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On Certain Multiple Series with Functional Equation in a Totally Imaginary Number Field I

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§1. Introduction.

In the recent paper [3], we considered a multiple series in a totally real number field, which is regarded as a generalization of the double series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}e^{-2\pi nm\tau} \qquad (\mathrm{Re}\,\tau>0)\,,$$

and proved that it satisfies functional equation.

In the present paper, we shall treat analogous problem in a totally imaginary number field. Our method will be similar to that of [3]; the proof is based on the transformation formula of Hecke-Rademacher, the expression of our series by integrals and the calculation of residues.

Let K be a totally imaginary number field of degree $n=2r$, $K^{(p)}$, $K^{(r+p)}=\bar{K}^{(p)}$ $(p=1,\cdots,r)$ the conjugates of K. Let $\mathfrak{d}$ be the differente ideal of K, $D=N(\mathfrak{d})$ the absolute value of the discriminant of K and R the regulator of K.

If μ is a number of K, then we denote by $\mu^{(q)}$ the conjugates of μ in $K^{(q)}$ $(q=1,\cdots,n)$. We define n-dimensional vector $\mu=(\mu^{(1)},\cdots,\mu^{(n)})$. More generally, we shall often use n-dimensional complex vector $\xi=(\xi_1,\cdots,\xi_n)$ such that $\xi_{r+p}=\bar{\xi}_p$ $(p=1,\cdots,r)$ and write

$$S(\xi)=\sum_{q=1}^{n}\xi_q\,,\qquad N(\xi)=\prod_{q=1}^{n}\xi_q\,.$$

Let $\tau_1,\cdots,\tau_n$ be positive numbers such that $\tau_{r+p}=\tau_p$ $(p=1,\cdots,r)$. Let $\xi=(\xi_1,\cdots,\xi_n)$ be a complex vector stated above. Let $\mathfrak{a}$ and $\mathfrak{b}$ be non-zero fractional ideals of K. For these τ, ξ, $\mathfrak{a}$ and $\mathfrak{b}$, we define the series $M(\tau,\xi\,;\mathfrak{a},\mathfrak{b})$ as follows:

$$M(\tau,\xi\,;\mathfrak{a},\mathfrak{b})=\sum_{\substack{(\mu)\subset\mathfrak{a}\\(\mu)\neq 0}}\frac{1}{N(\mu)^{1/2}}\sum_{\substack{\nu\in\mathfrak{b}\\\nu\neq 0}}\exp\{-2\pi S(|\nu\mu|\tau)+2\pi iS(\mu\nu\xi)\}\,, \tag{1.1}$$

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where the outer sum is taken over all non-zero principal ideals (μ) contained in $\mathfrak{a}$ and the inner sum is taken over all non-zero numbers of $\mathfrak{b}$. This series is well-defined, since the inner sum is independent of the choice of the generators of the ideal (μ). (Remark that the series has the square roots $N(\mu)^{1/2}$ as the denominators of terms.)

Now we shall introduce another series:

$$\zeta(s, \mathfrak{a}) = \sum_{\substack{(\mu)\subset \mathfrak{a} \\ (\mu)\neq 0}} \frac{1}{N(\mu)^s} \qquad (s=\sigma+it,\ \sigma>1)\,, \tag{1.2}$$

where s is complex variable, and the sum has the same meaning as the outer sum in (1.1). This series $\zeta(s, \mathfrak{a})$ has the analytic continuation over the whole s-plane (Lemma 2.1).

The purpose of this paper is to prove the following

THEOREM. *If we put*

$$\Phi(\tau, \xi\,;\, \mathfrak{a}, \mathfrak{b}) = M(\tau, \xi\,;\, \mathfrak{a}, \mathfrak{b}) + \zeta(1/2, \mathfrak{a}) + (-4\pi)^r \zeta(-1/2, \mathfrak{b})\tau_1 \cdots \tau_r\,,$$

then we have

$$N(\mathfrak{a}\mathfrak{b})^{1/2} \prod_{p=1}^{r} (\tau_p^2 + |\xi_p|^2)^{1/4} \cdot \Phi(\tau, \xi\,;\, \mathfrak{a}, \mathfrak{b})$$

$$= N(\mathfrak{a}^*\mathfrak{b}^*)^{1/2} \prod_{p=1}^{r} (\tau_p^{*2} + |\xi_p^*|^2)^{1/4} \cdot \Phi(\tau^*, \xi^*\,;\, \mathfrak{b}^*, \mathfrak{a}^*)\,,$$

where $\mathfrak{a}^* = (\mathfrak{a}\mathfrak{d})^{-1}$, $\mathfrak{b}^* = (\mathfrak{b}\mathfrak{d})^{-1}$ *and*

$$\tau_q^* = \frac{\tau_q}{\tau_q^2 + |\xi_q|^2} \qquad (q=1, \cdots, n)\,, \tag{1.3}$$

$$\xi_p^* = \frac{\xi_{r+p}}{\tau_p^2 + |\xi_p|^2}\,, \quad \xi_{r+p}^* = \frac{\xi_p}{\tau_p^2 + |\xi_p|^2} \qquad (p=1, \cdots, r)\,. \tag{1.4}$$

First we shall consider, in §2, the functions $\zeta(s, \lambda\,;\, \mathfrak{a})$ and summarize some properties of them in Lemmas 2.1, 2.2 and 2.3.

Next in §3, by using the transformation formula of Hecke-Rademacher we shall obtain the representation of $M(\tau, \xi\,;\, \mathfrak{a}, \mathfrak{b})$ as the series of the complex integrals:

$$M(\tau, \xi\,;\, \mathfrak{a}, \mathfrak{b}) = \sum_{\lambda} \frac{1}{2\pi i} \int_{(5/4)} H_\lambda(s, \tau, \xi\,;\, \mathfrak{a}, \mathfrak{b})\,ds\,. \tag{1.5}$$

The integrands $H_\lambda(s, \tau, \xi\,;\, \mathfrak{a}, \mathfrak{b})$ are the products of the gamma function, the $\zeta(s, \lambda\,;\, \mathfrak{a})$, the hypergeometric functions and some elementary functions (see (4.1) below). Using Lemma 2.3 and some results in [4], we shall have the estimate of $H_\lambda(s, \tau, \xi\,;\, \mathfrak{a}, \mathfrak{b})$, by which we shall be able to change the path of integration in (1.5). Then the functional equation satisfied by $H_\lambda(s, \tau, \xi\,;\, \mathfrak{a}, \mathfrak{b})$ (Lemma 4.2) will give the equation as follows:

$$M(\tau, \xi ; \mathfrak{a}, \mathfrak{b})=(DN(\mathfrak{a}\mathfrak{b}))^{-1} \prod_{p=1}^{r} (\tau_p^2+|\xi_p|^2)^{-1/2}M(\tau^*, \xi^* ; \mathfrak{b}^*, \mathfrak{a}^*)+R(\tau, \xi ; \mathfrak{a}, \mathfrak{b}),$$

where $R(\tau, \xi ; \mathfrak{a}, \mathfrak{b})$ is the sum of the residues of $H_1(s, \tau, \xi ; \mathfrak{a}, \mathfrak{b})$. Finally in §6, we shall calculate $R(\tau, \xi ; \mathfrak{a}, \mathfrak{b})$ and then we shall complete the proof of Theorem.

§2. Zeta functions with Grössencharacters.

Let $\varepsilon_1, \cdots, \varepsilon_{r-1}$ be the fundamental units of K, $\rho=e^{2\pi i/w}$ the primitive w-th root of 1, w being the number of the roots of unity in K. Let $e_p^{(j)}$ $(p=1, \cdots, r ; j=1, \cdots, r-1)$ be the numbers satisfying the following equations:

$$\begin{cases} \sum_{p=1}^{r} e_p^{(j)}=0 & (j=1, \cdots, r-1), \\ \sum_{p=1}^{r} e_p^{(i)} \log|\varepsilon_j^{(p)}|=\begin{cases}1 & (i=j) \\ 0 & (i\neq j)\end{cases} & (i, j=1, \cdots, r-1). \end{cases}$$

Let $a_1, \cdots, a_n$ be non-negative integers such that $a_p \cdot a_{r+p}=0$ $(p=1, \cdots, r)$. For such integers $a_1, \cdots, a_n$ and any rational integers $m_1, \cdots, m_{r-1}$ we put

$$\begin{aligned} v_p &= v_p(m_1, \cdots, m_{r-1} ; a_1, \cdots, a_n) \\ &= \sum_{j=1}^{r-1} e_p^{(j)}\left(2\pi m_j+\sum_{q=1}^{n} a_q \arg \varepsilon_j^{(q)}\right) \qquad (p=1, \cdots, r). \end{aligned} \tag{2.1}$$

Following Hecke [1], we define the Grössencharacter λ in K to be the function over complex vector $z=(z_1, \cdots, z_n)$:

$$\lambda(z)=\prod_{p=1}^{r} |z_p|^{-iv_p} \prod_{q=1}^{n} \left(\frac{z_q}{|z_q|}\right)^{a_q},$$

provided that $a_1, \cdots, a_n$ satisfy the additional condition

$$\prod_{q=1}^{n} \rho^{(q)a_q}=1. \tag{2.2}$$

Now we consider the series

$$\zeta(s, \lambda ; \mathfrak{a})=\sum_{\substack{(\mu)\subset \mathfrak{a} \\ (\mu)\neq 0}} \frac{\lambda(\mu)}{N(\mu)^s} \qquad (\sigma>1),$$

where the sum is taken over all non-zero principal ideals (μ) contained in $\mathfrak{a}$. This series is well-defined, since $\lambda(\varepsilon)=1$ for any unit ε of K.

If $\lambda=1$, then $\zeta(s, 1 ; \mathfrak{a})$ is the series $\zeta(s, \mathfrak{a})$ stated in §1. So we write, in the following, $\zeta(s, \mathfrak{a})$ instead of $\zeta(s, 1 ; \mathfrak{a})$.

LEMMA 2.1. (1) *$\zeta(s, \lambda; \mathfrak{a})$ has the analytic continuation over the whole s-plane and satisfies the functional equation as follows*:

$$\zeta(s, \lambda; \mathfrak{a})=\frac{(2\pi)^{n(s-1/2)}}{N(\mathfrak{a})\sqrt{D}}\frac{\Gamma(1-s;\bar{\lambda})}{\Gamma(s;\lambda)}\zeta(1-s, \bar{\lambda}; \mathfrak{a}^*), \tag{2.3}$$

where $\Gamma(s; \lambda)$ is the product of the gamma function:

$$\Gamma(s;\lambda)=\prod_{p=1}^{r}\Gamma\left(s+\frac{a_p+a_{p+r}}{2}+\frac{iv_p}{2}\right).$$

(2) *If $\lambda\neq 1$, then*

$$\Gamma(s;\lambda)\zeta(s, \lambda; \mathfrak{a})$$

is an entire function.

(3) *In the case $\lambda=1$,*

$$\Gamma(s)^r\zeta(s, \mathfrak{a})$$

is a meromorphic function with only two simple poles at $s=0$ and 1.

(4) *$\zeta(s, \mathfrak{a})$ is regular in the whole s-plane except at $s=1$, where $\zeta(s, \mathfrak{a})$ has simple pole with the residue*

$$\frac{(2\pi)^r R}{wN(\mathfrak{a})\sqrt{D}}.$$

PROOF. We can obtain these results from Hecke [1] in the same way as was stated in [3]. So we omit the proof. □

Lemma 2.1, (3) shows that $\zeta(s, \mathfrak{a})$ has the zero of order $r-1$ as $s=0$. Moreover, we have the following

LEMMA 2.2. *We have*

$$\zeta^{(r-1)}(0, \mathfrak{a})=-(r-1)!R/w. \tag{2.4}$$

PROOF. We see from Lemma 2.1, (4) that

$$\lim_{s\to 0} s\zeta(1+s, \mathfrak{a})=\operatorname*{Res}_{s=1}\zeta(s, \mathfrak{a})=\frac{(2\pi)^r R}{wN(\mathfrak{a})\sqrt{D}}. \tag{2.5}$$

On the other hand, by the functional equation

$$\zeta(1+s, \mathfrak{a})=\frac{(2\pi)^{n(s+1/2)}}{N(\mathfrak{a})\sqrt{D}}\frac{\Gamma(-s)^r}{\Gamma(1+s)^r}\zeta(-s, \mathfrak{a}^*),$$

which is obtained from (2.3), we have

(2.6)
$$\lim_{s\to 0} s\zeta(1+s, \mathfrak{a}) = \frac{(2\pi)^r}{N(\mathfrak{a})\sqrt{D}} \lim_{s\to 0} \{s\Gamma(-s)^r \zeta(-s, \mathfrak{a}^*)\}$$
$$= \frac{(2\pi)^r}{N(\mathfrak{a})\sqrt{D}} \frac{-1}{(r-1)!} \zeta^{(r-1)}(0, \mathfrak{a}^*) .$$

Comparing these two expressions (2.5) and (2.6), we have

$$\zeta^{(r-1)}(0, \mathfrak{a}^*) = -(r-1)!\, R/w .$$

Since this right-hand side is independent of the choice of $\mathfrak{a}$, we obtain (2.4). □

LEMMA 2.3. *In the strip* $-1/2 \leqq \sigma \leqq 3$, *we have*

$$\zeta(s, \lambda; \mathfrak{a})(s-1)^{e(\lambda)} \ll (1+|t|)^{3r} ,$$

where $\ll$ *is Vinogradov's symbol,*

$$e(\lambda) = \begin{cases} 1 & \text{if } \lambda = 1 , \\ 0 & \text{if } \lambda \neq 1 \end{cases}$$

and the constants implied in this estimation depend on λ *and* $\mathfrak{a}$.

PROOF. ([4, Hilfssatz 15].) □

§3. Representation by integrals.

Let $\varepsilon_1, \cdots, \varepsilon_{r-1}$ and ρ be the units of K stated in the previous section. We rewrite the inner sum of (1.1) as follows:

(3.1)
$$\sum_{\substack{\nu\in\mathfrak{b}\\ \nu\neq 0}} \exp\{-2\pi S(|\nu\mu|\tau) + 2\pi i S(\nu\mu\xi)\}$$
$$= \sum_{\substack{(\nu)\subset\mathfrak{b}\\ (\nu)\neq 0}} \sum_{b=1}^{w} \sum_{b_1,\cdots,b_{r-1}=-\infty}^{\infty} \exp\{-2\pi S(|\nu\mu\varepsilon_1^{b_1}\cdots\varepsilon_{r-1}^{b_{r-1}}|\tau)$$
$$+ 2\pi i S(\nu\mu\rho^b \varepsilon_1^{b_1}\cdots\varepsilon_{r-1}^{b_{r-1}}\xi)\} .$$

In this right-hand side, $b_1, \cdots, b_{r-1}$ run through all rational integers and the outer sum is taken over all non-zero principal ideals (ν) contained in $\mathfrak{b}$.

Now we quote the transformation formula of Hecke-Rademacher from Rademacher [4] as a lemma:

LEMMA 3.1. *Let* $W_1, \cdots, W_n$ *be positive numbers such that* $W_{p+r} = W_p$ $(p = 1, \cdots, r)$. *Let* $U_1, \cdots, U_n$ *be complex numbers such that* $U_{p+r} = \bar{U}_p$ $(p = 1, \cdots, r)$. *Then we have*

$$(3.2)\qquad \sum_{b_1,\cdots,b_{r-1}=-\infty}^{\infty} \exp\{-2\pi S(W|\varepsilon_1^{b_1}\cdots\varepsilon_{r-1}^{b_{r-1}}|)+2\pi i S(U\varepsilon_1^{b_1}\cdots\varepsilon_{r-1}^{b_{r-1}})\}$$

$$=\frac{2^r}{R}\sum_{m_1,\cdots,m_{r-1}=\infty}^{\infty}\ \sum_{\substack{a_1,\cdots,a_n\geqq 0\\ a_p\cdot a_{r+p}=0}}^{\infty}\frac{1}{2\pi i}\int_{(5/4)}(2\pi)^{-ns}$$

$$\times\prod_{p=1}^{r}\left\{\frac{(iU_p)^{a_p}(iU_{r+p})^{a_{r+p}}}{(W_p^2+|U_p|^2)^{(2s+l_p+iv_p)/2}}\frac{\Gamma(2s+l_p+iv_p)}{2^{2s+l_p+iv_p}\Gamma(l_p+1)}\right.$$

$$\left.\times F\left(s+\frac{l_p+iv_p}{2},\frac{1}{2}-s+\frac{l_p-iv_p}{2},l_p+1;\frac{|U_p|^2}{W_p^2+|U_p|^2}\right)\right\}ds,$$

where $m_1,\cdots,m_{r-1}$ run through all rational integers, $a_1,\cdots,a_n$ run through non-negative rational integers such that $a_p\cdot a_{r+p}=0$ $(p=1,\cdots,r)$. The v_p are the values defined by (2.1) *and the $F(\alpha,\beta,\gamma;x)$ are the Gauss hypergeometric functions. We put $l_p=a_p+a_{p+r}$ $(p=1,\cdots,r)$ and the integrals in* (3.2) *are the complex integrals taken along the vertical line $\sigma=5/4$.*

PROOF. ([4, Hilfssatz 14]). □

Let l be a non-negative rational integer, x a number such that $0<x<1$. We put

$$F(s,l,x)=F\left(\frac{s+l}{2},\frac{1-s+l}{2},l+1;x\right),\qquad G(s,l,x)=\frac{\Gamma(s+l)}{2^{s+l}\Gamma(l+1)}F(s,l,x)$$

([4, p. 368]). Since $F(\alpha,\beta,\gamma;x)=F(\beta,\alpha,\gamma;x)$, we have

$$F(1-s,l,x)=F(s,l,x).$$

Moreover, we easily see that

$$G(1-s,l,x)=\frac{\Gamma(1-s+l)}{\Gamma(s+l)}2^{2s-1}G(s,l,x),$$

which shows that $G(s,l,x)$ is meromorphic in the whole s-plane. (In the half plane $\sigma>0$, $G(s,l,x)$ is regular ([4, p. 368])).

Now applying Lemma 3.1 with

$$\begin{cases} W_q=|\nu^{(q)}\mu^{(q)}|\tau_q & (q=1,\cdots,n),\\ U_q=\nu^{(q)}\mu^{(q)}\rho^{(q)b}\xi_q & (q=1,\cdots,n)\end{cases}$$

to the sum over $b_1,\cdots,b_{r-1}$ in the right-hand side of (3.1) and putting

$$x_p=\frac{|\xi_p|^2}{\tau_p^2+|\xi_p|^2}\qquad(p=1,\cdots,r),$$

we have

$$(3.3)\qquad M(\tau, \xi\,;\,\mathfrak{a}, \mathfrak{b})$$

$$= \sum_{\substack{(\mu)\subset\mathfrak{a}\\(\mu)\neq 0}} \frac{2^r}{R} \sum_{\substack{(\nu)\subset\mathfrak{b}\\(\nu)\neq 0}} \sum_{\{m\}} \sum_{\{a\}} \sum_{b=1}^{w} \frac{1}{2\pi i}\int_{(5/4)} (2\pi)^{-ns} \prod_{q=1}^{n} \rho^{(q)ba_q}$$

$$\times \prod_{p=1}^{r}\left[\frac{(i\mu^{(p)}\nu^{(p)}\xi_p)^{a_p}(i\mu^{(r+p)}\nu^{(r+p)}\xi_{r+p})^{a_{r+p}}}{\{(|\,\mu^{(p)}\nu^{(p)}\,|\,\tau_p)^2+(|\,\mu^{(p)}\nu^{(p)}\xi_p\,|)^2\}^{(2s+l_p+iv_p)/2}}\right.$$

$$\left.\times\, G(2s+iv_p,\, l_p,\, x_p)\right]\frac{1}{N(\mu)^{1/2}}\,ds$$

$$= \sum_{\substack{(\mu)\subset\mathfrak{a}\\(\mu)\neq 0}} \frac{2^r}{R} \sum_{\substack{(\nu)\subset\mathfrak{b}\\(\nu)\neq 0}} \sum_{\{m\}} \sum_{\{a\}} \sum_{b=1}^{w} \frac{1}{2\pi i}\int_{(5/4)} (2\pi)^{-ns} \prod_{q=1}^{n} \rho^{(q)ba_q}$$

$$\times \prod_{p=1}^{r}\left\{\frac{(i\xi_p)^{a_p}(i\xi_{r+p})^{a_{r+p}}}{(\tau_p^2+|\,\xi_p\,|^2)^{(2s+l_p+iv_p)/2}}\,G(2s+iv_p,\, l_p,\, x_p)\right\}$$

$$\times \frac{1}{N(\mu)^{s+1/2}} \prod_{p=1}^{r} |\,\mu^{(p)}\,|^{-iv_p} \prod_{q=1}^{n}\left(\frac{\mu^{(q)}}{|\,\mu^{(q)}\,|}\right)^{a_q}$$

$$\times \frac{1}{N(\nu)^{s}} \prod_{p=1}^{r} |\,\nu^{(p)}\,|^{-iv_p} \prod_{q=1}^{n}\left(\frac{\nu^{(q)}}{|\,\nu^{(q)}\,|}\right)^{a_q} ds\,,$$

where we denote by $\sum_{\{m\}}$ and $\sum_{\{a\}}$ the sums in (3.2) over $m_1, \cdots, m_{r-1}$ and $a_1, \cdots, a_n$, respectively.

Here we see that

$$\sum_{b=1}^{w}\prod_{q=1}^{n}\rho^{(q)ba_q} = \begin{cases} w & \text{if } \prod_{q=1}^{n}\rho^{(q)a_q}=1\,, \\ 0 & \text{if not}\,. \end{cases}$$

Hence, by the definition of the Grössencharacters λ, we can rewrite (3.3) as follows:

$$(3.4)\qquad M(\tau, \xi\,;\,\mathfrak{a}, \mathfrak{b}) = \sum_{\substack{(\mu)\subset\mathfrak{a}\\(\mu)\neq 0}} \frac{2^r w}{R} \sum_{\substack{(\nu)\subset\mathfrak{b}\\(\nu)\neq 0}} \sum_{\{m\}} \sum_{\{a\}}{}^{*} \frac{1}{2\pi i}\int_{(5/4)} (2\pi)^{-ns}$$

$$\times \prod_{p=1}^{r}\left\{\frac{(i\xi_p)^{a_p}(i\xi_{r+p})^{a_{r+p}}}{(\tau_p^2+|\,\xi_p\,|^2)^{(2s+l_p+iv_p)/2}}\,G(2s+iv_p,\, l_p,\, x_p)\right\}$$

$$\times \frac{\lambda(\mu)}{N(\mu)^{s+1/2}}\,\frac{\lambda(\nu)}{N(\nu)^{s}}\,ds\,,$$

where $\sum^*$ means that $a_1, \cdots, a_n$ satisfy the condition (2.2). Therefore, the sum $\sum_{\{m\}}\sum^*_{\{a\}}$ over $m_1, \cdots, m_{r-1}$ and $a_1, \cdots, a_n$ is regarded as the sum $\sum_{\lambda}$ over all Grössencharacters λ.

If $0<\varepsilon\leqq\sigma\leqq 3+\varepsilon$, then we have

$$G(s, l, x) \ll \exp\left(-\frac{|t|}{4}\sqrt{1-x}\right)\frac{(1+l+|t|)^{\sigma-1/2}}{l+1}\ \frac{(1-x)^{-1/4}}{(1+(1/2)\sqrt{1-x})^{l/2}}$$

([4, Hilfssatz 19]). Hence, if $\sigma=5/4$, then

$$G(2s+iv_p, l_p+1, x_p) \ll \exp\left(-\frac{1}{4}|2t+v_p|\sqrt{1-x_p}\right)\frac{(1+l_p+|2t+v_p|)^2}{l_p+1}. \tag{3.5}$$

Putting

$$2\theta = \min_{1\leq p\leq r}\left(\frac{1}{4}\sqrt{1-x_p}\right) \tag{3.6}$$

and using (3.5), we have the estimation of the integrand in (3.4):

$$(2\pi)^{-ns}\prod_{p=1}^{r}\left\{\frac{(i\xi_p)^{a_p}(i\xi_{r+p})^{a_{r+p}}}{(\tau_p^2+|\xi_p|^2)^{(2s+l_p+iv_p)/2}}G(2s+iv_p, l_p, x_p)\right\}\frac{\lambda(\mu)}{N(\mu)^{s+1/2}}\ \frac{\lambda(\nu)}{N(\nu)^s}$$
$$\ll \frac{1}{N(\mu\nu)^{5/4}}\exp\left(-2\theta\sum_{p=1}^{r}|2t+v_p|\right)\prod_{p=1}^{r}\frac{(1+l_p+|2t+v_p|)^2}{l_p+1}x_p^{l_p/2}$$
$$\ll \frac{1}{N(\mu\nu)^{5/4}}\exp\left(-\theta\sum_{p=1}^{r}|2t+v_p|\right)\prod_{p=1}^{r}(1+l_p)x_p^{l_p/2}.$$

Further we can estimate $M(\tau, \xi; \mathfrak{a}, \mathfrak{b})$ as follows:

$$M(\tau, \xi; \mathfrak{a}, \mathfrak{b}) \ll \sum_{\{m\}}\sum_{\{l\}}\int_{-\infty}^{\infty}\exp\left(-\theta\sum_{p=1}^{r}|2t+v_p|\right)dt\cdot\prod_{p=1}^{r}(1+l_p)x_p^{l_p/2},$$

where $l_1, \cdots, l_n$ run through all non-negative rational integers.

Since this last sum is convergent ([2], p. 206), we see that the series in the right-hand side of (3.4) is absolutely convergent. Therefore we can change, in (3.4), the order of the summations over (ν), (μ) and λ. Moreover, we can invert the order of the summations over (ν), (μ) and the integration.

Thus we have

$$\begin{aligned} M(\tau, \xi; \mathfrak{a}, \mathfrak{b}) = \sum_{\lambda}\frac{1}{2\pi i}\int_{(5/4)}&\frac{2^r w}{R}(2\pi)^{-ns} \\ &\times\prod_{p=1}^{r}\left\{\frac{(i\xi_p)^{a_p}(i\xi_{r+p})^{a_{r+p}}}{(\tau_p^2+|\xi_p|^2)^{(2s+l_p+iv_p)/2}}G(2s+iv_p, l_p, x_p)\right\} \\ &\times\zeta(s, \lambda; \mathfrak{b})\zeta(1/2+s, \lambda; \mathfrak{a})ds, \end{aligned} \tag{3.7}$$

where λ runs through all Grössencharacters.

§4. Lemmas on integrands.

We shall denote the integrands in (3.7) by $H_\lambda(s, \tau, \xi; \mathfrak{a}, \mathfrak{b})$:

$$H_\lambda(s, \tau, \xi; \mathfrak{a}, \mathfrak{b}) = \frac{2^r w}{R}(2\pi)^{-ns} \times \prod_{p=1}^{r}\left\{\frac{(i\xi_p)^{a_p}(i\xi_{r+p})^{a_{r+p}}}{(\tau_p^2+|\xi_p|^2)^{(2s+l_p+iv_p)/2}} G(2s+iv_p, l_p, x_p)\right\} \times \zeta(s, \lambda; \mathfrak{b})\zeta(1/2+s, \lambda; \mathfrak{a}). \tag{4.1}$$

LEMMA 4.1. (1) *If $\lambda \neq 1$, then $H_\lambda(s, \tau, \xi; \mathfrak{a}, \mathfrak{b})$ is regular in the strip $-3/4 \leqq \sigma \leqq 5/4$.*
(2) *$H_1(s, \xi, \tau; \mathfrak{a}, \mathfrak{b})$ is regular in the strip above except at $s=1$, $1/2$, 0 and $-1/2$, where $H_1(s, \xi, \tau; \mathfrak{a}, \mathfrak{b})$ has simple poles.*

PROOF. In the right-hand side of (4.1), we replace $\Gamma(2s+l_p+iv_p)$ by

$$\frac{2^{2s+l_p+iv_p}}{2\sqrt{\pi}}\Gamma\left(s+\frac{l_p+iv_p}{2}\right)\Gamma\left(s+\frac{1}{2}+\frac{l_p+iv_p}{2}\right).$$

Then

$$H_\lambda(s, \tau, \xi; \mathfrak{a}, \mathfrak{b}) = \frac{2^r w}{R}(2\pi)^{-ns}(2\sqrt{\pi})^{-r} \times \prod_{p=1}^{r}\left\{\frac{(i\xi_p)^{a_p}(i\xi_{r+p})^{a_{r+p}}}{(\tau_p^2+|\xi_p|^2)^{(2s+l_p+iv_p)/2}} \frac{1}{\Gamma(l_p+1)} F(2s+iv_p, l_p, x_p)\right\} \times \prod_{p=1}^{r}\left\{\Gamma\left(s+\frac{l_p+iv_p}{2}\right)\Gamma\left(s+\frac{1}{2}+\frac{l_p+iv_p}{2}\right)\right\}\zeta(s, \lambda; \mathfrak{b})\zeta\left(\frac{1}{2}+s, \lambda; \mathfrak{a}\right),$$

or, we can write

$$H_\lambda(s, \tau, \xi; \mathfrak{a}, \mathfrak{b}) = \frac{2^r w}{R}(2\pi)^{-ns}(2\sqrt{\pi})^{-r} \times \prod_{p=1}^{r}\left\{\frac{(i\xi_p)^{a_p}(i\xi_{r+p})^{a_{r+p}}}{(\tau_p^2+|\xi_p|^2)^{(2s+l_p+iv_p)/2}} \frac{1}{\Gamma(l_p+1)} F(2s+iv_p, l_p, x_p)\right\} \times \Gamma(s; \lambda)\zeta(s, \lambda; \mathfrak{b})\Gamma(1/2+s; \lambda)\zeta(1/2+s, \lambda; \mathfrak{a}). \tag{4.2}$$

In view of Lemma 2.1, (2) and (3), the proof follows from (4.2) at once. □

LEMMA 4.2. *$H_\lambda(s, \tau, \xi; \mathfrak{a}, \mathfrak{b})$ satisfies the functional equation as follows:*

$$H_\lambda(s, \tau, \xi; \mathfrak{a}, \mathfrak{b}) = (DN(\mathfrak{a}\mathfrak{b}))^{-1}\prod_{p=1}^{r}(\tau_p^2+|\xi_p|^2)^{-1/2} H_{\bar\lambda}\left(\frac{1}{2}-s, \tau^*, \xi^*, \mathfrak{b}^*, \mathfrak{a}^*\right),$$

where $\mathfrak{a}^* = (\mathfrak{a}\mathfrak{d})^{-1}$, $\mathfrak{b}^* = (\mathfrak{b}\mathfrak{d})^{-1}$, and τ_p^* and ξ_p^* $(p=1, \cdots, n)$ are the numbers defined by

(1.3) *and* (1.4).

Proof. We apply the functional equation (2.4) to (4.2). Then we have

$$(4.3)\qquad H_\lambda(s,\tau,\xi\,;\mathfrak{a},\mathfrak{b})=\frac{1}{DN(\mathfrak{ab})}\,\frac{2^r w}{R}(2\pi)^{n(s-1/2)}(2\sqrt{\pi})^{-r}$$
$$\times\prod_{p=1}^{r}\left\{\frac{(i\xi_p)^{a_p}(i\xi_{r+p})^{a_{r+p}}}{(\tau_p^2+|\xi_p|^2)^{(2s+l_p+iv_p)/2}}\,\frac{1}{\Gamma(l_p+1)}F(2s+iv_p,l_p,x_p)\right\}$$
$$\times\Gamma(1/2-s\,;\bar{\lambda})\zeta(1/2-s,\bar{\lambda}\,;\mathfrak{a}^*)\Gamma(1-s\,;\bar{\lambda})\zeta(1-s,\bar{\lambda}\,;\mathfrak{b}^*)\,.$$

By the definitions of τ_p^* and ξ_p^* $(p=1,\cdots,n)$, we see that

$$\tau_p^{*2}+|\xi_p^*|^2=(\tau_p^2+|\xi_p|^2)^{-1}\qquad(p=1,\cdots,r)\,,$$
$$\frac{|\xi_p^*|^2}{\tau_p^{*2}+|\xi_p^*|^2}=\frac{|\xi_p|^2}{\tau_p^2+|\xi_p|^2}=x_p\qquad(p=1,\cdots,r)$$

and

$$\frac{(i\xi_p)^{a_p}(i\xi_{r+p})^{a_{r+p}}}{(\tau_p^2+|\xi_p|^2)^{(2s+l_p+iv_p)/2}}=\frac{(i\xi_{r+p}^*)^{a_p}(i\xi_p^*)^{a_{r+p}}}{(\tau_p^{*2}+|\xi_p^*|^2)^{(-2s+l_p-iv_p)/2}}$$
$$=(\tau_p^2+|\xi_p|^2)^{-1/2}\frac{(i\xi_{r+p}^*)^{a_p}(i\xi_p^*)^{a_{r+p}}}{(\tau_p^{*2}+|\xi_p^*|^2)^{(1-2s+l_p-iv_p)/2}}\qquad(p=1,\cdots,r)\,.$$

Hence, noting that $F(s,l,x)=F(1-s,l,x)$, we have, from (4.3),

$$H_\lambda(s,\tau,\xi\,;\mathfrak{a},\mathfrak{b})=DN(\mathfrak{ab})^{-1}\prod_{p=1}^{r}(\tau_p^2+|\xi_p|^2)^{-1/2}\frac{2^r w}{R}(2^r\pi)^{n(s-1/2)}(2\sqrt{\pi})^{-r}$$
$$\times\prod_{p=1}^{r}\left\{\frac{(i\xi_p^*)^{a_{r+p}}(i\xi_{r+p}^*)^{a_p}}{(\tau_p^{*2}+|\xi_p^*|^2)^{(1-2s+l_p-iv_p)/2}}\,\frac{1}{\Gamma(l_p+1)}F(1-2s-iv_p,l_p,x_p)\right\}$$
$$\times\Gamma(1/2-s\,;\bar{\lambda})\zeta(1/2-s,\bar{\lambda}\,;\mathfrak{a}^*)\Gamma(1-s\,;\bar{\lambda})\zeta(1-s,\bar{\lambda}\,;\mathfrak{b}^*)$$
$$=(DN(\mathfrak{ab}))^{-1}\prod_{p=1}^{r}(\tau_p^2+|\xi_p|^2)^{-1/2}H_{\bar{\lambda}}\left(\frac{1}{2}-s,\tau^*,\xi^*\,;\mathfrak{b}^*,\mathfrak{a}^*\right).\qquad\square$$

Lemma 4.3. *For* $-3/4\leqq\sigma\leqq 5/4$, *we have*

$$(4.4)\qquad H_\lambda(s,\tau,\xi\,;\mathfrak{a},\mathfrak{b})\ll\exp(-\theta|t|)$$

where θ *is the constant in* (3.6). *The constants implied in this estimate* (4.4) *depend on* λ, τ, ξ, $\mathfrak{a}$ *and* $\mathfrak{b}$.

Proof. In view of Lemma 4.2, it is sufficient to prove lemma under the assumption $1/4\leqq\sigma\leqq 5/4$. From (4.1), (3.5) and Lemma 2.3, it follows that

$$H_\lambda(s, \tau, \xi ; \mathfrak{a}, \mathfrak{b}) \ll \exp(-\theta \sum_{p=1}^{r} |2t+v_p|)(1+|t|)^{6r},$$

which gives the proof at once. □

§5. Functional equation.

By Lemma 4.3 we see that

$$\int_{5/4+iT}^{-3/4+iT} H_\lambda(s, \tau, \xi ; \mathfrak{a}, \mathfrak{b}) ds \to 0 \qquad (|T| \to \infty),$$

where the integral is taken along the horizontal line from $5/4+iT$ to $-3/4+iT$. Therefore by Lemma 4.1 and Cauchy's formula,

$$(5.1) \qquad \frac{1}{2\pi i}\int_{(5/4)} H_\lambda(s, \tau, \xi ; \mathfrak{a}, \mathfrak{b}) ds$$

$$= \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{(-3/4)} H_\lambda(s, \tau, \xi ; \mathfrak{a}, \mathfrak{b}) ds & (\text{if } \lambda \neq 1), \\ \dfrac{1}{2\pi i}\displaystyle\int_{(-3/4)} H_1(s, \tau, \xi ; \mathfrak{a}, \mathfrak{b}) ds + R(\tau, \xi ; \mathfrak{a}, \mathfrak{b}) & (\text{if } \lambda = 1), \end{cases}$$

where

$$R(\tau, \xi ; \mathfrak{a}, \mathfrak{b}) = \operatorname*{Res}_{s=1} H_1 + \operatorname*{Res}_{s=1/2} H_1 + \operatorname*{Res}_{s=0} H_1 + \operatorname*{Res}_{s=-1/2} H_1$$

is the sum of the residues of $H_1(s, \xi, \tau ; \mathfrak{a}, \mathfrak{b})$. Hence we have, by (3.7), (4.1) and (5.1),

$$(5.2) \qquad M(\tau, \xi ; \mathfrak{a}, \mathfrak{b}) = \sum_\lambda \frac{1}{2\pi i}\int_{(-3/4)} H_\lambda(s, \tau, \xi ; \mathfrak{a}, \mathfrak{b}) ds + R(\tau, \xi ; \mathfrak{a}, \mathfrak{b}).$$

By Lemma 4.2,

$$(5.3) \qquad \sum_\lambda \frac{1}{2\pi i}\int_{(-3/4)} H_\lambda(s, \tau, \xi ; \mathfrak{a}, \mathfrak{b}) ds$$

$$= (DN(\mathfrak{a}\mathfrak{b}))^{-1} \prod_{p=1}^{r} (\tau_p^2 + |\xi_p|^2)^{-1/2} \sum_\lambda \frac{1}{2\pi i}\int_{(5/4)} H_{\bar\lambda}(s, \tau^*, \xi^* ; \mathfrak{b}^*, \mathfrak{a}^*) ds.$$

In this sum, $\bar\lambda$ runs through all Grössencharacters. Hence the last sum is equal to

$$(5.4) \qquad \sum_\lambda \frac{1}{2\pi i}\int_{(5/4)} H_\lambda(s, \tau^*, \xi^* ; \mathfrak{b}^*, \mathfrak{a}^*) ds = M(\tau^*, \xi^* ; \mathfrak{b}^*, \mathfrak{a}^*).$$

Thus we have, by (5.2), (5.3) and (5.4), (5.5) $$M(\tau, \xi ; \mathfrak{a}, \mathfrak{b})=(DN(\mathfrak{a}\mathfrak{b}))^{-1} \prod_{p=1}^{r} (\tau_p^2+|\xi_p|^2)^{-1/2} M(\tau^*, \xi^* ; \mathfrak{b}^*, \mathfrak{a}^*)+R(\tau, \xi ; \mathfrak{a}, \mathfrak{b}) .$$

§6. Residues.

Now we shall calculate $R(\tau, \xi ; \mathfrak{a}, \mathfrak{b})$.
If $\lambda=1$, then (4.1) gives

$$H_1(s, \tau, \xi ; \mathfrak{a}, \mathfrak{b})=\frac{2^r w \Gamma(2s)^r}{R(4\pi)^{ns}} \prod_{p=1}^{r} \left\{\frac{1}{(\tau_p^2+|\xi_p|^2)^s} F\left(s, \frac{1}{2}-s, 1 ; x_p\right)\right\}$$
$$\times \zeta(s, \mathfrak{b}) \zeta(1/2+s, \mathfrak{a}) .$$

By this expression, we obtain

$$\operatorname*{Res}_{s=1} H_1=\frac{2^r w}{R(4\pi)^n} \prod_{p=1}^{r} \left\{\frac{1}{\tau_p^2+|\xi_p|^2} F\left(1, -\frac{1}{2}, 1 ; x_p\right)\right\} \zeta\left(\frac{3}{2}, \mathfrak{a}\right) \operatorname*{Res}_{s=1} \zeta(s, \mathfrak{b})$$
$$=\frac{1}{(4\pi)^r N(\mathfrak{b})\sqrt{D}} \prod_{p=1}^{r} \left\{\frac{1}{\tau_p^2+|\xi_p|^2} F\left(1, -\frac{1}{2}, 1 ; x_p\right)\right\} \zeta\left(\frac{3}{2}, \mathfrak{a}\right) .$$

Since

$$F(1, -1/2, 1 ; x)=(1-x)^{1/2}$$

and, by the functional equation

$$\zeta\left(\frac{3}{2}, \mathfrak{a}\right)=\frac{(-4)^r (2\pi)^{2r}}{N(\mathfrak{a})\sqrt{D}} \zeta\left(-\frac{1}{2}, \mathfrak{a}^*\right),$$

we have

(6.1) $$\operatorname*{Res}_{s=1} H_1=\frac{(-4\pi)^r}{DN(\mathfrak{a}\mathfrak{b})} \zeta\left(-\frac{1}{2}, \mathfrak{a}^*\right) \prod_{p=1}^{r} \tau_p(\tau_p^2+|\xi_p|^2)^{-3/2} .$$

As for $\operatorname{Res}_{s=-1/2} H_1$, it follows from Lemma 4.2 and (6.1) that

(6.2) $$\operatorname*{Res}_{s=-1/2} H_1= \lim_{s\to -1/2} \left(s+\frac{1}{2}\right) H_1(s, \tau, \xi ; \mathfrak{a}, \mathfrak{b})$$
$$=(DN(\mathfrak{a}\mathfrak{b}))^{-1} \prod_{p=1}^{r} (\tau_p^2+|\xi_p|^2)^{-1/2}$$
$$\times \lim_{s\to -1/2} \left(s+\frac{1}{2}\right) H_1\left(\frac{1}{2}-s, \tau^*, \xi^* ; \mathfrak{b}^*, \mathfrak{a}^*\right)$$

$$=-(DN(\mathfrak{ab}))^{-1}\prod_{p=1}^{r}(\tau_p^2+|\xi_p|^2)^{-1/2}\operatorname*{Res}_{s=1}H_1(s,\tau^*,\xi^*;\mathfrak{b}^*,\mathfrak{a}^*)$$

$$=-(DN(\mathfrak{ab}))^{-1}\prod_{p=1}^{r}(\tau_p^2+|\xi_p|^2)^{-1/2}$$

$$\times\frac{(-4\pi)^r}{DN(\mathfrak{a}^*\mathfrak{b}^*)}\zeta\left(-\frac{1}{2},\mathfrak{b}\right)\prod_{p=1}^{r}\tau_p^*(\tau_p^{*2}+|\xi_p^*|^2)^{-3/2}$$

$$=-(-4\pi)^r\zeta\left(-\frac{1}{2},\mathfrak{b}\right)\prod_{p=1}^{r}\tau_p\,.$$

Next we have

$$\operatorname*{Res}_{s=0}H_1=\frac{2^rw}{R}\zeta\left(\frac{1}{2},\mathfrak{a}\right)\prod_{p=1}^{r}F\left(0,\frac{1}{2},1;x_p\right)\operatorname*{Res}_{s=0}\{\Gamma(2s)^r\zeta(s,\mathfrak{b})\}\,.$$

By Lemma 2.2, we see that

$$\operatorname*{Res}_{s=0}(\Gamma(2s)^r\zeta(s,\mathfrak{b})\}=-\frac{R}{2^rw}\,.$$

Thus we obtain

$$\operatorname*{Res}_{s=0}H_1=-\zeta(1/2,\mathfrak{a})\,, \tag{6.3}$$

since

$$F(0,1/2,1;x)=1\,.$$

Finally we have

$$\operatorname*{Res}_{s=1/2}H_1=\lim_{s\to 1/2}\left(s-\frac{1}{2}\right)H_1(s,\tau,\xi;\mathfrak{a},\mathfrak{b}) \tag{6.4}$$

$$=(DN(\mathfrak{ab}))^{-1}\prod_{p=1}^{r}(\tau_p^2+|\xi_p|^2)^{-1/2}$$

$$\times\lim_{s\to 1/2}\left(s-\frac{1}{2}\right)H_1\left(\frac{1}{2}-s,\tau^*,\xi^*;\mathfrak{b}^*,\mathfrak{a}^*\right)$$

$$=-(DN(\mathfrak{ab}))^{-1}\prod_{p=1}^{r}(\tau_p^2+|\xi_p|^2)^{-1/2}\operatorname*{Res}_{s=0}H_1(s,\tau^*,\xi^*;\mathfrak{b}^*,\mathfrak{a}^*)$$

$$=(DN(\mathfrak{ab}))^{-1}\zeta\left(\frac{1}{2},\mathfrak{b}^*\right)\prod_{p=1}^{r}(\tau_p^2+|\xi_p|^2)^{-1/2}\,.$$

Collecting the values of the residues (6.1), (6.2), (6.3) and (6.4), and combining them with (5.5), we have

$$M(\tau, \xi; \mathfrak{a}, \mathfrak{b}) = (DN(\mathfrak{a}\mathfrak{b}))^{-1} \prod_{p=1}^{r} (\tau_p^2 + |\xi_p|^2)^{-1/2} M(\tau^*, \xi^*; \mathfrak{b}^*, \mathfrak{a}^*)$$
$$+ \frac{(-4\pi)^r}{DN(\mathfrak{a}\mathfrak{b})} \zeta\left(-\frac{1}{2}, \mathfrak{a}^*\right) \prod_{p=1}^{r} \tau_p (\tau_p^2 + |\xi_p|^2)^{-3/2}$$
$$+ (DN(\mathfrak{a}\mathfrak{b}))^{-1} \zeta\left(\frac{1}{2}, \mathfrak{b}^*\right) \prod_{p=1}^{r} (\tau_p^2 + |\xi_p|^2)^{-1/2}$$
$$- \zeta\left(\frac{1}{2}, \mathfrak{a}\right) - (-4\pi)^r \zeta\left(-\frac{1}{2}, \mathfrak{b}\right) \prod_{p=1}^{r} \tau_p . \tag{6.5}$$

If we put

$$\Phi(\tau, \xi; \mathfrak{a}, \mathfrak{b}) = M(\tau, \xi; \mathfrak{a}, \mathfrak{b}) + \zeta\left(\frac{1}{2}, \mathfrak{a}\right) + (-4\pi)^r \zeta\left(-\frac{1}{2}, \mathfrak{b}\right) \prod_{p=1}^{r} \tau_p ,$$

then we can rewrite (6.5) as follows:

$$\Phi(\tau, \xi; \mathfrak{a}, \mathfrak{b}) = (DN(\mathfrak{a}\mathfrak{b}))^{-1} \prod_{p=1}^{r} (\tau_p^2 + |\xi_p|^2)^{-1/2} \cdot \Phi(\tau^*, \xi^*; \mathfrak{b}^*, \mathfrak{a}^*) .$$

Thus we complete the proof of Theorem.

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