TOKYO J. MATH.
VOL. 16, No. 2, 1993

# Exponential Kummer Coverings and Determinants of Hypergeometric Functions

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## §1. Introduction.

In the papers of Aomoto [A1][A2], he discovered a generalization of hypergeometric function of Appell's hypergeometric function and studied the monodromy of the differential equation defined by this Aomoto-Gel'fand hypergeometric function. This generalized hypergeometric function is defined as an integral of differential form on some topological cycle. Recently, this integral is known to be closely related to a period analogue of $l$-adic representation of profinite braid group or generalized braid group. The explicit formula for the determinant of arithmetic Magnus representation is given in [O-T]. In this paper we treat the period analog of the above paper.

We explain the results of this paper. Let $n$ be an integer such that $n\geq 3$, $\lambda_1, \cdots, \lambda_n$ and $\alpha_1, \cdots, \alpha_n$ be real numbers such that $\lambda_1<\cdots<\lambda_n$ and $\alpha_i>0$ respectively. Let $a_{ij}$ $(1\leq i, j\leq n-1)$ be a singular integral of Jordan-Pochhammer type defined by

$$a_{ij}=\int_{\lambda_i}^{\lambda_{i+1}}\prod_{p=1}^{i}(x-\lambda_p)^{\alpha_p-1}\prod_{p=i+1}^{n}(\lambda_p-x)^{\alpha_p-1}x^{j-1}dx\,.$$

THEOREM 1. *The determinant of* $A=(a_{ij})_{1\leq i,j\leq n-1}$ *is given by*

$$\det A=\prod_{i=1}^{n}\{(-1)^{i-1}\prod_{j\neq i}(\lambda_j-\lambda_i)\}^{\alpha_i}\prod_{1\leq i<j\leq n}(\lambda_j-\lambda_i)^{-1}\cdot\frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_1+\cdots+\alpha_n)}\,.$$

This theorem is proved by Varchenko [Var]. In this paper, we give another direct proof of this determinant theorem. For the intermediate exterior product for Appell's hypergeometric functions, we have the following theorem.

THEOREM 2. *For an integer r such that* $1\leq r\leq n-1$, *and sets of indices*

$$I\in\{(i_1, \cdots, i_r)\mid 0\leq i_1<\cdots<i_r\leq n-2\}\,,\quad J\in\{(j_1, \cdots, j_r)\mid 0\leq j_1<\cdots<j_r\leq n-2\}\,,$$

*we define* $A_{I,J}$ *as the* $(I, J)$*-minor of the matrix* $(a_{i,j})_{i\in I, j\in J}$ *defined as above. Let* $\bar{\Omega}_I$ *be a differential form;*

Received September 22, 1992

$$\bar{\Omega}_I=\prod_{i=1}^{n}\left(\sum_{k=1}^{r}(-\lambda_i)^{k-1}z_{r-k+1}+(-\lambda_i)^r\right)^{\alpha_i-1}u_Y(z_1,\cdots,z_r)dz_1\wedge\cdots\wedge dz_r\,,$$

*where* $u_Y(z_1,\cdots,z_r)$ *is the Schur function of the Young diagram* $Y=(i_1,\cdots,i_r-r+1)$, *and* $\tilde{D}_J$ *be a domain in* $\boldsymbol{R}^r$ *defined by*

$$\tilde{D}_J=\{(z_1,\cdots,z_r)\in\boldsymbol{R}^r\,|\,(-1)^{j_k-k}L_{j_k}(z_1,\cdots,z_r)>0\,,$$
$$(-1)^{j_k-k+1}L_{j_k+1}(z_1,\cdots,z_r)>0\ \textit{for all}\ k=1,\cdots,r\}\,,$$

*such that* $w_J\in\boldsymbol{R}-\pi i\mu$, *where* $\mu=\#\{k\,|\,j_k<k\}$ ($w_J$ *is defined in* §5.). *Then we have*

$$A_{I,J}=\int_{\tilde{D}_J}\bar{\Omega}_I\,.$$

These determinants are interpreted as period matrices of exponential Kummer coverings of $\boldsymbol{P}^1$.

The contents of this paper is as follows. In Section 2, we introduce an exponential Kummer convering of $\boldsymbol{P}^1$ and interpret the integral $a_{ij}$ as the integral of the differential form on some complex manifold, so called an exponential Kummer covering. In Section 3, we construct complex analytic correspondences between exponential Kummer coverings of $\boldsymbol{P}^1$ and exponential Fermat hypersurfaces. In Section 4, we prove the main theorem by computing the integral of some differential form on the exponential Fermat hypersurface. In Section 5, we give some relations between Aomoto-Gel'fand hypergeometric functions of special type and the minor of Appell's hypergeometric functions. It is a great pleasure for the author to express his thanks to Prof. M. Yoshida who showed interests on his results and gave him patient encouragements.

## §2. Structure of homology of exponential Kummer covering of $\boldsymbol{P}^1$.

Let $\lambda_1,\cdots,\lambda_n$ be distinct complex numbers. We define the exponential Kummer covering $C$ as follows. The analytic Riemann surface $C$ is an analytic set of $\boldsymbol{C}\times\boldsymbol{C}^n$ defined by

$$C=\{(x,z_1,\cdots,z_n)\in\boldsymbol{C}\times\boldsymbol{C}^n\,|\,\exp(z_i)=x-\lambda_i\,(i=1,\cdots,n)\}\,.$$

We define the action of $(2\pi\sqrt{-1}\,\boldsymbol{Z})^n$ on $C$ by

$$(a_i)_{i=1,\cdots,n}\in(2\pi\sqrt{-1}\,\boldsymbol{Z})^n:\quad z_i\mapsto z_i+a_i\,.$$

This is a fixed point free action and $C/(2\pi\sqrt{-1}\,\boldsymbol{Z})^n\cong\boldsymbol{C}-\{\lambda_1,\cdots,\lambda_n\}$. Next, we compute the homology of $C$. Fix a base point $\bar{b}$ of $\boldsymbol{C}-\{\lambda_1,\cdots,\lambda_n\}$ and $\bar{\gamma}_i$ be a path rounding around $\lambda_i$ for $i=1,\cdots,n$. Let $b$ and $\gamma_i$ be liftings of $\bar{b}$ and $\bar{\gamma}_i$ such that $\partial\gamma_i=x_ib-b$, where $x_i$ is the action of $(0,\cdots,\overset{i}{2\pi\sqrt{-1}},0,\cdots,0)$. We can compute the homology of $C$ by the complex $K.$ below.

$$K_1=\bigoplus_{i=1}^{n} A\gamma_i \ni \gamma_i \mapsto u_i b \in Ab = K_0 ,$$

where $A=\boldsymbol{Z}[x_i, x_i^{-1}]$ is the Laurent polynomial ring of $n$-variables and $u_i=x_i-1$. Therefore we have the following

LEMMA 2.1. *The homology group $H_1(C, \boldsymbol{Z})$ is isomorphic to the kernel* $\mathrm{Ker}(\partial: K_1 \to K_0)$ and *$H_1(C, \boldsymbol{Z})$ is generated by $u_i\gamma_j - u_j\gamma_i$ $(i \neq j)$.*

Let $\alpha=(\alpha_1, \cdots, \alpha_n)$ be an element of $\boldsymbol{C}^n$ and $\omega_k$ the differential form defined by

$$\omega_k=\prod_{i=1}^{n} (x-\lambda_i)^{\alpha_i-1} x^{k-1}\, dx .$$

Here $(x-\lambda_i)^{\alpha_i-1}$ is interpreted as $\exp((\alpha_i-1)z_i)$. Then $\omega_k$ defines a linear form $\varphi_k$ on $H_1(C, \boldsymbol{Z})$ by

$$\varphi_k: \quad \gamma \in H_1(C, \boldsymbol{Z}) \to \int_\gamma \omega \in \boldsymbol{C}.$$

Then the linear form $\varphi_k$ is contained in

$$\mathrm{Hom}(H_1(C, \boldsymbol{Z}), \boldsymbol{C})(\alpha)=\{\varphi \mid \varphi(g\gamma)=\prod_{i=1}^{n} \exp(g_i\alpha_i)\varphi(\gamma)$$
$$\text{for all } \gamma \in H_1(C, \boldsymbol{Z}) \text{ and } (g_i) \in (2\pi\sqrt{-1}\,\boldsymbol{Z})^n\} .$$

We define an $A$-algebra $\boldsymbol{C}(\alpha)$ as follows. The underlying algebra is the complex number field $\boldsymbol{C}$ and the action of $x_i$ is given by the multiplication by $\exp(2\pi\sqrt{-1}\,\alpha_i) \in \boldsymbol{C}^\times$. We denote the natural homomorphism $A \to \boldsymbol{C}(\alpha)$ by the same letter $\alpha$. Then we have

$$\mathrm{Hom}(H_1(C, \boldsymbol{Z}), \boldsymbol{C})(\alpha)=\mathrm{Hom}_{\boldsymbol{C}}(H_1(C, \boldsymbol{Z}) \otimes_A \boldsymbol{C}(\alpha), \boldsymbol{C}) .$$

LEMMA 2.2. *The space $H_1(C, \boldsymbol{Z}) \otimes_A \boldsymbol{C}(\alpha)$ is isomorphic to the vector space generated by*

$$\alpha(u_{i+1})\gamma_i - \alpha(u_i)\gamma_{i+1} \qquad (i=1, \cdots, n-1)$$

*if the following condition*

$$\alpha_i \notin \boldsymbol{Z} \text{ for all } i\ (1 \leq i \leq n) \quad \text{and} \quad \alpha_0=-\sum_{i=1}^{n} \alpha_i \notin \boldsymbol{Z} \tag{2.1}$$

*holds.*

REMARK. The space $H_1(C, \boldsymbol{Z}) \otimes_A \boldsymbol{C}(\alpha)$ is isomorphic to the cohomology with the local coefficient $\boldsymbol{C}(\alpha)$ (cf. [S] p. 155, [H]) corresponding to the representation

$$\mathrm{Aut}(C/\boldsymbol{C}-\{\lambda_1, \cdots, \lambda_n\}) \simeq (2\pi\sqrt{-1}\,\boldsymbol{Z})^n \ni (g_i) \mapsto \prod_{i=1}^{n} \exp(g_i\alpha_i) \in \boldsymbol{C}^\times .$$

By the lemma, the linear functional $\varphi_k$ is determined by the value

$$\int_{\delta_1} \omega_k , \quad \int_{\delta_2} \omega_k , \quad \cdots, \quad \int_{\delta_{n-1}} \omega_k ,$$

where $\delta_i = u_{i+1}\gamma_i - u_i\gamma_{i+1}$. From now on we assume that $\lambda_i$'s are real numbers and $\lambda_1 < \cdots < \lambda_n$. Then we define a 1-chain $I_i(\varepsilon): I=[0, 1] \to C$ by $x = t(\lambda_i + \varepsilon) + (1-t)(\lambda_{i+1} - \varepsilon)$ and $z_j \in \boldsymbol{R}$ if $j \leq i$ and $z_j \in \boldsymbol{R} - \pi\sqrt{-1}$ if $j \geq i+1$. Let us define the lifting $C_i^+(\varepsilon)$ (resp. $C_i^-(\varepsilon)$) of a small circle around $\lambda_i$ (resp. $\lambda_{i+1}$) by

$$C_i^+(\varepsilon) = (x(\theta), z_i(\theta))$$
$$(\text{resp. } C_i^-(\varepsilon) = (x(\theta), z_i(\theta)))$$

with

$$x(\theta) = \lambda_i + \varepsilon \exp(\theta 2\pi\sqrt{-1}), \quad z_j(0) \in \boldsymbol{R} \ (j \leq i), \ z_j(0) \in \boldsymbol{R} - \pi\sqrt{-1} \ (j \geq i+1)$$
$$(\text{resp. } x(\theta) = \lambda_{i+1} - \varepsilon \exp(\theta 2\pi\sqrt{-1}), \quad z_j(0) \in \boldsymbol{R} \ (j \leq i), \ z_j(0) \in \boldsymbol{R} - \pi\sqrt{-1} \ (j \geq i+1)) .$$

Then $z_i(1) \in \boldsymbol{R} + 2\pi\sqrt{-1}$ and $z_{i+1}(1) \in \boldsymbol{R} + \pi\sqrt{-1}$ and the 1-cycle $\delta_i$ is homologous to

$$(x_i - 1)(x_{i+1} - 1)I_i(\varepsilon) + (x_{i+1} - 1)C_i^+(\varepsilon) - (x_i - 1)C_{i+1}^-(\varepsilon) .$$

If the real part of $\alpha_i$ is positive, then

$$\lim_{\varepsilon \to 0} \int_{C_i^+(\varepsilon)} \omega_k = 0 .$$

Therefore, we have

$$\int_{\delta_i} \omega_k = \lim_{\varepsilon \to 0} \int_{(x_i - 1)(x_{i+1} - 1)I_i(\varepsilon)} \omega_k = (\alpha(x_i) - 1)(\alpha(x_{i+1}) - 1) \lim_{\varepsilon \to 0} \int_{I_i(\varepsilon)} \omega_k ,$$

and by the definition of $I_i(\varepsilon)$, we have

$$\lim_{\varepsilon \to 0} \int_{I_i(\varepsilon)} \omega_k = \exp(-\sum_{j=i+1}^{n} \pi\sqrt{-1}\,\alpha_j) a_{ik} ,$$

where $a_{ik}$ is defined in the introduction. Now we define a singular integral:

$$\int_{I_i} \omega_k = \lim_{\varepsilon \to 0} \int_{I_i(\varepsilon)} \omega_k .$$

Then we have

$$\det\left(\int_{I_i} \omega_k\right)_{i,k} = \prod_{i=1}^{n} \prod_{j=i+1}^{n} \exp(-\alpha_j \pi \sqrt{-1}) \det(a_{ik}).$$

## §3. Symmetric construction.

In this section, we define a holomorphic map from the product $C \times \cdots \times C$ of $(n-1)$-copies of $C$ to a twisted exponential Fermat hypersurface. The similar construction was defined in [T]. First we prepare $(n-1)$ copies $C^{(1)}, \cdots, C^{(n-1)}$ of $C$. The coordinate of $C^{(i)}$ is denoted by $x^{(i)}, z_j^{(i)}$. Then the equation is given by $\exp(z_j^{(i)}) = x^{(i)} - \lambda_j$ $(j=1, \cdots, n, i=1, \cdots, n-1)$. The exponential Fermat hypersurface $X$ is an analytic subset in $\boldsymbol{C}^n = \{(w_1, \cdots, w_n)\}$ defined by

$$X: \quad \sum_{j=1}^{n} \prod_{k \neq j} (\lambda_k - \lambda_j)^{-1} \exp(w_j) = 1 .$$

We define a holomorphic map $\varphi$ from $C^{(1)} \times \cdots \times C^{(n-1)}$ to $X$ by

$$\varphi : C^{(1)} \times \cdots \times C^{(n-1)} \to X$$

$$(x^{(i)}, z_j^{(i)})_{i=1,\cdots,n-1,j=1,\cdots,n} \mapsto w_j = \sum_{i=1}^{n-1} z_j^{(i)} .$$

The morphism $\varphi$ is well defined, and it is easy to see that

$$\sum_{j=1}^{n} \prod_{k \neq j} (\lambda_k - \lambda_j)^{-1} \prod_{i=1}^{n-1} (x^{(i)} - \lambda_j) = 1$$

by the Lagrange interpolation formula. Let us define the group $G$ by $(2\pi\sqrt{-1}\,\boldsymbol{Z})^n$. Then we have a natural action of $G$ on $x$ by

$$(a_i) \in (2\pi\sqrt{-1}\,\boldsymbol{Z})^n \mapsto (w_i \mapsto w_i + a_i) \in \mathrm{Aut}(x) .$$

Let us define the group $N$ as the kernel $\mathrm{Ker}(G^n \to G)$ of the morphism determined by the summation. Then the groups $N$ and the symmetric group $S_{n-1}$ act on $C^{(1)} \times \cdots \times C^{(n-1)}$ and as a consequence the semi-direct product $N \rtimes S_{n-1}$ of $N$ and $S_{n-1}$ acts on $C^{(1)} \times \cdots \times C^{(n-1)}$.

PROPOSITION 3.1. *The holomorphic map $\varphi$ is invariant under the action of $N \rtimes S_{n-1}$ and it induces an isomorphism $\bar{\varphi}$*

$$C^{(1)} \times \cdots \times C^{(n-1)} / N \rtimes S_{n-1} \xrightarrow{\bar{\varphi}} X .$$

PROOF. This proposition is proved with the same argument as [T], [O-T]. Proof is left to the readers.

## §4. Computation of determinants.

In this section, we compute the determinant of

$$\left(\int_{I_i} \omega_k\right)_{i,k}$$

by using the holomorphic map $\varphi$ constructed in §3. We compute this determinant by

(1) Descending a differential form to $X$.

(2) Specifying the image of some topological chain in $C^{(1)} \times \cdots \times C^{(n-1)}$ under the map $\varphi$.

(3) Reducing the computation of the determinant to that of the integral of a form on the exponential Fermat hypersurface $X$.

(1) Descending a differential form to $X$.

Let $\omega_k^{(i)}$ be the differential form on $C^{(i)}$ defined as in §2. The form $\mathrm{pr}_i{}^*\omega_k^{(i)}$ induced by the projection $\mathrm{pr}_i$ is denoted by $\omega_k^{(i)}$ if there is no confusion. Consider an $(n-1)$-form $\Omega$ on $C^{(1)} \times \cdots \times C^{(n-1)}$ by

$$\Omega = \sum_{\sigma \in S_{n-1}} \omega_1^{(\sigma(1))} \wedge \cdots \wedge \omega_{n-1}^{(\sigma(n-1))} .$$

LEMMA 4.1. *Let $\bar{\Omega}$ be an $(n-1)$-form on $X$ defined by*

$$\bar{\Omega} = \left(\prod_{i=1}^{n} \exp((\alpha_i - 1)w_i)\right) dt_{n-1} \wedge \cdots \wedge dt_1 ,$$

*where $t_i$ is the $i$-th elementary symmetric polynomial in $x^{(1)}, \cdots, x^{(n-1)}$. Then we have $\Omega = \varphi^*\bar{\Omega}$.*

PROOF. Since

$$\varphi^*(\exp(\alpha_i - 1)w_i) = \exp\left((\alpha_i - 1)\sum_{j=1}^{n-1} z_i^{(j)}\right) = \prod_{j=1}^{n-1} \exp((\alpha_i - 1)z_i^{(j)}) ,$$

it is enough to show

$$\varphi^*(dt_1 \wedge \cdots \wedge dt_{n-1}) = \sum_{\sigma \in S_{n-1}} dx^{(\sigma(1))} \wedge (x^{(\sigma(2))})dx^{(\sigma(2))} \wedge \cdots \wedge (x^{(\sigma(n-1))})^{n-2} dx^{(\sigma(n-1))}$$

to prove the lemma. The right hand side is equal to

$$\sum_{\sigma \in S_{n-1}} \mathrm{sign}(\sigma)(x^{(\sigma(2))}) \cdots (x^{(\sigma(n-1))})^{n-2} dx^{(\sigma(1))} \wedge \cdots \wedge dx^{(\sigma(n-1))}$$

$$= V(x^{(1)}, \cdots, x^{(n-1)}) dx^{(1)} \wedge \cdots \wedge dx^{(n-1)} ,$$

where $V(x^{(1)}, \cdots, x^{(n-1)})$ is the Vandermonde matrix. On the other hand, the left hand side is equal to

$$(p_{n-2}^{(1)}dx^{(1)}+\cdots+p_{n-2}^{(n-1)}dx^{(n-1)})\wedge\cdots$$
$$\wedge(p_1^{(1)}dx^{(1)}+\cdots+p_1^{(n-1)}dx^{(n-1)})\wedge(dx^{(1)}+\cdots+dx^{(n-1)}),$$

where $p_k^{(j)}$ is the elementary symmetric function of $\{x^{(1)}, \cdots, x^{(j-1)}, x^{(j+1)}, \cdots, x^{(n-1)}\}$ of degree $k$. Therefore it is equal to $W(x^{(1)}, \cdots, x^{(n-1)})dx^{(1)}\wedge\cdots\wedge dx^{(n-1)}$, where

$$W(x^{(1)}, \cdots, x^{(n-1)})=\det\begin{pmatrix} p_{n-2}^{(1)} & \cdots & p_{n-2}^{(n-1)} \\ \vdots & & \vdots \\ p_1^{(1)} & \cdots & p_1^{(n-1)} \\ 1 & \cdots & 1 \end{pmatrix}.$$

Therefore it is enough to prove the following

LEMMA 4.2. $W(x^{(1)}, \cdots, x^{(n-1)})=V(x^{(1)}, \cdots, x^{(n-1)})$.

PROOF. See [T].

(2) Specifying the image of topological chain in $C^{(1)}\times\cdots\times C^{(n-1)}$ under the map $\varphi$.

In this paragraph, we assume that $\lambda_i$'s are contained in the set of real numbers and $\lambda_1<\cdots<\lambda_n$, and consider the chains $C_i^{\pm}(\varepsilon)$, $I_i(\varepsilon)$. We denote the copies of $I_i(\varepsilon)$ by $I_i^{(j)}(\varepsilon)$ $(j=1, \cdots, n-1)$. First, we compute the $t_{n-1}, \cdots, t_1$ coordinates of the points in $\varphi(I_1^{(1)}(\varepsilon)\times\cdots\times I_{n-1}^{(n-1)}(\varepsilon))$. Since $t_j$ is equal to the $j$-th elementary symmetric function of $x^{(1)}, \cdots, x^{(n-1)}$, where $\lambda_i+\varepsilon\leq x^{(i)}\leq\lambda_{i+1}-\varepsilon$, the linear form

$$L_i=(-1)^{i-1}\{t_{n-1}-\lambda_i t_{n-2}+\cdots+(-\lambda_i)^{n-1}\}=(-1)^{i-1}\prod_{k=1}^{n-1}(x^{(k)}-\lambda_i)$$

is positive.

LEMMA 4.3. *$\varphi$ induces a one to one mapping from $I_1^{(1)}(\varepsilon)\times\cdots\times I_{n-1}^{(n-1)}(\varepsilon)$ to the image $\varphi(I_1^{(1)}(\varepsilon)\times\cdots\times I_{n-1}^{(n-1)}(\varepsilon))$. Moreover the $(t_{n-1}, \cdots, t_1)$-coordinates of the points in the union $\tilde{D}=\bigcup_{\varepsilon>0}\varphi(I_1^{(1)}(\varepsilon)\times\cdots\times I_{n-1}^{(n-1)}(\varepsilon))$ coincides with the domain*

$$D=\{\boldsymbol{R}^{n-1}\ni(t_{n-1}, \cdots, t_1)\mid L_i(t_{n-1}, \cdots, t_1)>0 \text{ for all } i\}$$

*and $\tilde{D}$ is the lifting of $D$ such that*

$$w_j=\sum_{i=1}^{n-1}z_j^{(i)}\in\boldsymbol{R}-(j-1)\pi\sqrt{-1}.$$

PROOF. This is an immediate consequence from the definition of $I_i^{(j)}(\varepsilon)$.

(3) Reducing the computation of the integral to that of the exponential Fermat hypersurface.

First, we compute the integral

$$\int_{I_1^{(1)}\times\cdots\times I_{n-1}^{(n-1)}}\Omega=\lim_{\varepsilon\to 0}\int_{I_1^{(1)}(\varepsilon)\times\cdots\times I_{n-1}^{(n-1)}(\varepsilon)}\Omega .$$

By the definition,

$$\text{(4.1)}\qquad \int_{I_1^{(1)}\times\cdots\times I_{n-1}^{(n-1)}}\Omega=\sum_{\sigma\in S_{n-1}}\int_{I_1^{(1)}\times\cdots\times I_{n-1}^{(n-1)}}\omega_1^{(\sigma(1))}\wedge\cdots\wedge\omega_{n-1}^{(\sigma(n-1))}$$

$$=\sum_{\sigma\in S_{n-1}}\operatorname{sign}(\sigma)\int_{I_1^{(1)}\times\cdots\times I_{n-1}^{(n-1)}}\omega_{\sigma(1)}^{(1)}\wedge\cdots\wedge\omega_{\sigma(n-1)}^{(n-1)}$$

$$=\det\left(\int_{I_i}\omega_j\right)_{i,j}.$$

On the other hand,

$$\text{(4.2)}\qquad \int_{I_1^{(1)}\times\cdots\times I_{n-1}^{(n-1)}}\Omega=\int_{I_1^{(1)}\times\cdots\times I_{n-1}^{(n-1)}}\varphi^*\bar{\Omega}$$

$$=\int_{\varphi(I_1^{(1)}\times\cdots\times I_{n-1}^{(n-1)})}\bar{\Omega}=\int_{\bar{D}}\bar{\Omega}.$$

Therefore the computation of $\det(\int_{I_i}\omega_j)_{i,j}$ is reduced to that of the integral $\int_{\bar{D}}\bar{\Omega}$ on the exponential Fermat hypersurface $X$. Since $\exp((\alpha_i-1)w_i)\in\exp(-(i-1)\pi\sqrt{-1}\,\alpha_i)\boldsymbol{R}_+$, we have

$$\exp((\alpha_i-1)w_i)=\exp(-(i-1)\pi\sqrt{-1}\,\alpha_i)L_i^{\alpha_i-1},$$

and

$$\text{(4.3)}\qquad \int_{\bar{D}}\bar{\Omega}=\prod_{i=1}^{n}\exp(-(i-1)\pi\sqrt{-1}\,\alpha_i)\int_{L_i>0}\prod_{i=1}^{n}L_i^{\alpha_i-1}\,dt_{n-1}\wedge\cdots\wedge dt_1 .$$

By using the Jacobian matrix for $L_1,\cdots,L_{n-1}$ and $t_{n-1},\cdots,t_1$:

$$\frac{\partial(L_1,\cdots,L_{n-1})}{\partial(t_{n-1},\cdots,t_1)}=\det\begin{pmatrix}1 & \lambda_1 & \cdots & \lambda_1^{n-2}\\ \vdots & \vdots & & \vdots\\ 1 & \lambda_{n-1} & \cdots & \lambda_{n-1}^{n-2}\end{pmatrix}=\prod_{1\le i<j\le n-1}(\lambda_j-\lambda_i),$$

and the equation

$$\sum_{i=1}^{n}\frac{(-1)^{i-1}L_i}{\prod_{j\neq i}(\lambda_j-\lambda_i)}=1$$

derived from (3.1), the integral (4.3) is modified to

$$(4.4)\qquad \prod_{i=1}^{n}\exp(-(i-1)\pi\sqrt{-1}\,\alpha_i)\frac{1}{\prod_{1\le i<j\le n-1}(\lambda_j-\lambda_i)} \cdot\int_{L_1>0,\cdots,L_n>0}\prod_{i=1}^{n}L_i^{\alpha_i-1}\,dL_1\wedge\cdots\wedge dL_{n-1}.$$

By replacing $L_i$ by $(-1)^{i-1}\prod_{j\neq i}(\lambda_j-\lambda_i)l_i$ for the new variables $l_i$, we have

$$\int_{L_1>0,\cdots,L_n>0}\prod_{i=1}^{n}L_i^{\alpha_i-1}\,dL_1\wedge\cdots\wedge dL_{n-1}$$
$$=\prod_{i=1}^{n-1}((-1)^{i-1}\prod_{j\neq i}(\lambda_j-\lambda_i))^{\alpha_i}\left(\prod_{j=1}^{n-1}(-1)^{n-1}(\lambda_j-\lambda_n)\right)^{\alpha_n-1} \cdot\int_{l_1>0,\cdots,l_n>0}\prod_{i=1}^{n}l_i^{\alpha_i-1}\,dl_1\wedge\cdots\wedge dl_{n-1}.$$

Therefore the integral (4.4) is equal to

$$(4.5)\qquad \prod_{i=1}^{n}\exp(-(i-1)\pi\sqrt{-1}\,\alpha_i)\frac{1}{\prod_{1\le i<j\le n}(\lambda_j-\lambda_i)}\prod_{i=1}^{n}((-1)^{i-1}\prod_{j\neq i}(\lambda_j-\lambda_i))^{\alpha_i}$$
$$\cdot\int_{\substack{l_1>0,\cdots,l_{n-1}>0\\ 1-l_1-\cdots-l_{n-1}>0}}\prod_{i=1}^{n-1}l_i^{\alpha_i-1}(1-l_1-\cdots-l_{n-1})^{\alpha_n-1}\,dl_1\wedge\cdots\wedge dl_{n-1}$$
$$=\prod_{i=1}^{n}\exp(-(i-1)\pi\sqrt{-1}\,\alpha_i)\frac{1}{\prod_{1\le i<j\le n}(\lambda_j-\lambda_i)} \cdot\prod_{i=1}^{n}((-1)^{i-1}\prod_{j\neq i}(\lambda_j-\lambda_i))^{\alpha_i}\frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_1+\cdots+\alpha_n)}.$$

Combining (4.1), (4.2), (4.3), (4.4) and (4.5), we get the following theorem which is equivalent to Theorem 1 in §1.

THEOREM 4.4. *The determinant* $\det(\int_{I_i}\omega_k)_{i,k}$ *is equal to*

$$\prod_{i=1}^{n}\exp(-(i-1)\pi\sqrt{-1}\,\alpha_i)\frac{1}{\prod_{1\le i<j\le n}(\lambda_j-\lambda_i)} \cdot\prod_{i=1}^{n}((-1)^{i-1}\prod_{j\neq i}(\lambda_j-\lambda_i))^{\alpha_i}\frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_1+\cdots+\alpha_n)}.$$

COROLLARY 4.5. *The set of* $\varphi_1,\cdots,\varphi_{n-1}$ *on* $H_1(C,Z)\otimes C(\alpha)$ *forms a base for* $\mathrm{Hom}(H_1(C,Z)\otimes C(\alpha),C)$ *under the condition* (2.1).

## §5. Intermediate exterior product.

Let $r$ be an integer such that $1 \leq r \leq n-1$, then the exterior matrix for $a_{i,j}$ $(1 \leq i, j \leq n-1)$ can be expressed as a period matrix of the special complete intersection of the exponential Fermat hypersurface $X_r$ below. First, we define $X_r$ as the analytic submanifold of $C^r \times C^n$ defined by

$$X_r = \{(z_1, \cdots, z_r, w_1, \cdots, w_n) \mid$$
$$\exp(w_i) = \sum_{k=1}^{r} (-\lambda_i)^{k-1} z_{r-k+1} + (-\lambda_i)^r \text{ for } i=1, \cdots n)\} .$$

This manifold is also isomorphic to the analytic submanifold of $C^n = \{(w_1, \cdots, w_n)\}$ defined by

$$\sum_{i=1}^{n} \frac{\lambda_i^m}{\prod_{j \neq i}(\lambda_j - \lambda_i)} \exp(w_i) = 0 \qquad \text{for} \quad m = 0, \cdots, n-r-2 ,$$
$$\sum_{i=1}^{n} \frac{\lambda_i^{n-r-1}}{\prod_{j \neq i}(\lambda_j - \lambda_i)} \exp(w_i) = 1 .$$

To formulate the theorem, we introduce some notations. Let $I$, $J$ be the sets of indices such that

$$I \in \{(i_1, \cdots, i_r) \mid 0 \leq i_1 < \cdots < i_r \leq n-2\} ,$$
$$J \in \{(j_1, \cdots, j_r) \mid 0 \leq j_1 < \cdots < j_r \leq n-2\} .$$

We define $A_{I,J}$ as the determinant of the $(r \times r)$-matrix $(\int_{I_{i_p}} \omega_{j_q})_{p,q}$. Since $Y = (i_1, \cdots, i_r - r + 1)$ becomes a Young diagram, we can define a Schur function $s_Y(x^{(1)}, \cdots, x^{(r)})$ of $Y$ as

$$\det \begin{pmatrix} (x^{(1)})^{i_1} & \cdots & (x^{(r)})^{i_1} \\ \vdots & & \vdots \\ (x^{(1)})^{i_r} & \cdots & (x^{(r)})^{i_r} \end{pmatrix} \cdot \det \begin{pmatrix} (x^{(1)})^0 & \cdots & (x^{(r)})^0 \\ \vdots & & \vdots \\ (x^{(1)})^{r-1} & \cdots & (x^{(r)})^{r-1} \end{pmatrix}^{-1} .$$

By the fundamental theorem for symmetric polynomials, $s_Y$ can be expressed as a polynomial $u_Y(t_1, \cdots, t_r)$ in elementary symmetric functions of $t_1, \cdots, t_r$ of degree $1, \cdots, r$ respectively depending on $x^{(1)}, \cdots, x^{(r)}$.

LEMMA 5.1. *The total degree* $\deg(u_Y)$ *of* $u_Y$ *with respect to* $t_1, \cdots, t_r$ *is equal to* $i_r - (r-1)$. *Therefore we have the inequality* $\deg(u_Y) \leq n-r-1$ *and* $\{u_Y\}_Y$ *forms a base for polynomials in* $t_1, \cdots, t_r$ *of degree less than or equal to* $n-r-1$.

Now let us define a differential form $\bar{\Omega}_I$ on $X_r$ by

$$\bar{\Omega}_I = \prod_{i=1}^{n} (\exp(\alpha_i - 1)w_i)u_Y(z_1, \cdots, z_r)dz_1 \wedge \cdots \wedge dz_r .$$

We can define a map $\varphi_r$ from the product $C^{(1)} \times \cdots \times C^{(r)}$ of $C$ to $X_r$ by

$$C^{(1)} \times \cdots \times C^{(r)} \to X_r$$

$$(x^{(k)}, z_i^{(k)}) \mapsto w_i = \sum_{j=1}^{r} z_i^{(j)} .$$

We can prove the following proposition exactly in the same way as the proof of Lemma 4.1.

PROPOSITION 5.2. *The inverse image of $\bar{\Omega}_I$ under the map $\varphi_r$ is equal to*

$$\varphi_r{}^*\bar{\Omega}_I = \sum_{\sigma \in S_{r-1}} \operatorname{sign}(\sigma)\omega_{i_{\sigma(1)}}^{(1)} \wedge \cdots \wedge \omega_{i_{\sigma(r)}}^{(r)} .$$

Next we specify the image of the topological cycle $I_{j_1}^{(1)} \times \cdots \times I_{j_r}^{(r)}$ under the map $\varphi_r$. Let $L_i$ be the linear form defined by $(-1)^{i-1}(\sum_{k=1}^{n-1} t_k(-\lambda_i)^{n-k-1} + (-\lambda_i)^{n-1})$.

PROPOSITION 5.3. *The image $\tilde{D}_J$ of $I_{j_1}^{(1)} \times \cdots \times I_{j_r}^{(r)}$ under the map $\varphi_r$ is the lifting of $D_J$ in $\boldsymbol{R}^r = \{(z_1, \cdots, z_r)\}$ defined by*

$$D_J = \{(z_1, \cdots, z_r) \in \boldsymbol{R}^r \mid (-1)^{j_k - k} L_{j_k}(z_1, \cdots, z_r) > 0 ,$$
$$(-1)^{j_k - k + 1} L_{j_k + 1}(z_1, \cdots, z_r) > 0 \textit{ for all } k = 1, \cdots, r\} ,$$

*such that $w_j \in \boldsymbol{R} - \pi\sqrt{-1}\,\mu$, where $\mu = \#\{k \mid j_k < k\}$ .*

For the proof, imitate the proof of Lemma 4.3. As a consequence of the above two propositions, we have the following theorem.

THEOREM 5.4. *The determinant $A_{I,J}$ is expressed as*

$$A_{I,J} = \int_{\tilde{D}_J} \bar{\Omega}_I .$$

The integral on the right is a special case of Aomoto-Gel'fand's hypergeometric function.

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