TOKYO J. MATH.
VOL. 20, NO. 1, 1997

Contact Riemannian Manifolds with Constant φ-Sectional Curvature

Themis KOUFOGIORGOS*

University of Ioannina

(Communicated by N. Sasakura)

Abstract. In this paper a class of contact, non Sasakian, Riemannian manifolds of constant φ-sectional curvature is found and studied.

1. Introduction.

Let M be a Riemannian manifold. It is well known [1, p. 131] that the tangent sphere bundle T_1M admits a contact Riemannian structure (η, ξ, φ, g). T_1M together with this structure is a contact Riemannian manifold. If M is of constant sectional curvature $c=1$, then T_1M is a Sasakian manifold [9], i.e. its curvature tensor R satisfies $R(X, Y)\xi=\eta(Y)X-\eta(X)Y$ for all vector fields X, Y. If $c=0$, then the curvature tensor of T_1M satisfies the condition $R(X, Y)\xi=0$ [2]. Applying a D-homothetic deformation on a contact Riemannian manifold satisfying $R(X, Y)\xi=0$, we get a contact Riemannian manifold such that $R(X, Y)\xi=\kappa(\eta(Y)X-\eta(X)Y)+\mu(\eta(Y)hX-\eta(X)hY)$, where κ, μ are real constants and $2h$ is the Lie differentiation of φ in the direction of ξ. We call this kind of manifold (κ, μ)-*contact Riemannian manifold.* The above construction was done in [5] and the study of (κ, μ)-contact Riemannian manifolds has begun in [3]. Examples of such manifolds exist in all dimensions. The 3-dimensional non Sasakian, (κ, μ)-contact Riemannian manifolds have constant φ-sectional curvature, but for higher dimensions this is not, in general, true.

Our purpose in this paper is to find conditions, which characterize (κ, μ)-contact Riemannian manifolds with constant φ-sectional curvature. At first we prove that if the φ-sectional curvature at a point P of a $(2n+1)$-dimensional (κ, μ)-contact Riemannian manifold M $(n>1)$ is independent of the φ-section at P, then it is constant. This result is analogous to Schur's theorem and extends a corresponding result, which is valid on Sasakian manifolds. Our second result states that a non Sasakian, (κ, μ)-contact Riemannian manifold is of constant φ-sectional curvature if and only if $\mu=$

Received May 1, 1995
*Research supported by E.E.C. contract CHRX-CT94-0661

$\kappa+1$. Therefore in this case an explicit expression for the curvature tensor is given. As an application of the last statement we prove that the tangent sphere bundle, of a Riemannian manifold of constant sectional curvature c, is of constant φ-sectional curvature iff $c=2\pm\sqrt{5}$. Finally we give a method to construct (κ, μ)-contact Riemannian manifolds of constant φ-sectional curvature. It seems that these manifolds are the first examples of non Sasakian, contact Riemannian manifolds with constant φ-sectional curvature.

2. Contact Riemannian manifolds.

A differential 1-form η on a differentiable $(2n+1)$-dimensional manifold M is called a *contact form* if it satisfies $\eta\wedge(d\eta)^n\neq 0$ everywhere on M. By a *contact manifold* (M, η) we mean a manifold M together with a contact form η. Since $d\eta$ is of rank $2n$, there exists a global vector field ξ, called *the characteristic vector field*, such that $\eta(\xi)=1$ and $\mathscr{L}_\xi\eta=0$, where $\mathscr{L}_\xi$ denotes the Lie differentiation by ξ. Moreover it is well known that there exist a Riemannian metric g and a $(1, 1)$-tensor field φ satisfying

$$\varphi\xi=0\,,\quad \eta\circ\varphi=0\,,\quad g(X, \xi)=\eta(X) \tag{2.1}$$

$$\varphi^2=-I+\eta\otimes\xi\,,\quad d\eta(X, Y)=g(X, \varphi Y) \tag{2.2}$$

$$g(\varphi X, \varphi Y)=g(X, Y)-\eta(X)\eta(Y) \tag{2.3}$$

for all vector fields X, Y on M. The structure (η, ξ, φ, g) is called a *contact Riemannian structure* and the manifold M carrying such a structure is said to be a *contact Riemannian manifold.*

Following [1], we define the $(1, 1)$-type tensor field h by $2h=\mathscr{L}_\xi\varphi$. Then h satisfies the relations

$$h\xi=0\,,\quad \mathrm{Tr}\,h=\mathrm{Tr}\,h\varphi=0\,,\quad h\varphi+\varphi h=0\,. \tag{2.4}$$

The contact form η on M gives rise to an almost complex structure on the product $M\times R$. If this structure is integrable, then the contact Riemannian manifold is said to be *Sasakian*. Equivalently, a contact Riemannian manifold is Sasakian if and only if $R(X, Y)\xi=\eta(Y)X-\eta(X)Y$ for all vector fields X, Y. The sectional curvature $K(X, \varphi X)$ of a plane section spanned by a vector X orthogonal to ξ is called a *φ-sectional curvature*.

The tangent sphere bundle T_1M of a Riemannian manifold M admits a contact Riemannian structure, known as *the standard contact Riemannian structure*. From now on, when we refer to T_1M we will consider it equipped with the standard contact Riemannian structure.

For more details concerning contact Riemannian manifolds and related topics we refer the reader to [1].

3. (κ, μ)-contact Riemannian manifolds.

For real constants κ, μ, the (κ, μ)-nullity distribution of a contact Riemannian manifold $M(\eta, \xi, \varphi, g)$ is a distribution

$$N(\kappa, \mu) : P \to N_P(\kappa, \mu) = \{Z \in T_P M \mid R(X, Y)Z = \kappa(g(Y, Z)X - g(X, Z)Y) + \mu(g(Y, Z)hX - g(X, Z)hY)\}$$

where R is the curvature tensor of M. So, if the characteristic vector field ξ belongs to the (κ, μ)-nullity distribution we have

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)\,. \tag{3.1}$$

We call (κ, μ)-*contact Riemannian manifold*, a contact Riemannian manifold satisfying (3.1). The class of (κ, μ)-contact Riemannian manifolds contains the class of Sasakian manifolds, which we get for $\kappa = 1$ (and hence $h = 0$, by (3.2)). Characteristic examples of non Sasakian, (κ, μ)-contact Riemannian manifolds are the tangent sphere bundles of Riemannian manifolds of constant sectional curvature $c \neq 1$. Especially in the 3-dimensional case this class contains the Lie groups $SU(2)$, $SO(3)$, $SL(2, R)$, $O(1, 2)$, $E(2)$, $E(1, 1)$ with a left invariant metric. For more examples see [3].

From now on, we suppose $M(\eta, \xi, \varphi, g)$ is a $(2n+1)$-dimensional (κ, μ)-contact Riemannian manifold. In [3] the following formulas have been proved:

$$h^2 = (\kappa - 1)\varphi^2 \tag{3.2}$$

(so $\kappa \leq 1$ and $\kappa = 1$ iff M is a Sasakian manifold),

$$R(\xi, X)Y = \kappa(g(X, Y)\xi - \eta(Y)X) + \mu(g(hX, Y)\xi - \eta(Y)hX) \tag{3.3}$$

$$\begin{aligned} R(X, Y)\varphi Z = \varphi R(X, Y)Z &+ \{(1-\kappa)[\eta(X)g(\varphi Y, Z) - \eta(Y)g(\varphi X, Z)] \\ &+ (1-\mu)[\eta(X)g(\varphi hY, Z) - \eta(Y)g(\varphi hX, Z)]\}\xi \\ &- g(Y + hY, Z)(\varphi X + \varphi hX) + g(X + hX, Z)(\varphi Y + \varphi hY) \\ &- g(\varphi Y + \varphi hY, Z)(X + hX) + g(\varphi X + \varphi hX, Z)(Y + hY) \\ &- \eta(Z)\{(1-\kappa)[\eta(X)\varphi Y - \eta(Y)\varphi X] + (1-\mu)[\eta(X)\varphi hY - \eta(Y)\varphi hX]\}\,. \end{aligned} \tag{3.4}$$

Moreover in [3] the following results have been proved.

LEMMA 3.1. *If $\kappa < 1$, then M admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$, $D(-\lambda)$, defined by the eigenspaces of h, where $\lambda = \sqrt{1-\kappa}$.*

THEOREM 3.2. *If $\kappa < 1$, then*

$$R(X_\lambda, Y_\lambda)Z_{-\lambda} = (\kappa - \mu)[g(\varphi Y_\lambda, Z_{-\lambda})\varphi X_\lambda - g(\varphi X_\lambda, Z_{-\lambda})\varphi Y_\lambda]$$

$$R(X_{-\lambda}, Y_{-\lambda})Z_\lambda = (\kappa - \mu)[g(\varphi Y_{-\lambda}, Z_\lambda)\varphi X_{-\lambda} - g(\varphi X_{-\lambda}, Z_\lambda)\varphi Y_{-\lambda}]$$

$$R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = \kappa g(\varphi X_\lambda, Z_{-\lambda})\varphi Y_{-\lambda} + \mu g(\varphi X_\lambda, Y_{-\lambda})\varphi Z_{-\lambda}$$

$$R(X_\lambda, Y_{-\lambda})Z_\lambda = -\kappa g(\varphi Y_{-\lambda}, Z_\lambda)\varphi X_\lambda - \mu g(\varphi Y_{-\lambda}, X_\lambda)\varphi Z_\lambda$$

$$R(X_\lambda, Y_\lambda)Z_\lambda = [2(1+\lambda)-\mu][g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda]$$

$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = [2(1-\lambda)-\mu][g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}]$$

where X_λ, Y_λ, Z_λ and $X_{-\lambda}$, $Y_{-\lambda}$, $Z_{-\lambda}$ are the components of X, Y, Z on $D(\lambda)$ and $D(-\lambda)$ respectively.

THEOREM 3.3. *If $\kappa < 1$, then*

1) *The sectional curvature of a plane section (X, Y) orthogonal to ξ with $X \in D(\lambda)$ and $Y \in D(-\lambda)$ is given by*

$$K(X, Y) = -(\kappa+\mu)g(X, \varphi Y)^2 . \tag{3.5}$$

2) *The Ricci operator is given by*

$$Q = (2(n-1)-n\mu)I + (2(n-1)+\mu)h + (2(1-n)+n(2\kappa+\mu))\eta \otimes \xi . \tag{3.6}$$

THEOREM 3.4. *The tangent sphere bundle T_1M is a (κ, μ)-contact Riemannian manifold if and only if the base manifold M is of constant sectional curvature c. Moreover $\kappa = c(2-c)$ and $\mu = -2c$.*

4. Main results.

Let $M(\eta, \xi, \varphi, g)$ be a $(2n+1)$-dimensional (κ, μ)-contact Riemannian manifold. If $n=1$, and $\kappa \neq 1$, then it is well known [3] that M is of constant φ-sectional curvature. In the next theorem we consider the case $n>1$ and we give a necessary condition so that M is of constant φ-sectional curvature. This theorem extends two theorems of Ogiue (see [6] or [1]) and Endo [4], which are valid for $\kappa=1$ and $\mu=0$ respectively.

THEOREM 4.1. *Let $M(\eta, \xi, \varphi, g)$ be a $(2n+1)$-dimensional (κ, μ)-contact Riemannian manifold $(n>1)$. If the φ-sectional curvature of any point of M is independent of the choice of φ-section at the point, then it is constant on M and the curvature tensor is given by*

$$\begin{aligned}4R(X, Y)Z = {} & (H+3)\{g(Y, Z)X - g(X, Z)Y\} + (H+3-4\kappa)\{\eta(X)\eta(Z)Y \\ & - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \\ & + (H-1)\{2g(X, \varphi Y)\varphi Z + g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X\} \\ & - 2\{g(hX, Z)hY - g(hY, Z)hX + 2g(X, Z)hY - 2g(Y, Z)hX - 2\eta(X)\eta(Z)hY \\ & + 2\eta(Y)\eta(Z)hX + 2g(hX, Z)Y - 2g(hY, Z)X + 2g(hY, Z)\eta(X)\xi \\ & - 2g(hX, Z)\eta(Y)\xi - g(\varphi hX, Z)\varphi hY + g(\varphi hY, Z)\varphi hX\} \\ & + 4\mu\{\eta(Y)\eta(Z)hX - \eta(X)\eta(Z)hY + g(hY, Z)\eta(X)\xi - g(hX, Z)\eta(Y)\xi\}\end{aligned} \tag{4.1}$$

where H is the constant φ-sectional curvature. Moreover if $\kappa \neq 1$, then $\mu = \kappa + 1$ and $H = -2\kappa - 1$.

PROOF. For the Sasakian case $\kappa = 1$, the proof is known ([1], p. 97). So we have to prove the theorem for $\kappa \neq 1$. Let $P \in M$ and $X, Y \in T_P M$ orthogonal to ξ. Using the first identity of Bianchi, the basic properties of the curvature tensor, φ is antisymmetric, h is symmetric, (2.2) and (2.3) we get from (3.4), successively:

$$\begin{aligned} g(R(X, \varphi X)Y, \varphi Y) &= g(R(X, \varphi Y)Y, \varphi X) + g(R(X, Y)X, Y) \\ &- g(X, Y)^2 - g(hX, Y)^2 - 2g(X, Y)g(hX, Y) + g(X, X)g(Y, Y) \\ &+ g(X, X)g(hY, Y) + g(Y, Y)g(hX, X) + g(hX, X)g(hY, Y) \\ &- g(\varphi X, Y)^2 + g(\varphi hX, Y)^2 - g(\varphi hX, X)g(\varphi hY, Y) \end{aligned} \tag{4.2}$$

$$\begin{aligned} g(R(X, \varphi Y)X, \varphi Y) &= g(R(X, \varphi Y)Y, \varphi X) \\ &+ g(X, Y)^2 - g(hX, Y)^2 - g(\varphi hX, X)g(\varphi hY, Y) - g(X, X)g(Y, Y) \\ &- g(Y, Y)g(hX, X) + g(X, X)g(hY, Y) + g(hX, X)g(hY, Y) \\ &+ g(\varphi X, Y)^2 + g(\varphi hX, Y)^2 + 2g(\varphi X, Y)g(\varphi hX, Y) \end{aligned} \tag{4.3}$$

$$\begin{aligned} g(R(Y, \varphi X)Y, \varphi X) &= g(R(X, \varphi Y)Y, \varphi X) \\ &+ g(X, Y)^2 - g(hX, Y)^2 - g(\varphi hX, X)g(\varphi hY, Y) + g(\varphi X, Y)^2 \\ &+ g(\varphi hX, Y)^2 - 2g(\varphi X, Y)g(\varphi hX, Y) - g(X, X)g(Y, Y) \\ &- g(X, X)g(hY, Y) + g(Y, Y)g(hX, X) + g(hX, X)g(hY, Y) \end{aligned} \tag{4.4}$$

$$\begin{aligned} g(R(X, Y)\varphi X, \varphi Y) &= g(R(X, Y)X, Y) \\ &- g(X, Y)^2 - g(hX, Y)^2 - 2g(X, Y)g(hX, Y) + g(X, X)g(Y, Y) \\ &+ g(X, X)g(hY, Y) + g(Y, Y)g(hX, X) + g(hX, X)g(hY, Y) \\ &- g(\varphi X, Y)^2 + g(\varphi hX, Y)^2 - g(\varphi hX, X)g(\varphi hY, Y)\,. \end{aligned} \tag{4.5}$$

We now suppose that the φ-sectional curvature at P is independent of the φ-section at P, i.e. $K(X, \varphi X) = H(P)$ for any $X \in T_P M$ orthogonal to ξ. Let $X, Y \in T_P M$ and X, Y orthogonal to ξ. From

$$g(R(X+Y, \varphi X + \varphi Y)(X+Y), \varphi X + \varphi Y) = -H(P)g(X+Y, X+Y)^2$$

$$g(R(X-Y, \varphi X - \varphi Y)(X-Y), \varphi X - \varphi Y) = -H(P)g(X-Y, X-Y)^2$$

we get by a straightforward calculation

$$\begin{aligned} 2g(R(X, \varphi X)Y, \varphi Y) + g(R(X, \varphi Y)X, \varphi Y) + 2g(R(X, \varphi Y)Y, \varphi X) + g(R(Y, \varphi X)Y, \varphi X) \\ = -2H(P)\{2g(X, Y)^2 + g(X, X)g(Y, Y)\}\,. \end{aligned} \tag{4.6}$$

Combining (4.2), (4.3), (4.4) and (4.6) we get

$$\begin{aligned}&3g(R(X, \varphi Y)Y, \varphi X)+g(R(X, Y)X, Y)\\&-2g(hX, Y)^2-2g(X, Y)g(hX, Y)+g(X, X)g(hY, Y)+g(Y, Y)g(hX, X)\\&+2g(hX, X)g(hY, Y)+2g(\varphi hX, Y)^2-2g(\varphi hX, X)g(\varphi hY, Y)\\&=-H(P)\{2g(X, Y)^2+g(X, X)g(Y, Y)\}\,.\end{aligned} \tag{4.7}$$

Replacing Y by φY in (4.7) and using (2.3) and (2.4) we have

$$\begin{aligned}&-3g(R(X, Y)\varphi Y, \varphi X)+g(R(X, \varphi Y)X, \varphi Y)\\&-2g(\varphi hX, Y)^2+2g(X, \varphi Y)g(\varphi hX, Y)-g(X, X)g(hY, Y)+g(Y, Y)g(hX, X)\\&-2g(hX, X)g(hY, Y)+2g(hX, Y)^2+2g(\varphi hX, X)g(\varphi hY, Y)\\&=-H(P)\{2g(X, \varphi Y)^2+g(X, X)g(Y, Y)\}\,.\end{aligned} \tag{4.8}$$

Combining (4.8) with (4.3) and (4.5) we finally get

$$\begin{aligned}&3g(R(X, Y)X, Y)+g(R(X, \varphi Y)Y, \varphi X)\\&-2g(X, Y)^2-2g(hX, Y)^2-6g(X, Y)g(hX, Y)+2g(X, X)g(Y, Y)\\&+3g(X, X)g(hY, Y)+3g(Y, Y)g(hX, X)+2g(hX, X)g(hY, Y)\\&-2g(X, \varphi Y)^2+2g(\varphi hX, Y)^2-2g(\varphi hX, Y)g(\varphi hY, Y)\\&=-H(P)\{2g(X, \varphi Y)^2+g(X, X)g(Y, Y)\}\,.\end{aligned} \tag{4.9}$$

Now, (4.9) together with (4.7) yield

$$\begin{aligned}4g(R(X, Y)Y, X)=&(H(P)+3)\{g(X, X)g(Y, Y)-g(X, Y)^2\}+3(H(P)-1)g(X, \varphi Y)^2\\&-2\{g(hX, Y)^2+4g(X, Y)g(hX, Y)-2g(X, X)g(hY, Y)-2g(Y, Y)g(hX, X)\\&-g(hX, X)g(hY, Y)-g(\varphi hX, Y)^2+g(\varphi hX, X)g(\varphi hY, Y)\}\end{aligned} \tag{4.10}$$

for any $X, Y\in T_PM$ and X, Y orthogonal to ξ. Let $X, Y, Z\in T_PM$ and X, Y, Z orthogonal to ξ. Applying (4.10) in

$$g(R(X+Z, Y)Y, X+Z)=g(R(X, Y)Y, X)+g(R(Z, Y)Y, Z)+g(R(X, Y)Y, Z)$$

we finally get

$$\begin{aligned}4g(R(X, Y)Y, Z)=&(H(P)+3)\{g(X, Z)g(Y, Y)-g(X, Y)g(Y, Z)\}\\&+3(H(P)-1)g(X, \varphi Y)g(Z, \varphi Y)-2\{g(hX, Y)g(hZ, Y)+2g(X, Y)g(hZ, Y)\\&+2g(Z, Y)g(hX, Y)-2g(X, Z)g(hY, Y)-2g(Y, Y)g(hX, Z)\\&-g(hX, Z)g(hY, Y)-g(\varphi hX, Y)g(\varphi hZ, Y)+g(\varphi hX, Z)g(\varphi hY, Y)\}\,.\end{aligned} \tag{4.11}$$

Moreover, using (3.1), (2.1) and $h\varphi$ is symmetric, it is easy to check that (4.11) is valid for any Z and for X, Y orthogonal to ξ. Hence (4.11) is reduced to

$$\begin{aligned}R(X, Y)Y=&(H(P)+3)\{g(Y, Y)X-g(X, Y)Y\}+3(H(P)-1)g(X, \varphi Y)\varphi Y\\&-2\{g(hX, Y)hY+2g(X, Y)hY+2g(hX, Y)Y-2g(hY, Y)X\end{aligned}$$

$$-2g(Y,Y)hX-g(hY,Y)hX-g(\varphi hX,Y)\varphi hY+g(\varphi hY,Y)\varphi hX\} \tag{4.12}$$

for any X, Y orthogonal to ξ. Now let X, Y, Z be orthogonal to ξ. Replacing in (4.12) Y by $Y+Z$ and using (4.12) we get

$$\begin{aligned}4\{R(X,Y)Z+R(X,Z)Y\}&=(H(P)+3)\{2g(Y,Z)X-g(X,Y)Z-g(X,Z)Y\}\\&\quad+3(H(P)-1)\{g(X,\varphi Y)\varphi Z+g(X,\varphi Z)\varphi Y\}\\&\quad-2\{g(hX,Y)hZ+g(hX,Z)hY+2g(X,Y)hZ+2g(X,Z)hY+2g(hX,Y)Z\\&\quad+2g(hX,Z)Y-4g(hY,Z)X-4g(Y,Z)hX-2g(hY,Z)hX\\&\quad-g(\varphi hX,Y)\varphi hZ-g(\varphi hX,Z)\varphi hY+2g(\varphi hY,Z)\varphi hX\}\,.\end{aligned} \tag{4.13}$$

Replacing X by Y and Y by $-X$ in (4.13) we have

$$\begin{aligned}4\{R(X,Y)Z+R(Z,Y)X\}&=(H(P)+3)\{-2g(X,Z)Y+g(X,Y)Z+g(Y,Z)X\}\\&\quad+3(H(P)-1)\{-g(\varphi X,Y)\varphi Z-g(\varphi Z,Y)\varphi X\}\\&\quad-2\{-g(hY,X)hZ-g(hY,Z)hX-2g(X,Y)hZ-2g(Y,Z)hX-2g(X,hY)Z\\&\quad-2g(hY,Z)X+4g(hX,Z)Y+4g(X,Z)hY+2g(hX,Z)hY\\&\quad+g(\varphi hY,X)\varphi hZ+g(\varphi hY,Z)\varphi hX-2g(\varphi hX,Z)\varphi hY\}\,.\end{aligned} \tag{4.14}$$

Adding (4.13) and (4.14) and using Bianchi's first identity, φ is antisymmetric and φh is symmetric we get

$$\begin{aligned}4R(X,Y)Z&=(H(P)+3)\{g(Y,Z)X-g(X,Z)Y\}\\&\quad+(H(P)-1)\{2g(X,\varphi Y)\varphi Z+g(X,\varphi Z)\varphi Y-g(Y,\varphi Z)\varphi X\}\\&\quad-2\{g(hX,Z)hY+2g(X,Z)hY+2g(hX,Z)Y-2g(hY,Z)X\\&\quad-2g(Y,Z)hX-g(hY,Z)hX-g(\varphi hX,Z)\varphi hY+g(\varphi hY,Z)\varphi hX\}\end{aligned} \tag{4.15}$$

for any X, Y, Z orthogonal to ξ. Moreover, using (3.1) and $h\xi=\varphi\xi=0$, we conclude that (4.15) is valid for any Z and for X, Y orthogonal to ξ. Now, let X, Y, Z be arbitrary vectors of T_PM. Writing

$$X=X_T+\eta(X)\xi\,,\qquad Y=Y_T+\eta(Y)\xi$$

where $g(X_T,\xi)=g(Y_T,\xi)=0$, and using (3.1), (3.3) and $h\xi=0$, then (4.15) gives (4.1) after a straightforward calculation.

Now, we will prove that the φ-sectional curvature is constant. Let $\{X_i\}$, $i=1,\cdots,2n+1$, be a local orthonormal frame. Putting $Y=Z=X_i$ in (4.1), adding with respect to i and using (2.1)–(2.4) we get the following formula, for the Ricci operator, at any point of M:

$$\begin{aligned}2Q&=\{(n+1)H+3(n-1)+2\kappa\}I-\{(n+1)H+3(n-1)\\&\quad-2\kappa(2n-1)\}\eta\otimes\xi+2\{2(n-1)+\mu\}h\,.\end{aligned}$$

Comparing this with (3.6), which is valid on any (κ, μ)-contact Riemannian manifold with $\kappa \neq 1$, we get $(n+1)H = n-1-2n\mu-2\kappa$, i.e. H is constant. On the other hand, from (3.5) we have $H = -\kappa-\mu$. Comparing the two last equations we get $(n-1)(\mu-\kappa-1)=0$. Moreover, since $n>1$, we have $\mu=\kappa+1$ and so $H=-2\kappa-1$. This completes the proof of the theorem. □

In Theorem 4.1 we proved that $\mu=\kappa+1$, in the case where the non Sasakian, (κ, μ)-contact Riemannian manifold has constant φ-sectional curvature. Now we will prove the inverse, i.e. supposing $M(\eta, \xi, \varphi, g)$ is a $(2n+1)$-dimensional $(n>1)$, non Sasakian, (κ, μ)-contact Riemannian manifold with

$$\mu = \kappa + 1 \tag{4.16}$$

we will prove that M has constant φ-sectional curvature.

Let $X \in T_P M$ be a unit vector orthogonal to ξ. By Lemma 3.1 we can write

$$X = X_\lambda + X_{-\lambda} \qquad \text{where } X_\lambda \in D(\lambda) \text{ and } X_{-\lambda} \in D(-\lambda)\,.$$

Using Lemma 3.1, Theorem 3.2, (2.3), we get, after a long straightforward calculation,

$$K(X, \varphi X) = -(\kappa+\mu) + 4(\kappa-\mu+1)(g(X_\lambda, X_\lambda)g(X_{-\lambda}, X_{-\lambda}) - g(X_\lambda, \varphi X_{-\lambda})^2)$$

and hence by (4.16), $K(X, \varphi X) = -(\kappa+\mu) = \text{const}$. So we have proved the following theorem.

THEOREM 4.2. *Let $M(\eta, \xi, \varphi, g)$ be a $(2n+1)$-dimensional $(n>1)$, non Sasakian, (κ, μ)-contact Riemannian manifold. Then M has constant φ-sectional curvature if and only if $\mu=\kappa+1$.*

An immediate consequence of Theorems 4.2 and 3.4 is the following theorem.

THEOREM 4.3. *Let M be an n-dimensional Riemannian manifold, $n>2$, of constant sectional curvature c. The tangent sphere bundle T_1M has constant φ-sectional curvature (c^2) if and only if $c=2\pm\sqrt{5}$.*

REMARK. The tangent sphere bundle T_1M, of a 2-dimensional Riemannian manifold M of constant sectional curvature c, has constant φ-sectional curvature c^2 for any $c \neq 1$, as follows from (3.5) and Theorem 3.4.

5. Examples.

1. The first non-trivial example of a 3-dimensional, non Sasakian, contact Riemannian manifold of constant φ-sectional curvature was given in [5]. In [3] there exist more examples concerning the 3-dimensional case.

2. Theorem 4.3 gives two examples of $(2n+1)$-dimensional, $n>1$, non Sasakian, (κ, μ)-contact Riemannian manifolds of positive constant φ-sectional curvature, equal to $(2\pm\sqrt{5})^2$.

3. Now, we will give a method to construct non Sasakian, (κ, μ)-contact Riemannian manifolds of constant φ-sectional curvature. Let $M(\eta, \xi, \varphi, g)$ be a $(2n+1)$-dimensional, (κ, μ)-contact Riemannian manifold $(n>1, \kappa\neq 1, \mu<2)$. The existence of such a manifold follows from Theorem 3.4, taking $c>-1$ $(c\neq 1)$. By a D-homothetic deformation [7] we mean a change of structure tensors of the form

$$\bar{\eta}=a\eta\,,\quad \bar{\xi}=\tfrac{1}{a}\xi\,,\quad \bar{\varphi}=\varphi\,,\quad \bar{g}=ag+a(a-1)\eta\otimes\eta$$

where a is a positive constant. It is well known [3] that $M(\bar{\eta}, \bar{\xi}, \bar{\varphi}, \bar{g})$ is a new $(\bar{\kappa}, \bar{\mu})$-contact Riemannian manifold with

$$\bar{\kappa}=\frac{\kappa+a^2-1}{a^2} \quad \text{and} \quad \bar{\mu}=\frac{\mu+2a-2}{a}\,. \tag{5.1}$$

Choosing $a=(\kappa-1)/(\mu-2)>0$, $M(\bar{\eta}, \bar{\xi}, \bar{\varphi}, \bar{g})$ has constant φ-sectional curvature. In fact, substituting a in (5.1) we get $\bar{\mu}=\bar{\kappa}+1$ and so by Theorem 4.2, $M(\bar{\eta}, \bar{\xi}, \bar{\varphi}, \bar{g})$ has constant φ-sectional curvature

$$\bar{H}=-\bar{\kappa}-\bar{\mu}=1-2\bar{\mu}=(2(\mu-2)^2-3(1-\kappa))/(1-\kappa)\,.$$

So $\bar{H}$ is positive, negative or zero if

$$A>0\,,\ A<0\,,\ \text{or}\ A=0\,,\qquad \text{where}\quad A=2(\mu-2)^2-3(1-\kappa)\,,\quad (\mu<2)$$

respectively. The existence of a (κ, μ)-contact Riemannian manifold satisfying $A>0$ or $A<0$ or $A=0$ follows easily from Theorem 3.4.

The above examples give an answer (for $n>1$) to the following remark of Tanno (see [8], p. 445). "It seems to be an open problem if there exist contact Riemannian manifolds of constant φ-sectional curvature, which are not Sasakian".

References

[1] D. E. BLAIR, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math., **509** (1976), Springer.
[2] D. E. BLAIR, Two remarks on contact metric structures, Tôhoku Math. J. **29** (1977), 319–324.
[3] D. E. BLAIR, T. KOUFOGIORGOS and P. J. PAPANTONIOU, Contact metric manifolds satisfying a nullity condition, Israel J. of Math. **91** (1995), 189–214.
[4] H. ENDO, On the curvature tensor fields of a type of contact metric manifolds and of its certain submanifolds, submitted.
[5] T. KOUFOGIORGOS, Contact metric manifolds, Ann. Global Anal. Geom. **11** (1993), 25–34.
[6] K. OGIUE, On almost contact manifolds admiting axiom of planes or axiom of free mobility, Kodai Math. Sem. Rep. **16** (1964), 223–232.
[7] S. TANNO, The topology of contact Riemannian manifolds, Illinois J. Math. **12** (1968), 700–717.
[8] S. TANNO, Ricci curvatures of contact Riemannian manifolds, Tôhoku Math. J. **40** (1988), 441–448.
[9] Y. TASHIRO, On contact structure of tangent sphere bundles, Tôhoku Math. J. **21** (1969), 117–143.

Present Address:
Department of Mathematics, University of Ioannina,
Ioannina 451 10, Greece.
e-mail: tkoufog@cc.uoi.gr