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On a Local Embedding Theorem of Generalized-Mizohata Structures

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Introduction.

The purpose of this paper is to generalize the Hounie and Malagutti's local embedding theorem for Mizohata structures and to discuss their embedding theorem in the frame work of *Kuranishi.*

It is Treves who introduced the notion of Mizohata structures (cf. [Tr1]). Hounie and Malagutti developed Treves's theory and proved that; any formally integrable Mizohata structure is actually integrable if the Mizohata structure is strongly pseudoconvex and $\dim_{\mathbf{R}} M \geq 3$ (cf. [H-M]). This result reminds us of the CR-local embedding theorem (cf. [A2], [Ku3]), namely any formally integrable CR structure $(M, {}^0T'')$ is actually integrable if the CR structure $(M, {}^0T'')$ is strongly pseudoconvex and $\dim_{\mathbf{R}} M = 2n-1 \geq 7$. Furthermore, in many points, Mizohata structures quite resemble CR structures. Hence it seems quite reasonable to try to discuss both in one context. We, therefore, introduce a notion of a generalized complex manifold and consider a *regular* real hypersurface M, namely a submanifold with real codimension 1, which satisfies some conditions in a generalized complex manifold. Over this hypersurface, from the generalized complex manifold, naturally a structure (M, E_M) is induced as in the CR case, which we call a generalized-Mizohata structure. Like formally integrable CR structures, we introduce a notion of a formally integrable generalized-Mizohata structure, and consider the local embedding theorem. With these in mind, in a more general context, we would like to discuss a local embedding theorem of generalized-Mizohata structures, which covers Hounie and Malagutti's local embedding theorem, and the CR-local embedding theorem (see [Ku1], [Ku2], [Ku3]). For this purpose, we recall the proof of the local embedding theorem of CR-structures (cf. [A2]). The proof consists of the following three parts.

Part 1. Let f be a C^∞ local embedding of M into $\mathbf{C}^n$ at the reference point p_0. We set a neighborhood of p_0 by

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$$U_\varepsilon(f)=\{x\ ;\ x\in M,\ 2\mathrm{Re}(h\cdot f(x))<\varepsilon\}\ ,$$

where $\dim_{\mathbf{R}} M=2n-1$, and h is a fixed holomorphic function on $\mathbf{C}^n$ satisfying certain conditions (see in [A2]). Then, for the solvability of the D-Neumann problem over $U_\varepsilon(f)$, we see that it suffices to solve a nonlinear D_b-equation on $U_\varepsilon(f)$ (for the detail, see Chapter 8 in [A2]), where D-operator means the induced operator by the given CR structure ${}^0T''$.

Part 2. Let f be a C^∞ local embedding of M into $\mathbf{C}^n$, and $(M, {}^fT'')$ denotes the induced CR-structure and D_b^f denotes the tangential induced operator on $U_\varepsilon(f)$ by ${}^fT''$ via f (for the detail, see Chapter 2 in [A2]). For this D_b^f, we have an a priori estimate, if $2n-1\geq 7$.

Part 3. With the Neumann operator obtained in Part 2, by using the Nash iteration procedure, we obtain the local embedding theorem of CR structures.

We follow this approach. In the case of generalized-Mizohata structures, the part 1 and the part 3 are valid. We see the part 2. However, by following the proofs of Kuranishi's a priori estimate, this part is also valid with a slight change. Hence, we can discuss the generalized local embedding theorem in the frame work of *Kuranishi* [Ku1], [Ku2], [Ku3].

After completing this paper, I learned that Webster obtained a similar result (not the same one). However, I think that still our result is worth publication. And in the same letter, I also learned that A. Meziani proves the local integrability of all strictly pseudoconvex Mizohata structures in dimension two (see [Me]).

1. CR structures and Mizohata structures.

Let N be a complex manifold with complex dimension n (so the real dimension of N is $2n$). Let M be a real hypersurface in N. We assume that M is smooth. This means that; M is defined by a C^∞ function r on N by;

$$M=\{x\ ;\ x\in N,\ r(x)=0\}\ , \qquad dr\neq 0 \text{ at every point } p \text{ of } M\,.$$

Over M, a CR-structure is naturally induced from N. Namely, we set a subbundle of the complexified tangent bundle $\mathbf{C}\otimes TM$ by;

$${}^0T''=\mathbf{C}\otimes TM\cap T''N|_M\,.$$

Then,

$$\text{(1-1)}\qquad {}^0T''\cap\overline{{}^0T''}=0\,,\qquad \dim_{\mathbf{C}}\frac{\mathbf{C}\otimes TM}{{}^0T''+\overline{{}^0T''}}=1\,,$$

$$\text{(1-2)}\qquad [\Gamma(M,{}^0T''),\Gamma(M,{}^0T'')]\subset\Gamma(M,{}^0T'')\,.$$

Conversely, for an orientable C^∞ manifold M with real dimension $2n-1$, and for a subbundle E of the complexified tangent bundle $\mathbf{C}\otimes TM$, satisfying;

(1-1′) $$E\cap\bar{E}=0\,,\qquad \dim_{\mathbf{C}}\frac{\mathbf{C}\otimes TM}{E+\bar{E}}=1\,,$$

(1-2′) $$[\Gamma(M, E), \Gamma(M, E)]\subset\Gamma(M, E)\,,$$

the pair (M, E) is called a CR-structure. On the other hand, the Mizohata structure is defined as follows: Let M be a C^∞ manifold with real dimension $p+1$. We consider a subbundle E_M of the complexified tangent bundle $\mathbf{C}\otimes TM$ with $\dim_{\mathbf{C}}E_M=p$ satisfying;

(1-3) $$[\Gamma(M, E_M), \Gamma(M, E_M)]\subset\Gamma(M, E_M)\,.$$

We set

$$C(E_M)=E_M^\perp\cap T^*M\,,$$

$$E_M^\perp=\{u\,;\,u\in\mathbf{C}\otimes T^*M,\,u(L)=0 \text{ for } L\in E_M\}\,.$$

And for $(q, \xi)\in C(E_M)$, the Levi-form is defined by;

$$\Theta_{(q,\xi)}(u, v)=\frac{1}{2\sqrt{-1}}\,\xi([L_1, \overline{L_2}])(q)\,,$$

where L_1 and L_2 are C^∞ local sections of E_M defined in a neighborhood of q so that $L_1(q)=u$, $L_2(q)=v$. With these preparations, the pair (M, E_M) satisfying (1-3) is called a Mizohata structure, or a Mizohata manifold if and only if $C(E_M)\neq 0$ and the Levi form associated to E_M is non degenerate.

2. Generalized complex manifolds.

In this section, we introduce the notion of generalized complex manifolds and *regular* real hypersurfaces for generalized complex manifolds.

Let X be a differentiable manifold with real dimension $2n+p$. Let E be an $n+p$ dimensional subbundle of the complexified tangent bundle $\mathbf{C}\otimes TM$ satisfying;

(2-0) $$\dim_{\mathbf{C}}E\cap\bar{E}=p\,.$$

We assume that;

(2-1) $\mathbf{C}\otimes TM$ is generated by E and $\bar{E}$ at every point p of X,

(2-2) $$[\Gamma(X, E), \Gamma(X, E)]\subset\Gamma(X, E)\,.$$

If the pair (X, E) satisfies (2-1) and (2-2), then we call (X, E) a *generalized complex manifold.* For a generalized complex manifold (X, E), we can introduce *holomorphic functions* and *germs of holomorphic functions* by the standard way. That is to say, for a point q of X,

$$O_E(X)_q = \{u\ ;\ u \text{ is a complex function satisfying } Yu = 0 \text{ for an } E\text{-valued } C^\infty \text{ section on a neighborhood of } q\}\ .$$

We see some examples (see Example VII.1.2 and 1.3 and other important examples in [Tr]).

EXAMPLE 2-1. We take a real p-dimensional euclidean space $\mathbf{R}^p$, and let $(t_1, \cdots, t_p)$ be a real coordinate system of $\mathbf{R}^p$. We set E = the complexified tangent bundle $\mathbf{C}\otimes T\mathbf{R}^p$ itself. Then, this pair $(\mathbf{R}^p, E)$ obviously satisfies (2-1) and (2-2). And in this case, $O_E(\mathbf{R}^p) = \{c\ ;\ c \text{ is a constant}\}$.

EXAMPEL 2-2. More generally, we take $\mathbf{R}^p \times \mathbf{C}^n$, where $\mathbf{R}^p$ means the real p-dimensional euclidean space and $\mathbf{C}^n$ means the complex n-dimensional euclidean space. As a generalized complex structure, we set

$$E = \{\text{complex vector fields generated by } \partial/\partial t_1, \cdots, \partial/\partial t_p, \partial/\partial \overline{z_1}, \cdots, \partial/\partial \overline{z_n}\}\ ,$$

where $(t_1, \cdots, t_p)$ is a real coordinate system of $\mathbf{R}^p$, and $(z_1, \cdots, z_n)$ is a complex coordinate system of $\mathbf{C}^n$. Then, obviously, our $(\mathbf{R}^p \times \mathbf{C}^n, E)$ satisfies (2-0), (2-1) and (2-2). So $(\mathbf{R}^p \times \mathbf{C}^n, E)$ is a generalized complex manifold. In this case,

$$O_E(\mathbf{R}^p \times \mathbf{C}^n) = \{\text{ holomorphic functions of } z_1, \cdots, z_n\}\ .$$

On the other hand, if $p=0$, we have $E \cap \bar{E} = 0$. So by the Newlander-Nirenberg theorem, in this case, our generalized complex manifolds coincide with standard complex manifolds. More generally, we have that our generalized complex manifold (X^{2n+p}, E^{n+p}) is locally isomorphic to $(\mathbf{R}^p \times \mathbf{C}^n$, {complex vector fields generated by $\partial/\partial t_1, \cdots, \partial/\partial t_p$, $\partial/\partial \overline{z_1}, \cdots, \partial/\partial \overline{z_n}\})$, which was discussed in Example 2-2. This is easily proved by using the Newlander-Nirenberg theorem (our partial differential equation is elliptic).

Let (X, E) be a generalized complex manifold. We consider a smooth real hypersurface in X. Namely, we assume that M is defined by;

$$M = \{q\ ;\ q \in X, r(q) = 0\}\ ,$$

where r is a C^∞ function on X. In this paper, we assume more. That is to say, for every point q of M, there is an E-valued C^∞ section Y on a neighborhood of q satisfying;

$$Yr(q) \neq 0\ , \qquad Y_q \notin E_q \cap \overline{E_q}\ .$$

We call this real hypersurface a regular real hypersurface. Over this M, from E, naturally a structure is induced. Namely, we set a subbundle of E by

$$E_M = \{X\ ;\ X \in E, Xr = 0\}\ .$$

Then, obviously,

$$[\Gamma(M, E_M), \Gamma(M, E_M)] \subset \Gamma(M, E_M)\ .$$

We call this pair (M, E_M) a generalized-Mizohata structure or a generalized-Mizohata manifold. We see an example.

EXAMPLE 2-3. We consider a C^∞ embedding i from $\mathbf{R}^p \times \mathbf{R}$ to $\mathbf{R}^p \times \mathbf{C}$ defined by;

$$i : (t_1, \cdots, t_p, x) \longrightarrow \left((t_1, \cdots, t_p), x + \frac{\sqrt{-1}}{2} \sum_{i=1}^{p} t_i^2\right)$$

and consider the hypersurface $M = i(\mathbf{R}^p \times \mathbf{R})$. In $\mathbf{R}^p \times \mathbf{C}$, M is defined by the equation $\operatorname{Im} z = \frac{1}{2} \sum_{i=1}^{p} t_i^2$, where $(t_1, \cdots, t_p, z)$ is a coordinate system of $\mathbf{R}^p \times \mathbf{C}$. Then, by the definition of the induced strucure, E_M is generated by $L_1, \cdots, L_p$ where

$$L_i = \frac{\partial}{\partial t_i} - 2\sqrt{-1}\, t_i \frac{\partial}{\partial \bar{z}}, \qquad i = 1, \cdots, p .$$

Therefore our (M, E_M) becomes the Mizohata structure which Hounie and Malagutti discussed.

Let (X, E) be a generalized complex manifold with $\dim_{\mathbf{R}} X = 2n + p$, $\dim_{\mathbf{C}} E = n + p$, and $\dim_{\mathbf{C}} E \cap \bar{E} = p$. And let (M, E_M) be a regular hypersurface in X. For this (M, E_M), we set, by the same way as in Mizohata structures,

$$C(E_M) = E_M^{\perp} \cap T^*M ,$$

$$E_M^{\perp} = \{u \,;\, u \in \mathbf{C} \otimes T^*M,\ u(L) = 0 \text{ for } L \in E_M\} .$$

By the definition, for $(q, \xi) \in C(E_M)$, $\xi(\bar{L})(q) = 0$ for $L \in (E_M)_q$. The Levi form is defined by:

$$\Theta_{(q,\xi)}(u, v) = \frac{1}{2\sqrt{-1}}\, \xi([L_1, \overline{L_2}])(q) ,$$

where L_1, L_2 are C^∞ local sections of E_M defined in a neighborhood of q so that $L_1(q) = u$, $L_2(q) = v$.

DEFINITION 2.1. Let (M, E_M) be a generalized-Mizohata structure. If $C(E_M) \neq 0$ and the Levi form is positive or negative definite, the pair (M, E_M) is called strongly pseudoconvex.

We see this explicitly. By the definiton, we may assume that $(o, u) \in C(E_M)$, $u \neq 0$. By using the Newlander-Nirenberg theorem with the assumption; M being regular, for a point o of M, there is a local coordinate neighborhood $U(o)$ and a local coordinate system $(t_1, \cdots, t_p, z_1, \cdots, z_n)$ of $U(o)$ satisfying;

$$M \cap U(o) = \{q \,;\, q \in U(o),\ r(q) = 0\} , \qquad \frac{\partial r}{\partial z_n}(o) \neq 0 ,$$

(2-1) $$r(t_1, \cdots, t_p, z_1, \cdots, z_n) = \operatorname{Im} z_n - \psi(t_1, \cdots, t_p, z_1, \cdots, z_{n-1}, \operatorname{Re} z_n) .$$

Here $\operatorname{grad}\psi(t_1, \cdots, t_p, z_1, \cdots, z_{n-1}, \operatorname{Re} z_n)=0$ at o. And E is generated by

$$\frac{\partial}{\partial \overline{z_1}}, \quad \cdots, \quad \frac{\partial}{\partial \overline{z_n}}, \quad \frac{\partial}{\partial t_1}, \quad \cdots, \quad \frac{\partial}{\partial t_p},$$

and E_M is generated by

$$\frac{\partial}{\partial \overline{z_1}} - \frac{r_{\bar{1}}}{r_{\bar{n}}} \frac{\partial}{\partial z_{\bar{n}}}, \quad \cdots, \quad \frac{\partial}{\partial \overline{z_{n-1}}} - \frac{r_{\overline{n-1}}}{r_{\bar{n}}} \frac{\partial}{\partial z_{\bar{n}}},$$

$$\frac{\partial}{\partial t_1} - \frac{r_1}{r_{\bar{n}}} \frac{\partial}{\partial z_{\bar{n}}}, \quad \cdots, \quad \frac{\partial}{\partial t_p} - \frac{r_p}{r_{\bar{n}}} \frac{\partial}{\partial z_{\bar{n}}}.$$

Here $r_k = \partial r/\partial t_k$, $r_{\bar{i}} = \partial r/\partial z_{\bar{i}}$. We put

$$\zeta = \sqrt{-1}\left(r_{\bar{n}} \frac{\partial}{\partial z_n} - r_n \frac{\partial}{\partial \overline{z_n}}\right).$$

Obviously, $\zeta_o \notin (E_M)_o$, and $\mathbf{C} \otimes TM_o$ is generated by ζ_o, $(E_M)_o$, $(\overline{E_M})_o$. On the other hand, in general, we can infer neither that; $\zeta_o \notin (E_M)_o + (\overline{E_M})_o$, nor $\dim_{\mathbf{C}}((E_M)_q + (\overline{E_M})_q) = \text{const.}$ on a neighborhood of the origin o. Here $(E_M)_q + (\overline{E_M})_q$ means the subvector space of $\mathbf{C} \otimes TM_q$ which is generated by $(E_M)_q$, $(\overline{E_M})_q$. For example, in Example 2.3 in this paper,

$$\zeta_o \notin (E_M)_o + (\overline{E_M})_o \qquad \text{at the origin},$$

$$\zeta_q \in (E_M)_q + (\overline{E_M})_q \qquad \text{at } q \neq o.$$

In our case, if $r_k(q)=0$, $1 \le k \le p$,

$$\zeta_q \notin (E_M)_q + (\bar{E}_M)_q, \quad \text{and} \quad \dim_{\mathbf{C}}(E_M \cap \bar{E}_M)_q = p.$$

Since $(o, u) \in C(E_M)$, $u \neq 0$, we can infer $\zeta_o \notin (E_M)_o + (\bar{E}_M)_o$. So we set the decomposition of the vector space:

$$\mathbf{C} \otimes TM_o = (\mathbf{C}\zeta)_o + (E_M)_o + (\overline{E_M})_o. \tag{2-2}$$

Proposition 2.2. *(M, E_M) is called strongly pseudoconvex if and only if the Levi form $L(u, v)$ defined by;*

$$-\sqrt{-1}[L_1, \overline{L_2}]_\zeta = L(u, v)\zeta \qquad \textit{for} \quad u, v \in E_M,$$

where L_1, L_2 are C^∞ local sections of E_M defined in a neighborhood of the origin so that $L_1(o)=u$, $L_2(o)=v$ and $[L_1, \overline{L_2}]_\zeta$ means the ζ part of $[L_1, \overline{L_2}]$ according to (2.2), *is positive or negative definite.*

Namely, if our $p+n-1$ matrix;

$$\begin{pmatrix} \left(\dfrac{\partial^2 r}{\partial t_i \partial t_j}\right)_{1 \le i, j \le p} & \left(\dfrac{\partial^2 r}{\partial t_i \partial z_{\bar{k}}}\right)_{1 \le i \le p, 1 \le k \le n-1} \\ \left(\dfrac{\partial^2 r}{\partial z_k \partial t_j}\right)_{1 \le k \le p, 1 \le j \le n-1} & \left(\dfrac{\partial^2 r}{\partial z_l \partial z_{\bar{k}}}\right)_{1 \le l, k \le n-1} \end{pmatrix}$$

is positive or negative definite, then our (M, E_M) is strongly pseudoconvex. Now with these preparations, the following theorems are obvious.

THEOREM 2.3. *Let (M, E_M) be a generalized-Mizohata structure with* $\dim_{\mathbf{R}} M = 2n+p-1$, $\dim_{\mathbf{C}} E_M = n+p-1$. *We assume that (M, E_M) is strongly pseudoconvex and $n=1$, then our generalized-Mizohata structure becomes the Mizohata structure which Hounie and Malagutti introduced.*

For the case of $p=0$, we have

THEOREM 2.4. *Let (M, E_M) be a generalized-Mizohata structure. If $p=0$, then our structure becomes the standard CR-structure.*

3. An a priori estimate.

In this section, we introduce D-complex over a generalized-Mizohata manifold, and show an a priori estimate which plays an essential role in proving a general embedding theorem. Let (M, E_M) be a generalized-Mizohata structure. Namely, M is a complex $2n+p-1$ dimensional C^∞ manifold and E_M is a complex $n+p-1$ dimensional subbundle of the complexified tangent bundle $\mathbf{C} \otimes TM$ satisfying;

$$[\Gamma(M, E_M), \Gamma(M, E_M)] \subset \Gamma(M, E_M)\,.$$

For this (M, E_M), we set a first order differential operator D from $\Gamma(M, \mathbf{C})$ to $\Gamma(M, (E_M)^*)$ by the usual way. Namely, for u in $\Gamma(M, \mathbf{C})$,

$$Du(X) = Xu\,, \qquad X \in E_M\,.$$

Then by the same method as in the case of differential forms, we have a differential complex

$$0 \longrightarrow \Gamma(M, \mathbf{C}) \xrightarrow{D} \Gamma(M, (E_M)^*) \xrightarrow{D} \Gamma(M, \wedge^2 (E_M)^*) \longrightarrow$$

$$\longrightarrow \Gamma(M, \wedge^p (E_M)^*) \xrightarrow{D} \Gamma(M, \wedge^{p+1} (E_M)^*) \longrightarrow .$$

We see that; if (M, E_M) is strongly pseudoconvex, and if $n+p-1 \ge 3$, where $\dim_{\mathbf{R}} M = 2n+p-1$, $\dim_{\mathbf{C}} E_M = n+p-1$, Kuranishi's local a priori estimate holds and also "D_b-a priori estimate" holds (these are proved by the complete same method as in [A2], [Ku3]). Let $(x, z, z_n) \in \mathbf{R}^p \times \mathbf{C}^{n-1} \times \mathbf{C}$. We assume that M is a regular real hypersurface in $\mathbf{R}^p \times \mathbf{C}^{n-1} \times \mathbf{C} = \mathbf{R}^p \times \mathbf{C}^n$, $o \in M$. Then, we can assume that on a neighborhood of the origin, M is defined by

$$\operatorname{Im} z_n - \psi(x, z, \operatorname{Re} z_n) = 0\,,$$

where ψ is a real valued C^∞ function satisfying $\psi(o, o, o) = 0$ and

$$\psi(x, z, \operatorname{Re} z_n) = x^2 + |z|^2 + (\text{terms of degree} \geq 3 \text{ in } (x, z, \bar{z}, \operatorname{Re} z_n))\,,$$

where if necessary, we must change the coordinates. We set

$$\rho(x, z, z_n) = \operatorname{Im} z_n - \psi(x, z, \operatorname{Re} z_n)\,, \qquad h(x, z, z_n) = \frac{1}{2\sqrt{-1}} z_n + z_n^2\,.$$

Here

$$Y'_i := \frac{\partial}{\partial \bar{z}_i} - \left(\frac{\rho_{\bar{i}}}{\rho_{\bar{n}}}\right) \frac{\partial}{\partial \bar{z}_n}\,, \qquad 1 \leq i \leq n-1\,,$$

$$X'_j := \frac{\partial}{\partial x_j} - \left(\frac{\partial_j}{\rho_{\bar{n}}}\right) \frac{\partial}{\partial \bar{z}_n}\,, \qquad 1 \leq j \leq p\,,$$

where $\rho_{\bar{i}} = \partial \rho / \partial \bar{z}_i$, $\rho_j = \partial \rho / \partial x_j$.

We set $Y'_{n-1+j} := X'_j$. Then, as our generalized-Mizohata structure is strongly pseudoconvex, we can assume the $n+p-1$ matrix

$$(a_{lm})_{1 \leq l, m \leq n+p-1} > 0\,.$$

Here a_{lm} means the coefficient of ζ part of

$$-\sqrt{-1}[Y'_l, \overline{Y'_m}] \qquad \operatorname{mod}\{Y_k, \overline{Y_k}\,;\, 1 \leq k \leq n+p-1\}\,,$$

where

$$\zeta = \sqrt{-1}\left(r_{\bar{n}} \frac{\partial}{\partial z_n} - r_n \frac{\partial}{\partial \bar{z}_n}\right).$$

So by the Schmidt orthogonal procedure, we have an orthonormal base Y_j, $1 \leq j \leq n+p-1$, with $-\sqrt{-1}[Y_l, \overline{Y_m}]_\zeta = \delta_{lm}\zeta$, where $[Y_l, \overline{Y_m}]_\zeta$ means the ζ part of $[Y_l, \overline{Y_m}]$ according to (2.2). Now we follow the Kuranishi's method (cf. [A2], [Ku3]). We set

$$Y^o = \sum_{l=1}^{n+p-1} \frac{\overline{Y_l t}}{b} Y_l\,, \qquad W_l = Y_l - \frac{Y_l t}{b} Y^o \quad (1 \leq l \leq n+p-1)$$

where $t = 2\operatorname{Re} h(x, z, z_n)$, $b = \sqrt{\sum_{l=1}^{n+p-1} |Y_l t|^2}$. Then our W_l satisfies

$$\sum_{l=1}^{n+p-1} (\overline{Y_l t}) W_l = 0\,.$$

We set

$$U_\varepsilon(o)=\{q\ ;\ q\in M,\ 2\mathrm{Re}\,h(q)<\varepsilon\}\ .$$

Over $U_\varepsilon(o)$, we will introduce D_b-operator and D_b-complex. For this purpose, we set the characteristic curve

$$C=\{p\ ;\ p\in M,\ b(p)=0\}\ .$$

On $U_\varepsilon(o)-C$, we introduce a subbundle $(E_M)_b$ of E_M by

$$(E_M)_b=\{W\ ;\ W\in E_M,\ Wt=0\}\ .$$

Obviously this $(E_M)_b$ is generated by the above W_l's. Now we set

$$D_b: \Gamma(U_\varepsilon(o)-C,\ 1)\longrightarrow \Gamma(U_\varepsilon(o)-C,\ ((E_M)_b)^*)$$

by; for f in $\Gamma(U_\varepsilon(o)-C,\ 1)$,

$$D_bf(W)=Wf\ ,\qquad W\in(E_M)_b\ .$$

Then by the usual way,

$$0\longrightarrow \Gamma(U_\varepsilon(o)-C,\ 1)\xrightarrow{D_b}\Gamma(U_\varepsilon(o)-C,\ ((E_M)_b)^*)\xrightarrow{D_b}\Gamma(U_\varepsilon(o)-C,\ \wedge^2((E_M)_b)^*)\ .$$

We put the Levi metric on M. Then, with respect to this metric, we can treat Y_l, W_l by the complete same way as in [Ku3], [A2]. Hence we have

THEOREM 3.1. *If $n+p-1\geq 3$, then, for u in $\Gamma(U_\varepsilon(o)-C,\ ((E_M)_b)^*)$ with $(1/b)u$, W_iu $(1\leq i\leq n+p-1)$, D_bu, D_b^*u in L^2, we have*

$$\|D_bu\|^2+\|D_b^*u\|^2\geq c\left\|\left(\frac{1}{b}\right)\right\|^2,$$

where c is a positive constant, $\|\ \ \|$ means the L^2-norm defined by the Levi metric on $U_\varepsilon(o)-C$, and D^* means the adjoint operator with respect to the Levi metric.

4. Formally integrable generalized-Mizohata structures.

Let M be a C^∞ manifold with $\dim_{\mathbf{R}}M=2n+p-1$, and let E_M be a subbundle of the complexified tangent bundle $\mathbf{C}\otimes TM$ satisfying $[\Gamma(E_M),\ \Gamma(E_M)]\subset\Gamma(E_M)$, with $\dim_{\mathbf{C}}E_M=n+p-1$, $C(E_M)\neq 0$. We assume that for every $(p,u)\in C(E)$, $u\neq 0$, there is a vector field ζ of M, defined on a neighborhood of p such that $\mathbf{C}\otimes TM_p$ is generated by $(E_M)_p$, $(\overline{E_M})_p$, ζ_p.

DEFINITION 4.1. We assume that (M, E_M) satisfies the above. Then, the pair (M, E_M) is called a formally integrable generalized-Mizohata structure.

As we have shown in Sect. 2 in this paper, we have

PROPOSITION 4.2. *A regular real hypersurface in* $\mathbf{R}^p \times \mathbf{C}^n$ *is a formally integrable generalized-Mizohata structure.*

For a formally integrable Mizohata structure which Hounie and Malagutti introduced, we have

PROPOSITION 4.3. *Let* (M, E_M) *be a formally integrable Mizohata structure which Hounie and Malagutti stated. Then, this* (M, E_M) *becomes a formally integrable generalized-Mizohata structure in our sense.*

PROOF. We see that for every point $(p, u) \in C(E_M)$, $u \neq 0$, there is a real vector field ζ, defined on a neighborhood of p satisfying that $\mathbf{C} \otimes TM_p$ is generated by $(E_M)_p$, $(\overline{(E_M)})_p$, ζ. However, by the assumption, $\dim_{\mathbf{R}} M = \dim_{\mathbf{C}} E_M + 1$, and $(E_M)_p + \overline{(E_M)_p} \subsetneq \mathbf{C} \otimes TM$ (we note that (p, u) is a characteristic point). Then $(E_M)_p = (\overline{(E_M)})_p$ must hold. So by taking a supplement vector field of $(E_M)_p$ as ζ, our proposition is OK. Q.E.D.

Now we introduce the notion of strong pseudoconvexity. For a point (p, u) in $C(E_M)$, $u \neq 0$, we set a vector bundle decomposition

$$\mathbf{C} \otimes TM_p = \mathbf{C}\zeta_p + (E_M)_p + \overline{(E_M)_p}\,, \tag{4-1}$$

where $(E_M)_p + \overline{(E_M)_p}$ means the vector space generated by $(E_M)_p$ and $\overline{(E_M)_p}$. By using the decomposition (4-1), we set a Levi form by;

$$L(u, v)\zeta = -\sqrt{-1}\,[L_1, \overline{L_2}]_\zeta \qquad \text{for} \quad u, v \in E_M\,,$$

where L_1, L_2 are C^∞ local sections of E_M defined in a neighborhood of q so that $L_1(q) = u$, $L_2(q) = v$, and $[L_1, \overline{L_2}]_\zeta$ means the ζ part of $[L_1, \overline{L_2}]$ according to (4-1). If this Levi form is positive or negative definite at every point of $C(E_M)$, then our (M, E_M) is called a strongly pseudoconvex generalized-Mizohata structure.

5. Local embedding theorem.

Let (M, E_M) be a formally integrable generalized-Mizohata structure, which is strongly pseudoconvex. We see that this generalized-Mizohata structure can be locally embedded in $\mathbf{R}^p \times \mathbf{C}^n$ as a regular real hypersurface, where $\dim_{\mathbf{R}} M = 2n + p - 1$, $\dim_{\mathbf{C}} E_M = n + p - 1$. However, as you have already recognized, by the complete same line as in [A2], [Ku3], our local embedding theorem is proved. For the proof, we see the following corresponding parts, which are already obvious.

Part 1′. Let f be a C^∞ local embedding of M into $\mathbf{R}^p \times \mathbf{C}^n$ at the reference point p_o. We set a neighborhood of p_o by

$$U_\varepsilon(f) = \{x\ ;\ x \in M,\ 2\mathrm{Re}(h \cdot f(x)) < \varepsilon\}\,,$$

where h is a fixed holomorphic function on $\mathbf{R}^p \times \mathbf{C}^n$ satisfying certain conditions (see [A2]). Then, for the solvability of the D-Neumann problem over $U_\varepsilon(f)$, we see that it suffices to solve a nonlinear D_b-equation on $U_\varepsilon(f)$ (for the detail, see Chapter 8 in [A2], where D-operator means the induced operator by the given CR structure ${}^oT''$).

Part 2′. Let f be a C^∞ local embedding of M into $\mathbf{R}^p \times \mathbf{C}^n$, and $(M, {}^fT'')$ denotes the induced CR-structure and D_b^f denotes the tangential induced operator on $U_\varepsilon(f)$ by ${}^fT''$ via f (for the detail, see Chapter 2 in [A2]). For this D_b^f, we have an a priori estimate, if $n+p-1 \geq 3$.

Part 3′. With the Neumann operator obtained in Part 2, by using the Nash iteration procecure, we obtain the local embedding theorem of generalized-Mizohata structures.

Namely, we have

THEOREM 5.1. *If* (M, E_M) *is a formally integrable generalized-Mizohata structure, which is strongly pseudoconvex and* $n+p-1 \geq 3$, *where* $\dim_{\mathbf{R}} M = 2n+p-1$, $\dim_{\mathbf{C}} E_M = n+p-1$, *then there is a* C^∞ *local embedding of* M *into a generalized complex manifold* $\mathbf{R}^p \times \mathbf{C}^n$, *satisfying;*

$$Xf_i = 0\,, \qquad X \in E_M$$

where $f = (f_1, \cdots, f_p, f_{p+1}, \cdots, f_{p+n})$.

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