1\ 2)$ $\theta(\gamma_3)=(1)\ (2)$ $\theta(\gamma_4)=(1)\ (2)$	(0; 3, 3, 4, 4)

PROOF. It is sufficient to verify these results for each pair $(D(A), n)$ listed in Proposition 1. We shall give a brief proof of the theorem by taking the case $D(A)=2\cdot 3$, $n=6$ and the signature (0; 2, 2, 4, 4). By Theorem 5, we must find the integers $n_{ij}\in\{1, 2, 4, 6\}$ such that

$$6=\sum_{j=1}^{\rho_1}\frac{2}{n_{1_j}}=\sum_{j=1}^{\rho_2}\frac{4}{n_{2_j}}=\sum_{j=1}^{\rho_3}\frac{6}{n_{3_j}}, \qquad n_{1_j}|2\,, \quad n_{2_j}|4\,, \quad n_{3_j}|6\,.$$

In this case, we get the following 3 solutions:

$$\text{(i)} \qquad 6=\frac{2}{1}+\frac{2}{1}+\frac{2}{1}=\frac{4}{1}+\frac{4}{4}+\frac{4}{4}=\frac{6}{2}+\frac{6}{2}\,,$$

$$\text{(ii)} \qquad 6=\frac{2}{1}+\frac{2}{1}+\frac{2}{1}=\frac{4}{4}+\frac{4}{4}+\frac{4}{2}+\frac{4}{2}=\frac{6}{1}\,,$$

$$\text{(iii)} \qquad 6=\frac{2}{1}+\frac{2}{1}+\frac{2}{2}+\frac{2}{2}=\frac{4}{4}+\frac{4}{4}+\frac{4}{1}=\frac{6}{1}\,.$$

From this, we have the following result:

(i) $\theta(\gamma_1)$ is of type [2, 2, 2], $\theta(\gamma_2)$ is of type [1, 1, 4] and $\theta(\gamma_3)$ is of type [3, 3],

(ii) $\theta(\gamma_1)$ is of type [2, 2, 2], $\theta(\gamma_2)$ is of type [1, 1, 2, 2], $\theta(\gamma_3)$ is of type [6],

(iii) $\theta(\gamma_1)$ is of type [1, 1, 2, 2], $\theta(\gamma_2)$ is of type [1, 1, 4] and $\theta(\gamma_3)$ is of type [6],

where the permutation σ is of type $[n_1, n_2, \cdots, n_r]$ if σ is the product of disjoint r cycles of length n_j $(1\le j\le r)$. In the case (i), we may assume that $\theta(\gamma_1)=(1\ 2)(3\ 4)(5\ 6)$ and that $\theta(\gamma_2)$ fixes the letters 1 and 3. Then we have $\theta(\gamma_2)=(1)(3)(2\ 5\ 4\ 6)$. Otherwise we find that $\theta(\gamma_3)$ cannot be of type [3, 3], which is a contradiction. Hence we have $\theta(\gamma_3)=(1\ 6\ 2)(3\ 5\ 4)$. In the case (ii), we may also assume that $\theta(\gamma_1)=(1\ 2)(3\ 4)(5\ 6)$ and that $\theta(\gamma_2)$ fixes the letters 1 and 3. Then we have $\theta(\gamma_2)=(1)(3)(2\ 5)(4\ 6)$. Otherwise we have $\theta(\gamma_3)$ contain (5 6) and this contradicts the assumption that $\theta(\Gamma^*(A, O))$ is a transitive subgroup of S_n. So we have $\theta(\gamma_3)=(1\ 5\ 4\ 3\ 6\ 2)$. In the case (iii), we may assume that $\theta(\gamma_1)=(1\ 2)(3\ 4)(5)(6)$ and $\theta(\gamma_2)$ fixes the letter 1 and 3. Then we have $\theta(\gamma_2)=(1)(3)(2\ 4\ 5\ 6)$, $(1)(3)(2\ 5\ 6\ 4)$. This implies that $\theta(\gamma_3)=(1\ 6\ 5\ 4\ 3\ 2)$, $(1\ 6\ 4\ 3\ 5\ 2)$, respectively. Hence we have

	$\theta(\gamma_1)$	$\theta(\gamma_2)$	$\theta(\gamma_3)$
(i)	(1 2) (3 4) (5 6)	(1) (3) (2 5 4 6)	(1 6 2) (3 5 4)
(ii)	(1 2) (3 4) (5 6)	(1) (3) (2 5) (4 6)	(1 5 4 3 2 6)
(iii)	(1 2) (3 4) (5 6) (1 2) (3 4) (5 6)	(1) (3) (2 4 5 6) (1) (3) (2 5 6 4)	(1 6 5 4 3 2) (1 6 4 3 5 2)

Next we take the signature (0; 2, 3, 3, 3). In this case, there are no solutions n_{ij}. Therefore this case never occurs. We can verify the result for other cases just in a similar way.

Q.E.D.

References

[1] A. F. BEARDON, *The Geometry of Discrete Groups*, Graduate Text in Math. **91** (1983), Springer.

[2] B. IVERSEN, *Hyperbolic Geometry*, London Math. Soc. Student Text **25** (1992), Cambrige Univ. Press.

[3] J. LEHNER, *Discontinuous Groups and Automorphic Functions*, Math. Surveys **8** (1964), Amer. Math. Soc.

[4] J. F. MICHON, Courbes de Shimura hyperelliptiques, Bull. Soc. Math. France **109** (1981), 217–225.

[5] H. SHIMIZU, On discontinuous groups operating on the product of the upper half planes, Ann. of Math. **77** (1963), 33–71.

[6] H. SHIMIZU, On zeta functions of quaternion algebras, Ann. of Math. **81** (1965), 166–193.

[7] G. SHIMURA, Construction of class fields and zeta functions of algebraic curves, Ann. of Math. **85** (1967), 58–159.

[8] G. SHIMURA, On Dirichlet series and Abelian varieties attached to automorphic forms, Ann. of Math. **72** (1962), 237–294.

[9] D. SINGERMAN, Subgroups of Fuchsian groups and finite permutation groups, Bull. London Math. Soc. **2** (1970), 319–323.

[10] K. TAKEUCHI, Arithmetic triangle groups, J. Math. Soc. Japan **29** (1977), 91–106.

[11] K. TAKEUCHI, Arithmetic Fuchsian groups with signature $(1; e)$, J. Math. Soc. Japan **35** (1983), 381–407.

[12] K. TAKEUCHI, Subgroups of the modular group with signature $(0; e_1, e_2, e_3, e_4)$, Saitama Math. J. **14** (1996), 55–78.

[13] M. F. VIGNÉRAS, *Arithmétique des Algèbres des Quaternions*, Lecture Notes in Math. **800** (1980), Springer.

Present Address:
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, SAITAMA UNIVERSITY,
URAWA, SAITAMA, 338 JAPAN.
e-mail: jsunaga@rimath.saitama-u.ac.jp