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On the Cyclotomic Unit Group and the p-Ideal Class Group of a Real Abelian Number Field

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1. Introduction.

Let p be an odd prime number, which will be fixed throughout the present paper. For any real abelian number field K, let K_∞ denote the cyclotomic $\mathbf{Z}_p$-extension of K and K_n its n-th layer over K. Let A_n and $A'_n = A_n/(\langle$ideal classes of K_n which contain a prime ideal above $p\rangle \cap A_n)$ be the p-Sylow subgroups of the ideal class group and of the p-ideal class group, respectively, of K_n. Let E_n and C_n be the groups of units and of cyclotomic units in the sense of Sinnott, respectively, of K_n (cf. [7]). Denote by B_n the p-Sylow subgroup of the quotient group E_n/C_n. We write $\lambda_p(K)$ and $\mu_p(K)$ for the Iwasawa λ and μ invariants, respectively, of K_∞/K.

It is well known that the order of A_n and B_n are "almost" equal. For example, if $p \nmid [K:\mathbf{Q}]$ then $\sharp(A_n) = \sharp(B_n)$ (cf. [7]). Furthermore, the Iwasawa main conjecture proved by B. Mazur and A. Wiles implies that the characteristic ideals of $\mathbf{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$-modules $\varprojlim A_n$ and $\varprojlim B_n$ coincide, where the projective limits are taken with respect to the norm maps (cf. [6], [3]). So it arises a natural question: Is there any deeper relation between the Galois module structures of A_n and B_n?

In the present paper, we shall give an answer to the above question under the assumption that Greenberg's conjecture (cf. [2]) is valid. Specifically, we shall prove the following:

THEOREM 1. *Let K be a real abelian number field with $p \nmid [K:\mathbf{Q}]$. If we assume that Greenberg's conjecture is valid for K and p, namely, that the Iwasawa invariants $\lambda_p(K)$ and $\mu_p(K)$ vanish, then A'_n is embedded in B_n as a Galois module* (*namely,* $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$*-module*) *for all sufficiently large n.*

We remark that $\mu_p(K)$ always vanishes in the above theorem by the Ferrero-Washington theorem (cf. [1]).

We shall prepare some results about the Galois cohomology groups of cyclotomic unit groups in section 2, and give the proof of Theorem 1 in section 3.

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2. Galois cohomology groups of cyclotomic unit groups.

Let $\Gamma_{m,n}=\mathrm{Gal}(K_m/K_n)$ and $N_{m,n}=N_{K_m/K_n}$ for $m\geq n\geq 0$. In this section, we calculate the Tate cohomology groups $\hat{H}^i(\Gamma_{m,n}, C_m)$ for $i=-1,0$, $m\geq n\geq 0$ by the method of J. M. Kim in [5].

THEOREM 2. *Let K be a real abelian number field and p an odd prime with $p\nmid[K:\mathbf{Q}]$. For $m\geq n\geq 0$, we have*

$$\hat{H}^{-1}(\Gamma_{m,n}, C_m)\simeq(\mathbf{Z}/p^{m-n}\mathbf{Z})^{\oplus s} \quad \text{and} \quad \hat{H}^0(\Gamma_{m,n}, C_m)\simeq(\mathbf{Z}/p^{m-n}\mathbf{Z})^{\oplus s-1},$$

where s stands for the number of primes of K above p.

To prove the above theorem, we show the following:

PROPOSITION. *Let K be a real abelian number field with $p\nmid[K:\mathbf{Q}]$. Then*

$$C_m^{\Gamma_{m,n}}=C_n$$

for all $m\geq n\geq 0$.

PROOF. For a finite abelian group G with $p\nmid\#(G)$, and $\chi\in\mathrm{Hom}(G,\bar{\mathbf{Q}}_p^\times)$, we put

$$\varepsilon_\chi=\frac{1}{\#(G)}\sum_{\sigma\in G}Tr_{\mathbf{Q}_p(\chi(G))/\mathbf{Q}_p}(\chi(\sigma))\sigma^{-1}\in\mathbf{Z}_p[G].$$

Then, for any $\mathbf{Z}_p[G]$-module M, we have $M=\bigoplus\varepsilon_\chi M$, where χ runs over all elements of $\mathrm{Hom}(G,\bar{\mathbf{Q}}_p^\times)$ modulo $\mathbf{Q}_p$-conjugacy.

We put $G=\mathrm{Gal}(K/\mathbf{Q})$. Let $H_\chi=\mathrm{Ker}(\chi)$ for $\chi\in\mathrm{Hom}(G,\bar{\mathbf{Q}}_p^\times)$, and let $K^\chi=K^{H_\chi}$. We denote by K_n^χ the n-th layer of the cyclotomic $\mathbf{Z}_p$-extension over K^χ. Then $K_n^\chi/\mathbf{Q}$ is a cyclic extension with Galois group $G/H_\chi\times\Gamma_{n,0}$. Let $C_{\chi,n}$ be the cyclotomic unit group of K_n^χ for $n\geq 0$. Then $C_{\chi,n}\subseteq C_n$ and $N_{K_n/K_n^\chi}C_n\subseteq C_{\chi,n}$. Since $\chi(H_\chi)=1$, we see that

$$\varepsilon_\chi=\frac{1}{\#(G)}\sum_{\sigma\in G \bmod H_\chi}Tr_{\mathbf{Q}_p(\chi(G))/\mathbf{Q}_p}(\chi(\sigma))\sigma^{-1}\sum_{\tau\in H_\chi}\tau.$$

Hence $\varepsilon_\chi(C_n\otimes\mathbf{Z}_p)=\varepsilon_\chi(C_{\chi,n}\otimes\mathbf{Z}_p)$ for $n\geq 0$. Therefore,

$$C_m^{\Gamma_{m,n}}\otimes\mathbf{Z}_p=\bigoplus\varepsilon_\chi(C_m\otimes\mathbf{Z}_p)^{\Gamma_{m,n}}=\bigoplus\varepsilon_\chi(C_{\chi,m}\otimes\mathbf{Z}_p)^{\Gamma_{m,n}}=\bigoplus\varepsilon_\chi(C_{\chi,m}^{\Gamma_{m,n}}\otimes\mathbf{Z}_p).$$

Since $K_m^\chi/\mathbf{Q}$ is a cyclic extension, we have $C_{\chi,m}^{\Gamma_{m,n}}=C_{\chi,n}$ by Greither's theorem (cf. [4, Satz 2.1]). Hence we obtain

$$C_m^{\Gamma_{m,n}}\otimes\mathbf{Z}_p=\bigoplus\varepsilon_\chi(C_{\chi,n}\otimes\mathbf{Z}_p)=C_n\otimes\mathbf{Z}_p.$$

It follows from the above equation and $(C_m^{\Gamma_{m,n}})^{p^{m-n}}=N_{m,n}(C_m^{\Gamma_{m,n}})\subseteq C_n$ that $C_m^{\Gamma_{m,n}}=C_n$.
□

PROOF OF THEOREM 2. Let $f=p^\delta f'$ ($\delta=0$ or 1, and $p\nmid f'$) be the conductor of K and f_n the conductor of K_n. Then $f_n=p^{n+1}f'$ for $n\geq 1$. Put $\eta_{n,d}=N_{\mathbf{Q}(\zeta_d)/K_n\cap\mathbf{Q}(\zeta_d)}(1-\zeta_d)$ for $d\in\mathbf{N}$ and $n\geq 0$. Let

$$C_n' = \mathbf{Z}[\mathrm{Gal}(K_n/\mathbf{Q})]\eta_{n,p^{n+1}} \cap E_n ,$$
$$C_n'' = \mathbf{Z}[\mathrm{Gal}(K_n/\mathbf{Q})]\langle \eta_{n,dp^{n+1}} \mid d \mid f',\ d \neq 1\rangle,$$
$$C = \mathbf{Z}[\mathrm{Gal}(K/\mathbf{Q})]\langle -1, \eta_{0,d} \mid d \mid f', d \neq 1\rangle \cap E_0 .$$

Since $N_{m,n}\eta_{m,dp^{m+1}} = \eta_{n,dp^{n+1}}$ for $d \in \mathbf{N}$ and $m \geq n \geq 0$, we obtain

$$N_{m,n}C_m' = C_n' \quad \text{and} \quad N_{m,n}C_m'' = C_n'' , \tag{1}$$

in particular, we see that $C_n' \subseteq C_m'$ and $C_n'' \subseteq C_m''$. Then $C_n = CC_n'C_n''$ for $n \geq 0$ by [4, Folgerung 1.2]. We write K_T and $K_Z \subseteq K$ for the inertia and decomposition subfields, respectively, of K for p. Then $K_T = K \cap \mathbf{Q}(\zeta_{f'})$ and $C \subseteq K_T$.

CLAIM 1. $\mathrm{rank}_{\mathbf{Z}} N_{K_T/K_Z} C = s-1$.

PROOF. We denote by E_F the unit group of F for any algebraic number field F. Since the conductor of K_T is f', C is the cyclotomic unit group of K_T. It follows from [7] that E_{K_T}/C is finite. Hence $N_{K_T/K_Z}E_{K_T}/N_{K_T/K_Z}C$ is finite. Therefore $\mathrm{rank}_{\mathbf{Z}} N_{K_T/K_Z} C = \mathrm{rank}_{\mathbf{Z}} N_{K_T/K_Z} E_{K_T} = \mathrm{rank}_{\mathbf{Z}} E_{K_Z} = s-1$. □

CLAIM 2.

$$C^{(\sigma_p - 1)(p-1)} \subseteq C \cap C_n'C_n'' \subseteq \bar{C}$$

for $n \geq 0$, *where* $\sigma_p = \left(\frac{K_T/\mathbf{Q}}{p}\right)$ *and* $\bar{C} = \{\varepsilon \in C \mid N_{K_T/K_Z}\varepsilon = \pm 1\}$.

PROOF. Let $d \mid f'$, $d \neq 1$. Then $(1-\zeta_d)^{\sigma_p - 1} = \prod_{i=1}^{p-1}(1-\zeta_{pd}^{p+id})$ by [5, p. 516]. Taking the norm from $\mathbf{Q}(\zeta_{pd})$ to $\mathbf{Q}(\zeta_d) \cap K$, we find that $\eta_{0,d}^{(\sigma_p - 1)(p-1)} \in C \cap C_0'' \subseteq C \cap C_n''$. Hence we have $C^{(\sigma_p - 1)(p-1)} \subseteq C \cap C_n'C_n''$.

Let $\varepsilon = (\prod_{d \mid f', d \neq 1} \eta_{n,p^{n+1}d}^{\alpha_d})\eta_{n,p^{n+1}}^{\beta}$ be any element in $C \cap C_n'C_n''$, where α_d, $\beta \in \mathbf{Z}[\mathrm{Gal}(K_n/\mathbf{Q})]$. Taking the norm $N_{n,0}$, we have

$$\varepsilon^{p^n} = \left(\prod_{d \mid f', d \neq 1} \eta_{0,pd}^{\alpha_d}\right)\eta_{0,p}^{\beta} .$$

By [5, p. 516], it holds that $N_{\mathbf{Q}(\zeta_{pd})/\mathbf{Q}(\zeta_d)}(1-\zeta_{pd}) = \frac{1-\zeta_d^p}{1-\zeta_d}$. Hence

$$\begin{aligned} N_{K/K_Z}\eta_{0,pd} &= N_{K \cap \mathbf{Q}(\zeta_{pd})/K_Z \cap \mathbf{Q}(\zeta_d)} N_{\mathbf{Q}(\zeta_{pd})/K \cap \mathbf{Q}(\zeta_{pd})}(1-\zeta_{pd})^{[K:K_Z(K \cap \mathbf{Q}(\zeta_{pd}))]} \\ &= N_{K \cap \mathbf{Q}(\zeta_d)/K_Z \cap \mathbf{Q}(\zeta_d)} N_{\mathbf{Q}(\zeta_d)/K \cap \mathbf{Q}(\zeta_d)}(1-\zeta_d)^{(\sigma_p - 1)[K:K_Z(K \cap \mathbf{Q}(\zeta_{pd}))]} = 1 . \end{aligned}$$

Since $N_{K/K_Z}\eta_{0,p} = N_{K \cap \mathbf{Q}(\zeta_p)/\mathbf{Q}}\eta_{0,p}^{[K:K_Z(K \cap \mathbf{Q}(\zeta_p))]} = p^{[K:K_Z(K \cap \mathbf{Q}(\zeta_p))]}$ and $\eta_{0,p}^{\beta}$ is a unit, we have $N_{K/K_Z}\eta_{0,p}^{\beta} = 1$ by the above equation. Therefore $N_{K/K_Z}\varepsilon^{p^n} = N_{K_T/K_Z}\varepsilon^{p^n[K:K_T]} = 1$. Thus we obtain $C \cap C_n'C_n'' \subseteq \bar{C}$. □

By the Proposition and (1), we have

$$\begin{aligned} \hat{H}^0(\Gamma_{m,n}, C_m) &= C_n/N_{m,n}C_m = CC_n'C_n''/C^{p^{m-n}}C_n'C_n'' \\ &\simeq C/C \cap C^{p^{m-n}}C_n'C_n'' = C/C^{p^{m-n}}(C \cap C_n'C_n'') . \end{aligned} \tag{2}$$

CLAIM 3. $C^{p^{m-n}}(C\cap C_n'C_n'')=C^{p^{m-n}}\bar{C}$.

PROOF. Since $p\nmid[K:\mathbf{Q}]$, we see that $\#(\mathrm{Ker}(N_{K_T/K_Z}\colon C\to C)/C^{\sigma_p-1})=\#(\hat{H}^{-1}(\mathrm{Gal}(K_T/K_Z), C))$ is prime to p. So we find $p\nmid\#(\bar{C}/C^{(\sigma_p-1)(p-1)})$. Hence $C^{p^{m-n}}C^{(\sigma_p-1)(p-1)}=C^{p^{m-n}}\bar{C}$. The claim follows from Claim 2 and this equation. □

It follows from Claim 1 that $C\simeq\bar{C}\oplus\mathbf{Z}^{\oplus s-1}$. By (2), Claim 3, and this formula, we see $\hat{H}^0(\Gamma_{m,n}, C_m)\simeq C/C^{p^{m-n}}\bar{C}\simeq(\mathbf{Z}/p^{m-n}\mathbf{Z})^{\oplus s-1}$.

CLAIM 4. $\mathrm{Ker}(N_{m,n}\colon C_m\to C_n)\subseteq N_{l,m}C_l$ *for* $l\geq m\geq n\geq 0$.

PROOF. Let $C=\bar{C}\oplus C_f$, where $C_f\simeq\mathbf{Z}^{\oplus s-1}$. Let $\varepsilon=\varepsilon_1\varepsilon_2\varepsilon_3\in\mathrm{Ker}(N_{m,n}\colon C_m\to C_n)$, where $\varepsilon_1\in\bar{C}$, $\varepsilon_2\in C_f$, and $\varepsilon_3\in C_m'C_m''$. Then $1=N_{m,n}\varepsilon=(\varepsilon_1\varepsilon_2)^{p^{m-n}}N_{m,n}\varepsilon_3$, hence $N_{m,n}\varepsilon_3\in C\cap C_n'C_n''\subseteq\bar{C}$ by Claim 2. Therefore $\varepsilon_2=1$. It follows from (2) and Claim 3 that $\bar{C}\subseteq N_{l,m}C_l$. So we conclude $\varepsilon=\varepsilon_1\varepsilon_3\in\bar{C}C_m'C_m''\subseteq N_{l,m}C_l$. □

From $\hat{H}^0(\Gamma_{m,n}, C_m)\simeq(\mathbf{Z}/p^{m-n}\mathbf{Z})^{\oplus s-1}$, Claim 4, and the same argument as in section 3 of [5], we see that $\hat{H}^{-1}(\Gamma_{m,n}, C_m)\simeq(\mathbf{Z}/p^{m-n}\mathbf{Z})^{\oplus s}$. Thus we have completed the proof of Theorem 2. □

3. Proof of Theorem 1.

Throughout this section, we assume that a real abelian number field K and an odd prime p satisfy the assumption of Theorem 1.

Let I_n and P_n denote the ideal group and the principal ideal group, respectively, of K_n. Denote by $I_n^{(p)}$ the subgroup of I_n such that $A_n=I_n^{(p)}/P_n$ for $n\geq 0$. We write $\mathfrak{P}_{n,i}$ $(1\leq i\leq s)$ for the primes of K_n above p for $n\geq 0$, where s stands for the number of primes of K above p. Here, we note that all primes of K above p are totally ramified in K_∞ since $p\nmid[K:\mathbf{Q}]$. We write t_n for the non-p-part of the class number of K_n, and put $S_n=\langle\mathfrak{P}_{n,i}^{t_n}\mid 1\leq i\leq s\rangle\subseteq I_n^{(p)}$ for $n\geq 0$. We note $t_n|t_m$ for $m\geq n\geq 0$. Let $D_n=S_nP_n/P_n\subseteq A_n$. Then $A_n'\simeq A_n/D_n$.

Taking the cohomology sequence of the exact sequence of Galois modules

$$0\longrightarrow C_m\longrightarrow E_m\longrightarrow E_m/C_m\longrightarrow 0\,,$$

we get the exact sequence of Galois modules

$$0\longrightarrow B_n\longrightarrow B_m^{\Gamma_{m,n}}\longrightarrow\hat{H}^{-1}(\Gamma_{m,n}, C_m)\longrightarrow\hat{H}^{-1}(\Gamma_{m,n}, E_m)\longrightarrow\hat{H}^{-1}(\Gamma_{m,n}, B_m)$$

for $m\geq n\geq 0$. Since $\lambda_p(K)=\mu_p(K)=0$ from the assumption, and $\#(A_n)=\#(B_n)$ (cf. [7]), there exists a number $n_0\geq 0$ such that $B_m=B_{n_0}$ for all $m\geq n_0$, where we regard B_{n_0} as a subgroup of B_m by the injective map induced from the natural inclusion $E_{n_0}\subseteq E_m$: Note that the Proposition provides the injectivity of the map $B_n\to B_m$ for $m\geq n\geq 0$. Hence $\hat{H}^{-1}(\Gamma_{m,n}, B_m)\simeq B_n$ for all $m\geq n\geq n_0$ with $p^{m-n}B_{n_0}=0$. We will identify $\hat{H}^{-1}(\Gamma_{m,n}, E_m)$ with $P_m^{\Gamma_{m,n}}/P_n$ by the Galois module isomorphism

$$P_m^{\Gamma_{m,n}}/P_n\simeq\hat{H}^{-1}(\Gamma_{m,n}, E_m)\,,$$

$$(\alpha)\bmod P_n\mapsto\alpha^{\gamma_{m,n}-1}\bmod E_m^{\gamma_{m,n}-1}\,,$$

where $\gamma_{m,n}\in\Gamma_{m,n}$ is a fixed generator of $\Gamma_{m,n}$. So we have the exact sequence of Galois modules

$$0\longrightarrow \hat{H}^{-1}(\Gamma_{m,n}, C_m)\xrightarrow{f} P_m^{\Gamma_{m,n}}/P_n\longrightarrow B_n \tag{3}$$

for $m\geq n\geq n_0$ with $p^{m-n}B_{n_0}=0$. We shall show in the following that $\mathrm{Coker}(f)\simeq A'_n$ as Galois modules if $m\geq n$ is sufficiently large, which implies Theorem 1.

We shall prepare some lemmas to prove Theorem 1.

LEMMA 1. *Let $n\geq 0$. Then we have*

$$P_m^{\Gamma_{m,n}}/P_n=\langle I_n^{(p)}/P_n, (P_m\cap S_m)P_n/P_n\rangle$$

for sufficiently large $m\geq n$.

PROOF. Since $\lambda_p(K)=\mu_p(K)=0$, $I_n^{(p)}=I_n\cap P_m\subseteq P_m^{\Gamma_{m,n}}$ for sufficiently large $m\geq n$ by [2, Proposition 2]. Let (α) be any ideal in $P_m^{\Gamma_{m,n}}$. Then there exist $\mathfrak{A}\in I_n$ and $\mathfrak{P}\in\langle\mathfrak{P}_{m,i}\mid 1\leq i\leq s\rangle$ such that $(\alpha)=\mathfrak{A}\mathfrak{P}$. Since $\mathfrak{A}^{t_m}\in I_n^{(p)}=I_n\cap P_m$, we see $\mathfrak{P}^{t_m}\in P_m\cap S_m$. Thus we have $(P_m^{\Gamma_{m,n}})^{t_m}\subseteq\langle I_n^{(p)}, P_m\cap S_m\rangle$. Since $p\nmid t_m$ and $P_m^{\Gamma_{m,n}}/P_n$ is a finite p-group, we obtain $P_m^{\Gamma_{m,n}}/P_n\subseteq\langle I_n^{(p)}/P_n, (P_m\cap S_m)P_n/P_n\rangle$. This inclusion and $P_m\cap S_m\subseteq P_m^{\Gamma_{m,n}}$ imply Lemma 1. □

LEMMA 2. *Let $n\geq n_0$. Then*

$$(P_m\cap S_m)P_n/P_n\simeq(\mathbf{Z}/p^{m-n}\mathbf{Z})^{\oplus s},$$

$$((P_m\cap S_m)P_n/P_n)\cap I_n^{(p)}/P_n=D_n$$

for sufficiently large $m\geq n$.

PROOF. Since $I_n\cap S_m\subseteq I_n^{(p)}$ and $I_n^{(p)}\subseteq P_m$ for sufficiently large $m\geq n$ ([2, Proposition 2]), we have the following inclusions of groups for such $m\geq n$:

$$P_n\cap S_m\subseteq I_n\cap S_m\subseteq P_m\cap S_m\subseteq S_m.$$

We note that $I_n\cap S_m=S_m^{p^{m-n}}=S_n^{t_m/t_n}$, and that $S_m/P_m\cap S_m\simeq D_m$. Since $n\geq n_0$, it follows that $D_m\simeq D_n$ by the norm map $N_{m,n}$. Hence we have $S_m/P_m\cap S_m\simeq D_n$. So we find that $I_n\cap S_m/P_n\cap S_m\simeq S_n^{t_m/t_n}/P_n\cap S_n^{t_m/t_n}\simeq D_n$ and $S_m/I_n\cap S_m=S_m/S_m^{p^{m-n}}\simeq(\mathbf{Z}/p^{m-n}\mathbf{Z})^{\oplus s}$. Therefore we obtain

$$\#(P_m\cap S_m/P_n\cap S_m)=p^{(m-n)s}. \tag{4}$$

Let $\mathfrak{P}\in P_m\cap S_m$. Taking the norm operator $N_{m,n}$ from I_m to I_n, we have $\mathfrak{P}^{p^{m-n}}=N_{m,n}\mathfrak{P}\in P_n\cap S_m$. Hence

$$(P_m\cap S_m/P_n\cap S_m)^{p^{m-n}}=1. \tag{5}$$

It follows from $S_m\simeq\mathbf{Z}^{\oplus s}$ that

(6) $$p\text{-rank}(P_m \cap S_m/P_n \cap S_m) \leq s\,.$$

From (4), (5), and (6), we obtain $(P_m \cap S_m)P_n/P_n \simeq P_m \cap S_m/P_n \cap S_m \simeq (\mathbf{Z}/p^{m-n}\mathbf{Z})^{\oplus s}$. The second assertion of the lemma follows from $I_n^{(p)} \cap P_m \cap S_m = I_n^{(p)} \cap S_m = S_n^{t_m/t_n}$. □

LEMMA 3. *Let M be a finite abelian p-group. We assume that M has subgroups N and H such that $N \simeq (\mathbf{Z}/p^e\mathbf{Z})^{\oplus r}$, $p^{[e/2]}H = 0$, and $M = N + H$ for some $e \geq 0$, $r \geq 0$, where [ ] stands for the Gaussian symbol. Then, for any subgroup $N' \subseteq M$ with $N' \simeq (\mathbf{Z}/p^e\mathbf{Z})^{\oplus r}$, we have*

$$M = N' + H\,, \qquad N' \cap H = N \cap H\,.$$

PROOF. Write $N = \bigoplus_{i=1}^r \mathbf{Z}n_i$, and $N' = \bigoplus_{i=1}^r \mathbf{Z}n_i'$, where $\mathbf{Z}n_i \simeq \mathbf{Z}n_i' \simeq \mathbf{Z}/p^e\mathbf{Z}$. From the assumption of the lemma, there exist $(a_{ij}) \in M_r(\mathbf{Z})$ and $h_i \in H$, $1 \leq i \leq r$ such that

(7) $$n_i' = \sum_{j=1}^r a_{ij}n_j + h_i\,, \qquad 1 \leq i \leq r\,.$$

Multiplying (7) by p^{e-1}, we obtain

$$p^{e-1}n_i' = \sum_{j=1}^r a_{ij}p^{e-1}n_j\,, \qquad 1 \leq i \leq r\,,$$

since $p^{e-1}H = 0$ from the assumption of the lemma. Hence we find $p^{e-1}N' = p^{e-1}N$, and $(a_{ij} \bmod p) \in GL_r(\mathbf{Z}/p\mathbf{Z})$. Therefore there exists a matrix $(b_{ij}) \in M_r(\mathbf{Z})$ such that $(b_{ij})(a_{ij}) \equiv E_r \pmod{p^e}$, where $E_r \in M_r(\mathbf{Z})$ denotes the identity matrix. From (7), we have

$$\sum_{j=1}^r b_{ij}n_j' = n_i + h_i'\,, \qquad h_i' \in H\,, \quad 1 \leq i \leq r\,.$$

It follows from the above equation that $N \subseteq N' + H$. Hence we have $M = N' + H$. Since $(b_{ij} \bmod p^e) \in GL_r(\mathbf{Z}/p^e\mathbf{Z})$, it holds $N' = \bigoplus_{i=1}^r \mathbf{Z}(n_i + h_i')$. Let $x = \sum_{i=1}^r c_i(n_i + h_i')$ be any element in $N' \cap H$. Then $0 = p^{[e/2]}x = \sum_{i=1}^r c_i p^{[e/2]}n_i$. Hence we have $p^{[e/2]} \mid c_i$ for all $1 \leq i \leq r$. So we can see $x = \sum_{i=1}^r c_i n_i \in N$ by the assumption $p^{[e/2]}H = 0$. Therefore $N' \cap H \subseteq N \cap H$. Since $M = N' + H$, the same argument shows $N \cap H \subseteq N' \cap H$. Thus we conclude $N \cap H = N' \cap H$. □.

PROOF OF THEOREM 1. Let $n \geq n_0$. Put $M = P_m^{\Gamma_{m,n}}/P_n$, $N = (P_m \cap S_m)P_n/P_n$, $N' = \mathrm{Im}(f) \simeq \hat{H}^{-1}(\Gamma_{m,n}, C_m)$ and $H = I_n^{(p)}/P_n = A_n$ for $m \geq n$ with $p^{m-n}B_{n_0} = 0$, where f is the homomorphism in (3). By Theorem 2, Lemma 1, and Lemma 2, the above group M, N, N', and H satisfy the assumption of Lemma 3 if $m \geq n$ is sufficiently large. Therefore $\mathrm{Coker}(f) = M/N' = N + H/N' = N' + H/N' \simeq H/N' \cap H = H/N \cap H = A_n/D_n \simeq A_n'$ by Lemmas 1, 2 and 3. This completes the proof of Theorem 1. □

REMARK. The author wants to know if $\mathrm{Im}(f) = (P_m \cap S_m)P_n/P_n$ or not.

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