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Decomposition of S^4 As a Twisted Double of a Certain Manifold

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1. Introduction.

Throughout this paper, we will work in the *PL* category.

DEFINITION 1 (see [L]). Let N be a compact oriented 4-manifold with a boundary. We say that S^4 decomposes as a twisted double of N if $S^4 = N \cup_\partial -N$.

We use the word "twisted double" because we allow that the gluing map between the boundaries is not $id|_{\partial N}$. This conception is a kind of an extension of Heegaard splitting of S^3.

Let N_2 be a tubular neighborhood of a $(+)$-standard $\mathbf{R}P^2$ in S^4 ([M, P]). It is well known that the closure of $S^4 \setminus N_2$ is also homeomorphic to N_2 by an orientation reversing homeomorphism, i.e., $S^4 = N_2 \cup_\partial -N_2$. Thus S^4 decomposes as a twisted double of N_2. N_2 can be characterized as a total space of a 2-disk bundle over $\mathbf{R}P^2$ whose normal Euler number is 2 ([K2, L, M, P]). The boundary of N_2, which we call Q_2, is a rational homology 3 sphere ([P]). It is known that the 2 covering of S^4 branched along a $(-)$-standard $\mathbf{R}P^2$ is $\mathbf{C}P^2$ ([K1, K2, M]).

We extend these facts to the case of a certain 2-complex X_n $(n \geq 2)$ instead of $\mathbf{R}P^2$. The main theorem will be stated as: S^4 decomposes as a twisted double of N_n, where N_n is a regular neighborhood of a standard realization of X_n in S^4. We give the definitions of the complexes and manifolds, and state the main Theorem 1 in the next section. We prove the main Theorem 1 in section 3. In section 4, we study on Q_n the boundary of N_n, which is a Seifert rational homology 3-sphere. The author thinks Q_n as a typical example among prime 3-manifolds which can be embedded in S^4. In section 5, we also study a covering of S^4 branched along $-X_n$.

ADDITION. Using an S^1 action on S^4, we can construct some more Seifert 3 manifolds each of which decomposes S^4 as a twisted double. We will go into detail

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about it in another paper [Y].

2. Definitions and the main theorem.

First, we define a 2-complex X_n. For an integer n ($n \geq 2$), let X_n be a 2-complex defined as follows (Figure 1):

$$X_n = D^2/e^{2\pi\sqrt{-1}\theta} \sim e^{2\pi\sqrt{-1}(\theta+1/n)}, \qquad \text{where} \quad D^2 = \{|z| \leq 1 \mid z \in \mathbf{C}\} .$$

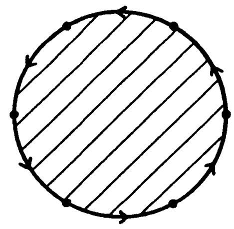


FIGURE 1. 2-complex X_n

Before defining a standard realization of X_n in S^4, we construct some subsets in S^3. We regard S^3 as the unit sphere of $\mathbf{C}^2$ and S^2 as $\mathbf{C}P^1$:

$$S^3 = \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1\} , \qquad S^2 = \mathbf{C}P^1 = \mathbf{C} \cup \{\infty\} .$$

Let p_n be a Seifert fibering of S^3 over S^2:

$$p_n : S^3 \to S^2$$
$$(z_1, z_2) \mapsto \frac{z_1^n}{(z_2 |z_1|^{n-1})} .$$

Let D_1 be a unit disk $\{z \in \mathbf{C} \mid |z| \leq 1\} \subset \mathbf{C} \subset S^2$ and V a standard solid torus $p_n^{-1}(D_1)$. $T_n = p_n^{-1}(1)$ and $T_n' = p_n^{-1}(-1)$ form a pair of parallel simple closed curves on ∂V, each of which represents $M_V + nL_V$ in $H_1(\partial V; \mathbf{Z})$ after changing its orientation if needed, where M_V, L_V are the classes of the meridian, longitude of ∂V respectively. Let Y_n and Y_n' be 2-complexes defined by $p_n^{-1}([0, 1])$, $p_n^{-1}([-1, 0])$ respectively, where we take the intervals in $\mathbf{R} \subset \mathbf{C}$. Here we note that $Y_n \cap \partial V = T_n$ and $Y_n' \cap \partial V = T_n'$. In fact, Y_n (Y_n', respectively) connects $p_n^{-1}(0)$ the core of V and T_n (T_n') in V.

Next we decompose S^2 into 4 parts: $D_{1/2}$, C_2, A_+ and A_-, and pull back them by p_n as a decomposition of S^3:

S^2	S^3
$D_{1/2} = \{z \in \mathbf{C} \mid \lvert z \rvert \leq 1/2\}$,	$V_0 = p_n^{-1}(D_{1/2})$,
$C_2 = \{z \in \mathbf{C} \mid \lvert z \rvert \geq 2\} \cup \{\infty\}$,	$V_1 = p_n^{-1}(C_2)$,
$A_+ = \{z \in \mathbf{C} \mid 1/2 \leq \lvert z \rvert \leq 2, \operatorname{Re} z \geq 0\}$,	$N(T_n) = p_n^{-1}(A_+)$,
$A_- = \{z \in \mathbf{C} \mid 1/2 \leq \lvert z \rvert \leq 2, \operatorname{Re} z \leq 0\}$.	$N(T_n') = p_n^{-1}(A_-)$.

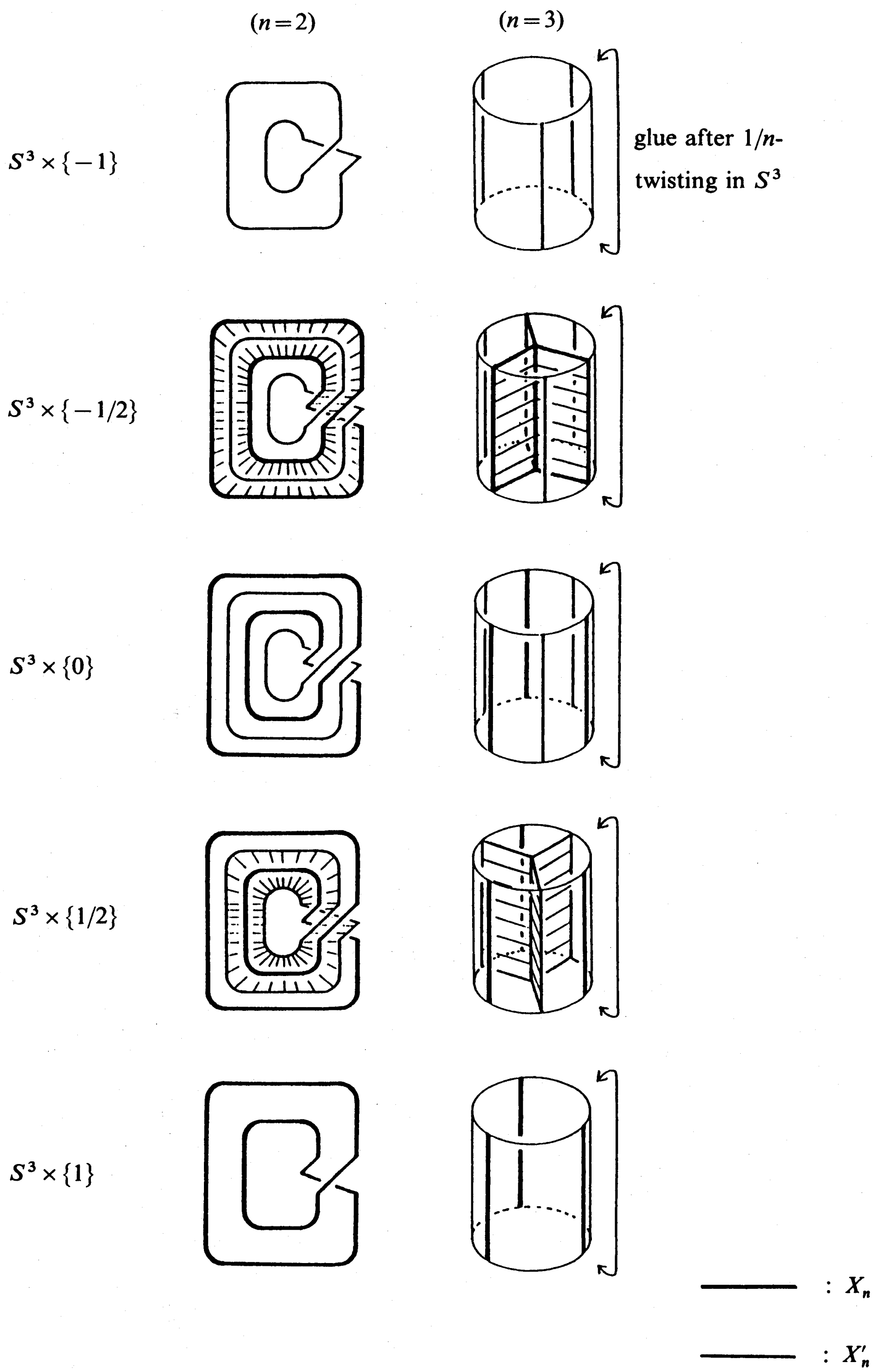


FIGURE 2. X_n and $-X_n$ in S^4

Finally we define a standard realization of X_n in S^4. We use the standard decomposition $S^4=B^4_- \cup_{\partial_-} S^3\times[-1, 1]\cup_{\partial_+} B^4_+$ (see Figure 2).

$$X_n\cap B^4_- =\varnothing\,, \qquad X_n\cap S^3\times[-1, -1/2)=\varnothing\,,$$
$$X_n\cap S^3\times\{-1/2\}=Y_n\times\{-1/2\}\,, \qquad X_n\cap S^3\times(-1/2, 1]=T_n\times(-1/2, 1]\,,$$
$$X_n\cap B^4_+ =\{\text{a disk } D^2_+ \subset B^4_+ \text{ such that}$$
$$D^2_+\cap\partial B^4_+ =\partial D^2_+ = T_n \text{ and } (B^4_+, D^2_+)\cong\text{“standard ball pair”}\}\,.$$

Here we note that T_n is a trivial knot in S^3 $(=\partial B^4_+)$.

Let N_n be a regular neighborhood of the standardly realized X_n in S^4. N_n is a connected oriented 4-manifold with a boundary.

We state the main theorem.

THEOREM 1. *For any n, S^4 decomposes as a twisted double of N_n.*

REMARK 1. In the case $n=2$, X_2 is homeomorphic to $\mathbf{R}P^2$ and the theorem is known ([K2, L, M, P]).

3. Proof of the main theorem.

We will show the decomposition explicitly.

Let $-X_n\subset S^4$ be another realization of X_n in S^4 defined as follows (Figure 2):

$$-X_n\cap B^4_- =\{\text{a disk } D^2_- \subset B^4_- \text{ such that}$$
$$D^2_-\cap\partial B^4_- =\partial D^2_- = T'_n \text{ and } (B^4_-, D^2_-)\cong\text{“standard ball pair”}\}\,,$$
$$-X_n\cap S^3\times[-1, 1/2)=T'_n\times[-1, 1/2)\,, \qquad -X_n\cap S^3\times\{1/2\}=Y'_n\times\{1/2\}\,,$$
$$-X_n\cap S^3\times(1/2, 1]=\varnothing\,, \qquad -X_n\cap B^4_+ =\varnothing\,.$$

It is casy to see that there is an orientation-reversing homeomorphism ρ of S^4 such that $\rho|_{X_n}$ is a homeomorphism from X_n to $-X_n$.

Using the notations defined in the last section, we construct N_n and $-N_n$ in S^4 simultaneously as follows (Figure 3):

$$N_n=V_0\times[-1, 0]\cup N(T_n)\times[-1, 1]\cup V_1\times[0, 1]\cup B^4_+\,,$$
$$-N_n=B^4_-\cup V_1\times[-1, 0]\cup N(T'_n)\times[-1, 1]\cup V_0\times[0, 1]\,.$$

It is easy to verify that $X_n\subset N_n$, $-X_n\subset -N_n$ and $N_n\cup -N_n=S^4$.

Next, we show that our N_n is in fact a regular neighborhood of X_n. The first half of the decomposition of N_n: $N^{(1)}_n=V_0\times[-1, 0]\cup N(T_n)\times[-1, 1]$, is a regular neighborhood of $X_n\cap N^{(1)}_n$.

Next, the other part $N^{(2)}_n=V_1\times[0, 1]\cup B^4_+$ is homeomorphic to a 4-ball B^4. Since T_n is also a trivial knot in ∂B^4, the pair $(B^4, X_n\cap B^4)(=(B^4, D^2_+))$ homeomorphic to the standard ball pair. In particular, this part B^4 is a regular neighborhood of $X_n\cap B^4$.

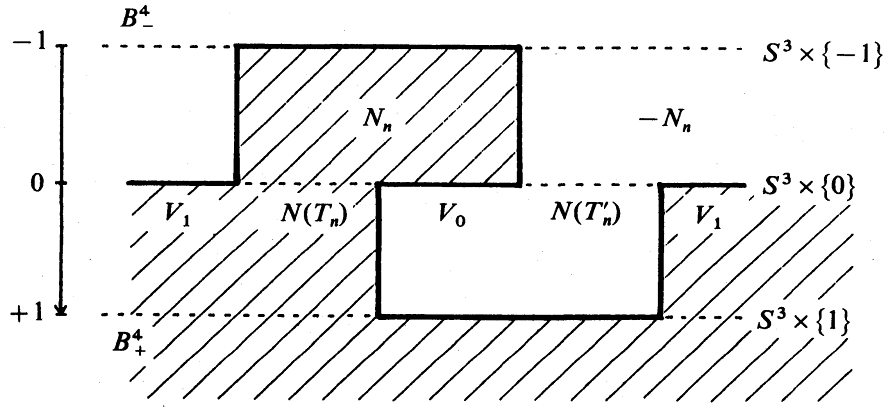


FIGURE 3. N_n and $-N_n$

Finally, we see the intersection of these two parts. Let A_2 be an annulus contained in $\partial N(T_n)$ defined by $N(T_n) \cap V_1$. By the construction, $N_n^{(1)} \cap N_n^{(2)}$ is $A_2 \times [0, 1] \cup N(T_n) \times \{1\}$. Clearly, it is homeomorphic to $S^1 \times D^2$ and is a regular neighborhood of $T_n \times \{1\}$, which is $X_n \cap (N_n^{(1)} \cap N_n^{(2)})$.

Thus, the union N_n of these two parts is also a regular neighborhood of X_n.

Similarly, $-N_n$ is that of $-X_n$, too. It is clear that our N_n and $-N_n$ are homeomorphic to each other by an orientation-reversing homeomorphism. We have the theorem. □

4. Some calculations on N_n and ∂N_n.

In this section, we study more about the manifolds N_n and ∂N_n. We let Q_n denote ∂N_n. First, we draw the framed links representing N_n and Q_n.

PROPOSITION 1. *N_n is described by the framed link $L(N_n)$ in Figure 4 and Q_n is the boundary of the 4-manifold described by the framed link $L(Q_n)$ in Figure 5.*

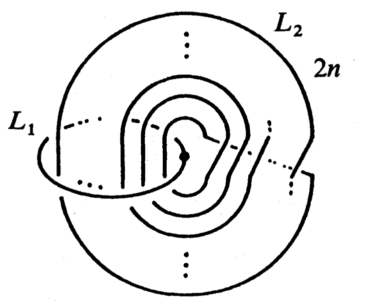


FIGURE 4. Framed link $L(N_n)$ of N_n

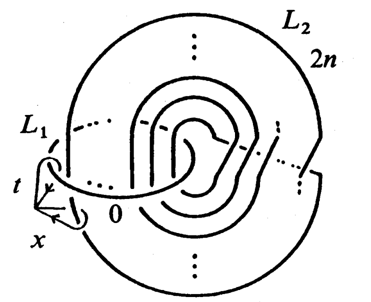


FIGURE 5. Framed link $L(Q_n)$ of Q_n

PROOF. We use the construction of N_n in the last section. The part $V_0 \times [-1, 0]$ is naturally identified with $S^1 \times D^3$. It is made of one 0-handle H^0 and one 1-handle

H^1, and it is described by an unknotted circle with a dot ([K1, p. 4]).

Let H^2 be the other part of N_n: $H^2 = N(T_n) \times [-1, 1] \cup V_1 \times [0, 1] \cup B^4_+$. Let Z_n be an annulus contained in Y_n defined by $p_n^{-1}([1/2, 1])$ which contains T_n as a component of its boundary, and let D, c denote the disk $X_n \cap H^2$ and its boundary:

$$D = Z_n \times \{-1/2\} \cup T_n \times (-1/2, 1] \cup D^2_+ , \qquad c = \partial D = p_n^{-1}(1/2) \times \{-1/2\} .$$

As we have seen in the last section, H^2 itself is a regular neighborhood of D and $H^2 \cong D \times D^2$. When we let $A_{1/2}$ denote an annulus $V_0 \cap N(T_n)$,

$$H^2 \cap S^1 \times D^3 = \partial H^2 \cap \partial(S^1 \times D^3) = A_{1/2} \times [-1, 0] \quad (\cong S^1 \times D^2) .$$

Since $A_{1/2}$ contains $p_n^{-1}(1/2)$ as a center circle, it is a regular neighborhood of c in the both sides of ∂H^2 and $\partial(S^1 \times D^3)$. Thus, we can regard H^2 as a 2-handle attached to $S^1 \times D^3$. The attaching circle of H^2 is c and drawn as L_2 in the framed link in Figure 4. We have a handlebody decomposition of N_n: $N_n = H^0 \cup H^1 \cup H^2$, where H^r is an r-handle.

The most troublesome step is to calculate the framing number of L_2. We can calculate it as follows:

Let c', a push-off of c in the attaching region, be $p_n^{-1}(\frac{1}{2}e^{i\varepsilon}) \times \{-1/2\}$, where ε (>0) is a sufficiently small number. The linking number $lk(c, c')$ in the framed link of $S^1 \times D^3$ (a dotted circle) is n.

On the other hand, since $lk(c, c') = n$ in $S^3(\times \{1/2\})$, the intersection number $D \circ D'$ is n, where D' is a push-off of D in H^2 bounded by c'.

Thus, in the side of ∂H^2, 0-framing of c is $-n$ twisted c' around c. But $-$-twisting in the side of ∂H^2 corresponds to a $+$-twisting in the side of $\partial(S^1 \times D^3)$, because the attaching map is orientation reversing. Thus the framing number of L_2 is $n + n = 2n$.

For the latter half of the theorem, see [K1, p. 7]. □

From the framed link $L(Q_n)$, we can calculate $\pi_1(Q_n)$ and $H_1(Q_n; Z)$:

$$\begin{aligned} \pi_1(Q_n) &= \langle x, t \mid x(xt)^n x^{-1}(xt)^{-n}, (xt)^n t^{-n}, x^n(xt)^n \rangle \\ &\cong \langle \alpha, \beta \mid \alpha^n = \beta^n = (\alpha\beta)^n \rangle , \end{aligned}$$

where the generators x, t are drawn in Figure 5, and $\alpha = x^{-1}$, $\beta = xt$. And

$$H_1(Q_n; \mathbf{Z}) \cong \pi_1(Q_n)/[\pi_1(Q_n), \pi_1(Q_n)] \cong \mathbf{Z}/n\mathbf{Z}\langle[\alpha]\rangle \oplus \mathbf{Z}/n\mathbf{Z}\langle[\beta]\rangle .$$

Thus, Q_n is a rational homology 3-sphere.

REMARK 2. In the case $n = 2$, it is known that $Q_2 \cong S^3/G_8$, where G_8 is the quaternion group and $\pi_1(Q_2) \cong G_8$ ([P]). $\pi_1(Q_n)$ is a finite group if and only if $n = 2$, because its quotient group $\langle \alpha, \beta \mid \alpha^n = \beta^n = (\alpha\beta)^n = 1 \rangle$ is a well-known triangle group, which is infinite if $n \geq 3$.

In the rest of this section, we study more about Q_n.

It is known that Q_2 admits a Seifert structure ([O]) whose invariants are $\{-1; (o_1, 0); (2, 1)(2, 1)(2, 1)\}$ ([P, O, p. 109]). We extend it to our 3-manifold Q_n.

PROPOSITION 2. *Q_n admits a Seifert structure whose invariants are* $\{-1; (o_1, 0); (n, 1)(n, 1)(n, n-1)\}$.

PROOF (see [Y]). At the beginning of the construction of N_n in section 2, we used a Seifert fibering p_n of S^3 over S^2. The Seifert invariants of p_n are $\{0; (o_1, 0); (n, 1)\}$ and its singular fiber is $p_n^{-1}(0)$ the core of V_0. The map $p_n \times id: S^3 \times [-1, 1] \to S^2 \times [-1, 1]$ defines a Seifert fibering of $S^3 \times [-1, 1]$ whose singular fiber lies over $\{0\} \times [-1, 1]$. By the construction of N_n in section 3 (see Figure 3), $Q_n = \partial N_n$ is contained in $S^3 \times [-1, 1] \subset S^4$ and Q_n is a union of fibers of $p_n \times id$. Thus the restriction $(p_n \times id)|_{Q_n}$ is a fibration. It is not hard to verify that the base space $(p_n \times id)(Q_n)$ is homeomorphic to S^2 which intersects $\{0\} \times [-1, 1]$ at 3 points. Thus the fibration of Q_n has 3 singular fibers: $p_n^{-1}(0) \times \{-1, 1, 0\}$. Because the neighborhood of each fiber is equivalent to $p_n|_{V_0}: V_0 \to D_{1/2}$, the singular types of the first two are both $(n, 1)$, since the orientation of the neighborhood agrees with that of V_0. On the other hand, the singular type of the third is $(n, -1)$, because the orientation induced as a boundary of N_n is opposite to that of V_0. After normalizing the Seifert invariants, we have the lemma:

$$\{0; (o_1, 0); (n, 1)(n, 1)(n, -1)\} \cong \{-1; (o_1, 0); (n, 1)(n, 1)(n, n-1)\} \qquad \square$$

5. Branched covering.

In this section, we study about a covering of S^4 branched along $-X_n$. The reason why we choose $-X_n$ will become clear soon.

In the case $n=2$, $-X_2 \subset S^4$ is pairwise homeomorphic to the $(-)$-standard embedding of $\mathbf{R}P^2$ into S^4, and its 2-fold branched covering is $\mathbf{C}P^2$ (see [K1, K2, M]). Here we note that the normal Euler number of the $(-)$-standard embedding is -2.

In the case $n>2$, $-X_n$ is not a manifold and has an S^1-singularity γ. Thus we consider a branched covering with the singularity removed, i.e., a covering of the exterior $S^4_\gamma = S^4 \setminus \mathrm{int}\, N(\gamma)$ branched along $-X_n \cap S^4_\gamma$, where $N(\gamma)$ is an open tubular neighborhood of γ in S^4. As we will see below, $\pi_1(S^4_\gamma \setminus -X_n \cap S^4_\gamma) \cong \mathbf{Z}/n\mathbf{Z}$. In this paper we only study an n-fold cyclic branched covering associated to it. It is a connected oriented 4-manifold with a boundary.

From Theorem 1: $S^4 = N_n \cup -N_n$ and a handlebody decomposition of $N_n: H^0 \cup H^1 \cup H^2$, we have a non-trivial handlebody decomposition

$$S^4 = N_n \cup -N_n = H^0_+ \cup H^1_+ \cup H^2_+ \cup (H^2_-)^\perp \cup (H^1_-)^\perp \cup (H^0_-)^\perp,$$

where $(H^r)^\perp$ is a dual $(4-r)$-handle. If we regard $(H^1_-)^\perp \cup (H^0_-)^\perp$ as $N(\gamma)$, we have a handlebody decomposition of $S^4_\gamma = S^4 \setminus \mathrm{int}\, N(\gamma)$:

$$S^4_\gamma = H^0_+ \cup H^1_+ \cup H^2_+ \cup (H^2_-)^\perp \quad (\cong S^2 \times D^2)\,.$$

LEMMA 1. *S^4_γ is described by the framed link in Figure* 6.

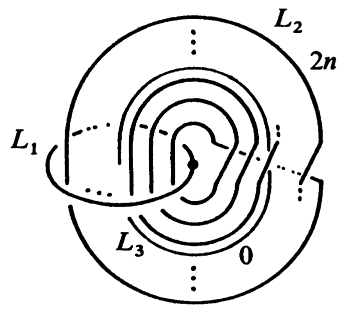


FIGURE 6. Framed link of S^4_γ

PROOF. We use the construction of N_n and its framed link which we have seen in the previous section:

$$N_n = V_0 \times [-1, 0] \cup N(T_n) \times [-1, 1] \cup V_1 \times [0, 1] \cup B^4_+\,,$$
$$(H^2_-)^\perp = B^4_- \cup V_1 \times [-1, 0] \cup N(T'_n) \times [-1, 1]\,.$$

From now on, we regard $V_0 \times [0, 1] \cup N(T'_n) \times [-1, 1]$ as $N(\gamma)$. On the other hand, it is easy to check that $\bar{N}_n = N_n \cup V_1 \times [-1, 0]$ is homeomorphic to N_n and that $\bar{N}_n$ can be described by the same framed link $L(N_n)$. Consequently, the only thing we must do is attaching B^4_- to $\bar{N}_n$. When we let $\bar{V}_0$ denote $V_0 \cup N(T_n) \cup N(T'_n) \subset S^3$, $B^4_- \cap \bar{N}_n = \bar{V}_0 \times \{-1\}$, which is a tubular neighborhood of a circle $l_3 = p_n^{-1}(0) \times \{-1\}$ in the both sides ∂B^4_- and $\partial \bar{N}_n$. Since $\bar{V}_0$ is a standard solid torus in ∂B^4_-, we can regard B^4_- as a 2-handle attached to $\bar{N}_n$ along $\bar{V}_0$.

In the framed link $L(N_n)$ (Figure 4), we can see that the part drawn as the exterior of L_1 is the side of $\bar{V}_0 \times \{-1\}$, by considering orientation. It is clear that the attaching circle l_3 is drawn as L_3 and its framing number is 0. We have the lemma. □

Before stating the next proposition, we introduce some notations and remarks.

Let C_n be a complex algebraic curve in $\mathbf{C}P^2$ of degree n defined by

$$\{[z_0 : z_1 : z_2] \in \mathbf{C}P^2 \,|\, z_0^n + z_1^n + z_2^n = 0\}\,.$$

C_n is a closed connected oriented surface of genus $\frac{1}{2}(n-1)(n-2)$, since C_n is an n-fold cyclic covering of $\mathbf{C}P^1 (= S^2)$ branched at n points in the equator of $\mathbf{C}P^1$:

$$C_n \to \mathbf{C}P^1$$
$$[z_0 : z_1 : z_2] \mapsto [z_1 : z_2]\,.$$

Let $\hat{\omega}_n$ be an n-periodic self-homeomorphism of C_n defind as follows:

$$\hat{\omega}_n : C_n \to C_n$$
$$[z_0 : z_1 : z_2] \mapsto [z_0 : z_1 : e^{2\pi\sqrt{-1}(1/n)} z_2] .$$

PROPOSITION 3. *Let M_n be the n-fold cyclic covering of S^4 branched along $-X_n$ with singularity removed.*

(1) *$-\partial M_n$ is a C_n-bundle over S^1 with monodromy $\hat{\omega}_n$.*

(2) *M_n is described by the framed link in Figure 7, which is a torus link $T(n, -n)$ each of whose components has framing number $n-1$.*

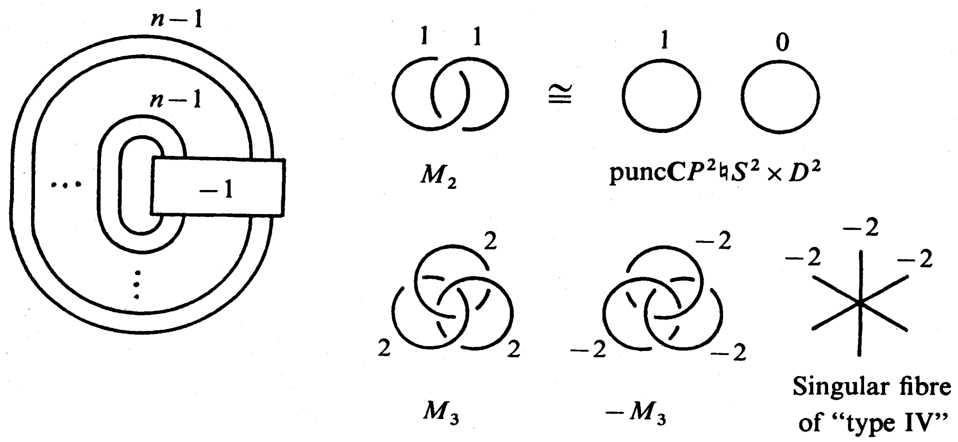


FIGURE 7. Framed link of M_n

PROOF. (1) We see ∂M_n from the side of $N(\gamma)$. ∂M_n is an n-fold cyclic covering of $\partial N(\gamma)$ branched along $\partial N(\gamma) \cap X_n$. Each of $\partial N(\gamma)$ and $\partial N(\gamma) \cap X_n$ is simultaneously regarded as a total space of a fibre bundle over S^1 with monodromy ω_n as follows:

$$\partial N(\gamma) = \partial D^3 \times [0, 1]/(x, 1) \sim (\omega_n(x), 0) : \partial D^3\text{-bundle} ,$$
$$\partial N(\gamma) \cap X_n = \{n \text{ points}\} \times [0, 1]/(x, 1) \sim' (\omega_n(x), 0) : \{n \text{ points}\}\text{-bundle} ,$$

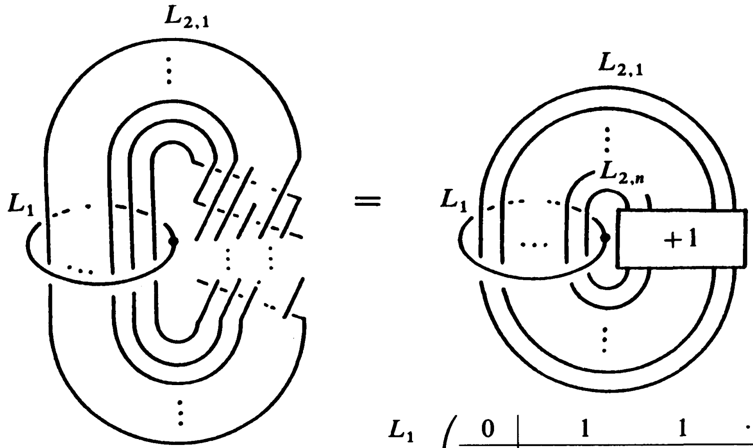
where

$$D^3 = \{(z, t) \in \mathbf{C} \times \mathbf{R} \mid |z|^2 + t^2 \leq 1\} ,$$
$$\{n \text{ points}\} = \{(z, t) \in D^3 \mid z^n = 1, t = 0\} \subset \text{the equator of } \partial D^3 ,$$

ω_n is a $(2\pi/n)$-rotation of D^3 along t-axis and $\sim'$ is a restriction of $\sim$ in the definition of the $\partial N(\gamma)$.

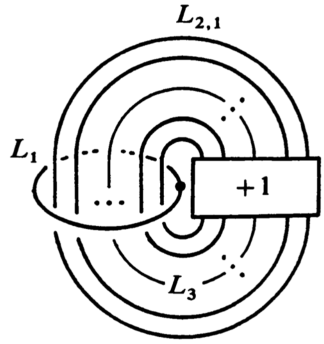
From the previous remark on C_n, it is clear that ∂M_n is the total space of the C_n-bundle over S^1 with monodromy ω_n.

(2) First, we construct n-fold cyclic unbranched covering $\tilde{N}_n$ of N_n. Since a generator of $\pi_1(N_n)$ ($\cong \mathbf{Z}/n\mathbf{Z}$) is represented by a circle which goes around the dotted circle once, $\tilde{N}_n$ is described by the framed link in Figure 8. The matrix added to the



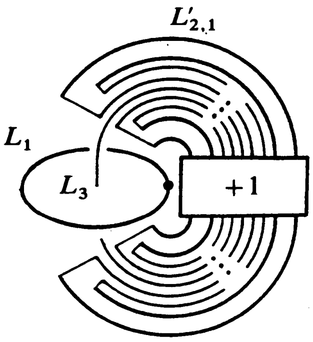
$$
\begin{array}{c}
L_1 \\ L_{2,1} \\ L_{2,2} \\ \vdots \\ L_{2,n}
\end{array}
\left(\begin{array}{c|cccc}
0 & 1 & 1 & \cdots & 1 \\ \hline
1 & n+1 & 1 & \cdots & 1 \\
1 & 1 & n+1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 1 \\
1 & 1 & \cdots & 1 & n+1
\end{array}\right)
$$

FIGURE 8. Framed link of $\hat{N}^n$



$$
\begin{array}{c}
L_1 \\ L_3 \\ L_{2,1} \\ \vdots \\ L_{2,n}
\end{array}
\left(\begin{array}{c|cccc}
0 & 1 & 1 & \cdots & 1 \\ \hline
1 & 0 & 1 & \cdots & 1 \\
1 & 1 & n+1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 1 \\
1 & 1 & \cdots & 1 & n+1
\end{array}\right)
$$

Handle slides
$L'_{2,i} = -(L_{2,i} - L_3)$



$$
\begin{array}{c}
L_1 \\ L_3 \\ L'_{2,1} \\ \vdots \\ L'_{2,n}
\end{array}
\left(\begin{array}{c|cccc}
0 & 1 & 0 & \cdots & 0 \\ \hline
1 & 0 & -1 & \cdots & -1 \\
0 & -1 & n-1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & -1 \\
0 & -1 & \cdots & -1 & n-1
\end{array}\right).
$$

Handle cancelation
(L_1 and L_3)

Figure 7

FIGURE 9. Kirby calculus on M_n

figure is the linking matrix. The action of the transformation group is easily shown.

Next, we will attach a 2-handle $(\tilde{H}^2_-)^\perp$ to $\tilde{N}_n$ $\mathbf{Z}/n\mathbf{Z}$-equivariantly. Since the co-core of the 2-handle $(H^2_-)^\perp : X_n \cap (H^2_-)^\perp$ is a branched locus, the attaching circle of $(\tilde{H}^2_-)^\perp$ is the same as that of $(H^2_-)^\perp$ in the previous claim and drawn as L_3 in the first figure of Figure 9.

Finally, using Kirby calculus, we cancel the 1-handle L_1. Those processes are left to the reader (Figure 9). We have the proposition. □

REMARK 3. It is pointed out by Professor Y. Matsumoto that $-M_n$ is diffeomorphic to a neighborhood of a singular fiber at 0 of Fermat-type surface V_{n+1} of degree $n+1$: $V_{n+1} = \{[z_0 : z_1 : z_2 : z_3] \in \mathbf{C}P^3 \mid z_0^{n+1} - z_1^{n+1} = z_2^{n+1} - z_3^{n+1}\}$,

$$p_{n+1} : V_{n+1} \to \mathbf{C}P^1 = \mathbf{C} \cup \{\infty\}$$

$$[z_0 : z_1 : z_2 : z_3] \mapsto \begin{cases} z_2^n / z_0^n & \text{if } z_0 = z_1 \text{ and } z_2 = z_3\,, \\ (z_0 - z_1)/(z_2 - z_3) & \text{otherwise} \end{cases}$$

and $-M_n \cong p_{n+1}^{-1}(D_{0,\varepsilon})$, where $D_{0,\varepsilon} = \{z \in \mathbf{C} \mid |z| \leq \varepsilon\}$ and $\varepsilon > 0$ is a sufficiently small number. See also [A] for the case $n=4$.

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References

[A] K. AHARA, On the topology of Fermat type surface of degree 5 and the numerical analysis of algebraic curves, Tokyo J. Math. **16** (1993), 321–340.

[K1] R. C. KIRBY, *The Topology of 4-Manifolds*, Lecture Notes in Math. **1374** (1989), Springer.

[K2] N. KUIPER, The quotient space of $\mathbf{C}P(2)$ by complex conjugation is the 4-sphere, Math. Ann. **208** (1974), 175–177.

[L] T. LAWSON, Splitting $S(4)$ on $\mathbf{R}P(2)$ via the branched cover of $\mathbf{C}P(2)$ over $S(4)$, Proc. Amer. Math. Soc. **86** (1982), 328–330.

[M] W. MASSAY, Imbeddings of projective planes and related manifolds in spheres, Indiana Univ. Math. J. **23** (1973), 791–812.

[O] P. ORLICK, *Seifert Manifolds*, Lecture Notes in Math. **291** (1972), Springer.

[P] T. PRICE, Homeomorphisms of quaternionic space and projective planes in four spaces, J. Austral. Math. Soc. Ser A **23** (1977), 112–128.

[Y] Y. YAMADA, Some Seifert 3-manifolds which decompose S^4 as a twisted double, preprint.

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