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Connecting Lemmas and Representing Homology Classes of Simply Connected 4-Manifolds

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Abstract. We consider surfaces in simply connected 4-manifolds. We estimate the normal Euler numbers of bounded non-orientable surfaces and consider the problem of representing characteristic homology classes by orientable surfaces. To do so, we develop techniques connecting the above problems for given surfaces with the problems for surfaces with fewer first Betti numbers.

1. Introduction.

Throughout this paper, we work in the smooth category, all 4-manifolds are compact, connected, simply connected and oriented, and all surfaces are compact. We shall assume that all orientable surfaces are oriented.

We investigate surfaces in 4-manifolds and we consider two problems. One is estimating the normal Euler numbers of bounded non-orientable surfaces. The other is the problem of representing characteristic homology classes by orientable surfaces. To attack these problems, we develop so-called "Connecting Lemma" ([14], [10]), which are geometric constructions connecting our problems with the problem of representing certain homology classes by 2-disks or 2-spheres. We actually obtain Connecting Lemmas I, II and III.

Applying Connecting Lemma I, we have the following theorem (cf. [15, Theorem in p. 40]).

THEOREM 1.1. *Let M be a simply connected 4-manifold with $\partial M \cong S^3$, and K a knot in ∂M. If K bounds a non-orientable surface N in M that represents zero in $H_2(M, \partial M; Z_2)$, then*

$$\left|\frac{e(N)}{2}-\sigma(M)-\sigma(K)\right| \le \beta_2(M)+\beta_1(N).$$

REMARK. P. M. Gilmer has pointed out that a similar but sharper inequality can

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be proved for non-simply connected M by algebraic topology techniques.

In Theorem 1.1, $\sigma(M)$ and $\sigma(K)$ are the signatures of M and K respectively, β_i is the i-th Betti number and $e(N)$ is the normal Euler number of N defined as follows. Let M be a 4-manifold with $\partial M \cong S^3$ and N a properly embedded surface in M with $\partial N \cong S^1 \cup \cdots \cup S^1$. Let ∂N be oriented. Take a section N' of the normal bundle of N that does not intersect N. Let $e(N, \partial N) = -\mathrm{lk}(\partial N, \partial N')$, where $\partial N'$ is to be oriented similarly to ∂N. We call $e(N, \partial N)$ the *normal Euler number* of the pair $(N, \partial N)$ (cf. [5]). It is the normal Euler number of a closed surface obtained by capping off ∂N with an orientable surface in ∂M and pushing into $\mathrm{Int}\, M$. The normal Euler number depends on the orientations of ∂M and ∂N. But, if $\partial N \cong S^1$, then $e(N, \partial N)$ is independent of the choice of orientation of ∂N. In this case, we use the notation $e(N)$ instead of $e(N, \partial N)$.

Let M and K be as in Theorem 1.1. Suppose that K bounds a non-orientable surface N in M that represents a characteristic homology class. Cap off the pair $(\partial M, K)$ with a pair (D^4, F) and we have a new surface $N_1 = N \cup F$ in $\hat{M} = M \cup D^4$, where F is a properly embedded, orientable surface in D^4 with $\partial F = K$. Note that N_1 represents a characteristic homology class in $H_2(\hat{M}; Z_2)$. By the Generalized Whitney's Congruence [16], we have $\sigma(\hat{M}) \equiv e(N_1) + 2(2 - \beta_1(N) - \beta_1(F)) \bmod 4$. Since $\sigma(\hat{M}) = \sigma(M)$, $e(N_1) = e(N)$, and both $\beta_1(F)$ and $\sigma(K)$ are even, $\sigma(M) + 2\sigma(K) \equiv e(N) - 2\beta_1(N) \bmod 4$. If M is a 4-ball, then $e(N) - 2\sigma(K) \equiv 2\beta_1(N) \bmod 4$. From this and Theorem 1.1, we have the following corollary.

COROLLARY 1.1.1. *Let K be a knot in ∂D^4. If K bounds a non-orientable surface N in D^4 with $\beta_1(N) = g$, then the integer $e(N) - 2\sigma(K)$ has one of the following values:*

$$-2g, \quad -2g+4, \quad -2g+8, \cdots, 2g-4, \quad 2g. \qquad \square$$

Using Connecting Lemma III, we prove

THEOREM 1.2. *Let M be a closed, simply connected 4-manifold with $b_2^+(M) = k$ and $b_2^-(M) = l$, and ξ a characteristic homology class in $H_2(M; Z)$ with $\xi \cdot \xi \equiv \sigma(M) \bmod 16$. Suppose that ξ is represented by an embedded, closed, orientable surface in M with genus g. If the 11/8-conjecture is true for the 4-manifolds with $b_2^+ \leq \max(k, l) + g - 1$ or $b_2^- \leq \max(k, l) + g - 1$, then*

$$|\xi \cdot \xi - \sigma(M)| \leq 16\left(\frac{\max(k, l) + g - 1}{3}\right),$$

where $\xi \cdot \xi$ is the self-intersection number of ξ and b_2^+ (resp. b_2^-) is the rank of positive (resp. negative) part of the intersection form of a manifold.

Note that if ξ is characteristic, $\xi \cdot \xi \equiv \sigma(M) \bmod 8$ (see [9, Lemma 3.4 on p-25]). The 11/8-conjecture states that for any closed spin 4-manifold M, the inequality $\beta_2(M) \geq 11/8 |\sigma(M)|$ holds (cf. [12], [3]).

Since S. K. Donaldson [1, Theorems B and C] shows that the 11/8-conjecture is true for any manifold with $b_2^+ \leq 2$ or $b_2^- \leq 2$, we have the following corollary.

COROLLARY 1.2.1. *Let M and ξ be as in above theorem.*

(i) *If $k \leq 1$, $l \leq 1$ and $g \leq 2$, then $\xi \cdot \xi = \sigma(M)$.*

(ii) *If $k \leq 2$, $l \leq 2$ and $g \leq 1$, then $\xi \cdot \xi = \sigma(M)$.* □

Let F be an embedded, closed, orientable surface in M that represents the characteristic homology class ξ in $H_2(M; Z)$ with $\xi \cdot \xi \equiv \sigma(M) + 8 \bmod 16$. It is not hard to see that, for any $\varepsilon \in \{-1, 1\}$, there exists an embedded torus T_ε in $S^2 \times S^2$ that represents the characteristic homology class $2\alpha + 2\varepsilon\beta$, where α and β are standard generators of $H_2(S^2 \times S^2; Z)$ such that $\alpha \cdot \alpha = \beta \cdot \beta = 0$ and $\alpha \cdot \beta = \beta \cdot \alpha = 1$. Let $(M', F_\varepsilon) = (M \# S^2 \times S^2, F \# T_\varepsilon)$. Clearly F_ε represents the characteristic homology class $\xi + 2\alpha + 2\varepsilon\beta$ and $(\xi + 2\alpha + 2\varepsilon\beta) \cdot (\xi + 2\alpha + 2\varepsilon\beta) \equiv \sigma(M') \bmod 16$. Apply Theorem 1.2 to (M', F_ε) for $\varepsilon = \pm 1$, and we have

COROLLARY 1.2.2. *Let M be as in Theorem 1.2 and ξ a characteristic homology class in $H_2(M; Z)$ with $\xi \cdot \xi \equiv \sigma(M) + 8 \bmod 16$. Suppose that ξ is represented by an embedded, closed, orientable surface in M with genus g. If the 11/8-conjecture is true for the 4-manifolds with $b_2^+ \leq \max(k, l) + g + 1$ or $b_2^- \leq \max(k, l) + g + 1$, then*

$$|\xi \cdot \xi - \sigma(M)| \leq 16\left(\frac{\max(k, l) + g + 1}{3}\right) - 8 .$$ □

In Section 2, we state Connecting Lemmas I, II and III and prove these lemmas. Section 3 is devoted to proving Theorems 1.1 and 1.2. In Section 4, by using Connecting Lemma III, we consider the problem of representing characteristic second homology classes of almost definite 4-manifolds by embedded tori. We give a necessary condition for characteristic homology classes to be represented by embedded tori. In particular, for characteristic homology classes of $CP^2 \# \overline{CP^2}$, we give a necessary and sufficient condition for them to be represented by embedded tori. In Section 5, we give two applications. Our first application is to give a necessary condition for a knot to bound a Möbius band in a 4-ball. This condition implies that neither $3_1 \# 3_1$ nor 4_1 can bound a Möbius band in a 4-ball (cf. [11]). Second one is to show that if the 11/8-conjecture is true, then for any nonnegative integer g, there exist infinitely many knots (in different knot concordance classes) with trivial Alexander polynomial which cannot bound orientable surface with genus g in a 4-ball. In particular, there exist infinitely many knots with trivial Alexander polynomial which cannot bound orientable surface with genus 2 in a 4-ball.

We conclude with some notation. If M is a closed 4-manifold, punc$M$ denotes M with an open 4-ball deleted; the orientation of $\partial(\text{punc}\,M)$ is the one induced from punc$M$. For a positive integer n, nM indicates the connected sum of n copies of M.

2. Connecting Lemmas.

In this section we introduce Connecting Lemmas. In particular Connecting Lemmas I and III are used to prove Theorems 1.1 and 1.2, respectively.

We recall the definition of the normal Euler number of a closed surface. Let M be a closed 4-manifold and N an embedded, closed surface in M. Take a section N' of the normal bundle of N that is transverse to N. At each point of $N \cap N'$ choose a local orientation of N. This determines a local orientation of N', and so an incidence number ± 1 for the intersection point. This is indepdent of the orientation choice. Then $e(N)$ is the sum of these induced numbers over all points of $N \cap N'$. We call $e(N)$ the *normal Euler number* of N.

By the definition of the normal Euler number, we have the following lemma.

LEMMA 2.1. *Let M_i be a simply connected 4-manifold with $\partial M_i \cong S^3$ and N_i a properly embedded surface in M_i $(i=1, 2)$. Let ∂N_i be an oriented link. If there exists an orientation reversing diffeomorphism f from the pair $(\partial M_1, \partial N_1)$ to the pair $(\partial M_2, \partial N_2)$, then we have a new pair $(M, N)=(M_1 \cup_f M_2, N_1 \cup_f N_2)$ and the following equality holds*:

$$e(N)=e(N_1, \partial N_1)+e(N_2, \partial N_2)\,. \qquad \square$$

CONNECTING LEMMA I. *Let M be a simply connected 4-manifold with $\partial M \cong S^3$ and N a properly embedded, non-orientable surface in M with $\partial N \cong S^1$. If N represents zero in $H_2(M, \partial M; Z_2)$, then for any $\varepsilon \in \{-1, 1\}$, there exist a 4-manifold M_1 and a properly embedded surface N_1 with $\partial N_1 = \partial N$ in $M \# M_1$ satisfying the following*:

(i) $M_1 \cong S^2 \times S^2$ *or* $\cong S^2 \tilde{\times} S^2$,

(ii) N_1 *is non-orientable, if* $\beta_1(N) \geq 2$,

(iii) N_1 *represents zero in* $H_2(M \# M_1, \partial(M \# M_1); Z_2)$,

(iv) $e(N_1)=e(N)+2\varepsilon$, *and*

(v) $\beta_1(N_1)=\beta_1(N)-1$.

REMARK. If we replace that $\partial M \cong S^3$ and $\partial N \cong S^1$ with that $\partial M \cong \emptyset$ and $\partial N \cong \emptyset$, then the above lemma still holds.

PROOF. Let C be an orientation reversing loop in N. Since M is a simply connected 4-manifold, C is null-homotopic. We note that in these dimensions (i.e., for 1-manifolds in 4-manifolds) every homotopy may be replaced by an isotopy. It follows that C bounds a 2-disk D in M. We can assume that D is transverse to N. Taking a neighborhood $V(D)$ of D in M suitably, we see that $N \cap V(D)$ consists of one Möbius band and some 2-disks $D_1, D_2, \cdots, D_l$ and that $\partial(N \cap V(D)) \subset \partial V(D)$ is a link as in Figure 1. Set $N' = N \cap V(D)$ and $L = \partial N'$. We orient $L \subset \partial V(D)$ so that the diagram of L in Figure 1 has only positive or negative crossings (Figure 2). Note that $e(N', L)=4k+4l+2$ if L is oriented as in Figure 2(a), and $e(N', L)= -4k-4l-2$ if L is oriented as in Figure 2(b).

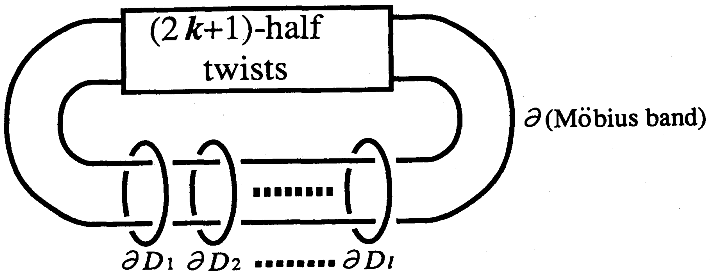

FIGURE 1

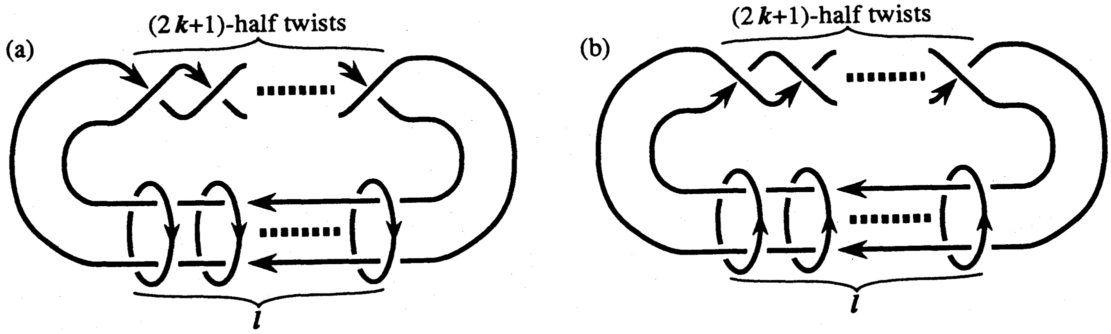

FIGURE 2

CLAIM 1. (a) If L is oriented as in Figure 2(a), then

(i) there exist mutually disjoint 2-disks in punc($S^2 \times S^2$) that represent $2\alpha+(k+l)\beta$ in H_2(punc($S^2 \times S^2$), ∂(punc($S^2 \times S^2$)); Z) and are bounded by L, and

(ii) there exist mutually disjoint 2-disks in punc($S^2 \times S^2$) that represent $2\alpha+(k+l+1)\beta$ in H_2(punc($S^2 \times S^2$), ∂(punc($S^2 \times S^2$)); Z) and are bounded by L.

(b) If L is oriented as in Figure 2(b), then

(i) there exist mutually disjoint 2-disks in punc($S^2 \times S^2$) that represent $2\alpha-(k+l)\beta$ in H_2(punc($S^2 \times S^2$), ∂(punc($S^2 \times S^2$)); Z) and are bounded by L, and

(ii) there exist mutually disjoint 2-disks in punc($S^2 \times S^2$) that represent $2\alpha-(k+l+1)\beta$ in H_2(punc($S^2 \times S^2$), ∂(punc($S^2 \times S^2$)); Z) and are bounded by L.

PROOF. There exist mutually disjoint $k+l+2$ 2-disks in punc($S^2 \times S^2$) that represent $2\alpha+(k+l)\beta$ and their boundary is as in Figure 3(a-i). It is not hard to see that $k+1$ strips $b_0, b_1, b_2, \cdots, b_k$ connecting the 2-disks can be chosen so that the boundary of the union of the 2-disks and the strips is L (Figure 4(a-i)). The resulting 2-disks are the required 2-disks.

The argument similar to that in the above proof and Figures 3 and 4 complete the proof. □

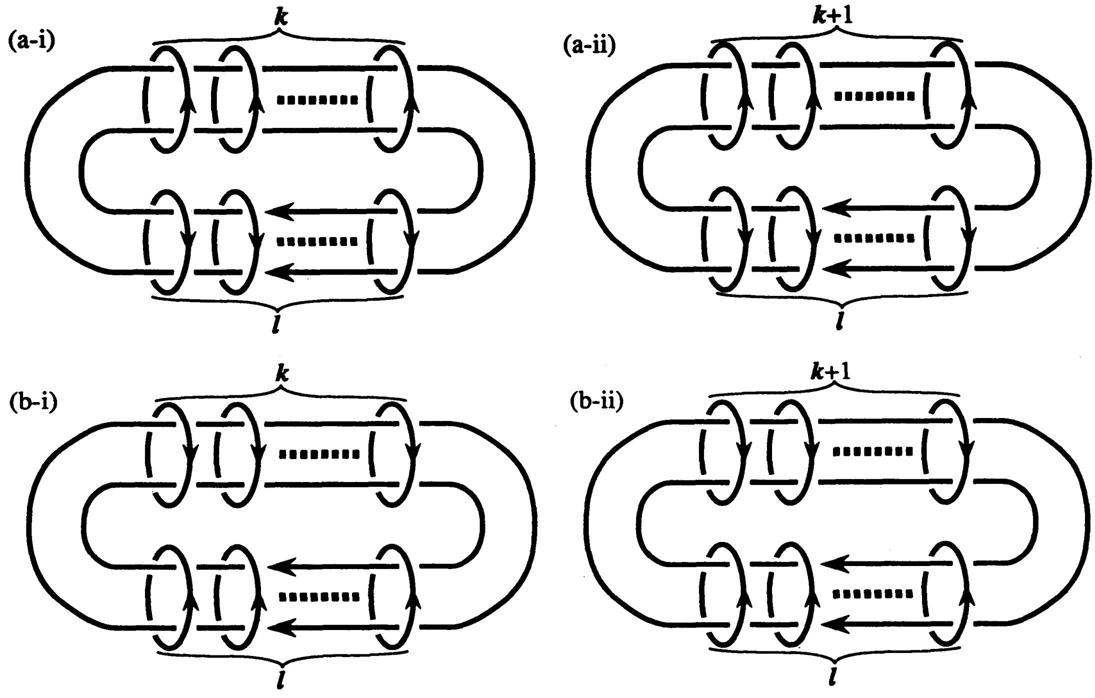

FIGURE 3

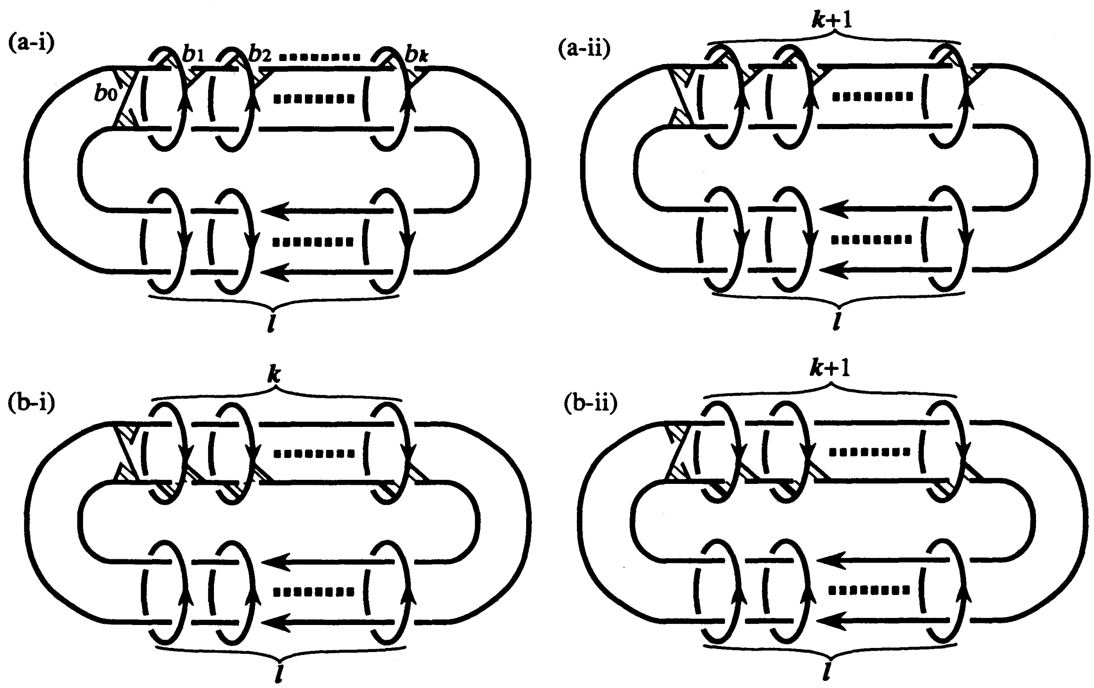

FIGURE 4

CLAIM 2. (a) If L is oriented as in Figure 2(a), then

(i) there exist mutually disjoint 2-disks in punc($S^2 \tilde{\times} S^2$) that represent $2\tilde{\alpha}+(k+l)\tilde{\beta}$ in H_2(punc($S^2 \tilde{\times} S^2$), ∂(punc($S^2 \tilde{\times} S^2$)); Z) and are bounded by L, and

(ii) there exist mutually disjoint 2-disks in punc($S^2 \tilde{\times} S^2$) that represent $2\tilde{\alpha}+(k+l-1)\tilde{\beta}$ in H_2(punc($S^2 \tilde{\times} S^2$), ∂(punc($S^2 \tilde{\times} S^2$)); Z) and are bounded by L.

(b) If L is oriented as in Figure 2(b), then

(i) there exist mutually disjoint 2-disks in punc($S^2 \tilde{\times} S^2$) that represent $2\tilde{\alpha}-(k+l+2)\tilde{\beta}$ in H_2(punc($S^2 \tilde{\times} S^2$), ∂(punc($S^2 \tilde{\times} S^2$)); Z) and are bounded by L, and

(ii) there exist mutually disjoint 2-disks in punc($S^2 \tilde{\times} S^2$) that represent $2\tilde{\alpha}-(k+l+1)\tilde{\beta}$ in H_2(punc($S^2 \tilde{\times} S^2$), ∂(punc($S^2 \tilde{\times} S^2$)); Z) and are bounded by L,

where $\tilde{\alpha}$ and $\tilde{\beta}$ are standard generators of H_2(punc($S^2 \tilde{\times} S^2$), ∂(punc($S^2 \tilde{\times} S^2$)); Z) such that $\tilde{\alpha}\cdot\tilde{\alpha}=1$, $\tilde{\alpha}\cdot\tilde{\beta}=\tilde{\beta}\cdot\tilde{\alpha}=1$ and $\tilde{\beta}\cdot\tilde{\beta}=0$.

PROOF. Let $O_0 \cup O_1$ be the Hopf link in ∂B^4. Attach 2-handles h_j^2 ($j=0$, 1) to B^4 along O_j with j-framing and 4-handle h^4 to $B^4 \cup h_0^2 \cup h_1^2$. The resulting 4-manifold $B^4 \cup h_0^2 \cup h_1^2 \cup h^4$ is deffeomorphic to $S^2 \tilde{\times} S^2$, and $h_0^2 \cup h_1^2 \cup h^4 \cong$ punc($S^2 \tilde{\times} S^2$). Let D_{01}, D_{02} be parallel copies of the core of h_1^2 and $D_{11}, D_{12}, \cdots, D_{1(k+l)}$ parallel copies of the core of h_0^2. Orienting $D_{01}, D_{02}, D_{11}, D_{12}, \cdots, D_{1(k+l)}$ suitably, we find that both D_{01} and D_{02} represent $\tilde{\alpha}$, each $D_{1j'}$ ($j'=1, 2, \cdots, k+l$) represents $\tilde{\beta}$ and $\bigcup \partial D_{jj'} \subset \partial$(punc($S^2 \tilde{\times} S^2$)) is the link as in Figure 5(a-i). It is not hard to see that $k+1$ strips $b_0, b_1, b_2, \cdots, b_k$ connecting the 2-disks can be chosen so that the boundary of the union of the 2-disks and the strips is L (Figure 6(a-i)). Note that the resulting 2-disks represent $2\tilde{\alpha}+(k+l)\tilde{\beta}$.

The argument similar to that in the above proof and Figures 5 and 6 complete the proof. □

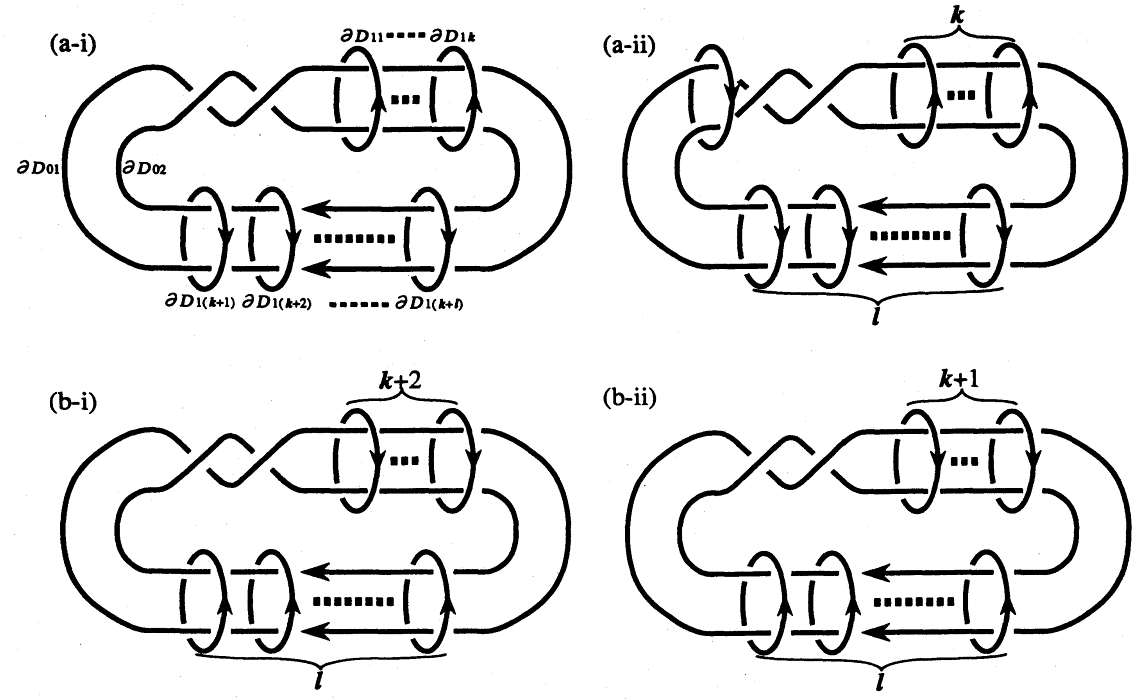

FIGURE 5

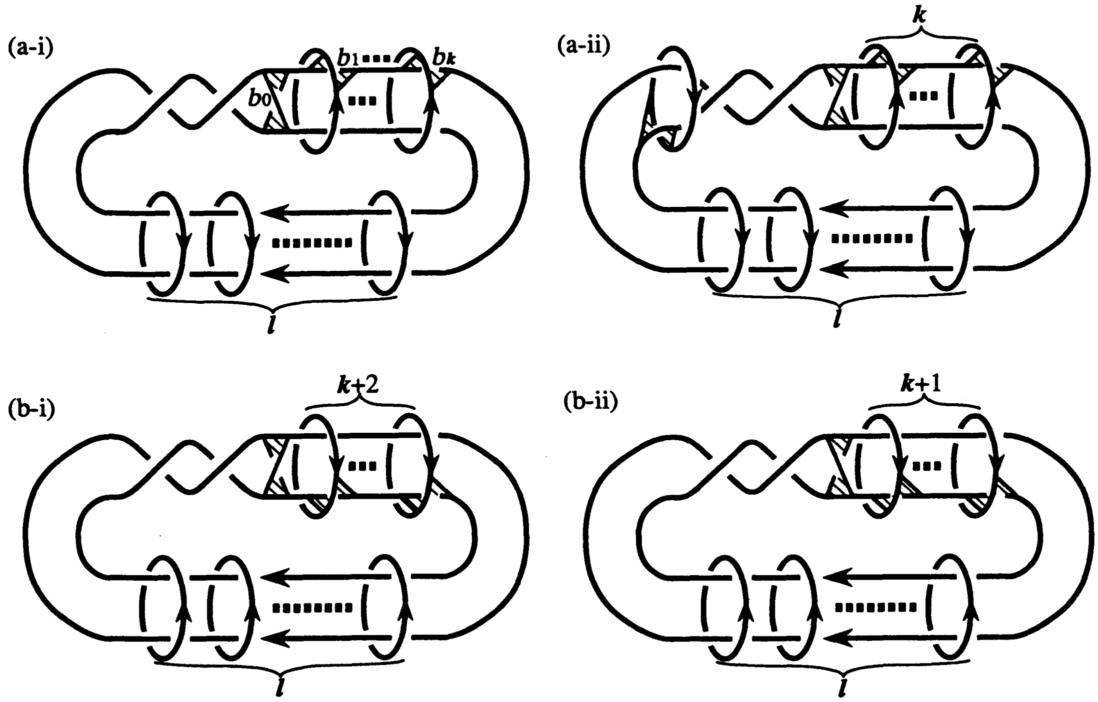

FIGURE 6

We consider the following four cases:

- L is as in Figure 2(a) and $k+l$ is even.
- L is as in Figure 2(a) and $k+l$ is odd.
- L is as in Figure 2(b) and $k+l$ is even.
- L is as in Figure 2(b) and $k+l$ is odd.

For any case and any $\varepsilon \in \{-1, 1\}$, by Claims 1 and 2, we find mutually disjoint 2-disks Δ in puncM_1 such that $M_1 \cong S^2 \times S^2$ or $S^2 \tilde{\times} S^2$, the second homology class $[\Delta, \partial\Delta]$ represented by Δ is divisible by 2 and $e(\Delta, \partial\Delta) = e(N', \partial N') + 2\varepsilon$. See Table 1. Cap off

TABLE 1

L	$k+l$	ε	M_1	$[\Delta, \partial\Delta]$	$e(\Delta, \partial\Delta)$	$e(N', L)$
Figure 2(a)	even	1	$S^2 \tilde{\times} S^2$	$2\tilde{\alpha}+(k+l)\tilde{\beta}$	$4k+4l+4$	$4k+4l+2$
		-1	$S^2 \times S^2$	$2\alpha+(k+l)\beta$	$4k+4l$	
	odd	1	$S^2 \times S^2$	$2\alpha+(k+l+1)\beta$	$4k+4l+4$	
		-1	$S^2 \tilde{\times} S^2$	$2\tilde{\alpha}+(k+l-1)\tilde{\beta}$	$4k+4l$	
Figure 2(b)	even	1	$S^2 \times S^2$	$2\alpha-(k+l)\beta$	$-4k-4l$	$-4k-4l-2$
		-1	$S^2 \tilde{\times} S^2$	$2\tilde{\alpha}-(k+l+2)\tilde{\beta}$	$-4k-4l-4$	
	odd	1	$S^2 \tilde{\times} S^2$	$2\tilde{\alpha}-(k+l+1)\tilde{\beta}$	$-4k-4l$	
		-1	$S^2 \times S^2$	$2\alpha-(k+l+1)\beta$	$-4k-4l-4$	

(Note: $e(\Delta, \partial\Delta) = [\Delta, \partial\Delta] \cdot [\Delta. \partial\Delta]$.)

the pair $(\partial(M-\operatorname{Int} V(D)), \partial(N-\operatorname{Int} N'))$ with a pair $(\operatorname{punc} M_1, \Delta)$, and we have a new surface $N_1=(N-\operatorname{Int} N')\cup\Delta$ in $M\# M_1$. Note that $\beta_1(N_1)=\beta_1(N)-1$, $\partial N_1=\partial N$ and N_1 represents zero in $H_2(M\# M_1, \partial(M\# M_1); Z_2)$. If $\beta_1(N)\geq 2$, then it is not hard to see that C can be chosen so that N_1 is non-orientable. Moreover by Lemma 2.1, $e(N_1)=e(N-\operatorname{Int} N', \partial(N-\operatorname{Int} N'))+e(\Delta, \partial\Delta)=e(N)-e(N', \partial N')+e(\Delta, \partial\Delta)$. Thus we have the following table:

TABLE 2

L	$k+l$	M_1	$e(N_1)$
Figure 2(a)	even	$S^2\tilde{\times}S^2$ $S^2\times S^2$	$e(N)+2$ $e(N)-2$
	odd	$S^2\times S^2$ $S^2\tilde{\times}S^2$	$e(N)+2$ $e(N)-2$
Figure 2(b)	even	$S^2\times S^2$ $S^2\tilde{\times}S^2$	$e(N)+2$ $e(N)-2$
	odd	$S^2\tilde{\times}S^2$ $S^2\times S^2$	$e(N)+2$ $e(N)-2$

This completes the proof. □

By the arguments similar to that in the above proof, we have the following known result [14], [10].

CONNECTING LEMMA II. *Let M be a simply connected 4-manifold with $\partial M\cong S^3$ (resp. $\cong\varnothing$) and N a properly embedded, non-orientable surface with $\partial N\cong S^1$ (resp. $\cong\varnothing$) that represents a characteristic homology class in $H_2(M, \partial M; Z_2)$. Then there exists a properly embedded surface N_1 with $\partial N_1=\partial N$ in $M\# S^2\times S^2$ such that*

(i) *N_1 is non-orientable, if $\beta_1(N)\geq 2$,*

(ii) *N_1 represents a characteristic homology class,*

(iii) *$e(N_1)=e(N)+2\varepsilon$ for some $\varepsilon=\pm 1$, and*

(iv) *$\beta_1(N_1)=\beta(N)-1$.* □

Let us recall the definition of the Arf invariant of surfaces in 4-manifolds representing characteristic homology classes. Let M be a 4-manifold with $\partial M\cong\varnothing$ or $\cong S^3$ and F a properly embedded, orientable surface in M with $\partial F\cong\varnothing$ or $\cong S^1$. Suppose that the homology class $[F, \partial F]\in H_2(M, \partial M; Z)$ is characteristic, then we can define a quadratic function $q: H_1(F; Z_2)\to Z_2$ as follows [16], [2], [13]. Let C be an embedded circle in F. Since M is simply connected, C bounds an embedded 2-disk D in M. We may assume that D is transverse to F at any point. The normal bundle ν_D of D is trivial. Note that any trivialization $\tau: \nu_D\cong D\times R^2$ induces a unique trivialization $\nu_D|\partial D\cong \partial D\times R^2$ on the boundary. The normal line bundle ν_C of C in F determines an orientable sub-line bundle in $\nu_D|\partial D$. Let $\mathcal{O}(D)$ be the number (mod 2) of the full twists of ν_C in $\nu_D|\partial D$ with respect to the unique trivialization above. Let $D\cdot F$ be the number of the intersection points of Int$D$ and F. Define $q(C)\in Z_2$ by

$$q(C)=\mathcal{O}(D)+D\cdot F \qquad \text{mod } 2\,.$$

This gives a well-defined function $q: H_1(F; Z_2)\to Z_2$ that is a quadratic function with respect to the intersection pairing $\cdot: H_1(F; Z_2)\otimes H_1(F; Z_2)\to Z_2$. Choose symplectic basis $a_1, a_2, \cdots, a_g, b_1, b_2, \cdots, b_g$ of $H_1(F; Z_2)$ by satisfyng $a_i\cdot a_j=b_i\cdot b_j=0$ and $a_i\cdot b_j=\delta_{ij}$ (Kronecker's delta). We define the *Arf invariant* Arf(F) of F to be $\sum_{i=1}^{g} q(a_i)q(b_i)$ mod 2. It is known that Arf(F) depends only on the relative integral homology class $[F, \partial F]$ and the knot concordance class of the embedding $\partial F\to\partial M$. In fact, we have the following theorem, which is a generalization of Rohlin's Theorem [16].

THEOREM 2.2. *Let M be a simply connected 4-manifold with $\partial M\cong\emptyset$ or $\cong S^3$ and F a properly embedded, orientable surface in M with $\partial F\cong\emptyset$ or $\cong S^1$ that represents a characteristic homology class. Then we have*

$$\mathrm{Arf}(\partial F)+\mathrm{Arf}(F)\equiv\frac{[F, \partial F]\cdot[F, \partial F]-\sigma(M)}{8} \qquad \text{mod } 2\,,$$

where Arf(∂F) *is the Arf invariant of the knot $\partial F\subset\partial M$ if $\partial F\neq\emptyset$, and* Arf(∂F)$=0$ *if* $\partial F=\emptyset$. □

The above theorem implies that for any embedded, closed, orientable surface F in M representing a characteristic homology class, Arf(F)$=0$ if and only if $[F]\cdot[F]\equiv\sigma(M)$ mod 16.

We state the third Connecting Lemma.

CONNECTING LEMMA III. *Let M be a closed, simply connected 4-manifold and F an embedded, closed, orientable surface in M that represents a characteristic homology class. If Arf(F)$=0$, i.e., $[F]\cdot[F]\equiv\sigma(M)$* mod 16, *then there exists an embedded, closed, orientable surface F_1 in $M\# S^2\times S^2$ such that $[F_1]$ is a characteristic homology class,* Arf(F_1)$=0$, $[F_1]\cdot[F_1]=[F]\cdot[F]$ *and* genus(F_1)$=$genus(F)-1.

PROOF. Set genus(F)$=g$. Since Arf(F)$=0$, there exist symplectic basis $a_1, a_2, \cdots, a_g, b_1, b_2, \cdots, b_g$ of $H_1(F; Z_2)$ such that $q(a_i)=q(b_j)=0$ for any $i=1, 2, \cdots, g$. It follows that there exists an embedded essential loop C in F with $q(C)=0$. Since M is simply connected, C bounds a 2-disk D in M that is transverse to F. Taking a neighborhood $V(D)$ of D suitably, we see that $F\cap V(D)$ consists of one annulus and some 2-disks $D_1, D_2, \cdots, D_l$ and that $\partial(F\cap V(D))\subset\partial V(D)$ is a link as in Figure 7. Note that, in Figure 7, k is equal to the times of the full twists of ν_C in $\nu_D|\partial D$ with respect to the unique trivialization $\nu_D|\partial D\cong\partial D\times R^2$, and l is equal to the number $D\cdot F$ of the intersection points of Int$D$ and F. It follows from $q(C)=0$ that $k+l$ is even. By the arguments similar to that in the proof of Claim 1 in the proof of Connecting Lemma I, there exist mutually disjoint 2-disks Δ in punc$(S^2 \times S^2)$ that represent $0\alpha + 2m\beta$ ($m \in Z$) and $\partial\Delta \subset \partial(\text{punc}(S^2 \times S^2))$ is $\partial(F \cap V(D)) \subset \partial V(D)$. Cap off the pair $(\partial(M - \text{Int}\, V(D)),$ $\partial(F - \text{Int}\, V(D) \cap F))$ with the pair $(\text{punc}(S^2 \times S^2), \Delta)$, we have a new closed, orientable surface $F_1 = (F - \text{Int}\, V(D) \cap F) \cup \Delta$ in $M \# S^2 \times S^2$. By the above construction, we find that $[F_1]$ is a characteristic homology class, $[F_1] \cdot [F_1] = [F] \cdot [F]$ and genus(F_1) = genus$(F) - 1$. Since $[F_1] \cdot [F_1] = [F] \cdot [F]$ and $\sigma(M \# S^2 \times S^2) = \sigma(M)$, by Theorem 2.2, we have $\text{Arf}(F_1) = \text{Arf}(F) = 0$. □

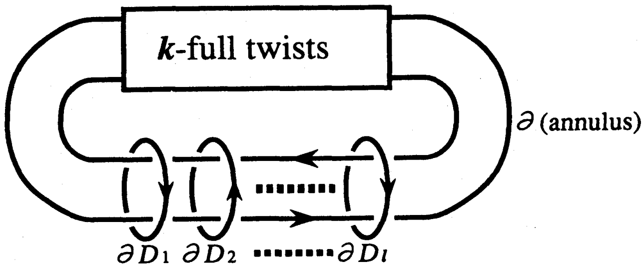

FIGURE 7

3. Proofs of Theorems 1.1 and 1.2.

We use the following theorem for proving Theorem 1.1.

THEOREM 3.1 (Viro [17], Gilmer [4]). *Let M be a simply connected 4-manifold with $\partial M \cong S^3$ and K a knot in ∂M. Suppose that K bounds an orientable surface F in M. If $[F, \partial F]$ is divisible by 2, then*

$$\left| \frac{[F, \partial F] \cdot [F, \partial F]}{2} - \sigma(M) - \sigma(K) \right| \leq \beta_2(M) + \beta_1(F) .$$

PROOF OF THEOREM 1.1. Set $\beta_1(N) = g$. Using Connecting Lemma I repeatedly, for any $\varepsilon \in \{-1, 1\}$, we have 4-manifolds $M_{1\varepsilon}, M_{2\varepsilon}, \cdots, M_{g\varepsilon}$ and a properly embedded 2-disk D_ε in $M \# M_{1\varepsilon} \# M_{2\varepsilon} \# \cdots \# M_{g\varepsilon}$ such that $M_{i\varepsilon} \cong S^2 \times S^2$ or $\cong S^2 \tilde{\times} S^2$, $[D_\varepsilon, \partial D_\varepsilon]$ is divisible by 2, $[D_\varepsilon, \partial D_\varepsilon] \cdot [D_\varepsilon, \partial D_\varepsilon] = e(N) + 2\varepsilon g$ and $\partial D_\varepsilon = K$. It follows from Theorem 3.1 that for any $\varepsilon \in \{-1, 1\}$

$$\left| \frac{e(N) + 2\varepsilon g}{2} - \sigma(M \# M_{1\varepsilon} \# M_{2\varepsilon} \# \cdots \# M_{g\varepsilon}) - \sigma(K) \right| \leq \beta_2(M \# M_{1\varepsilon} \# M_{2\varepsilon} \# \cdots \# M_{g\varepsilon}) .$$

This implies that, for any $\varepsilon \in \{-1, 1\}$,

$$\left| \frac{e(N) + 2\varepsilon g}{2} - \sigma(M) - \sigma(K) \right| \leq \beta_2(M) + 2g .$$

These two inequalities imply the required inequality

$$\left|\frac{e(N)}{2}-\sigma(M)-\sigma(K)\right| \leq \beta_2(M)+g\,.$$

This completes the proof. □

In order to prove Theorem 1.2, we show one implication of the 11/8-conjecture. This proposition is proved by the arguments similar to Kikuchi's [8, Proof of Lemma 3.4].

PROPOSITION 3.2. *Let M be a closed, simply connected 4-manifold with $b_2^+(M)=k$ and $b_2^-(M)=l$. Suppose that S is an embedded 2-sphere in M that represents a characteristic homology class. If the 11/8-conjecture is true for the 4-manifolds with $b_2^+ \leq \max(k,l)-1$ or $b_2^- \leq \max(k,l)-1$, then*

$$|[S]\cdot[S]-\sigma(M)| \leq 16\left(\frac{\max(k,l)-1}{3}\right).$$

PROOF. Set $\max(k,l)=m$. Since $[S]$ is a characteristic homology class, by [7, Theorem 1], $[S]\cdot[S]\equiv\sigma(M)$ mod 16. Set $[S]\cdot[S]-\sigma(M)=16x$ $(x\in Z)$. It is sufficient to prove that

$$|x| \leq \frac{m-1}{3}\,.$$

Let

$$(M',S')=\begin{cases}(M,S)\,\#\,(k-l)(\overline{CP^2},CP^1) & \text{if} \quad k\geq l\,,\\ (M,S)\,\#\,(l-k)(CP^2,CP^1) & \text{if} \quad k<l\,.\end{cases}$$

Note that $b_2^+(M')=b_2^-(M')=m$, $[S']$ is a characteristic homology class in $H_2(M';Z)$ and $[S']\cdot[S']=16x$.

In case that $x>0$, taking the connected sum $(M',S')\,\#\,(16x-1)(\overline{CP^2},CP^1)$, we have a new manifold pair (M'',S''). Clearly $[S'']$ is a characteristic homology class in $H_2(M'';Z)$ and $[S'']\cdot[S'']=1$. Let $U(S'')$ be a tubular neighborhood of S'' in M''. Since $[S'']\cdot[S'']=1$, we have a new manifold $M_1=(M''-U(S''))\cup D^4$ with $b_2^+(M_1)=m-1$. Note that $M''=M_1\,\#\,CP^2$. The fact that $[S'']$ is a characteristic homology class implies that M_1 is a spin 4-manifold. The 11/8-conjecture says that

$$8\beta_2(M_1)\geq 11\,|\sigma(M_1)|\,.$$

Since $\beta_2(M_1)=2m+16x-2$ and $\sigma(M_1)=-16x$, we have

$$x\leq\frac{m-1}{3}\,.$$

In case that $x<0$, the similar arguments give us

$$-x \le \frac{m-1}{3}. \qquad \square$$

PROOF OF THEOREM 1.2. *Let* ξ be as in Theorem 1.2. Using Connecting Lemma III repeatedly, we have an embedded 2-sphere S in $M \# g(S^2 \times S^2)$ such that $[S]$ is a characteristic homology class and $[S]\cdot[S]=\xi\cdot\xi$. From Proposition 3.2, if the 11/8-conjecture is true for the 4-manifolds with $b_2^+ \le \max(k+g, l+g)-1$ or $b_2^- \le \max(k+g, l+g)-1$, then

$$|[S]\cdot[S]-\sigma(M \# g(S^2 \times S^2))| \le 16\left(\frac{\max(k+g, l+g)-1}{3}\right).$$

This completes the proof. $\square$

4. Tori in almost definite 4-manifolds.

Almost all in this section, we study which characteristic second homology classes are representable by embedded tori for almost definite 4-manifolds where $b_2^+=1$ or $=2$. And we have the following results.

THEOREM 4.1. *Let M be a closed, simply connected 4-manifold with $b_2^+(M)=b_2^-(M)=1$ and ξ a characteristic homology class in $H_2(M; Z)$. If ξ is represented by an embedded torus in M, then $|\xi\cdot\xi|=0$ or $=8$.*

Let γ and $\bar{\gamma}$ be standard generators of $H_2(CP^2 \# \overline{CP^2}; Z)$ with $\gamma\cdot\gamma=-\bar{\gamma}\cdot\bar{\gamma}=1$ and let $\xi=x\gamma+y\bar{\gamma}$ $(x, y \in Z)$. Note that if ξ is characteristic and $|\xi\cdot\xi|=0$ or $=8$, then $|x|=|y|$, $(|x|, |y|)=(3, 1)$, or $=(1, 3)$. It is not hard to see that ξ is represented by an embedded torus in $CP^2 \# \overline{CP^2}$ when $|x|=|y|$, $(|x|, |y|)=(3, 1)$ or $=(1, 3)$. Thus Theorem 4.1 gives a necessary and sufficient condition for characteristic homology classes of $CP^2 \# \overline{CP^2}$ to be represented by embedded tori, i.e., we have the following corollary.

COROLLARY 4.1.1. *Let ξ be a characteristic homology class in $H_2(CP^2 \# \overline{CP^2}; Z)$. Then ξ is represented by an embedded torus in $CP^2 \# \overline{CP^2}$ if and only if $|\xi\cdot\xi|=0$ or $=8$.* $\square$

Theorem 4.1 also gives a necessary and sufficient condition for characteristic homology classes of $S^2 \times S^2$ to be represented by embedded tori. However this is not remarkable because Rohlin's genus theorem [15] and Rohlin's signature theorem [16] give the same condition, too.

THEOREM 4.2. *Let M be a closed, simply connected 4-manifold with $b_2^+(M)=k$ and $b_2^-(M)=l$, and ξ a characteristic homology class in $H_2(M; Z)$. Suppose that ξ is represented by an embedded torus in M.*

(i) *If* $k=1$, $l\ge 3$ *and* $\xi\cdot\xi\equiv\sigma(M) \bmod 16$, *then* $\xi\cdot\xi<-1$.

(ii) *If* $k=1$, $l\ge 2$ *and* $\xi\cdot\xi\equiv\sigma(M)+8 \bmod 16$, *then* $\xi\cdot\xi<10$.

(iii) *If* $k=2$, $l\geq 3$ *and* $\xi\cdot\xi\equiv\sigma(M)$ mod 16, *then* $\xi\cdot\xi<1$.
(iv) *If* $k\leq 2$, $l\leq 2$ *and* $\xi\cdot\xi\equiv\sigma(M)$ mod 16, *then* $\xi\cdot\xi=\sigma(M)$.

Part (iv) follows directly from Corollary 1.2.1, (ii).

THEOREM 4.3. *Let M be a closed, simply connected* 4*-manifold with* $b_2^+(M)=1$ *and* $b_2^-(M)\geq 1$, *and* ξ *a characteristic homology class in* $H_2(M; Z)$ *with* $\xi\cdot\xi\equiv\sigma(M)$ mod 16. *If* ξ *is represented by an embedded, orientable surface in M with genus* 2, *then* $\xi\cdot\xi<1$.

In the above theorem, if $b_2^-(M)=1$, then by reversing orientations, we have $\xi\cdot\xi=0$ (cf. Corollary 1.2.1, (i)).

In order to prove Theorems 4.1, 4.2 and 4.3, we need the following theorem.

THEOREM 4.4 (Lawson [10], Kikuchi [8]). *Let M be a closed, simply connected* 4*-manifold with* $b_2^+(M)=k$ *and* $b_2^-(M)=l$, *and* ξ *a characteristic homology class in* $H_2(M; Z)$. *Suppose that* ξ *is represented by an embedded sphere in M*.
(i) *If* $k=3$ *and* $l\geq 3$, *then* $\xi\cdot\xi<1$.
(ii) *If* $k=2$ *and* $l\geq 4$, *then* $\xi\cdot\xi<-1$.

The above theorem follows from [8, Theorem 1.3, (1)(2)]. For part (ii), Lawson has a result [10, Theorem 3, (ii)] concerning 4-manifolds with odd intersection form. But, from [10, Proof of Theorem 3], we note that his result holds for 4-manifolds without odd intersection form. Connecting Lemma III connects our results Theorems 4.2 and 4.3 with the above theorem.

Let us start to prove the theorems.

PROOF OF THEOREM 4.3. By using Connecting Lemma III twice, we have an embedded sphere in $M\# 2(S^2\times S^2)$ that represents the characteristic homology class η with $\eta\cdot\eta=\xi\cdot\xi$. Since $b_2^+(M\# 2(S^2\times S^2))=3$ and $b_2^-(M\# 2(S^2\times S^2))=b_2^-(M)+2\geq 3$, by Theorem 4.4, (i), we have $\xi\cdot\xi=\eta\cdot\eta<1$. □

PROOF OF THEOREM 4.2. Part (i) (resp. part (iii)) follows from Connecting Lemma III and Theorem 4.4, (ii) (resp. (i)) by the argument similar to that in the proof of Theorem 4.3.

For part (ii), take a connected sum with $(\overline{CP^2}, T)$ where T is an embedded torus representing the characteristic homology class $3\bar{\gamma}$. As a result, we obtain an embedded, orientable surface with genus 2 in $M\#\overline{CP^2}$ that represents the characteristic homology class $\xi+3\bar{\gamma}$. Applying Theorem 4.3, we have $(\xi+3\bar{\gamma})\cdot(\xi+3\bar{\gamma})=\xi\cdot\xi-9<1$.

Part (iv) follows from Corollary 1.2.1, (ii). This completes the proof. □

PROOF OF THEOREM 4.1. Note that $\xi\cdot\xi\equiv 0$ or $\equiv 8$ mod 16. If $\xi\cdot\xi\equiv 0$ mod 16, then it follows from part (iv) of Theorem 4.2 that $\xi\cdot\xi=0$.

We consider the case that $\xi\cdot\xi\equiv 8$ mod 16. Set $\xi\cdot\xi=16x+8$ $(x\in Z)$ and suppose $x\geq 1$. We can easily see that there exists an embedded torus T' in $\overline{CP^2}$ that represents the characteristic homology class $3\bar{\gamma}$. Let T be an embedded torus in M that represents the characteristic homology class ξ and let $(M', F)=(M, T)\#(\overline{CP^2}, T')$. Since $b_2^+(M')=1$, $b_2^-(M')=2$, genus$(F)=2$ and F represents the characteristic homology class $\xi+3\bar{\gamma}$ with self-intersection number $16x-1$, by Theorem 4.3, we have $16x-1<1$. This contradicts $x\geq 1$.

If we set $\xi\cdot\xi=16x-8$ $(x\in Z)$ and suppose $x\leq -1$, then we obtain a contradiction by reversing orientations. It follows that $\xi\cdot\xi=8$ or $=-8$. This completes the proof. □

5. Applications.

Our first application, a consequence of Theorem 1.1, is as follows.

PROPOSITION 5.1. *If a knot K bounds a Möbius band in a 4-ball, then there exists an integer x such that*

$$|8x+4\mathrm{Arf}(K)-\sigma(K)|\leq 2 .$$

The above proposition implies that neither $3_1\#3_1$ nor 4_1 bounds a Möbius band in a 4-ball (cf. [11]).

PROOF. If K bounds a Möbius band N in a 4-ball B^4, then by Connecting Lemma II, there exits a properly embedded 2-disk D in $B^4\#S^2\times S^2$ such that $[D, \partial D]$ is a characteristic homology class, $\partial D=K$ and $[D, \partial D]\cdot[D, \partial D]=e(N)+2\varepsilon$ for some $\varepsilon=\pm 1$. By Theorem 2.2, we have

$$e(N)+2\varepsilon\equiv 8\,\mathrm{Arf}(K) \qquad \bmod 16 .$$

Set $e(N)+2\varepsilon=16x+8\mathrm{Arf}(K)$ $(x\in Z)$ and apply Theorem 1.1 to K, N and B^4, we have

$$\left|\frac{16x+8\mathrm{Arf}(K)-2\varepsilon}{2}-\sigma(K)\right|\leq 1 .$$

This implies

$$|8x+4\mathrm{Arf}(K)-\sigma(K)|\leq 2 .$$

This completes the proof. □

Before stating the second application, we need some preliminaries.

Let K_0 be a knot in S^3 and D^2 a 2-disk intersecting K_0 in its interior. Let $w=\mathrm{lk}(\partial D^2, K_0)$. A $1/n$-Dehn surgery along ∂D^2 changes K_0 into a new knot K_n in S^3. We say that K_n is obtained from K_0 by an *(n, w)-twisting* on D^2. A. J. Casson states the following theorem.

THEOREM 5.2 (Casson [6, Remark in p. 56]). *Any knot with trivial Arf invariant is concordant to a knot that can be obtained from a knot with trivial Alexander polynomial by a $(-1, -1)$-twisting.*

This theorem gives the following lemma.

LEMMA 5.3. *For any odd integer x with $x^2 \equiv 1 \bmod 16$, there exists a properly embedded 2-disk Δ in $\mathrm{punc}(CP^2 \# \overline{CP^2})$ such that $\partial\Delta$ is a knot with trivial Alexander polynomial and $[\Delta, \partial\Delta]$ is the characteristic homology class $x\gamma + \bar{\gamma}$, where γ and $\bar{\gamma}$ are standard generators such that $\gamma \cdot \gamma = -\bar{\gamma} \cdot \bar{\gamma} = 1$.*

PROOF. Let Δ be a properly embedded 2-disk in $\mathrm{punc}CP^2$ such that $[\Delta, \partial\Delta] = x\gamma$. Suppose $x^2 \equiv 1 \bmod 16$, then by Theorem 2.2, the Arf invariant of the knot $\partial\Delta \subset \partial(\mathrm{punc}CP^2)$ is zero. By Theorem 5.2, there exists a 2-disk D in $\partial(\mathrm{punc}CP^2)$ such that $\mathrm{lk}(\partial D, \partial\Delta) = -1$ and a -1-Dehn surgery along ∂D changes the knot $\partial\Delta$ into a knot with trivial Alexander polynomial, say K_x. Hence, by attaching a 2-handle to $\mathrm{punc}CP^2$ with framing -1 along ∂D, we find Δ a properly embedded 2-disk in $\mathrm{punc}(CP^2 \# \overline{CP^2})$ such that $\partial\Delta = K_x$ and $[\Delta, \partial\Delta] = x\gamma + \bar{\gamma}$. □

Let x be an odd integer with $x^2 \equiv 1 \bmod 16$. By Lemma 5.3, we have a knot K_x in $\partial(\mathrm{punc}(CP^2 \# \overline{CP^2}))$ bounding a 2-disk that represents the characteristic homology class $x\gamma + \bar{\gamma}$. If $|x| \neq |x'|$, then K_x is not concordant to $K_{x'}$. (In fact, if K_x is concordant to $K_{x'}$, then we have an embedded 2-sphere in $2(CP^2 \# \overline{CP^2})$ that represents $x\gamma_1 + \bar{\gamma}_1 - (\gamma_2 + x'\bar{\gamma}_2)$. This together with Corollary 1.2.1, (ii) imply that $|x| = |x'|$.) Suppose that K_x bounds an orientable surface with genus g in a 4-ball, then we have an embedded, orientable surface F in $CP^2 \# \overline{CP^2}$ with genus g that represents the characteristic homology class $x\gamma + \bar{\gamma}$ in $H_2(CP^2 \# \overline{CP^2}; Z)$. Moreover $[F] \cdot [F] = x^2 - 1 \equiv 0 = \sigma(CP^2 \# \overline{CP^2}) \bmod 16$. This fact and Theorem 1.2 imply that

$$|x^2 - 1| \leq 16 \cdot \frac{g}{3},$$

if the 11/8-conjecture is true for the 4-manifolds with $b_2^+ \leq g$ or $b_2^- \leq g$. So there are infinitely many x which does not satisfy the above inequality. Thus we obtain

PROPOSITION 5.4. *For any nonnegative integer g, there exist infinitely many knots with trivial Alexander polynomial which cannot bound orientable surface with genus g in a 4-ball, if the 11/8-conjecture is true for the 4-manifolds with $b_2^+ \leq g$ or $b_2^- \leq g$.* □

In particular, by [1, Theorem C] the following corollary holds.

COROLLARY 5.4.1. *There exist infinitely many knots with trivial Alexander polynomial which cannot bound orientable surface with genus 2 in a 4-ball.* □

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