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The Ergodic Closing Lemma for C^1 Regular Maps

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In order to solve the C^1 Structural Stability Conjecture by Palis and Smale, Mañé [1] established the ergodic closing lemma for diffeomorphisms. In 1987 he gave in [2] an answer to the conjecture by using original ideas. The ergodic closing lemma is a result that captures the asymptotic behaviour of orbits in the ergodic theory. This result is based on Pugh's closing lemma [3].

Our aim is to prove the ergodic closing lemma for regular maps of closed manifolds. Let M be a closed C^∞ manifold and $C^1(M)$ be the set of C^1 maps of M with the C^1-topology. $f \in C^1(M)$ is called to be *regular* if for every $x \in M$ the derivative $D_x f: T_x M \to T_{f(x)} M$ is surjective. $R^1(M)$ denotes the set of regular maps in $C^1(M)$ with the relative topology of $C^1(M)$. For $f \in R^1(M)$, $\Omega(f)$ denotes the nonwandering set; $\Omega(f) = \{x \in M$: for every neighborhood U of x there is $n > 0$ such that $f^n(U) \cap U \neq \emptyset\}$, and $\mathrm{Per}(f)$ denotes the set of periodic points.

Recently, in [4] L. Wen showed the C^1 closing lemma for regular maps as follows.

THEOREM (L. Wen). *Let $f: M \to M$ be a regular map and p be a nonwandering point of f. Then for any C^1 neighborhood $\mathfrak{U}$ of f in $R^1(M)$ and any neighborhood U of p in M there is $g \in \mathfrak{U}$ such that* $\mathrm{Per}(g) \cap U \neq \emptyset$.

The idea of the proof is in Proposition 1 below. Let $f \in R^1(M)$ and $q_0 \in M$ be a non periodic point of f. We define an infinite sequence $\boldsymbol{Q} = \{Q_n : n \geq 0\}$ of disjoint non-empty finite sets $Q_n = \{f^{-n}(q_0)\}$ for $n = 0, 1, 2, \cdots$. Then $f: \boldsymbol{Q}' - Q_0 \to \boldsymbol{Q}'$, where $\boldsymbol{Q}' = \bigcup_{n=0}^{\infty} Q_n$, such that f maps Q_n onto Q_{n-1} for $n \geq 1$. An infinite sequence $q_0, q_1, \cdots, q_n, \cdots$ is called a *branch* of $(\boldsymbol{Q}, f)$ if $q_n \in Q_n$ for $n \geq 0$.

PROPOSITION 1 ([4]). *Under the above notations, for $\varepsilon_0 > 0$ there are a number $\rho_0 > 2$ and an integer $\mu_0 \geq 1$ such that for any finite ordered set $P = \{p_0, p_1, \cdots, p_t\}$ in $T_{q_0}M$, there is $y \in P \cap U(p_t, \rho_0 |p_0 - p_t|)$ such that for any branch $\Sigma = \{q_0, q_1, \cdots, q_n, \cdots\}$ of $(\boldsymbol{Q}, f)$, there is $w \in P \cap U(p_t, \rho_0 |p_0 - p_t|)$, where w is before y in the order of P, together with $\mu_0 + 1$ points $c_0, c_1, \cdots, c_{\mu_0}$ in $U(p_t, \rho_0 |p_0 - p_t|)$, not necessarily distinct, satisfying the following two conditions* (a) *and* (b),

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(a) $c_0=w$ *and* $c_{\mu_0}=y$,

(b) $|T_n^{-1}(c_n)-T_n^{-1}(c_{n+1})|\leq\varepsilon_0 d'(T_n^{-1}(c_{n+1}), T_n^{-1}(A))$ *for* $0\leq n\leq\mu_0-1$,

where $|\cdot|$ *denotes a Riemannian norm of* TM, $U(x,r)=\{z\in T_{q_0}M : |x-z|<r\}$, $T_n=D_{q_n}f^n : T_{q_n}M\to T_{q_0}M$ $(n\geq 0)$, $A=P(w,y)\cup\partial U(p_t, \rho_0|p_0-p_t|)$, $P(w,y)=\{p\in P : p$ *is after* w *and before* $y\}$, *and* d' *is the distance on* $T_{q_0}M$ *induced by* $|\cdot|$.

From Proposition 1 we can derive the following

PROPOSITION 2. *Given* $f\in R^1(M)$, $p\in M$, $\varepsilon>0$ *and a neighborhood* $\mathfrak{U}$ *of* f *there exist* $r>0$ *and* $\rho>2$ *such that if* $x\in U_{\tilde r}(p)$ *with* $0<\tilde r\leq r$ *and* $f^m(x)\in U_{\tilde r}(p)$ *for some* $m>0$ *then there exist* $0\leq m_1<m_2\leq m$ *and* $g\in\mathfrak{U}$ *such that*

(i) $f^{m_1}(x)\in U_{\rho\tilde r}(p)$ *and* $f^{m_2}(x)\in U_{\rho\tilde r}(p)$,

(ii) $g^{m_2-m_1}(f^{m_1}(x))=f^{m_1}(x)$,

(iii) $g=f$ *on* $\{M-B_\varepsilon(f^{-1}, p, m_2-m_1)\}\cup\{M-B_\varepsilon(f^{-1}, f^{m_1}(x), m_2-m_1)\}$,

(iv) $d(f^n(f^{m_1}(x)), g^n(f^{m_1}(x)))\leq\varepsilon$ *for* $0\leq n\leq m_2-m_1$,

where $U_\varepsilon(p)=\{z\in M : d(p,z)<\varepsilon\}$, $B_\varepsilon(f^{-1},x,m)=\bigcup_{j=0}^m\bigcup_{z\in f^{-j}(x)}B_\varepsilon(z)$ *and* $B_\varepsilon(z)=\overline{U_\varepsilon(z)}$.

This proposition is a result corresponding to Lemma I.2 of Mañé [1] and it will play an essential role to prove Theorem stated below.

Let $f\in R^1(M)$ and $\mathfrak{M}(f)$ be the set of all f-invariant probability measures defined on the Borel sets on M. We define $\Sigma(f)$ as the set of points $x\in M$ such that for every neighborhood $\mathfrak{U}$ of f and every $\varepsilon>0$ there exist $g\in\mathfrak{U}$ and $y\in M$ such that $y\in\mathrm{Per}(g)$, $g=f$ on $M-B_\varepsilon(f^{-1},x,m)$ and $d(f^n(x), g^n(y))\leq\varepsilon$ for all $0\leq n\leq m$, where m is the minimal period of y for g.

If $\mathfrak{U}$ is a neighborhood of f and $\varepsilon>0$, we let $\Sigma'(\mathfrak{U},\varepsilon)$ be the set of points $x\in M$ such that there exist $g\in\mathfrak{U}$, $y\in M$ and $m>0$ satisfying $g^m(y)=y$, $g=f$ on $M-B_\varepsilon(f^{-1},x,m)$ and $d(f^n(x), g^n(y))\leq\varepsilon$ for all $0\leq n\leq m$.

From Proposition 2 it follows that the interior of $\Sigma'(\mathfrak{U},\varepsilon)$, $\Sigma(\mathfrak{U},\varepsilon)$, is non-empty. Indeed, let $f^{m_1}(x)$ be as in Proposition 2. Then $f^{m_1}(x)\in\Sigma'(\mathfrak{U},\varepsilon)\subset\Sigma(\mathfrak{U},2\varepsilon)$ if $y=f^{m_1}(x)$ and $m=m_1-m_2$. Choose a basis $\{\mathfrak{U}_n\}$ of neighborhoods of f and a sequence $\{\varepsilon_n\}$ converging to 0. Then we have

$$\Sigma(f)=\bigcap_{n\geq 1}\Sigma(\mathfrak{U}_n,\varepsilon_n). \tag{1}$$

If we establish (1), then $\Sigma(f)$ is a Borel set. (1) is checked as follows.

By the definition it is clear that $\Sigma(f)\supset\bigcap_{n\geq 1}\Sigma(\mathfrak{U}_n,\varepsilon_n)$. Let $x\in\Sigma(f)$ and $n\geq 1$. Then there exist $g\in\mathfrak{U}_n$, $y\in M$ and $m>0$ such that $g^m(y)=y$, $g=f$ on $M-B_{\varepsilon_n/2}(f^{-1},x,m)$ and $d(f^j(x), g^j(y))\leq\varepsilon_n/2$ for $0\leq j\leq m$. Take $\delta>0$ such that if $d(w,z)\leq\delta$ then for $0\leq j\leq m$ and $\tilde z\in f^{-j}(z)$ there exists $\tilde w\in f^{-j}(w)$ satisfying $d(\tilde w,\tilde z)\leq\varepsilon_n/2$ and $d(f^j(w), f^j(z))\leq$

$\varepsilon_n/2$ for $0 \le j \le m$. Then, for $w \in U_\delta(x)$ we have

$$B_{\varepsilon_n}(f^{-1}, w, m) \supset B_{\varepsilon_n/2}(f^{-1}, x, m),$$

and hence

$$f = g \quad \text{on} \quad M - B_{\varepsilon_n}(f^{-1}, w, m).$$

Moreover we have

$$\begin{aligned} d(f^j(w), g^j(y)) &\le d(f^j(w), f^j(x)) + d(f^j(x), g^j(y)) \\ &\le \varepsilon_n \qquad (0 \le j \le m). \end{aligned}$$

Thus $U_\delta(x) \subset \Sigma'(\mathfrak{U}_n, \varepsilon_n)$ and so $x \in \Sigma(\mathfrak{U}_n, \varepsilon_n)$. Since n is arbitrary, we have $x \in \bigcap_{n \ge 1} \Sigma(\mathfrak{U}_n, \varepsilon_n)$.

Notice that $\Sigma(f)$ is not necessarily f-invariant. The following proposition is based on the remarkable proof of the ergodic closing lemma for diffeomorphisms by Mañé ([1]).

PROPOSITION 3. *For every $f \in R^1(M)$, every neighborhood $\mathfrak{U}$ of f and every $\varepsilon > 0$ the following holds:*

$$\mu(\Sigma(\mathfrak{U}, \varepsilon)) = 1$$

for every ergodic $\mu \in \mathfrak{M}(f)$.

It is well known that if a set has total measure for every ergodic measure in $\mathfrak{M}(f)$ then it is total for every measure in $\mathfrak{M}(f)$. Therefore we have the following theorem which is an aim of this paper.

THEOREM. *If $f \in R^1(M)$, then $\Sigma(f)$ has total measure for every measure in $\mathfrak{M}(f)$.*

To obtain Theorem it remains only to prove Propositions 2 and 3.

First we give the proof of Proposition 2. If $p \in \mathrm{Per}(f)$, then the conclusion of Proposition 2 is clear. Thus we prove the case when $p \notin \mathrm{Per}(f)$. Let $\varepsilon > 0$, $\mathfrak{U}$ be a neighborhood of f and d_1 be a distance of $R^1(M)$. Then there exists $0 < \eta < \varepsilon/2$ such that $d_1(f, g) < 2\eta$ implies $g \in \mathfrak{U}$. Denote $T_xM(\xi) = \{u \in T_xM : |u| \le \xi\}$ for $x \in M$ and take $\xi > 0$ such that $\exp_x\colon T_xM(\xi) \to M$ is an embedding for $x \in M$. Then we can find $\varepsilon_0 > 0$ such that for $g \in R^1(M)$, $x \in M$ and $c_1, c_2 \in T_xM$ with $B(c_2, |c_1 - c_2|/\varepsilon_0) \subset T_xM(\xi)$ there exists a diffeomorphism $h\colon M \to M$ satisfying

$$(2) \qquad \begin{cases} \text{(i)} & h(\exp_x(c_2)) = \exp_x(c_1), \\ \text{(ii)} & \mathrm{supp}(h) \subset \exp_x(B(c_2, |c_1 - c_2|/\varepsilon_0)), \\ \text{(iii)} & d_1(h \circ g, g) < \eta. \end{cases}$$

Put $q_0 = p$. Let (Q, f), $\rho_0 > 2$ and $\mu_0 \ge 1$ be as in Proposition 1 for $\varepsilon_0 > 0$. Then there exists $0 < r_0 < \varepsilon/4$ such that the following (3) holds. For $0 < r'_0 \le r_0$

(3)
- (a) if W is a connected component of $\bigcup_{n=0}^{\mu_0+1} f^{-n}(B_{r'_0}(p))$, then there is a unique $q\in\bigcup_{n=0}^{\mu_0+1} f^{-n}(p)$ satisfying $q\in W$, and so we write $W_{r'_0}(q)=W$, since W depends on q and r'_0,
- (b) $f^n|W_{r'_0}(q): W_{r'_0}(q)\to B_{r'_0}(p)$ is a diffeomorphism if $f^n(q)=p$ for some $0\le n\le\mu_0+1$,
- (c) $W_{r_0}(q)\subset U_{\varepsilon/4}(q)$ for every $q\in\bigcup_{n=0}^{\mu_0+1} f^{-n}(p)$.

Furthermore, by Lemma 4.2 of [4] there exist $0<\lambda\le r_0$ and $f_1\in R^1(M)$ with $d_1(f, f_1)<\eta$ satisfying

(4)
- (d) $f_1=\exp_{f(q)}\circ(D_qf)\circ\exp_q^{-1}$ on $W_{\lambda/4}(q)$ for $q\in\bigcup_{n=1}^{\mu_0} f^{-n}(p)$,
- (e) $f_1=\exp_{f(q)}\circ(D_{f(q)}f^{\mu_0})^{-1}\circ\exp_p^{-1}\circ f^{\mu_0+1}$ on $W_{\lambda/4}(q)$ for $q\in f^{-\mu_0-1}(p)$,
- (f) $f_1^{\mu_0+1}=f^{\mu_0+1}$ on $W_\lambda(q)$ for $q\in f^{-\mu_0-1}(p)$,
- (g) $f_1=f$ on $M-\bigcup\{W_\lambda(q) : q\in\bigcup_{n=1}^{\mu_0+1} f^{-n}(p)\}$.

Define a metric d' of $B_{r_0}(p)$ by

$$d'(w, z)=|\exp_p^{-1}(w)-\exp_p^{-1}(z)| \qquad (w, z\in B_{r_0}(p)) .$$

Then we can check the existence of $r>0$ and $\rho>2$ satisfying for $0<\tilde r\le r$ and $w, z\in U_{\tilde r}(p)$

$$U(w, \rho_0 d'(w, z) ; d')\subset U_{\rho\tilde r}(p)\subset B_{\lambda/4}(p) .$$

Indeed, put $r=\lambda/(8\rho_0+4)<r_0/(2\rho_0+1)$. Since $d(y, p)=d'(y, p)$ for $y\in B_{r_0}(p)$, we have $d'(w, z)\le 2\tilde r$ and so

$$U(w, \rho_0 d'(w, z) ; d')\subset U(w, 2\rho_0\tilde r ; d') .$$

Since $d'(w, p)\le\tilde r$,

$$U(w, \rho_0 d'(w, z) ; d')\subset U(p, (2\rho_0+1)\tilde r ; d')=U_{(2\rho_0+1)\tilde r}(p)$$

and so $\rho=2\rho_0+1$ is our requirement.

Let us take $0<\tilde r\le r$, $x\in U_{\tilde r}(p)$ and $m>0$ with $f^m(x)\in U_{\tilde r}(p)$, and $P=\{x, f(x), \cdots, f^m(x)\}\cap B_{\lambda/4}(p)$ is represented as $P=\{p_0, p_1, \cdots, p_t\}$ where $p_0=x$ and $p_t=f^m(x)$ (since $U_{\tilde r}(p)\subset B_{\lambda/4}$). Then we have

$$U(p_t, \rho_0 d'(p_0, p_t) ; d')\subset U_{\rho\tilde r}(p)\subset B_{\lambda/4}(p) .$$

Letting $P'=\exp_p^{-1}(P)$ and $p'_i=\exp_p^{-1}(p_i)$ for $0\le i\le t$, we have $P'=\{p'_0, p'_1, \cdots, p'_t\}$ and

$$U(p'_t, \rho_0|p'_0-p'_t|)\subset\exp_p^{-1}(U_{\rho\tilde r}(p))\subset\exp_p^{-1}(B_{\lambda/4}(p)) .$$

By Proposition 1 there is $y'\in P'\cap U(p'_t, \rho_0|p'_0-p'_t|)$ such that for any branch $\Sigma=\{\bar q_0, \bar q_1, \cdots\}$ of (Q, f) there is $w'(\Sigma)\in P'\cap U(p'_t, \rho_0|p'_0-p'_t|)$. Let $w(\Sigma)=\exp_p(w'(\Sigma))$ and $y=\exp_p(y')$. Then $f^\psi(w(\Sigma))=y$ for some $\psi>0$. Since $\lambda\le r_0$ and (3) holds, we have $\psi>\mu_0+1$. Put $z=f^{\psi-\mu_0-1}(w)$, then $f^{\mu_0+1}(z)=y$ and $z\in W_{\lambda/4}(q_{\mu_0+1})$ for some

$q_{\mu_0+1} \in f^{-\mu_0-1}(p)$. Choose and fix a branch $\Gamma=\{\tilde{q}_0, \tilde{q}_1, \cdots\}$ of (Q,f) satisfying $\tilde{q}_{\mu_0+1}=q_{\mu_0+1}$, then for Γ there exist $w' \in P' \cap U(p'_t, \rho_0|p'_0-p'_t|)$ and $c'_0, c'_1, \cdots, c'_{\mu_0} \in U(p'_t, \rho_0|p'_0-p'_t|)$ satisfying (a) and (b) of Proposition 1. Then $f^{\varphi}(w)=y$ for some $\varphi>\mu_0+1$ where $w=\exp_p(w')$. Thus, by Proposition 1 (b) and (2) for $0 \le n \le \mu_0-1$ there exists a diffeomorphism $h_n: M \to M$ with $d_1(h_n \circ f_1, f_1)<\eta$ satisfying

$$h_n(\exp_{\tilde{q}_n}((D_{\tilde{q}_n}f^n)^{-1}(c'_{n+1})))=\exp_{\tilde{q}_n}((D_{\tilde{q}_n}f^n)^{-1}(c'_n)),$$

and so define a map $g \in R^1(M)$ by

$$g=\begin{cases} h_n \circ f_1 & \text{on} \quad W_\lambda(\tilde{q}_{n+1}) \qquad (0 \le n \le \mu_0-1), \\ f_1 & \qquad\qquad\qquad\qquad \text{otherwise}. \end{cases}$$

Since $d_1(g,f)<2\eta$, we have $g \in \mathfrak{U}$, and $g^{\varphi}(w)=w$ by the definition of g. Since $w=f^{m_1}(x)$ and $y=f^{m_2}(x)$ for some $0 \le m_1 < m_2 \le m$ and $w, y \in U(p_t, \rho d'(p_0, p_t); d') \subset U_{\rho\tilde{r}}(p)$, (i) is satisfied, and $g^{m_2-m_1}(f^{m_1}(x))=g^{\varphi}(w)=f^{m_1}(x)$ ensures that (ii) is satisfied. By the definition of g we have

$$\{y \in M : f(y) \ne g(y)\} \subset \bigcup_{i=0}^{\mu_0+1} \bigcup_{q \in f^{-i}(p)} W_\lambda(q) \subset B_\varepsilon(f^{-1}, p, m_2-m_1). \tag{5}$$

Since $f^{m_1}(x) \in B_{\lambda/4}(p)$, for every $q \in \bigcup_{i=0}^{\mu_0+1} f^{-i}(p)$ there exists $x' \in \bigcup_{i=0}^{\mu_0+1} f^{-i}(f^{m_1}(x))$ such that $x' \in W_{\lambda/4}(q)$, and so

$$\bigcup_{i=0}^{\mu_0+1} \bigcup_{q \in f^{-i}(p)} W_\lambda(q) \subset \bigcup_{i=0}^{\mu_0+1} \bigcup_{x' \in f^{-i}(f^{m_1}(x))} B_\varepsilon(x')$$
$$\subset B_\varepsilon(f^{-1}, f^{m_1}, m_2-m_1)$$

from which we have (iii). It only remains to prove (iv). Let us choose a sequence $0<n_1<n_2<\cdots<n_k<n_{k+1}=m_2-m_1-\mu_0-1<m_2-m_1$ satisfying $f^{n_i}(f^{m_1}(x)) \in W_\lambda(q_i)$ for some $q_i \in f^{-\mu_0-1}(p)$ and $f^n(f^{m_1}(x)) \notin \bigcup\{W_\lambda(q) : q \in f^{-\mu_0-1}(p)\}$ if $n \ne n_i$ $(1 \le i \le k+1)$. Since $f^n(f^{m_1}(x)) \notin \bigcup_{0<j\le\mu_0+1} \bigcup_{q \in f^{-j}(p)} W_\lambda(q)$ $(0 \le n \le n_1-1)$, we have

$$f^n(f^{m_1}(x))=g^n(f^{m_1}(x)) \qquad \text{for} \quad 0 \le n \le n_1.$$

Since $f^{n_1}(f^{m_1}(x))=g^{n_1}(f^{m_1}(x)) \in W_\lambda(q_1)$, we have $f^n(f^{m_1}(x)), g^n(f^{m_1}(x)) \in W_\lambda(f^{n-n_1}(q_1))$ for $n_1+1 \le n \le n_1+\mu_0+1$, and so by (5)

$$d(f^n(f^{m_1}(x)), g^n(f^{m_1}(x)))<2\lambda<\varepsilon \qquad \text{for} \quad n_1+1 \le n \le n_1+\mu_0+1.$$

From Proposition 1 (b) and (2) we have $g^n(f^{m_1}(x))=f_1^n(f^{m_1}(x))$ for $n_1+1 \le n \le n_1+\mu_0+1$ and so by (4) (f)

$$g^{n_1+\mu_0+1}(f^{m_1}(x))=f^{n_1+\mu_0+1}(f^{m_1}(x)).$$

Repeating this process we obtain (iv). The proof of Proposition 2 is completed.

For the proof of Proposition 3 Mañé [1] prepared a measure theoretical property about certain partitions of the s-torus $T^s = S^1 \times S^1 \times \cdots \times S^1$ for the case of diffeomorphisms. Our proof is of course in the framework of Mañé. We repeat the techniques described in [1], though it seems that the description is helpful for us.

We say that a set $A \subset T^s$ is a cube if it can be written as $A = I_1 \times \cdots \times I_s$ where the sets I_i are intervals in S^1 with equal lengths. If p_i is the middle point of I_i we say that the point $(p_1, \cdots, p_s)$ is the center of the cube A. The length of the intervals I_i is called the side of the cube. For each $k \in Z^+$ let $\mathscr{P}_1^{(k)} \leq \mathscr{P}_2^{(k)} \leq \cdots$ be a sequence of partitions of T^s each one containing a finite set of disjoint cubes whose union is T^s. Suppose also that the side of the atoms of $\mathscr{P}_j^{(k)}$ is $2\pi/k^j$. For every atom Q of $\mathscr{P}_j^{(k)}$ we can associate cubes $\hat{Q}$ and $\tilde{Q}$ having the same center as Q and sides $2\pi/k^{j-1}$ and $6\pi/k^{j-1}$ respectively. If $x \in T^s$, we denote by $\mathscr{P}_j^{(k)}(x)$ the atom of $\mathscr{P}_j^{(k)}$ containing x.

For $k \geq 1, j \geq 1, \delta > 0$ and a Borel probability measure μ of T^s define

$$\tilde{B}_\delta(j, k)_\mu = \{x \in T^s : \mu(\mathscr{P}_j^{(k)}(x)) \geq \delta \mu(\tilde{\mathscr{P}}_j^{(k)}(x))\} .$$

LEMMA 1 (Mañé [1]). *$\tilde{B}_\delta(j, k)_\mu$ is a Borel set. If k is odd, then $\mu(\tilde{B}_\delta(j, k)_\mu) \geq 1 - \delta 3^s k^s$.*

Let $l = k^{sj}$. Then there exists $\{x_1, \cdots, x_l\} \subset T^s$ such that for every $Q \in \mathscr{P}_j^{(k)}$ there is a unique point $x \in \{x_1, \cdots, x_l\}$ with $x \in Q$. Thus $\tilde{B}_\delta(j, k)_\mu = \bigcup_{i \notin S} \mathscr{P}_j^{(k)}(x_i)$ where $S = \{1 \leq i \leq l : \mu(\mathscr{P}_j^{(k)}(x_i)) < \delta \mu(\tilde{\mathscr{P}}_j^{(k)}(x_i))\}$, and so $\tilde{B}_\delta(j, k)_\mu$ is a Borel set. Since k is odd, the sets $\tilde{\mathscr{P}}_j^{(k)}(x_i)$ $(1 \leq i \leq l)$ cover each atom of $\mathscr{P}_j^{(k)}$ exactly. Thus we have

$$\begin{aligned} \mu(\{x \in T^s : \mu(\mathscr{P}_j^{(k)}(x)) < \delta \mu(\tilde{\mathscr{P}}_j^{(k)}(x))\}) &= \sum_{i \in S} \mu(\mathscr{P}_j^{(k)}(x_i)) \\ &< \delta \sum_{i \in S} \mu(\tilde{\mathscr{P}}_j^{(k)}(x_i)) \\ &\leq \delta \sum_{i=1}^{l} \mu(\tilde{\mathscr{P}}_j^{(k)}(x_i)) \\ &= \delta \sum_{i=1}^{l} 3^s k^s \mu(\mathscr{P}_j^{(k)}(x_i)) \\ &= \delta 3^s k^s \end{aligned}$$

and so

$$\begin{aligned} \mu(\tilde{B}_\delta(j, k)_\mu) &= 1 - \mu(\{x \in T^s : \mu(\mathscr{P}_j^{(k)}(x)) < \delta \mu(\tilde{\mathscr{P}}_j^{(k)}(x))\}) \\ &\geq 1 - \delta 3^s k^s . \end{aligned}$$

The proof of Lemma 1 is completed.

Let us put

$$K = \bigcup \{\partial \hat{A} \cup \partial \tilde{A} : A \in \mathscr{P}_j^{(k)}, k \geq 1, j \geq 1\} .$$

Then for every Borel probability measure μ of T^s there is $y \in T^s$ such that $\mu(\tau_y(K))=0$ where $\tau_y: M \to M$ is a translation defined by $\tau_y(x)=y+x$ for $x \in T^s$. This is checked as follows: Let us put

$$K_{k,j}=\bigcup\{\partial\hat{A} \cup \partial\tilde{A} : A \in \mathscr{P}_j^{(k)},\ k \geq 1,\ j \geq 1\},$$
$$K_k=\bigcup\{K_{k,j} : j \geq 1\}.$$

Then $K=\bigcup\{K_k : k \geq 1\}$ and so K is a Borel set. For $z=(z_1, \cdots, z_s) \in T^s$ and $0 \leq t \leq 1$ we denote by $z(t)=(tz_1, \cdots, tz_s) \in T^s$. Take and fix $z=(z_1, \cdots, z_s) \in T^s$ with $z_i \neq 0, 1$ $(1 \leq i \leq s)$. We claim that there exists a sequence $\{t_n\}_{n \geq 1}$ with $(1/2)^n \leq t_n \leq (1/2)^{n-1} - (1/2)^{n+1}$ such that $\mu(\tau_{z(t_l)}(K) \cap \tau_{z(t_n)}(K))=0$ if $l \neq n$. Indeed, for every $\varepsilon>0$ put

$$\varepsilon_{l_1}=\varepsilon(1/2)^{l_1},$$
$$\varepsilon_{l_1,i_1}=\varepsilon_{l_1}(1/2)^{i_1},$$
$$\varepsilon_{l_1,i_1,l_2}=\varepsilon_{l_1,i_1}(1/2)^{l_2},$$
$$\varepsilon_{l_1,i_1,l_2,i_2}=\varepsilon_{l_1,i_1,l_2}(1/2)^{i_2},$$

for $l_1, i_1, l_2, i_2 \geq 1$ and take a set

$$C=\{c_i\}=\{(a(i), b(i)) : a(i) \in N,\ b(i) \in N\} \qquad \text{with} \quad c_1=(1, 1)$$

such that every pair $(a, b) \in N \times N$ is contained in C and $c_i \neq c_j$ if $i \neq j$. Put $I_1=[(1/2), 1-(1/2)^2]$ where [,] denotes a closed interval. Then $t \in I_1$ implies $\mu(K_{c_{a(1)}} \cap \tau_{z(t)}(K_{c_{b(1)}}))=(K_{1,1} \cap \tau_{z(t)}(K_{1,1}))=0$ since $K_{1,1}=\varnothing$. Since $K_{c_{a(2)}}$ and $K_{c_{b(2)}}$ are finite lattices, there exists a closed interval $I_1' \subset I_1$ with $\operatorname{int} I_1' \neq \varnothing$ such that for $t, t' \in I_1'$ with $t \neq t'$

$$(K_{c_{a(2)}} \cap \tau_{z(t)}(K_{c_{b(2)}})) \cap (K_{c_{a(2)}} \cap \tau_{z(t')}(K_{c_{b(2)}}))=\varnothing,$$

from which

$$\#\{t \in I_1' : \mu(K_{c_{a(2)}} \cap \tau_{z(t)}(K_{c_{b(2)}})) \geq \varepsilon_{c_{a(2)}, c_{b(2)}}\} \leq \frac{1}{\varepsilon_{c_{a(2)}, c_{b(2)}}}.$$

Therefore we have that there is a closed interval $I_2 \subset I_1' \subset I_1$ with $\operatorname{int} I_2 \neq \varnothing$ such that for every $t \in I_2$

$$\mu(K_{c_{a(2)}} \cap \tau_{z(t)}(K_{c_{b(2)}})) < \varepsilon_{c_{a(2)}, b(2)}.$$

Repeating this process we have a sequence of closed intervals $I_1 \supset I_2 \supset \cdots \supset I_i \supset \cdots$ such that

$$I_i \subset \{t \in I_{i-1} : \mu(K_{c_{a(i)}} \cap \tau_{z(t)}(K_{c_{b(i)}})) < \varepsilon_{c_{a(i)}, b(i)}\}$$

for $i \geq 2$. Take $t \in \bigcap_{i \geq 1} I_i$. Then we have

$$\begin{aligned}\mu(K\cap\tau_{z(t)}(K))&=\mu\Bigl(\bigcup_{i\geq 1}(K_{c_{a(i)}}\cap\tau_{z(t)}(K_{c_{b(i)}})\Bigr)\\&\leq\sum_{i\geq 1}\mu(K_{c_{a(i)}}\cap\tau_{z(t)}(K_{c_{b(i)}}))\\&<\varepsilon\,.\end{aligned}$$

From this result, for every $n\geq 1$ there is $t(n)\in[(1/2),\ 1-(1/2)^2]$ such that $\mu(K\cap\tau_{z(t(n))}(K))<1/n$. Assume that $t(n)\to t_1$ as $n\to\infty$, then we have

$$\mu(K\cap\tau_{z(t_1)}(K))=0\,.$$

Repeating this process we obtain the claim. By the claim, for every $n>0$ there exists $y(n)\in T_s$ such that $\mu(\tau_{y(n)}(K))<1/n$. Indeed, if it is false, then there is $\varepsilon>0$ such that $\mu(\tau_y(K))>\varepsilon$ for every $y\in T^s$. Let z and $\{t(i)\}$ be as in the claim. Then we have

$$\begin{aligned}1=\mu(T^s)&\geq\mu\Bigl(\bigcup_{i\geq 1}\tau_{z(t(i))}(K)\Bigr)\\&=\sum_{i\geq 1}\mu(\tau_{z(t(i))}(K))=\infty\,,\end{aligned}$$

which is a contradiction. Assume that $y(n)\to y\in T^s$ as $n\to\infty$, then we have $\tau_y(K)=0$.

Therefore we may assume that $\mu(K)=0$. Since M is isometrically embedded in T^s for large s, we suppose that $M\subset T^s$ and we consider that an ergodic measure of M is a Borel probability measure on T^s.

To show Proposition 3 Mañé prepared furthermore the following three lemmas. First define $\Sigma(\mathfrak{U},\varepsilon,r,\rho)$, whose $r>0$, $\rho>2$, as the set of point $x\in M$ such that if $y\in U_{\tilde r}(x)$ for some $0<\tilde r\leq r$ and $f^m(y)\in U_{\tilde r}(x)$ for some $m>0$ then there exist $0\leq m_1<m_2\leq m$ and $g\in\mathfrak{U}$ satisfying (i), (ii), (iii) and (iv) of Proposition 2. We can not check that $\Sigma(\mathfrak{U},\varepsilon,r,\rho)$ is closed in M. However, if $r_n>0$ and $\rho_n>2$ are monotone sequences converging to 0 and $+\infty$ respectively then

$$M=\bigcup_{n\geq 1}\bigcup_{m\geq 1}\Sigma(\mathfrak{U},\varepsilon,r_n,\rho_m)\tag{6}$$

for every neighborhood $\mathfrak{U}$ of f and every $\varepsilon>0$. Remark here that for integers $n\geq 1$ and $m\geq 1$ there exist an odd integer $k=k(n,m)>0$ and an integer $j(n,m)>0$ such that if $j\geq j(n,m)$ and $x\in T^s$ then there is $0<r\leq r_n$ satisfying

$$\begin{cases}\text{(i)} & \overline{\mathscr{P}_j^{(k)}(x)}\subset U_r(x)\,,\\ \text{(ii)} & \operatorname{int}\hat{\mathscr{P}}_j^{(k)}(x)\supset B_{\rho_m r}(x)\,.\end{cases}\tag{7}$$

LEMMA 2 (Mañé [1], Lemma 1.6). *If* $x\in\overline{\Sigma(\mathfrak{U},\varepsilon,r_n,\rho_m)}$, $j\geq j(n,m)$, $k=k(n,m)$ *and* $\mu(\mathscr{P}_j^{(k)}(x))\geq\delta\mu(\hat{\mathscr{P}}_j^{(k)}(x))$ *then*

$$\mu(\hat{\mathscr{P}}_j^{(k)}(x)\cap\Sigma(\mathfrak{U},2\varepsilon))\geq\delta\mu(\hat{\mathscr{P}}_j^{(k)}(x)).$$

Since μ is ergodic, there exists $y\in M$ such that

$$(8)\qquad\begin{cases}\lim_{l\to\infty}\dfrac{1}{l}\#\{1\leq i\leq l: f^i(y)\in\Sigma(\mathfrak{U},2\varepsilon)\cap\hat{\mathscr{P}}_j^{(k)}(x)\}=\mu(\Sigma(\mathfrak{U},2\varepsilon)\cap\hat{\mathscr{P}}_j^{(k)}(x)),\\ \lim_{l\to\infty}\dfrac{1}{l}\#\{1\leq i\leq l: f^i(y)\in\mathscr{P}_j^{(k)}(x)\}=\mu(\mathscr{P}_j^{(k)}(x)).\end{cases}$$

Since $x\in\overline{\Sigma(\mathfrak{U},\varepsilon,r_n,\rho_m)}$, we have that if $f^{i_1}(y)$ and $f^{i_2}(y)$ belong to $\mathscr{P}_j^{(k)}(x)$ for some $i_1<i_2$ then there exists $i_1\leq i_3<i_2$ satisfying

$$f^{i_3}(y)\in\hat{\mathscr{P}}_j^{(k)}(x)\cap\Sigma(\mathfrak{U},2\varepsilon).$$

Indeed, let $0<r\leq r_n$ be as in (7). Then $f^{i_1}(y),f^{i_2}(y)\in\mathscr{P}_j^{(k)}(x)\subset\overline{\mathscr{P}_j^{(k)}(x)}\subset U_r(x)$. Put $\delta_1=d(\overline{\mathscr{P}_j^{(k)}(x)},U_r(x)^c)>0$ where E^c denotes the complement of E. Since $B_{\rho_m r}(x)\subset\operatorname{int}\hat{\mathscr{P}}_j^{(k)}(x)$, we have $\delta_2=d(B_{\rho_m r}(x),(\operatorname{int}\hat{\mathscr{P}}_j^{(k)}(x))^c)>0$. Take $z\in\Sigma(\mathfrak{U},\varepsilon,r_n,\rho_m)$ such that $d(x,z)<\min\{\delta_1,\delta_2\}$. Then

$$d(z,f^{i_j}(y))\leq d(z,x)+d(x,f^{i_j}(x))<r\qquad(j=1,2)$$

and hence by Proposition 2 there exists $i_1\leq i_3<i_2$ satisfying $f^{i_3}(y)\in B_{\rho_m r}(z)$ and $f^{i_3}(x)\in\Sigma(\mathfrak{U},2\varepsilon)$. Since $d(x,f^{i_3}(y))<\delta_2+\rho_m r$, we have $f^{i_3}(y)\in\hat{\mathscr{P}}_j^{(k)}(x)\cap\Sigma(\mathfrak{U},2\varepsilon)$, from which

$$\#\{1\leq i\leq l: f^i(y)\in\hat{\mathscr{P}}_j^{(k)}(x)\cap\Sigma(\mathfrak{U},2\varepsilon)\}\geq\#\{1\leq i\leq l: f^i(y)\in\mathscr{P}_j^{(k)}(x)\}-1.$$

From (8)

$$\begin{aligned}\mu(\hat{\mathscr{P}}_j^{(k)}(x)\cap\Sigma(\mathfrak{U},2\varepsilon))&=\lim_{l\to\infty}\#\{1\leq i\leq l: f^i(y)\in\hat{\mathscr{P}}_j^{(k)}(x)\cap\Sigma(\mathfrak{U},2\varepsilon)\}\\&\geq\lim_{l\to\infty}\#\{1\leq i\leq l: f^i(y)\in\mathscr{P}_j^{(k)}(x)\}\\&=\mu(\mathscr{P}_j^{(k)}(x))\geq\delta\mu(\hat{\mathscr{P}}_j^{(k)}(x)).\end{aligned}$$

Now define $\Lambda_\delta^0(n,m)$, for $\delta>0$, as the set of point $x\in\boldsymbol{T}^s$ such that for $k=k(n,m)$

$$\mu(\mathscr{P}_{j_i}^{(k)}(x))\geq\delta\mu(\tilde{\mathscr{P}}_{j_i}^{(k)}(x))$$

holds for an infinite sequence $v(x)=\{j_i\}\subset\{j\geq j(n,m)\}$. Let us put

$$\Lambda_\delta(n,m)=\Lambda_\delta^0(n,m)\cap\overline{\Sigma(\mathfrak{U},\varepsilon,r_n,\rho_m)}.$$

LEMMA 3. (a) *$\Lambda_\delta^0(n,m)$ and $\Lambda_\delta(n,m)$ are Borel sets,*
(b) *$\mu(\Lambda_\delta^0(n,m))\geq 1-\delta 3^s k^s$,*
(c) *$\bigcup_{r\geq 1}\Lambda_{1/r}(n,m)=\overline{\Sigma(\mathfrak{U},\varepsilon,r_n,\rho_m)}$ μ-a.e.*

Since $\tilde{B}_\delta(k, j)_\mu = \{x \in T^s : \mu(\mathscr{P}_j^{(k)}(x)) \geq \delta\mu(\tilde{\mathscr{P}}_j^{(k)}(x))\}$ is a Borel set by Lemma 1, we have that

$$\Lambda_\delta^0(n, m) = \bigcap_{i \geq j(n,m)} \bigcup_{j \geq i} \tilde{B}_\delta(k, j)_\mu = \overline{\lim_{j\to\infty}} \tilde{B}_\delta(k, j)_\mu$$

is also a Borel set, and by Lemma 1

$$\mu(\Lambda_\delta^0(n, m)) = \mu\left(\overline{\lim_{j\to\infty}} \tilde{B}_\delta(k, j)\right) \geq \overline{\lim}\, \mu(\tilde{B}_\delta(k, j)) \geq 1 - \delta 3^s k^s .$$

Therefore (a) and (b) are proved. To obtain (c) suppose that there exists a Borel set V with $\mu(V) > 0$ such that

$$V \subset \overline{\Sigma(\mathfrak{U}, \varepsilon, r_n, \rho_m)} - \bigcup_{r \geq 1} \Lambda_{1/r}(n, m) .$$

Since $V \cap \bigcup_{r \geq 1} \Lambda_{1/r}(n, m) = \varnothing$, we have

$$1 \geq \mu(V) + \mu(\Lambda_{1/r}(n, m)) \geq \mu(V) + \left(1 - \frac{1}{r} 3^s k^s\right)$$

for every $r \geq 1$ and so $1 \geq \mu(V) + 1 > 1$, thus contradicting.

LEMMA 4 (Mañé [1], Lemma I.7). *Given an open set U considering $\Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c$ there exist sequences $\{x_i\} \subset \Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c$ and $\{j_i\}$ with $j_i \in \nu(x_i)$, not necessarily infinite sequences, such that*

(I) $\{\hat{\mathscr{P}}_{j_i}^{(k)}(x_i)\}$ *are disjoint and* $\bigcup_i \overline{\hat{\mathscr{P}}_{j_i}^{(k)}(x_i)} \subset U$,

(II) $\mu(\{\Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c\} - \bigcup_i \hat{\mathscr{P}}_{j_i}^{(k)}(x_i)) = 0$,

where $k = k(n, m)$.

Denote by $\mathscr{F}$ the family of sets $\mathscr{P}_j^{(k)}(x)$ with $x \in \Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, \varepsilon)^c$ and $j \in \nu(x)$. Take $A_1 \in \mathscr{F}$ satisfying $\operatorname{diam} A_1 = \max\{\operatorname{diam} A : A \in \mathscr{F}$ and $\bar{\hat{A}} \subset U\}$. If $A_1 = \mathscr{P}_{j_1}^{(k)}(x_1)$ for some x_1 and $\bar{\hat{A}} \supset \Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c$, then we have

$$\mu(\Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c - \hat{\mathscr{P}}_{j_1}^{(k)}(x_1))$$

$$= \mu(\Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon) - \overline{\hat{\mathscr{P}}_{j_1}^{(k)}(x_1)}) = 0 .$$

For the case when $\bar{\hat{A}}_1 \not\supset \Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c$, $U - \bar{\hat{A}}_1$ is a nonempty open set such that $(U - \bar{\hat{A}}_1) \cap \{\Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c\} \neq \varnothing$ since U is open and $\bar{\hat{A}} \subset U$. Thus we can find $A_2 \in \mathscr{F}$ satisfying $\operatorname{diam} A_2 = \max\{\operatorname{diam} A : A \in \mathscr{F},\ \bar{\hat{A}} \subset U$ and $\hat{A} \cap \hat{A}_1 = \varnothing\}$. Indeed, let $x \in (U - \bar{\hat{A}}_1) \cap \{\Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c\}$. Then $\overline{\hat{\mathscr{P}}_j^{(k)}(x)} \subset U - \bar{\hat{A}}_1$ for some $j \in \nu(x)$ since $U - \bar{\hat{A}}_1$ is open and $\# \nu(x) = \infty$. Thus $\{A \in \mathscr{F} : \bar{\hat{A}} \subset U$ and $\hat{A} \cap \hat{A}_1 = \varnothing\} \neq \varnothing$. If $\bar{\hat{A}}_1 \cup \bar{\hat{A}}_2 \supset$

$\Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c$, then $\{x_1, x_2\}$ and $\{j_1, j_2\}$ are sequences satisfying (I) and (II) where $A_2 = \mathscr{P}_{j_2}^{(k)}(x)$. For the case when $\bar{\hat{A}}_1 \cup \bar{\hat{A}}_2 \not\supset \Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c$, inductively we have the following (i) or (ii):

(i) There exist finite sequences $\{x_i\}_{i=1}^l \subset \Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c$ and $\{j_i\}_{i=1}^l$ with $j_i \in \nu(x_i)$ such that

$$\bigcup_{i=1}^{l} \bar{\hat{\mathscr{P}}}_{j_i}^{(k)}(x_i) \supset \Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c .$$

(ii) There exists an infinite sequence $\{A_i\}_{i \geq 1} \subset \mathscr{F}$ such that

(a) $\bar{\hat{A}}_i \subset U$,

(b) $\hat{A}_i \cap \hat{A}_l = \varnothing$ if $i \neq l$,

(c) $\operatorname{diam} A_i = \max\{\operatorname{diam} A : A \in \mathscr{F},\ \bar{\hat{A}} \subset U \text{ and } \hat{A} \cap \hat{A}_l = \varnothing\ (1 \leq l \leq i)\}$.

When (i) holds, the sequences $\{x_i\}_{i=1}^l$ and $\{j_i\}_{i=1}^l$ satisfy (I) and (II). For the case when we have (ii), it follows that

$$\lim_{i \to \infty} \operatorname{diam} A_i = 0 \tag{9}$$

by (c), and

$$\sum_{i \geq 1} \mu(A_i) = \mu\left(\bigcup_{i \geq 1} A_i\right) \leq 1 \tag{10}$$

by (b). From (ii) it is clear that $\Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c - \bigcup_{i=1}^N \bar{\hat{A}}_i \neq \varnothing$ for $N > 0$. Fix $N > 0$ and take x from the set, then there exists $A \in \mathscr{F}$ such that $x \in A$ and $\bar{\hat{A}} \cap \bigcup_{i=1}^N \bar{\hat{A}}_i = \varnothing$ and $\bar{\hat{A}} \subset U$. If $\bar{\hat{A}} \cap \bar{\hat{A}}_{\tilde{N}} = \varnothing$ for $\tilde{N} > N$, then there exists $N_1 > N$ such that $\operatorname{diam} A > \operatorname{diam} A_{N_1}$ by (9), which is inconsistent with (c). Therefore we have that there exists $N_1 > N$ such that $\bar{\hat{A}} \cap \bar{\hat{A}}_i = \varnothing$ $(1 \leq i < N_1)$ and $\bar{\hat{A}} \cap \bar{\hat{A}}_{N_1} \neq \varnothing$. Since $\operatorname{diam} A \leq \operatorname{diam} A_{N_1}$ by (c), clearly $\operatorname{diam} \hat{A} \leq \operatorname{diam} \hat{A}_{N_1}$ and hence $\bar{A} \subset \bar{\bar{A}}_{N_1}$. Therefore we have $x \in A \subset \bar{\bar{A}}_{N_1} \subset \bigcup_{i \geq N} \bar{\bar{A}}_i$, which shows that

$$\Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c - \bigcup_{i=1}^{N} \bar{\hat{A}}_i \subset \bigcup_{i > N} \bar{\bar{A}}_i .$$

From this relation we have

$$\begin{aligned}
\mu\left((\Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c) - \bigcup_{i=1}^{N} \hat{A}_i\right) &= \mu\left((\Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)) - \bigcup_{i=1}^{N} \bar{\hat{A}}_i\right) \\
&\leq \mu\left(\bigcup_{i > N} \bar{\bar{A}}_i\right) \\
&= \mu\left(\bigcup_{i > N} \tilde{A}_i\right)
\end{aligned}$$

$$\leq \sum_{i>N} \mu(\tilde{A}_i) \leq \delta^{-1} \sum_{i>N} \mu(A_i).$$

Since $\sum_{i>N} \mu(A_i) \to 0$ as $N \to \infty$ by (10), we have the conclusion.

To obtain Proposition 3 given an open set U as in Lemma 4 there exist sequences $\{x_i\}$ and $\{j_i\}$ satisfying (I) and (II). Since $x_i \in \Lambda_\delta(n, m) = \Lambda_\delta(n, m) \cap \overline{\Sigma(\mathfrak{U}, \varepsilon, r_n, \rho_m)}$, we have $j_i \geq j(n, m)$ and $\mu(\mathscr{P}_{j_i}^{(k)}(x_i)) \geq \delta \mu(\tilde{\hat{\mathscr{P}}}_{j_i}^{(k)}(x_i)) \geq \delta \mu(\hat{\mathscr{P}}_{j_i}^{(k)}(x_i))$ and so by Lemma 3

$$\mu(\hat{\mathscr{P}}_{j_i}^{(k)}(x_i) \cap \Sigma(\mathfrak{U}, 2\varepsilon)) \geq \delta \mu(\hat{\mathscr{P}}_{j_i}^{(k)}(x_i)).$$

Since $\mu(\hat{\mathscr{P}}_{j_i}^{(k)}(x_i)) = \mu(\mathscr{P}_{j_i}^{(k)}(x_i) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c) + \mu(\mathscr{P}_{j_i}^{(k)}(x_i) \cap \Sigma(\mathfrak{U}, 2\varepsilon))$, we have

$$\mu(\hat{\mathscr{P}}_{j_i}^{(k)}(x_i)) \geq \frac{1}{1-\delta} \mu(\hat{\mathscr{P}}_{j_i}^{(k)}(x_i) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c)$$

and so

$$\begin{aligned}\mu(U) \geq \sum_{i\geq 1} \mu(\hat{\mathscr{P}}_{j_i}^{(k)}(x_i)) &\geq \frac{1}{1-\delta} \sum_{i\geq 1} \mu(\hat{\mathscr{P}}_{j_i}^{(k)}(x_i) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c) \\ &= \frac{1}{1-\delta} \mu\Big(\bigcup_{i\geq 1} \hat{\mathscr{P}}_{j_i}^{(k)}(x_i) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c\Big) \\ &= \frac{1}{1-\delta} \mu(\Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c).\end{aligned}$$

If $\mu(\Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c) > 0$, then we have $\mu(U) > (1/(1-\delta))\mu(\Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c)$ for some open set U, thus contradicting. Therefore $\mu(\Lambda_\delta(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c) = 0$. We claim that for $n \geq 1$ and $m \geq 1$, $\Sigma(\mathfrak{U}, 2\varepsilon) \supset \overline{\Sigma(\mathfrak{U}, \varepsilon, r_n, \rho_m)}$ μ-a.e. Indeed, if $\mu(V) > 0$ where $V = \overline{\Sigma(\mathfrak{U}, \varepsilon, r_n, \rho_m)} - \Sigma(\mathfrak{U}, 2\varepsilon)^c$, then $V \subset \bigcup_{r\geq 1} \Lambda_{1/r}(n, m)$ μ-a.e. by Lemma 3 (c). Since $\Lambda_\delta(n, m) \subset \Lambda_{\delta'}(n, m)$ when $\delta > \delta'$, there exists $r > 0$ such that $V \subset \Lambda_{1/r}(n, m)$ μ-a.e. and so $\mu(\Lambda_{1/r}(n, m) \cap \Sigma(\mathfrak{U}, 2\varepsilon)^c) \geq \mu(V) > 0$, which is a contradiction. Therefore

$$\Sigma(\mathfrak{U}, 2\varepsilon) = \bigcup_{n\geq 1} \bigcup_{m\geq 1} \overline{\Sigma(\mathfrak{U}, \varepsilon, r_n, \rho_m)} = M \qquad \mu\text{-a.e.} \quad \text{(by (6))}$$

and so $\mu(\Sigma(\mathfrak{U}, 2\varepsilon)) = 1$. The proof of Proposition 3 is completed.

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