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On the Triviality Index of Knots

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§1. Introduction.

In [6] the first author derived a new numerical invariant, denoted by $O(K)$, of knots from their diagrams and showed that if the Conway polynomial of a knot K is not one, then $O(K)$ is finite ([6] Corollary 2.4). In this paper, we call $O(K)$ the triviality index of K. It arises a problem as to whether or not there exists a knot K such that $O(K)=n$ for any natural number n.

In this paper, we show the following theorems.

THEOREM A. *If a knot K has a $2n$-trivial diagram ($n>1$), the coefficient of z^{2n} of the Conway polynomial of K is even.*

THEOREM B. *For any natural number n with $n>1$, there exist infinitely many knots K's with $O(K)=n$.*

Moreover in the case $O(K)=3$ we show the following.

THEOREM C. *Let $f(z)=1+\sum_{i=2}^{l} a_{2i}z^{2i}$, where a_{2i} ($2\leqq i\leqq l$) are integers. If a_4 is odd, there is a knot K such that $O(K)=3$ and the Conway polynomial of K is $f(z)$.*

Throughout this paper, we work in PL-category and refer to Burde and Zieschang [1] and Rolfsen [8] for the standard definitions and results of knots and links.

§2. Definitions and facts.

The Conway polynomial $\nabla_L(z)$ ([2]) and the Jones polynomial $V_L(t)$ ([3]) are invariants of the isotopy type of an oriented knot or link in a 3-sphere S^3. The Conway polynomial is defined by the following formulas:

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$$\nabla_U(z)=1 \quad \text{for the trivial knot } U\,,$$
$$\nabla_{L_+}-\nabla_{L_-}=z\nabla_{L_0}\,.$$

And the Jones polynomial is defined by the followings:

$$V_U(t)=1 \quad \text{for the trivial knot } U\,,$$
$$t^{-1}V_{L_-}(t)-tV_{L_+}(t)=(t^{1/2}-t^{-1/2})V_{L_0}(t)\,,$$

where L_+, L_- and L_0 are identical except near one point where they are as in Fig. 2-1.

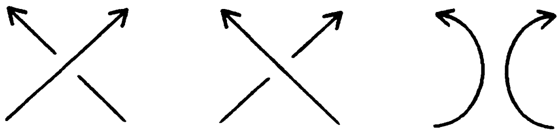

FIGURE 2-1

We defined the following number in [6].

NOTATION. Let L be a link, and $\tilde{L}$ a diagram of L with the set of crossing points $D(\tilde{L})=\{c_1, c_2, \cdots, c_n\}$. For a subset $D=\{c_{k_1}, c_{k_2}, \cdots, c_{k_m}\}$ of $D(\tilde{L})$, we denote by $\tilde{L}_D$ the diagram obtained from $\tilde{L}$ by changing the crossing at all points of D.

DEFINITION. Let K be a knot and $\tilde{K}$ a diagram of K with the set of crossing points $D(\tilde{K})$. Let $A_1, A_2, \cdots, A_n$ be nonempty subsets of $D(\tilde{K})$ with $A_i \cap A_j=\emptyset$ for $i\neq j$. For any nonempty subfamily $\mathscr{A}=\{A_{j_1}, A_{j_2}, \cdots, A_{j_l}\}$ of $\{A_1, A_2, \cdots, A_n\}$, we denote the set $A_{j_1}\cup A_{j_2}\cup\cdots\cup A_{j_l}$ by $\mathscr{A}$ for convenience. We say that $\tilde{K}$ is an n-trivial diagram of K with respect to $\{A_1, A_2, \cdots, A_n\}$ if for any nonempty (not necessarily proper) subfamily $\mathscr{A}$ of $\{A_1, A_2, \cdots, A_n\}$, $\tilde{K}_{\mathscr{A}}$ is a diagram of the trivial knot.

If a knot K has an n-trivial diagram and has no $(n+1)$-trivial diagrams, we denote the number n by $O(K)$, and call it the triviality index of K. If a knot K has an n-trivial diagram for any natural number n, we define $O(K)=\infty$.

In our notation, Lemma 2 of Yamamoto [9] is stated as follows.

PROPOSITION. *For any knot K, $O(K)\geqq 2$.*

In [6], we showed the following theorem and corollary.

THEOREM 2. *If a knot K has an n-trivial diagram, then the Conway*

polynomial $\nabla_K(z)$ *of* K *is of the following form;*
(1) *if* n *is odd, then*

$$\nabla_K(z)=1+a_{n+1}z^{n+1}+a_{n+3}z^{n+3}+\cdots,$$

and
(2) *if* n *is even, then*

$$\nabla_K(z)=1+a_n z^n+a_{n+2}z^{n+2}+\cdots.$$

COROLLARY. *If the Conway polynomial of* K *is not one, then* $O(K)$ *is finite.*

Theorem 2 gives an upper bound of $O(K)$ for a knot K, but it makes no difference between the knot K with $O(K)=2m-1$ and the knot K' with $O(K')=2m$. It arises a problem as to whether or not there exists a knot K with $O(K)=n$ for any natural number n with $n>1$.

At first we show Theorem A to distinguish between the knot K with $O(K)=2m-1$ and the knot K' with $O(K')=2m$.

§ 3. Proof of Theorem A.

Step 1. We define the following model. Let K be a knot, $\tilde{K}$ a diagram of K, and $\hat{K}$ the projection of K associated to $\tilde{K}$, i.e. $\hat{K}$ has no information of over and under crossings. And let $C=\{c_1, c_2, \cdots, c_{2n}\}$ be a subset of the set of crossing points $D(\tilde{K})$. Since $\hat{K}$ is a knot projection, there is an immersion f of S^1 in R^2 such that $f(S^1)=\hat{K}$. By c_i, we denote also a point of $\hat{K}$ associated to c_i of $\tilde{K}$. Let $f^{-1}(c_i)=\{d_i, d_i'\}$ and $S^1=\sigma=\partial D^2$. We have the model σ as shown in Fig. 3-1.

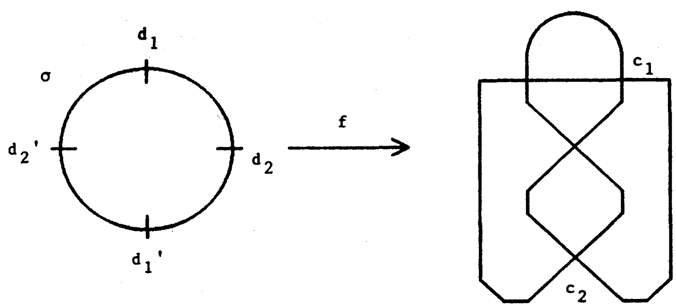

FIGURE 3-1

Let δ_i, δ_i' be regular neighborhoods of d_i, d_i' in σ and mutually disjoint ($1\leq i\leq 2n$). Let B_i be a band and $\partial B_i=\alpha_i\cup\alpha_i'\cup\beta_i\cup\beta_i'$ as shown in Fig. 3-2.

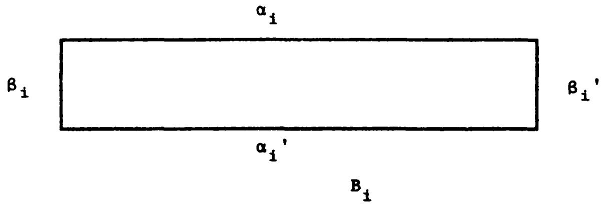

FIGURE 3-2

We make B_i full twisted and attach β_i and β_i' to δ_i and δ_i' in D^2, then we have an orientable surface $S=(\cup_{i=1}^{2n} B_i)\cup D^2$ as shown in Fig. 3-3.

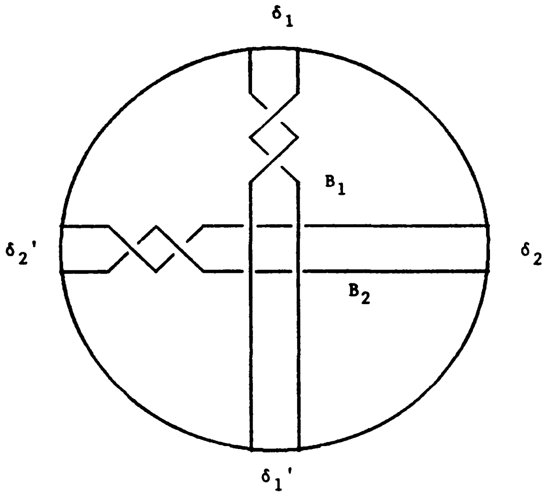

FIGURE 3-3

And let $\partial S=L$. Then L is a link or a knot. We call L a band model of $\tilde{K}$ with respect to C. Let $\tilde{L}$ be a diagram of L and a_i one of two crossing points of the boundary of the full-twisted band B_i in $\tilde{L}$. For any subset $C'=\{x_1, x_2, \cdots, x_q\}$ of C, we denote the link diagram and also link type obtained from $\tilde{K}$ smoothing at the points of C' by $\tilde{K}(C')$ or $\tilde{K}(x_1, x_2, \cdots, x_q)$ and denote the number of components of the link L by μL. Then we have Proposition 3.1.

PROPOSITION 3.1. *Let $M=\{1, 2, \cdots, 2n\}$ and N be a subset of M. For a knot K and the band model L of $\tilde{K}$ with respect to $C=\{c_1, c_2, \cdots, c_{2n}\}$, we have*

$$\mu\tilde{L}(\{a_i \mid i\in M-N\})=\mu\tilde{K}(\{c_i \mid i\in N\}) .$$

Step 2. For a set X, we denote the number of elements of X by $\# X$. Let $\tilde{K}$ be a knot diagram with the set of crossing points $D(\tilde{K})$, and $C=\{c_1, c_2, \cdots, c_{2n}\}$ a subset of $D(\tilde{K})$. We show the following lemma.

LEMMA 3.2. *Let* $M=\{1, 2, \cdots, 2n\}$, $\nu=\{\{M_1, M_2, \cdots, M_n\} \mid M_i \subset M, \#M_i=2\ (i=1, 2, \cdots, n), \cup_{i=1}^{n} M_i=M\}$. *And let* κ_C *be a subset of* ν *such that for any* i $(1 \leqq i \leqq n)$ $\mu K(\{\{c_j, c_{j'}\} \mid M_i=\{j, j'\}\})=1$, *then we have that* $\mu\tilde{K}(C)=1$ *if and only if* $\#\kappa_C$ *is odd.*

PROOF. We prove Lemma 3.2 by the induction on n. In the case $n=1$, $C=\{c_1, c_2\}$, $M=\{1, 2\}$ and $\nu=\{\{M\}\}$. If $\tilde{K}(C)$ is a knot, we have $\#\kappa_C=1$ since $\kappa_C=\{\{M\}\}$. If $\tilde{K}(C)$ is a link, $\#\kappa_C=0$ since $\kappa_C=\emptyset$. Then we have Lemma 3.2.

Let $n>1$ and C' be a subset of C where $\#C'=2m$ $(n>m)$. It is supposed that $\mu\tilde{K}(C')=1$ if and only if $\#\kappa_{C'}$ is odd. We consider the band model L of K with respect to C as defined in Step 1. Let B_i be an outermost band in $B_1, B_2, \cdots, B_{2n}$, namely when we separate σ into two parts σ_1, σ_2 where $\sigma_1 \cup \sigma_2=\sigma$, $\sigma_1 \cap \sigma_2=\{d_i, d'_i\}$, and one of σ_i $(i=1, 2)$ does not contain both d_j and d'_j for any j $(j \neq i,\ j=1, 2, \cdots, 2n)$. Let σ_1 be a part of σ satisfying the above condition as shown in Fig. 3-4.

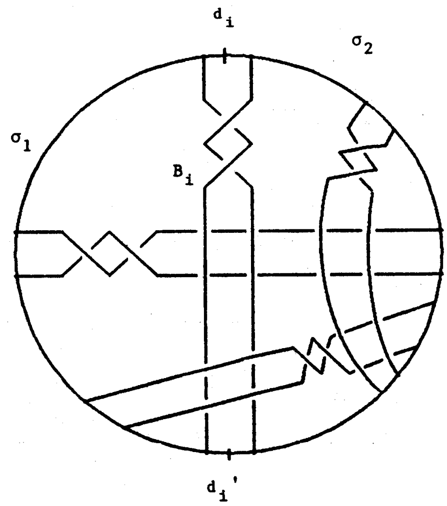

FIGURE 3-4

Let $N=\{j \in M \mid$ there is d_j or d'_j on the $\sigma_1\}$. Since $\mu\tilde{K}(c_i, c_k)=3$ for $k \in M-N-\{i\}$ by Proposition 3.1, any element of κ_C has $\{i, j\}$ $(j \in N)$ as an element. Let $C(j)=\{c_q\}$ $(q \in M-\{i, j\})$, then we have

$$\#\kappa_C=\sum_{j \in N} \#\kappa_{C(j)} . \tag{3.1}$$

By the hypothesis of induction, we have $\mu\tilde{K}(C(j))=1$ if and only if $\#\kappa_{C(j)}$ is odd. Then we show the relation between $\#\kappa_C$ and $\mu\tilde{K}(C)$ by considering $\mu\tilde{K}(C(j))$ and $\#N$. By Proposition 3.1, we have $\mu\tilde{K}(C(j))=\mu\tilde{L}(a_i, a_j)$. We consider two cases on the number of components of $\tilde{L}(a_i)$. We note that, since $\mu\tilde{L}(a_i)=\mu\tilde{K}(\{c_k \mid k\in M-\{i\}\})$, $\mu\tilde{L}(a_i)$ is even.

Case 1. $\mu\tilde{L}(a_i)\geqq 4$. Since $\mu\tilde{L}(a_i, a_j)\geqq 3$ for any $j\in N$, we have $\mu\tilde{K}(C(j))\geqq 3$. By the hypothesis of induction, $\#\kappa_{C(j)}$ is even. By (3.1), we have $\#\kappa_C$ is even. And since $\mu\tilde{L}\geqq 3$, we have $\mu\tilde{K}(C)\geqq 3$. Therefore we have that $\tilde{K}(C)$ is a link and $\#\kappa_C$ is even.

Case 2. $\mu\tilde{L}(a_i)=2$. Let $N'=\{j\in N \mid \alpha_j$ and α'_j are contained in different components on $\tilde{L}(a_i)\}$. We have by (3.1)

$$\begin{aligned}\#\kappa_C&=\sum_{j\in N}\#\kappa_{C(j)}\\&=\sum_{j\in N'}\#\kappa_{C(j)}+\sum_{j\in N-N'}\#\kappa_{C(j)}\,.\end{aligned}\tag{3.2}$$

Since $\mu\tilde{L}(a_i, a_j)=1$ for any $j\in N'$, we have $\mu\tilde{K}(C(j))=1$ and by the hypothesis of induction $\#\kappa_{C(j)}$ is odd. Since $\mu\tilde{L}(a_i, a_j)\geqq 3$ for any $j\in N-N'$, we have $\mu\tilde{K}(C(j))\geqq 3$ and $\#\kappa_{C(j)}$ is even. Therefore we have by (3.2)

$$\begin{aligned}\#\kappa_C&\equiv\sum_{j\in N'}1+\sum_{j\in N-N'}0\\&\equiv\#N'\qquad(\mathrm{mod}\,2)\,.\end{aligned}\tag{3.3}$$

In the case $\tilde{K}(C)$ is a knot, considering there is two points d_i, d'_i on the $\tilde{L}(a_i)$, d_i and d'_i are contained in different components of $\tilde{L}(a_i)$. Moreover for $j\in N'$, α_j and α'_j ($\alpha_j, \alpha'_j\in\partial B_j$) are contained in different components of $\tilde{L}(a_i)$. Therefore $\#N'$ is odd when $\mu\tilde{K}(C)=1$. In the same way when $\tilde{K}(C)$ is a link and $\mu\tilde{L}=3$, d_i and d'_i are contained in the same component in $\tilde{L}(a_i)$. Therefore we have $\#N'$ is even when $\tilde{K}(C)$ is a link. By (3.3), we have when $\tilde{K}(C)$ is a knot $\#\kappa_C$ is odd, and when $\tilde{K}(C)$ is a link $\#\kappa_C$ is even.

By Case 1 and Case 2, we have that when $\tilde{K}(C)$ is a knot $\#\kappa_C$ is odd, and when $\tilde{K}(C)$ is a link $\#\kappa_C$ is even. This completes the proof of Lemma 3.2.

Step 3. In this Step, we complete the proof of Theorem A by making use of Lemma 3.2 and the following Lemma 3.3.

Let $\tilde{K}$ be an n-trivial diagram of K with respect to $\{A_1, A_2, \cdots, A_n\}$.

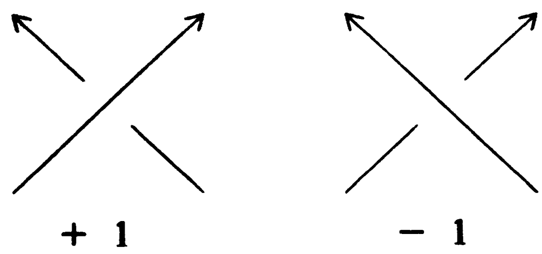

FIGURE 3-5

Let $A_i=\{c_{i1}, c_{i2}, \cdots, c_{i\,\alpha(i)}\}$, and ε_{ij} the sign of c_{ij} defined as shown in Fig. 3.5 ($i=1, 2, \cdots, n$).

By $K\begin{pmatrix}1 & 2 \cdots k \\ i_1 & i_2 \cdots i_k\end{pmatrix}$, we denote the link which is obtained from K by changing the crossing at $c_{11}, c_{12}, \cdots, c_{1\,i_1-1}, c_{21}, c_{22}, \cdots, c_{2\,i_2-1}, \cdots, c_{k1}, c_{k2}, \cdots, c_{k\,i_k-1}$ and smoothing at $c_{1\,i_1}, c_{2\,i_2}, \cdots, c_{k\,i_k}$. In [6], we showed the following lemma.

LEMMA 3.3. *If a knot K has an n-trivial diagram with respect to $\{A_1, A_2, \cdots, A_n\}$, then the Conway polynomial $\nabla_K(z)$ of K is of the following form.*

$$\nabla_K(z)=1+z^n \sum_{\substack{1\leqq i_j\leqq \alpha(j)\\ j=1,2,\cdots,n}} \varepsilon_{1\,i_1}\varepsilon_{2\,i_2} \cdots \varepsilon_{n\,i_n}\nabla_{K\left(\begin{smallmatrix}1 & 2 & \cdots & n\\ i_1 & i_2 & \cdots & i_n\end{smallmatrix}\right)}(z) . \tag{3.4}$$

Let $\tilde{K}$ be a $2n$-trivial diagram with respect to $\{A_1, A_2, \cdots, A_{2n}\}$ of K, and $A_i=\{c_{i1}, c_{i2}, \cdots, c_{i\,\alpha(i)}\}$ ($i=1, 2, \cdots, 2n$). We note that $\tilde{K}$ is a 2-trivial diagram with respect to $\{A_j, A_k\}$ for any j, k ($j<k$, $j, k=1, 2, \cdots, 2n$). Let a_{2m} be the coefficient of z^{2m} of Conway polynomial of K ($m=1, 2, \cdots$), then we have by Lemma 3.3

$$\begin{aligned} a_2 &\equiv \#\left\{K\begin{pmatrix}j & k\\ i_j & i_k\end{pmatrix} \;\middle|\; \mu K\begin{pmatrix}j & k\\ i_j & i_k\end{pmatrix}=1,\ 1\leqq i_j\leqq \alpha(j),\ 1\leqq i_k\leqq \alpha(k)\right\} \\ &\equiv \#\{\tilde{K}(c_{i_j}, c_{i_k}) \mid \mu\tilde{K}(c_{i_j}, c_{i_k})=1,\ 1\leqq i_j\leqq\alpha(j),\ 1\leqq i_k\leqq \alpha(k)\} \qquad (\mathrm{mod}\ 2) . \end{aligned}$$

Therefore we have

$$a_2\equiv \#\{(d_j, d_k)\in A_j\times A_k \mid \mu\tilde{K}(d_j, d_k)=1\} \qquad (\mathrm{mod}\ 2) .$$

Since $\tilde{K}$ is a $2n$-trivial diagram ($n>1$), we have $a_2=0$, then we have for any j, k ($j<k$, $j, k=1, 2, \cdots, 2n$)

$$\#\{(d_j, d_k)\in A_j\times A_k \mid \mu\tilde{K}(d_j, d_k)=1\}\equiv 0 \qquad (\mathrm{mod}\ 2) . \tag{3.5}$$

Similarly, we have by Lemma 3.3

$$\begin{aligned} a_{2n} &\equiv \#\left\{K\begin{pmatrix}1 & 2\cdots 2n\\ i_1 & i_2\cdots i_{2n}\end{pmatrix} \;\middle|\; \mu K\begin{pmatrix}1 & 2\cdots 2n\\ i_1 & i_2\cdots i_{2n}\end{pmatrix}=1 ,\right. \\ &\qquad\qquad \left. 1\leqq i_j\leqq\alpha(j),\ j=1, 2, \cdots, 2n\right\} \\ &\equiv \#\{\tilde{K}(c_{i_1}, c_{i_2}, \cdots, c_{i_{2n}}) \mid \mu\tilde{K}(c_{i_1}, c_{i_2}, \cdots, c_{i_{2n}})=1 , \\ &\qquad\qquad 1\leqq i_j\leqq \alpha(j),\ j=1, 2, \cdots, 2n\} \qquad (\mathrm{mod}\ 2) . \end{aligned}$$

Then we have

$$(3.6)\qquad a_{2n}\equiv \#\{(d_1, d_2, \cdots, d_{2n})\in A_1\times A_2\times\cdots\times A_{2n} \mid \mu\tilde{K}(d_1, d_2, \cdots, d_{2n})=1\} \pmod 2 .$$

By Lemma 3.2, we have $\mu\tilde{K}(d_1, d_2, \cdots, d_{2n})=1$ if and only if $\#\kappa_{\{d_1,d_2,\cdots,d_{2n}\}}$ is odd for $\{d_1, d_2, \cdots, d_{2n}\}$. Therefore we have

$$a_{2n}\equiv \#\{(d_1, d_2, \cdots, d_{2n})\in A_1\times A_2\times\cdots\times A_{2n} \mid \mu\tilde{K}(d_1, d_2, \cdots, d_{2n})=1\}$$
$$\equiv \sum_{(d_1,d_2,\cdots,d_{2n})\in A_1\times A_2\times\cdots\times A_{2n}} \#\kappa_{\{d_1,d_2,\cdots,d_{2n}\}} \pmod 2 .$$

Let $M_i=\{m(i), m'(i)\}$ $(1\leqq i\leqq n)$, then we have

$$(3.7)\qquad a_{2n}\equiv \sum_{(d_1,d_2,\cdots,d_{2n})\in A_1\times A_2\times\cdots\times A_{2n}} \#\{(M_1, M_2, \cdots, M_n) \mid \mu K(d_{m(i)}, d_{m'(i)})=1,\ 1\leqq i\leqq n\}$$
$$\equiv \sum_{\{\nu_1,\nu_2,\cdots,\nu_n\}\in\nu} \#\{(d_{m(1)}, d_{m'(1)}, d_{m(2)}, \cdots, d_{m(n)}, d_{m'(n)})\in A_{m(1)}\times A_{m'(1)}\times \cdots\times A_{m'(n)} \mid \mu\tilde{K}(d_{m(i)}, d_{m'(i)})=1, 1\leqq i\leqq n\} \pmod 2 .$$

We fix one of $\{M_1, M_2, \cdots, M_n\}\in\nu$, then

$$(3.8)\qquad \#\{(d_{m(1)}, d_{m'(1)}, d_{m(2)}, \cdots, d_{m(n)}, d_{m'(n)})\in A_{m(1)}\times A_{m'(1)}\times \cdots\times A_{m'(n)} \mid \mu\tilde{K}(d_{m(i)}, d_{m'(i)})=1,\ 1\leqq i\leqq n\}$$
$$=\prod_{i=1}^{n} \#\{(d_{m(i)}, d_{m'(i)})\in A_{m(i)}\times A_{m'(i)} \mid \mu\tilde{K}(d_{m(i)}, d_{m'(i)})=1\} .$$

By (3.5), we have

$$(3.9)\qquad \prod_{i=1}^{n} \#\{(d_{m(i)}, d_{m'(i)})\in A_{m(i)}\times A_{m'(i)} \mid \mu\tilde{K}(d_{m(i)}, d_{m'(i)})=1\} \equiv 0 \pmod 2 .$$

By (3.7), (3.8) and (3.9), we have

$$a_{2n}\equiv 0 \pmod 2 .$$

This completes the proof of Theorem A.

§4. Proof of Theorem B.

The knot K_n in Fig. 4-1 has an n-trivial diagram ([6]). It is not hard to see that it is an alternating knot. The Conway polynomial of the knot K_n in Fig. 4-1 is of the following form:

If $n=2m$ $(m\geqq 1)$, $\nabla_{K_n}(z)=1-2z^{2m}+\cdots$.

If $n=2m-1$ $(m\geqq 2)$, $\nabla_{K_n}(z)=1-(2m-1)z^{2m}+\cdots$.

By Theorem 2, if a knot K has a $2m$-trivial diagram and $a_{2m}\neq 0$,

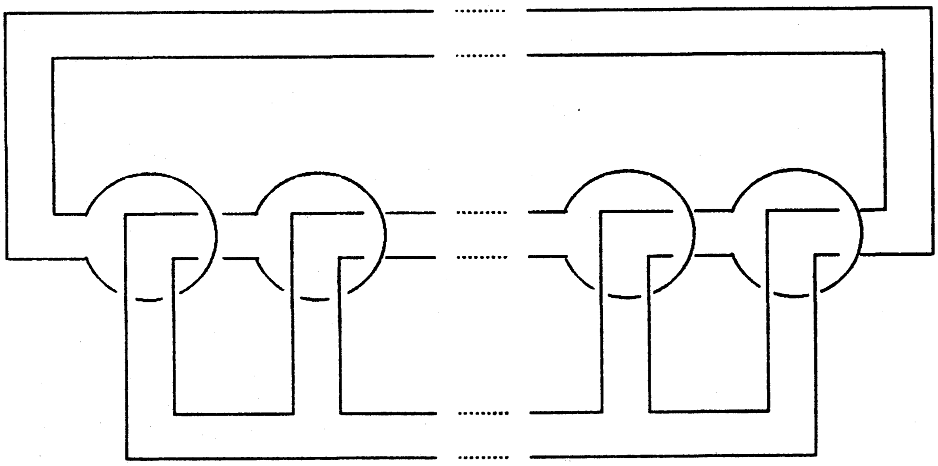

FIGURE 4-1

$O(K)=2m$. And by Theorem A, if K has a $(2m-1)$-trivial diagram and a_{2m} is odd, $O(K)=2m-1$. Therefore we have

$$O(K_n)=n \qquad (n\geqq 2) .$$

Let K_n^l be the knot as in Fig. 4-2, where the rectangle labelled l stands for a 2-string integral tangle with l full twists as shown in Fig. 4-3. Since the Conway polynomial of K_n^l is the same as that of K_n, and K_n^l has an n-trivial diagram, we have

$$O(K_n^l)=n \qquad (n\geqq 2) .$$

The relation between the Jones polynomial of K_n^l, $V_{K_n^l}(t)$, and that of K_n, $V_{K_n}(t)$, is calculated as follows in [4]:

$$V_{K_n^l}(t)=(t^2-1)(V_{K_n}(t)-1)\sum_{i=0}^{l-1} t^{2i}+V_{K_n}(t) .$$

The knot K_n is an alternating knot and the minimal crossing number of K_n is $3n$ by Murasugi [5]. And the reduced degree of $V_K(t)$ is equal to the minimal crossing number of K for an alternating knot K ([5]). Then we have

$$V_{K_n}(t)\neq 1 .$$

Therefore we have for l and l' $(l<l')$

$$V_{K_n^l}(t)\neq V_{K_n^{l'}}(t) .$$

This completes the proof of Theorem B.

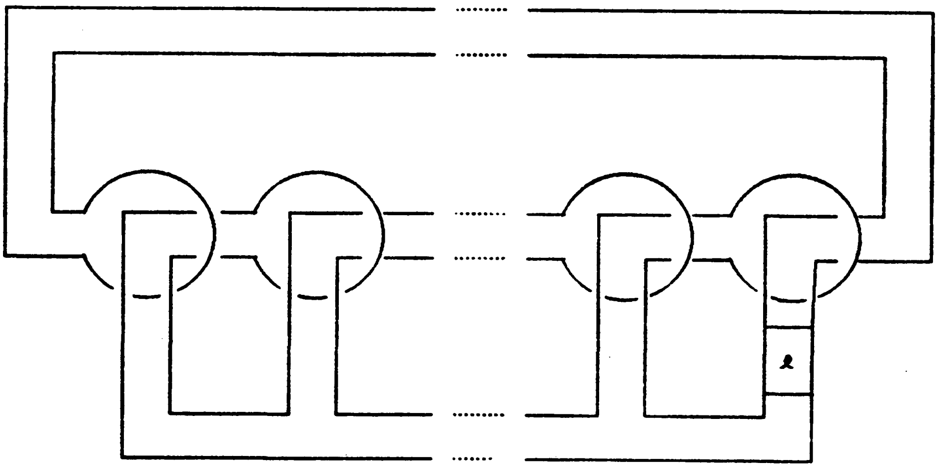

FIGURE 4-2

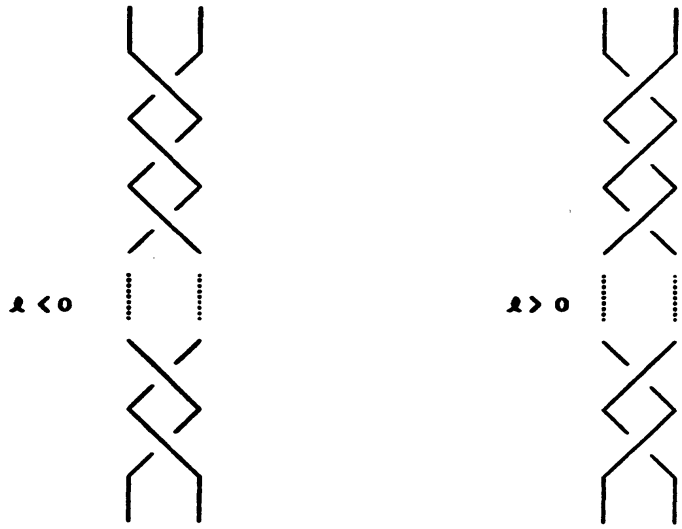

FIGURE 4-3

§ 5. Proof of Theorem C.

We consider the knot $K_{p_1,p_2,\cdots,p_l}$ as shown in Fig. 5-1 ([7]). By rectangle labelled p_i $(i=1, 2, \cdots, l)$, we denote the integral 2-string tangle as shown in Fig. 4-3. The Conway polynomial $\nabla_{K_{p_1,p_2,\cdots,p_l}}$ of $K_{p_1,p_2,\cdots,p_l}$ is the following:

$$\begin{aligned}\nabla_{K_{p_1,p_2,\cdots,p_l}} &= 1+\sum_{i=1}^{l}(-1)^{i-1}p_{l+1-i}z^{2i}\\ &= 1+p_l z^2-p_{l-1}z^4+\cdots+(-1)^{l-1}p_1 z^{2l}\,.\end{aligned}$$

Let $p_l=0$ and p_{l-1} be an odd integer, then we have

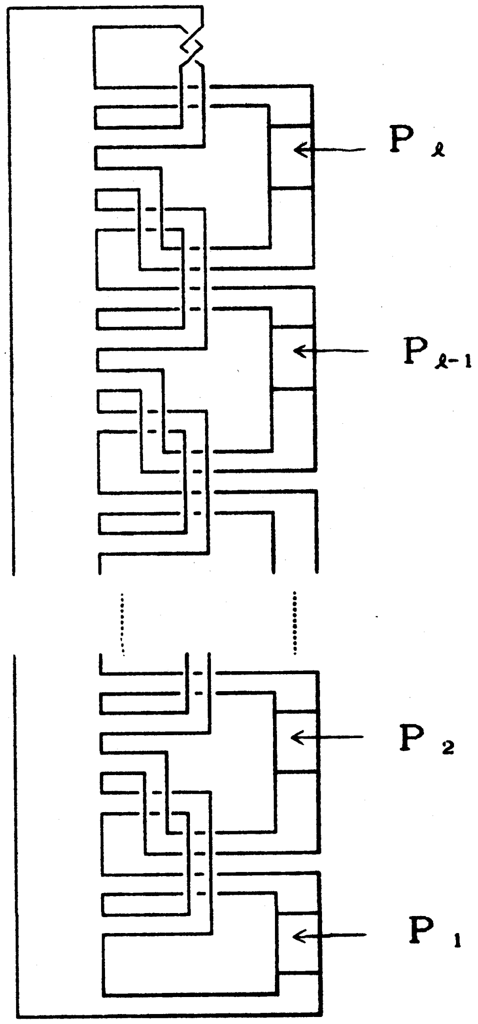

FIGURE 5-1

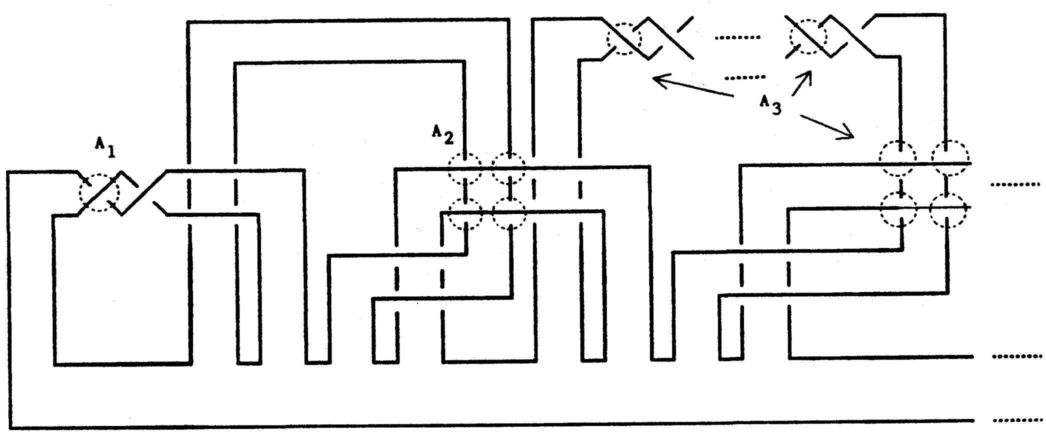

FIGURE 5-2

$$\nabla_{K_{p_1,p_2,\cdots,0}}=1-p_{l-1}z^4+p_{l-2}z^6+\cdots+(-1)^{l-1}p_1z^{2l}.$$

Let $-p_{l-1}=a_4$ and $(-1)^{i-1}p_{l+1-i}=a_{2i}$ $(i=3, 4, \cdots, l)$. Therefore we have

$$\nabla_{K_{p_1,p_2,\cdots,0}}=f(z)\ .$$

And $K_{p_1,p_2,\cdots,0}$ has a 3-trivial diagram with respect to $\{A_1, A_2, A_3\}$ as shown in Fig. 5-2. This completes the proof of Theorem C.

REMARK. For prime knots whose minimal crossing numbers are less than or equal to 9, the triviality indices of them are 2 except for the following knots; $O(8_2)=2$, or 3. $O(8_{14})=3$, or 4. $O(8_{21})=3$. $O(9_8)=2$, 3, or 4. $O(9_{25})=2$, or 3. $O(9_{26})=O(9_{27})=O(9_{41})=O(9_{44})=3$.

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