The Extension of Dominated Splittings for C^1 -Regular Maps

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§0. Introduction.

Let M be a closed C^∞ manifold with a Riemannian metric $\|\cdot\|$ and $C^1(M)$ be the set of all C^1 maps with the C^1 -topology. We say that $f \in C^1(M)$ is *regular* if $D_x f: T_x M \rightarrow T_{f(x)} M$ is injective for all $x \in M$. Denote by $\mathcal{R}^1(M)$ the set of all regular maps in $C^1(M)$. Clearly $\mathcal{R}^1(M)$ is open and $f \in \mathcal{R}^1(M)$ is a local diffeomorphism. For $f \in \mathcal{R}^1(M)$ we let

$$M_f = \{(x_i) \mid x_i \in M \text{ and } f(x_i) = x_{i+1}, i \in \mathbb{Z}\}.$$

Then M_f is a subset of the product space $M = \prod_{i \in \mathbb{Z}} M_i$ where each M_i is a replica of M . We give a metric $\tilde{d}$ for M by

$$\tilde{d}(x, y) = \sum_{-\infty}^{\infty} \frac{d(x_i, y_i)}{2^{|i|}}$$

where d is a metric for M induced by $\|\cdot\|$. Then M_f is closed. Define a homeomorphism $f^*: M_f \rightarrow M_f$ by $f^*((x_i)) = (f(x_i))$. If $P_i: M_f \rightarrow M$ is a projection defined by $(x_i) \rightarrow x_i$ ($i \in \mathbb{Z}$), then $P_i \circ f^* = f \circ P_i$ holds. We say that (M_f, f^*) is the *inverse limit* of (M, f) . Given a tangent bundle $\pi: TM \rightarrow M$ we write

$$TM^* = \{(x, v) \in M_f \times TM \mid P_0(x) = \pi(v)\}$$

and define a projection

$$\pi^*: TM^* \rightarrow M_f \quad \text{by } \pi^*((x, v)) = x$$

for $(x, v) \in TM^*$. Then $\pi^*: TM^* \rightarrow M_f$ is the continuous vector bundle (see [1]). A norm $\|\cdot\|$ of TM^* is given by $\|(x, v)\|^* = \|v\|$ for $(x, v) \in TM^*$ and a linear bundle automorphism $Df^*: TM^* \rightarrow TM^*$ covering f^* is given by

$$Df^*(x, v) = (f^*(x), D_{x_0} f(v))$$

for $(x, v) \in TM^*$. If Λ is an f -invariant subset of M ($f(\Lambda) = \Lambda$), then we write $\Lambda_f = \{(x_i) \in M_f \mid x_i \in \Lambda, i \in \mathbb{Z}\}$ and $T_{\Lambda_f}M^* = \bigcup_{x \in \Lambda_f} T_x M^*$ where $T_x M^* = \pi^{*-1}(x)$. To simplify the notations TM^* , $\|\cdot\|^*$, f^* and Df^* are regarded as

$$TM^* = TM, \quad \|\cdot\|^* = \|\cdot\|, \quad f^* = f \quad \text{and} \quad Df^* = Df.$$

Given a compact invariant set Λ of $f \in \mathcal{R}^1(M)$ (in general $f^{-1}(\Lambda) \neq \Lambda$) we say that Λ is *hyperbolic* if there exist a Df -invariant continuous splitting (a hyperbolic splitting) $T_{\Lambda_f}M^* = E^s \oplus E^u$ and constants $c > 0$ and $0 < \lambda < 1$ such that $\|Df^n|E^s\| \leq c\lambda^n$ and $\|Df^{-n}|E^u\| \leq c\lambda^n$ for all $n \geq 0$. A splitting $T_{\Lambda_f}M^* = E \oplus F$ is called *dominated* if it is continuous, Df -invariant and there exist $c > 0$ and $0 < \lambda < 1$ such that for all $x \in \Lambda_f$ and $n > 0$

$$\|Df^n|E(x)\| \cdot \|Df^{-n}|F(f^n(x))\| \leq c\lambda^n$$

where $E(x) = E \cap T_x M^*$ and $F(x) = F \cap T_x M^*$. It is easy to see that $E(x) = E(y)$ for $x, y \in \Lambda_f$ with $x_0 = y_0$. It is clear that the dominated splitting is an extension of the hyperbolic splitting.

The purpose of this paper is to prove the following

THEOREM 1. *Suppose $f \in \mathcal{R}^1(M)$ has a dominated splitting. Then there exist $m_1 > 0$, a neighborhood U of Λ_f and an extended splitting $T_U M^* = \bar{E} \oplus \bar{F}$ satisfying*

- (a) $Df^{m_1}(\bar{E}(x)) = \bar{E}(f^{m_1}(x))$, $Df^{m_1}(\bar{F}(x)) = \bar{F}(f^{m_1}(x))$ for $x \in U \cap f^{-m_1}(U)$,
- (b) $\bar{E}(x) = E(x)$, $\bar{F}(x) = F(x)$ for $x \in \Lambda_f$,
- (c) *there exists $0 < \lambda_1 < 1$ such that $\|Df^{m_1}|\bar{E}(x)\| \cdot \|Df^{-m_1}|\bar{F}(f^{m_1}(x))\| \leq \lambda_1$ for $x \in U \cap f^{-m_1}(U)$.*

For the proof we need the following

THEOREM 2. *Under the notations and the assumptions, there exist a neighborhood V of Λ_f and a continuous splitting $T_V M^* = \tilde{E} \oplus \tilde{F}$ such that*

- (a) $\tilde{E}(x) = E(x)$, $\tilde{F}(x) = F(x)$ for $x \in \Lambda_f$,
- (b) $\tilde{E}(x) = \tilde{E}(y)$ for $x, y \in V$ with $x_0 = y_0$.

The following is obtained as a corollary of Theorem 1.

COROLLARY. *Let $f \in \mathcal{R}^1(M)$ and Λ be a hyperbolic set. For $\delta > 0$ small there exist a neighborhood N of Λ_f in M_f and continuous u -disks $\{\tilde{W}_\delta^u(x) \mid x \in N\}$ such that*

- (i) $\tilde{W}_\delta^u(x) = W_\delta^u(x)$ for $x \in \Lambda_f$,
- (ii) $\tilde{W}_\delta^u(f(x)) \subset f(\tilde{W}_\delta^u(x))$ for $x \in N \cap f^{-1}(N)$.

Here $W_\delta^u(x)$ is defined as $W_\delta^u(x) = \{y \in M \mid \text{there exists } y_0 = y \text{ such that } d(x_{-n}, y_{-n}) \leq \delta \text{ for all } n \geq 0\}$.

In 1970, Hirsch, Palis, Pugh and Shub obtained an extension theorem of hyperbolic sets for diffeomorphisms as follows

THEOREM ([2]). Let $f: M \rightarrow M$ be a diffeomorphism and Λ be a hyperbolic set. For $\delta > 0$ small there exist a neighborhood N of Λ and continuous u -disks $\{\tilde{W}_\delta^u(x) \mid x \in N\}$ such that

- (i) $\tilde{W}_\delta^u(x) = W_\delta^u(x)$ for $x \in \Lambda$,
- (ii) $\tilde{W}_\delta^u(f(x)) \subset f(\tilde{W}_\delta^u(x))$ for $x \in N \cap f^{-1}(N)$.

Here $W_\delta^u(x)$ is usually defined as $W_\delta^u(x) = \{y \in M \mid d(f^{-n}(x), f^{-n}(y)) \leq \delta \text{ for all } n \geq 0\}$.

To prove this theorem they used essentially the following

LEMMA (4.4 of [2]). Let M and E be topological manifolds and $\pi: E \rightarrow M$ be a fiber bundle. For a closed set X , if $s: X \rightarrow E$ is a continuous section over X , then there exist a neighborhood N of X in M and a continuous section $\bar{s}$ over N satisfying $\bar{s}|_X = s$.

Since a subbundle E of $T_{A_f}M^*$ agrees with E of $T_A M$, by Lemma 4.4 of [2], a continuous extension $\tilde{E}$ of E on a certain neighborhood is obtained. However $F \subset T_{A_f}M^*$ is a subbundle on $\Lambda_f \subset M_f$ and M_f is not ANR. Thus we can not obtain an extension $\tilde{F}$ of F by the technique in [2]. The extension of F will be obtained by Theorem 2.

§1. Proof of Theorem 1.

Let X be a compact metric space and $\pi: E \rightarrow X$ be a continuous euclidean vector bundle with $\|\cdot\|$. For K a compact subset, denote by $T: E|_K \rightarrow E|_{f(K)}$ a bundle automorphism covering a homeomorphism $f: X \rightarrow X$. Suppose T has continuous splittings $E|_K = E_1 \oplus F_1$ and $E|_{f(K)} = E_2 \oplus F_2$ such that there exists $0 < \lambda < 1$ satisfying

- (i) $T(E_1(x)) = E_2(f(x))$, $T(F_1(x)) = F_2(f(x))$ for $x \in K$,
- (ii) $\|T|_{E_1(x)} \cdot \|T^{-1}|_{F_2(f(x))} \| \leq \lambda$ for $x \in K$.

We denote by $\Gamma(E_1)$ and $\Gamma(F_1)$ the set of bounded sections over E_1 and F_1 . Then $\Gamma(E_1)$ and $\Gamma(F_1)$ become Banach spaces under supremum norm $\|\cdot\|_1$. Put $\Gamma(E|_K) = \Gamma(E_1) \times \Gamma(F_1)$ and define a norm $\|(\cdot, \cdot)\|_1$ of $\Gamma(E|_K)$ by

$$\|(\xi, \eta)\|_1 = \max\{\|\xi\|_1, \|\eta\|_1\} \quad (\xi \in \Gamma(E_1), \eta \in \Gamma(F_1)).$$

Clearly $\Gamma(E|_K)$ is a Banach space. Let us define

$$\Sigma(F_1, E_1) = \left\{ s \mid \begin{array}{l} s: \Gamma(F_1) \rightarrow \Gamma(E_1) \text{ is a map satisfying} \\ \text{(i) } s(\xi)(x) = 0 \text{ when } \xi(x) = 0 \text{ for } \xi \in \Gamma(F_1) \text{ and } x \in K, \\ \text{(ii) } s(\xi)(x) = s(\eta)(x) \text{ when } \xi(x) = \eta(x) \text{ for } \xi, \eta \in \Gamma(F_1) \\ \text{and } x \in K, \end{array} \right\}$$

and put $\xi(K) = \{x \in K \mid \xi(x) \neq 0\}$ for $\xi \in \Gamma(F_1)$. Then a subset

$$\Sigma_0(K) = \{s \in \Sigma(F_1, E_1) \mid \|s\|_* < \infty\},$$

where

$$\|s\|_* = \sup_{\xi \in \Gamma(F_1)} \sup_{x \in \xi(K)} \frac{\|s(\xi)(x)\|}{\|\xi(x)\|},$$

is a Banach space under the norm $\|\cdot\|_*$. Moreover the subset $\Sigma_0^1(K)$ defined by

$$\Sigma_0^1(K) = \{s \in \Sigma_0(K) \mid L_{\text{fib}}(s) \leq 1\},$$

where $L_{\text{fib}}(s) = \sup_{x \in K} L_x(s)$ and $L_x(s) = \min\{L \geq 0 \mid \|s(\xi)(x) - s(\eta)(x)\| \leq L\|\xi(x) - \eta(x)\| \text{ for } \xi, \eta \in \Gamma(F_1)\}$, is a closed subset. Similarly $\Sigma_0^1(f(K))$ is defined for $\Gamma(E_2)$ and $\Gamma(F_2)$.

The following is very similar to the analogous result of [3] and [4].

GRAPH TRANSFORMATION THEOREM. *Under the notations and the assumptions as above, if $0 < \varepsilon_0 < \|T^{-1}\|^{-1}$ and if $G: E|_K \rightarrow E|_{f(K)}$ is a homeomorphism satisfying*

$$(i) \quad G(E(x)) = E(f(x)),$$

$$(ii) \quad G(0_x) = 0_{f(x)},$$

$$(iii) \quad L(G - T|E(x)) < \varepsilon_0,$$

then there exists a graph transformation $\Gamma_G: \Sigma_0^1(K) \rightarrow \Sigma_0^1(f(K))$ such that for $s \in \Sigma_0^1(K)$ $\xi \in \Gamma(F_2)$ and $x \in f(K)$,

$$\Gamma_G(s)(\xi)(x) = (G_2 \circ (1, s)) \circ (G_1 \circ (1, s))^{-1}(\xi(x)),$$

where $G(u) = (G_1(u), G_2(u)) \in E_2 \times F_2$, and $L(\Gamma_G) < 1$ is satisfied.

To prove Theorem 1, let c and λ be as above. Take $m_1 > 0$ satisfying $c\lambda^{m_1} < \lambda < 1$. Then it is easily checked that f^{m_1} has the dominated splitting $T_{\Lambda, f^{m_1}} M = E \oplus F$ satisfying $\|Df^{m_1}|E(x)\| \cdot \|Df^{-m_1}|F(f^{m_1}(x))\| \leq \lambda$ for all $x \in \Lambda_{f^{m_1}}$. To simplify we may suppose $m_1 = 1$ and give the proof of Theorem 1. Use here Theorem 2 (proved in the next section). Let V be as in Theorem 2 and take a neighborhood W of Λ_f such that $W \cup f(W) \subset V$. Then we have that for $x \in W$

$$D_x f = \begin{pmatrix} A_x & B_x \\ C_x & K_x \end{pmatrix} : \tilde{E}(x) \oplus \tilde{F}(x) \longrightarrow \tilde{E}(f(x)) \oplus \tilde{F}(f(x)).$$

For W sufficiently small, define a linear isomorphism $T_x: \tilde{E}(x) \oplus \tilde{F}(x) \rightarrow \tilde{E}(f(x)) \oplus \tilde{F}(f(x))$ by

$$T_x = \begin{pmatrix} A_x & 0 \\ 0 & K_x \end{pmatrix}$$

such that $\|A_x\| \cdot \|K_{f^{-1}(x)}\| \leq \lambda'$ for some $0 < \lambda < \lambda' < 1$, and define a bundle isomorphism $T: \tilde{E}_1 \oplus \tilde{F}_1 \rightarrow \tilde{E}_2 \oplus \tilde{F}_2$ by $T|_{\tilde{E}(x) \oplus \tilde{F}(x)} = T_x$ for $x \in W$ where $\tilde{E}_1 = \tilde{E}|_W$, $\tilde{F}_1 = \tilde{F}|_W$, $\tilde{E}_2 = \tilde{E}|_{f(W)}$ and $\tilde{F}_2 = \tilde{F}|_{f(W)}$. Obviously we have

$$\|T|\tilde{E}(x)\| \cdot \|T^{-1}|\tilde{F}(f(x))\| \leq \lambda' \quad (x \in W).$$

Let $0 < \varepsilon_0 < \|T^{-1}\|^{-1}$. If W is sufficiently small, then we have $L(T - Df|T_x M^*) < \varepsilon_0$ for all $x \in W$. By graph transformation theorem, we can define a graph transformation

$\Gamma_{Df}: \Sigma_0^1(W) \rightarrow \Sigma_0^1(f(W))$ satisfying $L(\Gamma_{Df}) < 1$.

Take U a neighborhood of Λ_f such that $U \subset f(W) \cap W$ and let a continuous map $\varphi: M_f \rightarrow [0, 1]$ satisfy

$$\varphi(x) = \begin{cases} 0 & \text{if } x \notin f(W) \\ 1 & \text{if } x \in U. \end{cases}$$

For $\xi \in \Gamma(\tilde{F}_1)$ define $\xi' \in \Gamma(\tilde{F}_2)$ by

$$\xi'(x) = \begin{cases} \xi(x) & \text{if } x \in W \cap f(W) \\ 0 & \text{if } x \in f(W) - W \end{cases}$$

and

$$H(s)(\xi)(x) = \begin{cases} 0 & \text{if } x \in f(W) - W \\ \varphi(x)\Gamma_{Df}(s)(\xi')(x) & \text{if } x \in W \cap f(W). \end{cases}$$

Then $H: \Sigma_0^1(W) \rightarrow \Sigma_0^1(W)$ is a contraction. This follows from the fact that

$$\begin{aligned} \|H(s) - H(s')\|_* &= \sup_{\xi \in \Gamma(\tilde{F}_1)} \sup_{x \in \xi(W)} \frac{\|H(s)(\xi)(x) - H(s')(\xi)(x)\|}{\|\xi(x)\|} \\ &= \sup_{\xi \in \Gamma(\tilde{F}_1)} \sup_{x \in \xi(W \cap f(W))} \frac{\|\varphi(x)\Gamma_{Df}(s)(\xi')(x) - \varphi(x)\Gamma_{Df}(s')(\xi')(x)\|}{\|\xi(x)\|} \\ &\leq \sup_{\xi \in \Gamma(\tilde{F}_1)} \sup_{x \in \xi(W \cap f(W))} \frac{\|\Gamma_{Df}(s)(\xi')(x) - \Gamma_{Df}(s')(\xi')(x)\|}{\|\xi(x)\|} \\ &\leq \sup_{\xi' \in \Gamma(\tilde{F}_2)} \sup_{x \in \xi'(f(W))} \frac{\|\Gamma_{Df}(s)(\xi')(x) - \Gamma_{Df}(s')(\xi')(x)\|}{\|\xi'(x)\|} \\ &= \|\Gamma_{Df}(s) - \Gamma_{Df}(s')\|_* \\ &\leq L(\Gamma_{Df})\|s - s'\|_* . \end{aligned}$$

Thus there exists a unique $s^* \in \Sigma_0^1(W)$ satisfying $H(s^*) = s^*$. Put $\bar{F}(x) = \text{graph}(s^*(\tilde{F}(x)))$ for $x \in U \subset W \cap f(W)$. Then by the definition of Γ_{Df} , we have that $\bar{F}(x)$ is a linear subspace of $T_x M^*$, and for $x \in U \cap f^{-1}(U)$

$$\begin{aligned} \text{graph}(s^*|\tilde{F}(f(x))) &= \text{graph}(H(s^*)|\tilde{F}(f(x))) \\ &= \text{graph}(\Gamma_{Df}(s^*)|\tilde{F}(f(x))) \\ &= Df(\text{graph}(s^*|\tilde{F}(x))) . \end{aligned}$$

Therefore $Df(\bar{F}(x)) = \bar{F}(f(x))$. The uniqueness of s^* guarantees that

- (i) $\bar{F}(x) = F(x)$ for $x \in \Lambda_f$,
- (ii) $\bar{F} = \bigcup_{x \in U} \bar{F}(x)$ is continuous.

By the same argument the continuous extension $\bar{E}$ of E is obtained.

Finally we check the part (c). Take and fix λ_1 with $\lambda < \lambda_1 < 1$ and choose $\varepsilon > 0$ such that $e^{2\varepsilon}\lambda \leq \lambda_1$. Since $\bar{E}$ and $\bar{F}$ are continuous, there exists $\delta > 0$ such that $\tilde{d}(x, y) < \delta$ ($x, y \in U$) implies

$$\|Df|_{\bar{E}(x)}\| \leq e^\varepsilon \|Df|_{\bar{E}(y)}\|, \quad \|Df^{-1}|_{\bar{F}(x)}\| \leq e^\varepsilon \|Df^{-1}|_{\bar{F}(y)}\|.$$

Since U is arbitrary, we can choose U such that $U \subset \{y \in M_f : \tilde{d}(A_f, y) \leq \delta'\}$ where $0 < \delta' \leq \delta$ is a number satisfying $\tilde{d}(f(x), f(y)) < \delta$ whenever $d(x, y) < \delta'$ (the neighborhood U is our requirement). Then, for $x \in U \cap f^{-1}(U)$ we have

$$\begin{aligned} \|Df|_{\bar{E}(x)}\| \cdot \|Df^{-1}|_{\bar{F}(f(x))}\| &\leq e^{2\varepsilon} \|Df|_{\bar{E}(y)}\| \cdot \|Df^{-1}|_{\bar{F}(f(y))}\| \\ &\leq e^{2\varepsilon} \lambda \leq \lambda_1 \end{aligned}$$

for $y \in A_f$ with $\tilde{d}(x, y) \leq \delta'$.

§2. Proof of Theorem 2.

Let c and $0 < \lambda < 1$ be as above. Take $m > 0$ such that $c\lambda^m = \lambda_0 < 1$. Let A_f be a compact invariant subset having the dominated splitting $T_{A_f}M^* = E \oplus F$. Since $E(x) = E(y)$ for $x, y \in A_f$ with $x_0 = y_0$, we can suppose, without loss of generality, that E is a continuous subbundle over A . Then, by Lemma 4.4 of [2], there exist an open neighborhood V of A and a continuous splitting $\tilde{E}$ over V such that $\tilde{E}(x) = E(x)$ for all $x \in A$.

Define an open neighborhood V of A_f by $V = (\cdots \times M \times M \times V \times M \times M \times \cdots) \cap M_f$ and put $\tilde{E}(x) = \tilde{E}(x_0)$. Then $\tilde{E} = \bigcup_{x \in V} \tilde{E}(x)$ is a continuous subbundle satisfying $\tilde{E}(x) = E(x)$ for $x \in A_f$. Take and fix $\delta > 0$, and put $V_0 = V \cap U_\delta(A_f)$ where $U_\delta(A_f) = \{y \in M_f \mid \tilde{d}(y, A_f) < \delta\}$.

Now let $\{V_n\}_{n \geq 0}$ be a collection of open neighborhoods of A_f such that

$$(2.1) \quad \begin{aligned} (i) \quad & f^{-m}(V_n) \cap V_n \cap f^m(V_n) \supset \bar{V}_{n+1} \quad (n \geq 0), \\ (ii) \quad & \bigcap_{n \geq 0} V_n = A_f. \end{aligned}$$

Then $f^{mi}(x) \in V_0$ for $-n \leq i \leq n$ when $x \in V_n$. Moreover, Urysohn's lemma guarantees that there exist continuous functions $\gamma_n: M_f \rightarrow [0, 1]$ satisfying

$$\gamma_n(x) = \begin{cases} 0 & \text{if } x \in M_f - V_n, \\ 1 & \text{if } x \in \bar{V}_{n+1}. \end{cases}$$

Let $\dim F = u$ and denote by $G^u(V_0)$ the Grassmann bundle over V_0 . By (2.1) we can find $n(x) \geq 0$ such that $x \in V_{n(x)}$ and $x \notin V_{n(x)+1}$ for $x \in V_0 - A_f$. Then a section $g: V_0 \rightarrow G^u(V_0)$ is defined by

$$g(x) = \begin{cases} g_0(x) & \text{if } x \in V_0 - V_1, \\ (1 - \gamma_{n(x)}(x))g_{n(x)-1}(x) + \gamma_{n(x)}(x)g_{n(x)}(x) & \text{if } x \in V_1 - A_f, \\ F(x) & \text{if } x \in A_f, \end{cases}$$

where

$$g_n(x) = D_{f^{-n}(x)} f^n(\tilde{E}(f^{-n}(x))^\perp) \quad \text{for } x \in V_n.$$

Here $\tilde{E}(y)^\perp$ denotes the orthocomplement of $\tilde{E}(y)$. Notice that for $x \in V_1 - A_f$

$$\begin{aligned} & (1 - \gamma_n(x))g_{n-1}(x) + \gamma_n(x)g_n(x) \\ &= \{v_1 + \gamma_n(x)v_2 \mid v \in g_n(x) \text{ and } v = v_1 + v_2 \in g_{n-1}(x) \oplus g_{n-1}(x)^\perp\}. \end{aligned}$$

For the proof of Theorem 2 it is enough to prove that g is continuous. In fact the continuity of $g|_{V_0 - A_f}$ and $g|_{A_f}$ are clear. Thus it only remains to see that for $\{x^n\} \subset V_0 - A_f$, if $x^n \rightarrow x \in A_f$ as $n \rightarrow \infty$ then

$$(2.2) \quad g(x^n) \rightarrow g(x).$$

For $x \in A_f$ let (φ, U) be a chart with $x_0 \in U$. To simplify we regard as $T_U M = \varphi(U) \times \mathbb{R}^{\dim M}$. Then we have that for $y \in U$ and $v \in D_y \varphi(T_y M)$

$$v = v_x^1 + v_x^2 \in D_{x_0} \varphi(E(x)) \oplus D_{x_0} \varphi(F(x)).$$

Thus (2.2) is equivalent to the following:

$$(2.3) \quad \begin{aligned} & \text{there exists a chart } (\varphi, U) \text{ with } x_0 \in U \text{ such that for } \varepsilon > 0, \\ & \text{there exists } N > 0 \text{ so that for } n \geq N \text{ and } v \in D_{x_0} \varphi(g(x^n)) \text{ with } |v| = 1, \\ & |v_x^1|/|v_x^2| < \varepsilon, \end{aligned}$$

where $|\cdot|$ denotes the euclidean norm of $\mathbb{R}^{\dim M}$. Therefore, to obtain the conclusion it is enough to see (2.3). For the proof of (2.3) we prepare some notations and results.

Take $\varepsilon_0 > 0$ such that $[(1 + \varepsilon_0)^2 / (1 - \varepsilon_0)^2] \lambda_0 = \lambda_1 < 1$. Then we can easily find an atlas $\{(\varphi_i, W_i) : 1 \leq i \leq l\}$ such that for $1 \leq i \leq l$, $x \in W_i$ and $v \in T_x M$

$$(1 - \varepsilon_0) |D_x \varphi_i(v)| \leq \|v\| \leq (1 + \varepsilon_0) |D_x \varphi_i(v)|$$

(see [5]). Let $\{U_i : 1 \leq i \leq l\}$ be an open cover of M such that $\bar{U}_i \subset W_i$ and denote by $\delta_0 > 0$ a Lebesgue number of $\{U_i\}$.

From the property of the dominated splitting $T_{A_f} M = E \oplus F$, we have

$$\frac{\|Df^m|E(x)\|}{m(Df^m|F(x))} \leq \lambda_0$$

for $x \in A_f$, where $m(Df)$ denotes the minimal norm of Df . When $x_0 \in U_i$ and $x_m \in U_j$ for $x \in M_f$, we write

$$D_{x_0}f_{ij}^m = D_{x_m}\varphi_i \circ D_{x_0}f^m \circ D_{x_0}\varphi_j.$$

Then we have

$$\frac{|D_{x_0}f_{ij}^m|D_{x_0}\varphi_i(E(x))|}{m(D_{x_0}f_{ij}^m|D_{x_0}\varphi_i(F(x)))} \leq \frac{(1+\varepsilon_0)^2}{(1-\varepsilon_0)^2} \lambda_0 = \lambda_1.$$

Therefore we have the following

LEMMA 2.1. *Let $0 < \lambda_1 < \lambda_2 < 1$. Then there exists $0 < \delta_1 < \delta_0$ such that for $x \in \Lambda_f$ and $y \in M_f$ with $d(x_0, y_0) \leq \delta_1$,*

- (i) *$d(x_m, y_m) \leq \delta_0$ and so there exist $1 \leq i, j \leq l$ so that $x_0, y_0 \in U_i$ and $x_m, y_m \in U_j$,*
- (ii)

$$\frac{|D_{y_0}f_{ij}^m|D_{x_0}\varphi_i(E(x))|}{m(D_{y_0}f_{ij}^m|D_{x_0}\varphi_i(F(x)))} \leq \lambda_2.$$

Let $\tilde{E} \subset T_V M$ be as above and $W \subset V$ be a closed neighborhood of Λ in M . Then the following is easily checked.

LEMMA 2.2. *Under the notations as above, there exists $0 < \delta_2 \leq \delta_0$ such that, if $d(x, y) \leq \delta_2$ ($x, y \in W$), then*

$$D_x\varphi_i(\tilde{E}(x)) \cap D_y\varphi_i(\tilde{E}(y)^\perp) = \{0\}$$

where $x, y \in U_i$.

LEMMA 2.3. *Let $\delta_2 > 0$ be as in Lemma 2.2. Then there exists $T_1 > 0$ such that for $x \in \Lambda_f$, $y \in W$ with $d(x_0, y) \leq \delta_2$, if $x_0, y \in U_i$ for some $U_i \in \{U_i\}$ and $v \in D_y\varphi_i(\tilde{E}(y)^\perp)$ with $|v| = 1$, then*

$$\frac{|v_x^1|}{|v_x^2|} < T_1.$$

PROOF. If this is false, for $n > 0$ there exist $x^n \in \Lambda_f$, $y^n \in W$ satisfying $d(x_0^n, y^n) \leq \delta_2$ and $x_0^n, y^n \in U_{i_n}$, and $v(n) \in D_{y^n}\varphi_{i_n}(\tilde{E}(y^n)^\perp)$ satisfying $|v(n)| = 1$, such that

$$\frac{|v(n)_{x^n}^1|}{|v(n)_{x^n}^2|} > n.$$

For simplicity write $i = i_n$ and suppose that $x^n \rightarrow x \in \Lambda_f$, $y^n \rightarrow y \in W$ and $v(n) \rightarrow v \in D_y\varphi_i(\tilde{E}(y)^\perp)$. Since $1 + |v(n)_{x^n}^2| \geq |v(n)_{x^n}^1| > n|v(n)_{x^n}^2|$, we have $v_{x^n}^2 = 0$. Thus $v \in D_{x_0}\varphi_i(E(x)) = D_{x_0}\varphi_i(E(x_0))$. By Lemma 2.2, $D_{x_0}\varphi_i(E(x_0)) \cap D_y\varphi_i(\tilde{E}(y)^\perp) = \{0\}$ and so $v = 0$. This is a contradiction since $|v| = 1$.

LEMMA 2.4. For $\varepsilon > 0$ there exists $0 < \delta_3 < \delta_0$ such that, for $x \in \Lambda_f$ and $y \in M$ satisfying $d(x_0, y) \leq \delta_3$, if $x_0, y \in U_i$ and $x_m, f^m(y) \in U_j$, then

(i) for $v_x^1 \in D_{x_0} \varphi_i(E(x))$ with $|v_x^1| = 1$

$$|D_y f_{ij}^m(v_x^1) - (D_y f_{ij}^m(v_x^1))_{f^m(x)}^1| = |(D_y f_{ij}^m(v_x^1))_{f^m(x)}^2| \leq \varepsilon,$$

(ii) for $v_x^2 \in D_{x_0} \varphi_i(F(x))$ with $|v_x^2| = 1$

$$|D_y f_{ij}^m(v_x^2) - (D_y f_{ij}^m(v_x^2))_{f^m(x)}^2| = |(D_y f_{ij}^m(v_x^2))_{f^m(x)}^1| \leq \varepsilon.$$

PROOF. If (i) is false, then there exists $\varepsilon > 0$ such that for $n > 0$ there exist $x^n \in \Lambda_f$, $y^n \in M$ satisfying $d(x_0^n, y^n) < 1/n$ and $x_0^n, y^n \in U_{i_n}$, and $v_{x^n}^1 \in D_{x_0^n} \varphi_{i_n}(E(x^n))$ satisfying $|v_{x^n}^1| = 1$, such that, letting $x_m^n, f^m(y^n) \in U_{j_n}$,

$$|D_{y^n} f_{i_n j_n}^m(v_{x^n}^1) - (D_{y^n} f_{i_n j_n}^m(v_{x^n}^1))_{f^m(x^n)}^1| \geq \varepsilon.$$

To simplify we may suppose that $x^n \rightarrow x$ and $v_{x^n}^1 \rightarrow v_x^1$ and put $i_n = i$ and $j_n = j$. Since $y^n \rightarrow x_0$, we have

$$|D_{x_0} f_{ij}^m(v_x^1) - (D_{x_0} f_{ij}^m(v_x^1))_{f^m(x)}^1| \geq \varepsilon.$$

But $D_{x_0} f_{ij}^m(v_x^1) = (D_{x_0} f_{ij}^m(v_x^1))_{f^m(x)}^1$ because E is Df -invariant, thus contradicting. Similary (ii) is proved.

The following lemmas are easily proved and we omit the proof.

LEMMA 2.5. For every $\varepsilon > 0$ there exist $0 < \delta_4 < \delta_0$ and $r_1 > 0$ such that, if $\tilde{d}(x, y) \leq \delta_4$ ($x, y \in \Lambda_f$) and if $v \in \mathbb{R}^{\dim M}$ satisfying $|v| = 1$ and $|v_y^1| \leq r_1$, then

$$\frac{|v_x^1|}{|v_x^2|} < \varepsilon.$$

LEMMA 2.6. For $K > 1$ and $0 < r_2, r_3 < 1$ there exists $0 < \delta_5 < \delta_0$ such that, if $\tilde{d}(x, y) \leq \delta_5$ ($x, y \in \Lambda_f$) and if $v \in \mathbb{R}^{\dim M}$ satisfying $|v| = 1$, $|v_y^2| \geq r_2$ and $|v_y^1| \geq r_3$, then

$$\frac{|v_x^1|}{|v_x^2|} \leq K \frac{|v_y^1|}{|v_y^2|}.$$

PROOF OF THEOREM 2. Let $\delta_1 > 0$ be as in Lemma 2.1 and $\{U_i\}$ be as above. Take $0 < \delta_2 \leq \delta_1$ and $T_1 > 0$ as in Lemma 2.3. By Lemma 2.1, for $\lambda_2 < \lambda_3 < 1$ and $0 < T_2 < T_1/2$ we see that there exists $\varepsilon_1 > 0$ such that for $x \in \Lambda_f$ and $y \in M$ with $d(x_0, y) \leq \delta_2$, if $x_0, y \in U_i$ and $x_m, f^m(y) \in U_j$ for some U_i and U_j , then

$$(i) \quad \frac{|Df^m|D\varphi(E(x))| + (1 + 1/T_2)\varepsilon_1}{m(Df^m|D\varphi(F(x))) - (1 + T_1)\varepsilon_1} \leq \lambda_3.$$

$$(ii) \quad \frac{\varepsilon_1}{m(Df^m|D\varphi(F(x))) - (1 + T_1)\varepsilon_1} \leq \lambda_3 T_2,$$

where $Df^m = D_y f_{i,j}^m$ and $D\varphi = D_{x_0} \varphi_i$. Take $0 < \delta_3 < \delta_2$ as in Lemma 2.4 for ε_1 . Then we have the following

CLAIM 1. For $x \in A_f$, $y \in M$ satisfying $d(x_0, y) \leq \delta_3$ and $x_0, y \in U_i$, and for $v \in D_y \varphi_i(T_y M)$ with $|v_x^1|/|v_x^2| \leq T_1$, the following inequality holds:

$$\frac{|(Df^m(v))_{f^m(x)}^1|}{|(Df^m(v))_{f^m(x)}^2|} \leq \lambda_3 T_1.$$

PROOF. By Lemma 2.4 we have

$$\begin{aligned} & |(Df^m(v))_{f^m(x)}^2| \\ &= |(Df^m(v_x^2))_{f^m(x)}^2 + (Df^m(v_x^1))_{f^m(x)}^2| \\ &\geq |Df^m(v_x^2)| - |Df^m(v_x^2) - (Df^m(v_x^2))_{f^m(x)}^2| - |(Df^m(v_x^1))_{f^m(x)}^2| \\ &\geq \{m(Df^m|D\varphi(F(x))) - \varepsilon_1\} |v_x^2| - \varepsilon_1 |v_x^1|. \end{aligned}$$

Similarly

$$|(Df^m(v))_{f^m(x)}^1| \leq \{|Df^m|D\varphi(E(x))| + \varepsilon_1\} |v_x^1| + \varepsilon_1 |v_x^2|.$$

If $T_2 \leq |v_x^1|/|v_x^2| \leq T_1$, then

$$\frac{|(Df^m(v))_{f^m(x)}^1|}{|(Df^m(v))_{f^m(x)}^2|} \leq \frac{\{|Df^m|D\varphi(E(x))| + (1 + \varepsilon_1/T_2)\} |v_x^1|}{\{m(Df^m|D\varphi(F(x))) - (1 + T_1)\varepsilon_1\} |v_x^2|} \leq \lambda_3 T_1.$$

If $|v_x^1|/|v_x^2| < T_2$, then

$$\begin{aligned} \frac{|(Df^m(v))_{f^m(x)}^1|}{|(Df^m(v))_{f^m(x)}^2|} &< \frac{\{|Df^m|D\varphi(E(x))| + \varepsilon_1\} |v_x^1|}{\{m(Df^m|D\varphi(F(x))) - (1 + T_2)\varepsilon_1\} |v_x^2|} \\ &\quad + \frac{\varepsilon_1}{\{m(Df^m|D\varphi(F(x))) - (1 + T_2)\varepsilon_1\}} \\ &< \lambda_3 \cdot \frac{|v_x^1|}{|v_x^2|} + \lambda_3 T_2 < 2\lambda_3 T_2 < \lambda_3 T_1. \end{aligned}$$

Therefore Claim 1 is concluded.

Take $0 < \delta_4 < \delta_3$ and $r_1 > 0$ as in Lemma 2.5 for T_1 , and $0 < \delta_5 < \delta_4$ as in Lemma 2.6 under the conditions

$$1 < K < \frac{1}{\lambda_3}, \quad r_2 = \frac{1}{1 + T_1} \quad \text{and} \quad r_3 = r_1.$$

Moreover choose $0 < \delta < \delta_5/2$ such that

- (a) $U_\delta(A) \subset W$,
- (b) if $\tilde{d}(x, y) \leq \delta$ then $\tilde{d}(f^m(x), f^m(y)) \leq \delta_5/2$.

Put $V_0 = V \cap U_\delta(A_f)$ and as above define a section $g: V_0 \rightarrow G^u(V_0)$.

CLAIM 2. For $y \in V_0 - A_f$, $0 \leq n \leq n(y)$, $x \in A_f$ with $\tilde{d}(f^{-(n(y)-n)m}(y), x) \leq \delta$ ($y(n)$, $x_0 \in U_i$) and $v \in D_{y(n)}\varphi_i(D_{y(0)}f^{nm}(\tilde{E}(f^{-n(y)m}(y))^\perp)$ with $|v|=1$, the following holds:

$$\frac{|v_x^1|}{|v_x^2|} \leq T_1,$$

where $y(n) = y_{-(n(y)-n)m}$ ($0 \leq n \leq n(y)$).

PROOF. Let $x^0 \in A_f$ with $\tilde{d}(f^{-n(y)m}(y), x^0) \leq \delta$. If $x_0^0, y(0) \in U_i$, then we have by Lemma 2.3 that, for $v \in D_{y(0)}\varphi_i(\tilde{E}(f^{-n(y)m}(y))^\perp)$ with $|v|=1$,

$$\frac{|v_{x^0}^1|}{|v_{x^0}^2|} \leq T_1,$$

and by Claim 1

$$(2.4) \quad \frac{|(D_{y(0)}f_{ij}^m(v))_{f^m(x^0)}^1|}{|(D_{y(0)}f_{ij}^m(v))_{f^m(x^0)}^2|} \leq \lambda_3 T_1$$

where $y(1)$, $x_m^0 \in U_j$. For $x^1 \in A_f$ with $\tilde{d}(f^{-(n(y)-1)m}(y), x^1) \leq \delta$ ($\leq \delta_5/2$), we have

$$\tilde{d}(f^m(x^0), x^1) \leq \delta_5 < \delta_4$$

since $\tilde{d}(f^{-(n(y)-1)m}(y), f^m(x^0)) < \delta_5/2$. When $|(D_{y(0)}f_{ij}^m(v))_{f^m(x^0)}^1| \leq r_1$, then by Lemma 2.5

$$\frac{|(D_{y(0)}f_{ij}^m(v))_{x^1}^1|}{|(D_{y(0)}f_{ij}^m(v))_{x^1}^2|} < T_1.$$

When $|(D_{y(0)}f_{ij}^m(v))_{f^m(x^0)}^1| > r_1$, by (2.4)

$$|(D_{y(0)}f_{ij}^m(v))_{f^m(x^0)}^2| \geq \frac{1}{1 + \lambda_3 T_1} > \frac{1}{1 + T_1}$$

and so by Lemma 2.6

$$\begin{aligned} \frac{|(D_{y(0)}f_{ij}^m(v))_{x^1}^1|}{|(D_{y(0)}f_{ij}^m(v))_{x^1}^2|} &\leq K \cdot \frac{|(D_{y(0)}f_{ij}^m(v))_{f^m(x^0)}^1|}{|(D_{y(0)}f_{ij}^m(v))_{f^m(x^0)}^2|} \\ &\leq K\lambda_3 T_1 < T_1. \end{aligned}$$

Therefore we obtain the conclusion for $n=1$. By using induction step Claim 2 is concluded.

Let $\{x^n\} \subset V_0 - A_f$ be a sequence converging to $x \in A_f$. For $\varepsilon > 0$ we can find $\varepsilon'_1 > 0$ such that, for $x \in A_f$, $y \in M$ with $d(x_0, y) < \delta_2$ and for $1 \leq i, j \leq l$ satisfying $x_0, y \in U_i$ and $x_m, f^m(y) \in U_j$,

$$(i) \quad \frac{|Df^m|D\varphi(E(x))| + (1 + 2/\varepsilon)\varepsilon'_1}{m(Df^m|D\varphi(F(x))) - (1 + T_1)\varepsilon'_1} \leq \lambda_3,$$

$$(ii) \quad \frac{\varepsilon'_1}{m(Df^m|D\varphi(F(x))) - (1 + \varepsilon/2)} < \varepsilon/2,$$

where $Df^m = D_x f^m_{ij}$ and $D\varphi = D_{x_0} \varphi_i$. Let $\delta'_3 < \delta_2$ be as in Lemma 2.4 for $\varepsilon = \varepsilon'_1$. Then, as in Claim 2 we have that for $x \in A_f$, $y \in M$ with $d(x_0, y) \leq \delta'_3$, if $x_0, y \in U_i$, $x_m, f^m(y) \in U_j$ for some U_i and U_j and $v \in D_y \varphi_i(T_y M)$ with $|v^1_x|/|v^2_x| \leq T_1$, then

$$(2.5) \quad (i) \quad \frac{|(Df^m(v))_{f^m(x)}^1|}{|(Df^m(v))_{f^m(x)}^2|} \leq \lambda_3 \frac{|v^1_x|}{|v^2_x|} \quad \text{if} \quad \frac{\varepsilon}{2} \leq \frac{|v^1_x|}{|v^2_x|} \leq T_1,$$

$$(ii) \quad \frac{|(Df^m(v))_{f^m(x)}^1|}{|(Df^m(v))_{f^m(x)}^2|} \leq \varepsilon \quad \text{if} \quad \frac{|v^1_x|}{|v^2_x|} < \frac{\varepsilon}{2}.$$

Take $k > 0$ such that $\lambda_3^k T_1 < \varepsilon$, and take $N > 0$ such that for $n \geq N$

$$\tilde{d}(f^{-l}(x), f^{-l}(x^n)) \leq \delta'_3 \quad (0 \leq l \leq km) \quad \text{and} \quad n(x^n) \geq k.$$

Then, by Claim 2 we have that, for a fixed $n \geq N$ and $v(0) \in D_{\bar{x}_0} \varphi_{i_0}(D_{\bar{x}_0} f^{(n(x^n)-k)m}(\tilde{E}(f^{-n(x^n)m}(x^n))^\perp))$ with $|v(0)| = 1$,

$$\frac{|v(0)_{f^{-km}(x)}^1|}{|v(0)_{f^{-km}(x)}^2|} \leq T_1$$

where $\bar{x}_0 = x_{-km}^n$, $\tilde{x}_0 = x_{-n(x^n)m}^n$ and $\bar{x}_0, x_{-km} \in U_{i_0}$, and by (2.5)

$$\frac{|(D_{\bar{x}_0} f^m_{i_0 i_1}(v(0)))_{f^{-(k-1)m}(x)}^1|}{|(D_{\bar{x}_0} f^m_{i_0 i_1}(v(0)))_{f^{-(k-1)m}(x)}^2|} \leq \max\{\lambda_3 T_1, \varepsilon\}.$$

Put

$$v(1) = \frac{D_{\bar{x}_0} f^m_{i_0 i_1}(v(0))}{|D_{\bar{x}_0} f^m_{i_0 i_1}(v(0))|}.$$

Then we have

$$v(1) \in D_{\bar{x}_1} \varphi_{i_1}(D_{\bar{x}_1} f^{(n(x^n)-k+1)m}(\tilde{E}(f^{-n(x^n)m}(x^n))^\perp)) \quad \text{and} \quad |v(1)| = 1,$$

where $\bar{x}_1 = x_{-(k-1)m}^n$, $\tilde{x}_1 = \tilde{x}_0$ and $\bar{x}_1, x_{-(k-1)m} \in U_{i_1}$. By (2.5)

$$\frac{|(D_{\bar{x}_1} f^m_{i_1 i_2}(v(1)))_{f^{-(k-2)m}(x)}^1|}{|(D_{\bar{x}_1} f^m_{i_1 i_2}(v(1)))_{f^{-(k-2)m}(x)}^2|} \leq \max\{\lambda_3^2 T_1, \varepsilon\},$$

and inductively

$$\frac{|(Df^{km}(v(0)))_{\bar{x}}^1|}{|(Df^{km}(v(0)))_{\bar{x}}^1|} \leq \max\{\lambda_3^k T_1, \varepsilon\} = \varepsilon$$

where $Df^{km} = D_{\bar{x}_0} f_{i_0 i_k}^{km}$ and $x_0^n, x_0 \in U_{i_k}$. Notice that

$$Df^{km}(D_{\bar{x}_0} f^{(n(x^n)-k)m}(\tilde{E}(f^{-n(x^n)m}(x^n))^\perp)) = D_{\bar{x}_0} f^{n(x^n)m}(\tilde{E}(f^{-n(x^n)m}(x^n))^\perp).$$

Then $g_{n(x)}(x)$ is continuous. Similarly we can prove that $g_{n(x)-1}(x)$ is continuous. The continuity of g was proved.

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